handed out: April 28, 2020
handing in: May 7, 2020
presentation/discussion: May 8, 2020

Preliminary Comment

Recall that the exercises are an offer – beans are given quite generously – and a playground for you to explore. Your solutions will not be scrutinized in great detail by the tutors. Hence, highlight whatever you judge as remarkable and be active in the exercise group.
Operationally, do not just submit a bunch of figures, but add some comments for every graph, need not be wordy. This helps to (i) focus your own attention while scanning your result for peculiarities and (ii) demonstrate that you realized the important issues.

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1. **Pedestrian's Solution of ODE System** present ☐

   Implement the *explicit Euler scheme* and demonstrate it by calculating trajectories for

   $$\begin{aligned}\dot{u}_1 &= u_2 - [1.8u_1]^2 \exp(-u_1) \\ \dot{u}_2 &= [0.7 + 0.5u_1]^{-2} - u_2\end{aligned}$$

   Two aspects are to be learned:

   (a) The key parameter in this numerical scheme is the step-size Δt. Explore the influence of the integration time step $\Delta\tau$ on $u(\tau)$. (It controls the stability and accuracy of the solution and the required computational effort.)

   (b) Drawing trajectories is often nontrivial and their choice can produce artifacts that lead to misinterpretations. For illustration, draw trajectories that start at regular distances on (i) the boundary of domain $\{u_1, u_2\} \in [0, 2] \times [0, 2]$ and (ii) on a circle with center (1,1) and radius 1.

   Further approaches [optional]: If you know the fixpoint and know that it is attractive, you may choose a circle with a small radius and centered at that fixpoint. Then solve the *time-reversed* system, which turns the attractive fixpoint into a repelling one.

   If you use a high-level tool, do not *employ its built-in ODE solver. You may want to use it for comparison.*

2. **Glycolysis Model** present ☐

   Using an ODE-solver of your high-level tool, or the Runge-Kutta Cash-Karp algorithm given in Appendix A, implement the numerical solution of the glycolysis model

   $$\begin{aligned}\dot{u}_1 &= -u_1 + u_2[\alpha + u_1^2] \\ \dot{u}_2 &= \beta - u_2[\alpha + u_1^2]\end{aligned}$$

   with system parameters α and β, and explore its flow by plotting an appropriate set of parametric curves $(u_1(t), u_2(t))$ together with the two nullclines. In addition, plot some typical time series, $u_1(t)$ and $u_2(t)$.

   Hint: This model is given as (2.22) in the lecture notes. The flow for one choice of parameters is shown in Fig. 2.3 and the stability diagram Fig. 2.5 offers some help for choosing the parameters. The aim of this exercise is to get better acquainted with your tools of choice.

3. **Three-Dimensional Dynamical System** present ☐

   Imagine a 3d-system with a fixpoint that is oscillatory and unstable.

   (a) The oscillation must involve a two-dimensional subspace. How is it determined?

   (b) What are the possibilities for the remaining subspace?

4. **Saddle Node Bifurcation** present ☐

   Study the system $\dot{u} = \mu - u^2$ (Lecture notes: (2.48) and Figures 2.12 and 2.13).

   (a) Choose some interesting values of μ (< 0, 0, > 0) and initial states $u(0)$ and, *before* running a numerical solver, qualitatively draw the corresponding trajectories $u(t)$.

   (b) Verify your drawings by calculating the trajectories (analytically or numerically).

5. **Globally Stable Subcritical Pitchfork Bifurcation** present ☐

   Study the system
   $$\dot{u} = f(u; \mu) = -u[[u^2 - 1]^2 - \mu - 1]$$
   (Lecture notes: (2.53) and Figure 2.19).

   (a) Determine the fixpoints u_0 of this system and their stability, both as functions of the system parameter μ.
   Hint: Use Mathematica, sympy, or similar to find $u_0 \in \left\{0, \pm\sqrt{1 \pm \sqrt{1+\mu}}\right\}$. *If you have no access to such a tool, use the result as given here (do not waste time by calculating by hand).*

   (b) Choose some interesting values of μ and initial states $u(0)$ and, *before* running a numerical solver, qualitatively draw the corresponding trajectories.

   (c) Verify your drawings by calculating the trajectories numerically.

6. **The Last Question**

   (a) What are the key messages you took home from the lecture?

   (b) What are still open questions?

   (c) What associated issues did you miss?