



Solutions to Problem Set 10

Fundamentals of Simulation Methods 8 ECTS
Heidelberg University WiSe 20/21

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February 3, 2021

1 Isothermal 1D hydrodynamics system (7 pts)

In this problem, we study the Eulerian equations of hydrodynamics in one dimension, as per

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \quad (1)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial[\rho u^2 + P]}{\partial x} = 0, \quad (2)$$

where ρ, u and P are the density, velocity and pressure respectively. In order to close this system of equations, we assume that pressure is related to density as per $P = \rho c_s^2$, where c_s is constant. To analyse the stability and behaviour of this system, we introduce a small linear perturbation $\delta\rho, \delta u$ of the density and velocity to some steady solution ρ_0, u_0 as per

$$\rho(x, t) = \rho_0 + \delta\rho(x, t), \quad (3)$$

$$u(x, t) = u_0 + \delta u(x, t). \quad (4)$$

Plugging these perturbed states into Eq.(1-2) and only considering the first order, linear terms, and remembering that ρ_0, u_0 corresponds to a solution of Eq.(1-2), we arrive at

$$\partial_t(\delta\rho) + u_0\partial_x(\delta\rho) + \rho_0\partial_x(\delta u) = 0, \quad (5)$$

$$u_0\partial_t(\delta\rho) + \rho_0\partial_t(\delta u) + 2\rho_0u_0\partial_x(\delta u) + u_0^2\partial_x(\delta\rho) + \partial_x(\delta P) = 0, \quad (6)$$

where $\delta P = \delta\rho c_s^2$ is the perturbed pressure. If we substitute Eq.(5) into Eq.(6) we can simplify the system to

$$\partial_t(\delta\rho) + u_0\partial_x(\delta\rho) + \rho_0\partial_x(\delta u) = 0, \quad (7)$$

$$\rho_0\partial_t(\delta u) + \rho_0u_0\partial_x(\delta u) + \partial_x(\delta P) = 0. \quad (8)$$

By defining the material derivative, i.e. the Lagrangian derivative in a co-moving frame of reference, as per

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}, \quad (9)$$

good!
I think this way is elegant; alternatively you could explicitly continue first and then see it "appear" in equation as shown in tutorial

we can rewrite the system of equations in the co-moving frame of reference relative to the steady solution ρ_0, u_0 as

$$D_t(\delta\rho) + \rho_0\partial_x(\delta u) = 0, \quad (10)$$

$$\rho_0 D_t(\delta u) + \partial_x(\delta P) = 0. \quad (11)$$

If we then take the material derivative of Eq.(10) and the spatial derivative of Eq.(11), we can use the commutative property of partial derivatives and simplify the system of equations to

$$D_t^2(\delta\rho) - \partial_x^2(\delta P) = 0, \quad (12)$$

which also can be written as

$$D_t^2(\delta\rho) - c_s^2\partial_x^2(\delta\rho) = 0, \quad (13)$$

using our earlier assumption on the pressure, which specifies the perturbed pressure as $\delta P = \delta\rho c_s^2$. Since Eq.(13) corresponds to the well known wave equation, we can interpret this result as the description on how the density perturbation will propagate with velocities $\pm c_s$ in each direction in the steady state solution's frame of reference, or with velocities $u_0 \pm c_s$ in the inertial frame of reference. Thus, the constant c_s specifies the speed at which 'information' traverses the domain, which is equivalent to the isothermal sound speed of the medium.

If we do a Fourier decomposition of the density perturbation, we can handle each wave mode independently, since any arbitrary perturbation can be attained from a linear superposition of individual modes. It is thus sufficient to only consider a single mode of the type

$$\delta\rho(x, t) = Ae^{i(kx - \omega t)}, \quad (14)$$

which we can substitute into Eq.(13) to obtain

$$\left[[-i\omega + u_0 ik]^2 + c_s^2 k^2\right] Ae^{i(kx - \omega t)} = 0. \quad (15)$$

Since this equation has to hold true for all x and t , for a nontrivial solution where $A \neq 0$, the large parenthesis has to equal zero. Simplifying this expression further and solving for ω gives the dispersion relation

$$\omega = k(u_0 \pm c_s), \quad (16)$$

which is linear in k , as expected for planar waves.

Furthermore, we can write the system of equations of Eq.(1-2) in vector notation as per

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q})}{\partial x} = 0, \quad (17)$$

where we define the state vector \mathbf{q} as

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \quad (18)$$

and the corresponding flux functions \mathbf{F} as

$$\mathbf{F} = \begin{bmatrix} q_2 \\ \frac{q_2^2}{q_1} + q_1 c_s^2 \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + P \end{bmatrix}. \quad (19)$$

Taking the PDE in Eq.(17), which is written in conservative form, and transforming it into primitive form as per

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q})}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial x} = 0, \quad (20)$$

we can notice that the derivative of \mathbf{F} with respect to \mathbf{q} corresponds to the Jacobian \mathbf{J} of the dynamical system as per

$$\mathbf{J} = \frac{\partial \mathbf{F}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} 0 & 1 \\ \left(c_s^2 - \frac{q_2^2}{q_1^2}\right) & 2\frac{q_2}{q_1} \end{bmatrix}. \quad (21)$$

The characteristic polynomial of the matrix \mathbf{J} is

$$\det[\mathbf{J} - \lambda \mathbf{1}] = \lambda^2 - [\text{tr } \mathbf{J}] \lambda + \det \mathbf{J} = 0, \quad (22)$$

which can be solved as

$$\lambda_{\pm} = \frac{1}{2} \text{tr } \mathbf{J} \pm \frac{1}{2} \sqrt{[\text{tr } \mathbf{J}]^2 - 4 \det \mathbf{J}}, \quad (23)$$

to obtain the eigenvalues

$$\lambda_{\pm} = \frac{q_2}{q_1} \pm c_s = u \pm c_s, \quad (24)$$

which can be substituted back into the original eigenvalue problem

$$\mathbf{J} \mathbf{v} = \lambda \mathbf{v}, \quad (25)$$

to obtain the corresponding eigenvectors

$$\mathbf{v}_{\pm} = \begin{bmatrix} 1 \\ \lambda_{\pm} \end{bmatrix} = \begin{bmatrix} 1 \\ u \pm c_s \end{bmatrix}. \quad (26)$$

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Ex. 2 missing