handed out: May 26, 2020
handing in: June 4, 2020
presentation/discussion: June 5, 2020

1. **L63 System – Basics** present ☐

Implement a numerical solver for the L63 system

$$\begin{aligned}\dot{u}_1 &= \Pr[u_2 - u_1] \ , \\ \dot{u}_2 &= -u_1 u_3 + r\, u_1 - u_2 \ , \\ \dot{u}_3 &= u_1 u_2 - b\, u_3\end{aligned}$$

as given by (4.42) in the lecture notes.

(a) Check your implementation by reproducing Figure 4.33 from the lecture notes.

(b) Choose parameters (use Figure 4.28 for orientation) and initial states, and draw all simple projections, i.e., $\big(u_i(t), u_j(t)\big)$ with $\{i,j\} \in \{1,2,3\}$ and $i \neq j$.
Optional: Add the projections to a representation of the three-dimensional trajectory in analogy to Figure 4.32.

2. **L63 System – Ensemble** present ☐

Enhance your code such that it can propagate an ensemble of states. To achieve this, add an additional iteration that (i) sets the initial state, (ii) propagates it to the desired times, and (iii) processes the results.

Keeping the classical parameters $\Pr = 10$ and $b = \frac{8}{3}$ and recalling the transitions at $r_0 = 13.926$, $r_1 = 24.06$, and $r_c = 24.74$, choose some interesting value(s) of r to further study the L63 system with ensembles. Suggestions for initial ensembles are

- spheres of radius R at one of the fixpoints, hence, for $\mathbf{u}_+$, (u_1, u_2, u_3) with

$$[u_1 - \sqrt{b[r-1]}]^2 + [u_2 - \sqrt{b[r-1]}]^2 + [u_3 - r + 1]^2 < R^2 \ ,$$

- Disks in u_1u_2-plane at $u_1 = u_2 = 0$, $u_3 \in \{0, r-1, 2[r-1]\}$ and with radius R,
- or, instead of a full disk, just its boundary $(u_1, u_2) = \big(R\cos(2\pi k/N), R\sin(2\pi k/N)\big)$ with N points on the circumference.

To setup such an initial ensemble (i) choose a regular grid with resolution Δ to cover the circle/sphere and (ii) select only those states as initial points that lie within the circle/sphere.

(a) Plot the ensemble after some appropriate times (get orientation from Figure 4.33). Such plots are easiest and most quantitative to read as (u_i, u_j)-projections as done for task 1 b), nicer but much harder and less quantitative to read as a 3d graph, and nicest as a movie (if you are up to it). Use a computational precision of 10^{-8}.

(b) Without changing anything else, do the above with a computational precision of 10^{-12}. Discuss, illuminating the issue of deterministic chaos. (Depending on whether you chose the fixpoint $\mathbf{u}_+$ or the u_3-axis for your initial ensemble, you can highlight different aspects.)

3. **L63 System – Decomposition of State Space by Attracting Set** present ☐

For $r < r_1 \approx 24.06$, any initial state will eventually converge to one of the fixpoints $\mathbf{u}_0$, $\mathbf{u}_+$, or $\mathbf{u}_-$. Choose a two-dimensional section $\mathfrak{S}$ of the state space, choose three colors, one for each of the fixpoints, and mark the points $\mathbf{u} \in \mathfrak{S}$ by a small square in the color of the fixpoint that is approached by the trajectory starting at $\mathbf{u}$. Define "approached" if the distance to the fixpoint is smaller than 10^{-4}.

As a start, you may choose $\mathfrak{S}$ as $(u_1, u_2) \in [-a, a] \otimes [-a, a]$ and $u_3 = c \geq 0$ with, say $a = 30$ and $c = 0$. Explore the situation for your choice of r (probably no need to do a plot for $r < 1$ ☺). Specifically look at $(\Pr, b) = (10, \frac{8}{3})$ for which we know the transitions – $r_0 = 13.926$, $r_1 = 24.06$, and $r_c = 24.74$ – and discuss the results.

Hints/Comments:

- *This exercise consumes a lot of computational resources. Recall the advice on exercise sheet 4.*
- *The resulting graphics looks nice if you cover $\mathfrak{S}$ by a bitmap and fill each of its pixels with the corresponding color.*
- *The convergence time diverges as $r \to r_1$. Hence, set an upper limit for the simulation time – if it is reached, mark the pixel as white – and choose r sufficiently small.*
- *You might be tempted to look at the interval $r_1 < r < r_c$, where in addition to $\mathbf{u}_\pm$ also the strange attractor is stable, or to some of the windows with stable limit cycles. The challenge there is to detect when such a stable manifold has been reached and, if there is more than one, which one of them.*

4. **The Last Question** want to discuss ☐

   (a) What are the key messages you took home from the lecture?

   (b) What are still open questions?

   (c) What associated issues did you miss?