Problem Set 10

Exercises for the lecture Fundamentals of Simulation Methods, WS 2020

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Hand in until Wednesday, 03.02, 23:59
Tutorials on Friday, 05.02
(Group 1, Giovanni Leidi 09:15)
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(Group 3, Siddhant Deshmukh 11:15)
(Group 4, Fabian Kutzki 14:15)

1. Isothermal 1D hydrodynamics system [7 pt.]

In this exercise you are going to write and use an Eulerian finite-volume code (based on the Godunov method) to solve the 1D equations of isothermal hydrodynamics:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \, , \tag{1}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial \left[\rho u^2 + P\right]}{\partial x} = 0 \tag{2}$$

where ρ is the density, u the velocity, and P the pressure. The system is written in conservative form. The set of conservative variables is:

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} \tag{3}$$

and the fluxes are:

$$\mathbf{F}_{Euler} = \begin{bmatrix} \rho u \\ \rho u^2 + P \end{bmatrix} \tag{4}$$

Consider a small linear perturbation of the density,

$$\rho(x,t) = \rho_0 + \delta\rho(x,t) \; . \tag{5}$$

We can Fourier decompose $\delta\rho(x,t)$ into modes of the type

$$\delta\rho(x,t) = A e^{i(kx-\omega t)} \; , \tag{6}$$

1. Derive the dispersion relation between ω and k.
2. Explain the meaning of the solution.
3. Argue, on the basis of these solutions, that the *isothermal sound speed* is $c_s = \sqrt{P/\rho}$, so that $P = \rho c_s^2$.
4. Show that the system of isothermal Euler equations has the following eigenvalues:

$$\lambda_1 = u - c_s \;\; \lambda_2 = u + c_s \tag{7}$$

associated with the following eigenvectors:

$$\mathbf{K_1} = \begin{bmatrix} 1 \\ u - c_s \end{bmatrix}, \;\; \mathbf{K_2} = \begin{bmatrix} 1 \\ u + c_s \end{bmatrix} \tag{8}$$

2. Isothermal 1D hydrodynamics solver: convergence rate [13 pt.]

1. Based on the code that you have implemented for the advection problem, construct a finite-volume isothermal hydrodynamics solver in 1D following the **pseudo-code provided at the end of the sheet**.

   Additional remarks:

   - Use *constant reconstruction* to get the left and right states at each interface $i + 1/2$:

   $$\mathbf{q}_{\mathbf{L},i+1/2} = \mathbf{q}_i, \;\; \mathbf{q}_{\mathbf{R},i+1/2} = \mathbf{q}_{i+1} \tag{9}$$

   - Use the *HLL* flux to solve the Riemann problem at each interface $i + 1/2$:

   $$\mathbf{F}_{HLL} = \begin{cases} \mathbf{F_L} & if\ S_L \geq 0 \\ \frac{S_R \mathbf{F_L} - S_L \mathbf{F_R} + S_L S_R (\mathbf{q_R} - \mathbf{q_L})}{S_R - S_L} & if\ S_L < 0\ and\ S_R > 0 \\ \mathbf{F_R} & if\ S_R \leq 0 \end{cases}$$

   where $S_L = min(u_L, u_R) - c_s$, $S_R = max(u_L, u_R) + c_s$[1], $\mathbf{F_L} = \mathbf{F}_{Euler}(\mathbf{q_L})$ and $\mathbf{F_R} = \mathbf{F}_{Euler}(\mathbf{q_R})$.
   - Use the *RK1* algorithm to integrate in time: remember that the time step is limited by the CFL condition:

   $$\Delta t = \text{CFL} \cdot \frac{\Delta x}{|u|_{max} + c_s} \tag{10}$$

   - Implement periodic boundary conditions using the *ghost cells* technique.

[1] S_L and S_R are estimates of the wave speeds (given by the eigenvalues of the system) that are needed to solve the Riemann Problem at each interface in order to get the Godunov flux.

- The conservative variables are stored at cell centers ($N_{cc} = N_x + 2$ ghost cells), the L,R states and HLL fluxes are stored at face centers ($N_{fc} = N_x + 1$).
- For Python users: if you want to reduce the run time, avoid (the inner) explicit loops and use *numpy.ndarray* operations instead.

2. Use your code to solve the following 1D isothermal hydrodynamics problem: the x-grid goes from $x = 0$ to $x = 1$ and the isothermal sound speed is $c_s = 1$. Boundary conditions are periodic. Excite a left-going sound wave by setting the following initial conditions:

$$\rho(x, t = 0) = 1 + A \cdot \mathbf{K_1}[0] \sin(2\pi x) \quad (11)$$

$$\rho u(x, t = 0) = A \cdot \mathbf{K_1}[1] \sin(2\pi x) \quad (12)$$

where $A = 10^{-6}$ to stay in the linear regime, $\mathbf{K_1}[0] = 1$ and $\mathbf{K_1}[1] = -1$. Run the simulations until $t_{max} = 1$ using CFL=0.4.

3. What is the expected order of convergence of this finite-volume scheme?

4. Store the initial ($\mathbf{q}^0$) and final ($\mathbf{q}^n$) state vectors and compute the L_1 error for different resolutions $N_x = (100, 200, 300, 400, 500)$:

$$||\Delta\mathbf{q}||_1 = \left[\sum_k (\Delta\mathbf{q}_k)^2\right]^{1/2} \quad (13)$$

where k is the set of indices associated with the conservative variables and

$$\Delta\mathbf{q}_k = \sum_i |\mathbf{q}^n_{k,i} - \mathbf{q}^0_{k,i}|/N_x \quad (14)$$

(i refers to the $i - th$ cell). Plot the numerical relation $L_1(N_x)$ in log-log scale and show that it is consistent with what expected.

5. What would you do to increase the order of accuracy of the scheme?

Algorithm 1 Finite Volume 1D code

create grid
set initial conditions
t=0
while $t < t_{max}$ **do**
    store (ρ, ρu) in output
    get Δt using Eq. 10
    apply periodic BCs
    for each interface $i + 1/2$ **do**
        get L,R states at interface using constant reconstruction
        $\mathbf{F}_{i+1/2}$ = $\mathbf{F}_{HLL}(\mathbf{q_L}, \mathbf{q_R})$
    end for
    for each cell i **do**
        get residuals: $\mathbf{R}_i = \frac{\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}}{\Delta x} \Delta t$
        update (RK1): $\mathbf{q}_i = \mathbf{q}_i - \mathbf{R}_i$
    end for
    t = t + Δt
end while