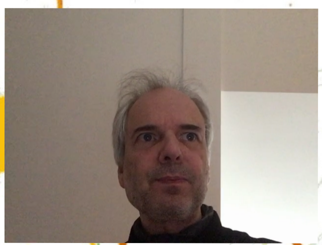


TIME SERIES ANALYSIS & RECURRENT NEURAL NETWORKS #7

- Poisson state space models
- Intro into nonlinear dynamics

<u>Main lecture:</u> **Daniel Durstewitz**

<u>Exercises:</u> **Leonard Bereska, Manuel Brenner, Daniel Kramer, Georgia Koppe**

Heidelberg University

Poisson state space models

$c_t = (c_{1t} \dots c_{Nt})^T, \quad t = 1 \dots T$

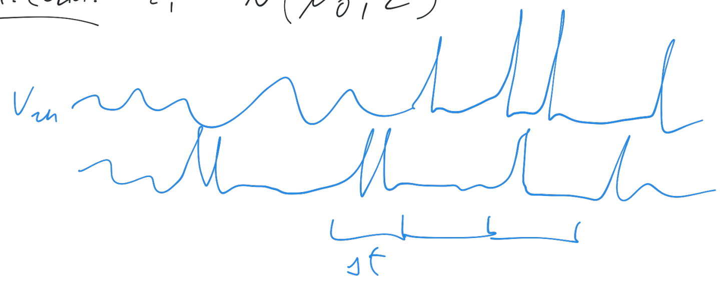
$c_{nt} \in \mathbb{N}$

obs. model: $c_{nt} \mid z_t \sim \text{Poisson}(\mu_{nt}) = \frac{\mu_{nt}^{c_{nt}}}{c_{nt}!} e^{-\mu_{nt}}$

$$\log \mu_{nt} = b_0^{(n)} + \underset{1 \times M}{b_1^{(n)}} \underset{M \times 1}{z_t} \rightarrow \overset{\text{elem.-wise}}{\log} \underset{\begin{pmatrix} \mu_{1t} \\ \vdots \\ \mu_{Nt} \end{pmatrix}}{\mu_t} = b_0 + \underset{N \times M \text{ mtx.}}{B} z_t$$

latent DS: $z_t = A z_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \Sigma)$

ini. cond.: $z_1 \sim N(\mu_0, \Sigma)$



EM-algo. for Poisson-linear st. sp. model

M-step: max $E_{q(z|x)}[\log p_\theta(z, C)]$ $\quad C = \{c_t\}$

w.r.t. $\theta = \{B, b_0, A, \mu_0, \Sigma\}$

$$E_q[\log p_\theta(C|z) + \log p_\theta(z)]$$

$$= E_q\left[-\frac{M}{2}\log 2\pi - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(z_1 - \mu_0)^T \Sigma^{-1}(z_1 - \mu_0)\right] +$$

ini. cond.

$$E_q\left[-\frac{M(T-1)}{2}\log 2\pi - \frac{T-1}{2}\log|\Sigma| - \frac{1}{2}\sum_{t=2}^{T}(z_t - A z_{t-1})^T \Sigma^{-1}(z_t - A z_{t-1})\right] +$$

trans. model

$$E_q\left[\sum_{t=1}^{T}\sum_{n=1}^{N}\log\left(\frac{\mu_{nt}^{c_{nt}}}{c_{nt}!} e^{-\mu_{nt}}\right)\right]$$

obs. model

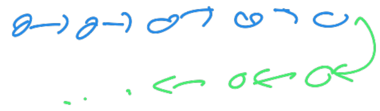
$$E_q[\bullet] = E_q\left[\sum^T\sum^N\left(c_{nt}\log\mu_{nt} - \cancel{\log c_{nt}!} - \mu_{nt}\right)\right]$$

$$= E_q\left[\sum_t\sum_n c_{nt}\left(b_0^{(n)} + b_1^{(n)} z_t\right) - e^{b_0^{(n)} + b_1^{(n)} z_t}\right] + \text{const.}$$

we required: $E[z_t], E[z_t z_t^T], E[z_t z_{t-1}^T]$

$$= \sum_t \sum_n \left[c_{nt} b_0^{(n)} + c_{nt} b_1^{(n)} E[z_t] - e^{b_0^{(n)}} E\left[e^{b_1^{(n)} z_t}\right]\right]$$

E-step: $p(z|c)$



Kalman filter Poisson

trans. distr.

$$p(z_t|c_{1:t}) = \frac{p(c_t|z_t)\int p(z_t|z_{t-1})\,p(z_{t-1}|c_{1:t-1})\,dz_{t-1}}{p(c_t|c_{1:t-1})}$$

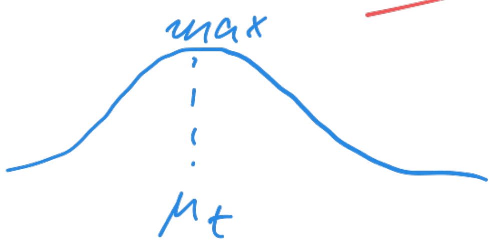
assume: $p(z_t|c_{1:t}) \approx N(\mu_t, V_t)$

$$\Rightarrow \int p(z_t|z_{t-1})\,p(z_{t-1}|c_{1:t-1})\,dz_{t-1}$$
$$= N(A\mu_{t-1}, C_{t-1})$$

$$\begin{pmatrix} \frac{\partial^2}{\partial z_1^2} & \frac{\partial^2}{\partial z_1 \partial z_2} & \cdots & \frac{\partial^2}{\partial z_1 \partial z_n} \\ \vdots & & & \vdots \\ \frac{\partial^2}{\partial z_n \partial z_1} & \cdots & \cdots & \frac{\partial^2}{\partial z_n^2} \end{pmatrix}$$

Hessian

$$\log\left\{\frac{\mathrm{Poisson}(c_t|z_t)\,N(A\mu_{t-1}, C_{t-1})}{\cancel{p(\ldots)}}\right\} =: Q(z)$$



$$\frac{\partial^2 \log f(z)}{\partial z\,\partial z^T} = \frac{\partial^2}{\partial z\,\partial z^T}\left(-\frac{1}{2}(z-\mu)^T\Sigma^{-1}(z-\mu)\right)$$
$$= -\Sigma^{-1}$$

$E[z] = \arg\max_z f(z)$

if f is Gaussian

max. $Q(z)$ w.r.t. z

$$\rightarrow \mu_t = z^{max}$$
$$V_t := \left[-\frac{\partial^2 Q}{\partial z\,\partial z^T}\right]^{-1}\Bigg|_{z^{max}}$$

$$\rightarrow p(z_t|c_1 \ldots c_t) \approx N(\mu_t, V_t)$$

- M-step: moment-gen. func. of Gaussian
  NR
- E-step: Gaussian approx.

Smith ... Brown (2003) Neural Comp.

Dynamical Systems Theory

recursive map $\underbrace{x_t = F_\theta(x_{t-1})}_{\text{discrete-time DS}} + \cancel{\varepsilon_t}$

$\dot{x} = f_\theta(x, t)$

contin. time DS

F was linear or affine: AR / VAR / LDS

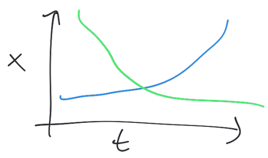
$x_t = \alpha x_{t-1} + c$, "determ. AR(1)"

ini. cond. x_0

$t \to \infty$:

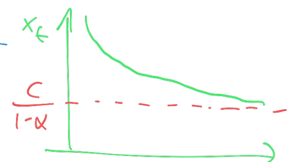
$x_t = \alpha(\alpha x_{t-2} + c) + c = \alpha(\alpha(\alpha x_{t-3} + c) + c) + c$

$= \ldots = \alpha^t x_0 + c \underbrace{\sum_{i=0}^{t-1} \alpha^i}$



$c \frac{1-\alpha^t}{1-\alpha}$ for $|\alpha| < 1$

$\lim_{t \to \infty} = \frac{c}{1-\alpha}$



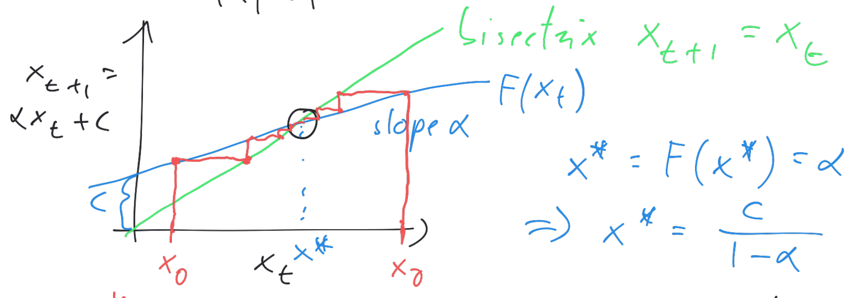
→ forget expon. fast about x_0, its past

1^{st}-order return map

$(x_t, x_{t+1}) = (x_t, F(x_t))$

p^{th}-order return map (x_t, x_{t+p})

$|\alpha| < 1$



"cobwebs"

$x^* = F(x^*) = \alpha x^* + c$

$\Rightarrow x^* = \frac{c}{1-\alpha}$ fixed point

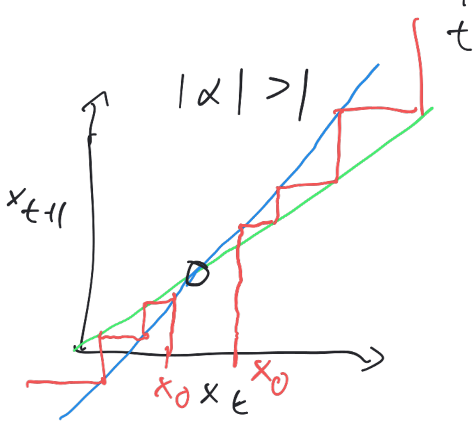
→ stable fixed point

$|\alpha| < 1$

"FP attractor" — "basin of attraction"

$x_0 \in C$

$\lim_{t \to \infty} x_t \to x^*$



FP $\frac{c}{1-\alpha}$

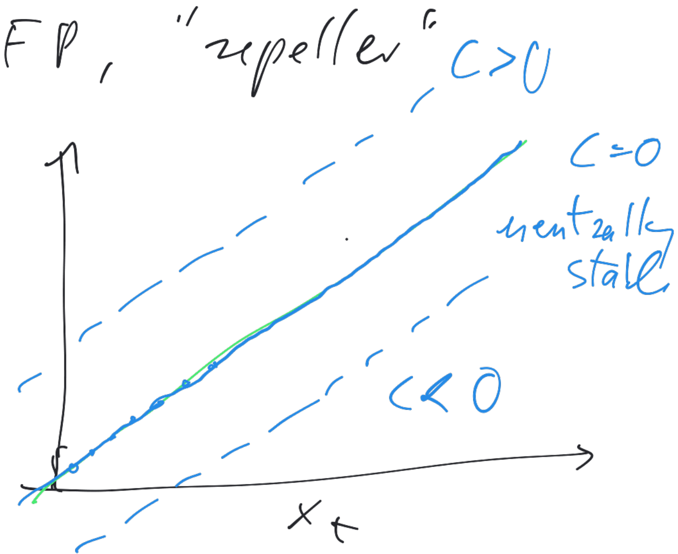
unstable FP, "repeller"

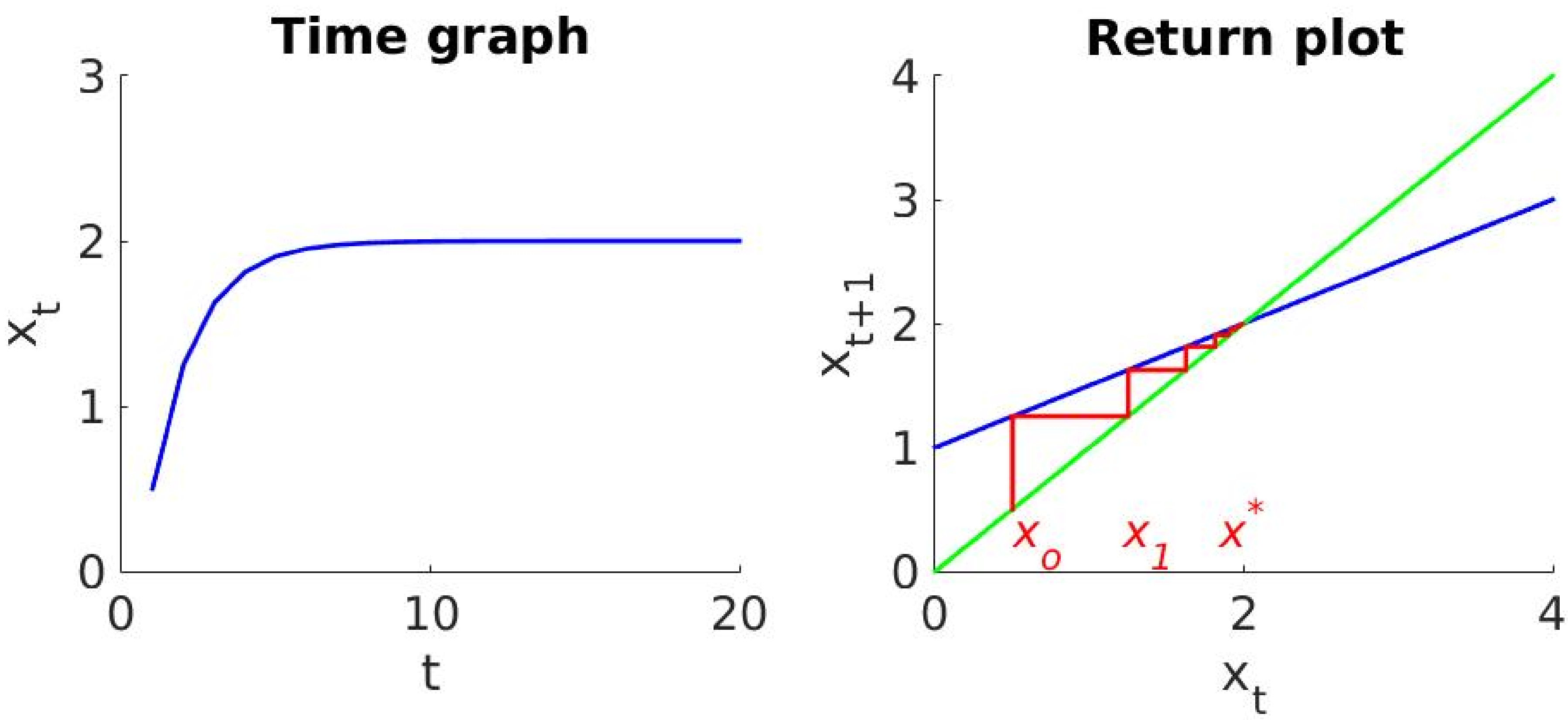
$\alpha = 1$

$x_{t+1} = x_t + c$

$x_{t+1} = x_t$

"line attractors"




Time graph
3
2
1
0
x_t
0
10
20
t
Return plot
4
3
2
1
0
x_{t+1}
x_o
x_1
x^*
0
2
4
x_t