Solutions to Problem Set 10

Fundamentals of Simulation Methods 8 ECTS
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1 Isothermal 1D hydrodynamics system (7 pts)

In this problem, we study the Eulerian equations of hydrodynamics in one dimension, as per

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial\left[\rho u^2 + P\right]}{\partial x} = 0, \tag{2}$$

where ρ, u and P are the density, velocity and pressure respectively. In order to close this system of equations, we assume that pressure is related to density as per $P = \rho c_s^2$, where c_s is constant. To analyse the stability and behaviour of this system, we introduce a small linear perturbation $\delta\rho$, δu of the density and velocity to some steady solution ρ_0, u_0 as per

$$\rho(x,t) = \rho_0 + \delta\rho(x,t), \tag{3}$$

$$u(x,t) = u_0 + \delta u(x,t). \tag{4}$$

Plugging these perturbed states into Eq.(1-2) and only considering the first order, linear terms, and remembering that ρ_0, u_0 corresponds to a solution of Eq.(1-2), we arrive at

$$\partial_t(\delta\rho) + u_0\partial_x(\delta\rho) + \rho_0\partial_x(\delta u) = 0, \tag{5}$$

$$u_0\partial_t(\delta\rho) + \rho_0\partial_t(\delta u) + 2\rho_0 u_0\partial_x(\delta u) + u_0^2\partial_x(\delta\rho) + \partial_x(\delta P) = 0, \tag{6}$$

where $\delta P = \delta\rho c_s^2$ is the perturbed pressure. If we substitute Eq.(5) into Eq.(6) we can simplify the system to

$$\partial_t(\delta\rho) + u_0\partial_x(\delta\rho) + \rho_0\partial_x(\delta u) = 0, \tag{7}$$

$$\rho_0\partial_t(\delta u) + \rho_0 u_0\partial_x(\delta u) + \partial_x(\delta P) = 0. \tag{8}$$

By defining the material derivative, i.e. the Lagrangian derivative in a co-moving frame of reference, as per

$$\frac{D}{Dt} \coloneqq \frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}, \tag{9}$$

we can rewrite the system of equations in the co-moving frame of reference relative to the steady solution ρ_0, u_0 as

$$D_t(\delta\rho) + \rho_0 \partial_x(\delta u) = 0, \tag{10}$$

$$\rho_0 D_t(\delta u) + \partial_x(\delta P) = 0. \tag{11}$$

If we then take the material derivative of Eq.(10) and the spatial derivative of Eq.(11), we can use the commutative property of partial derivatives and simplify the system of equations to

$$D_t^2(\delta\rho) - \partial_x^2(\delta P) = 0, \tag{12}$$

which also can be written as

$$D_t^2(\delta\rho) - c_s^2 \partial_x^2(\delta\rho) = 0, \tag{13}$$

using our earlier assumption on the pressure, which specifies the perturbed pressure as $\delta P = \delta\rho c_s^2$. Since Eq.(13) corresponds to the well known wave equation, we can interpret this result as the description on how the density perturbation will propagate with velocities $\pm c_s$ in each direction in the steady state solution's frame of reference, or with velocities $u_0 \pm c_s$ in the inertial frame of reference. Thus, the constant c_s specifies the speed at which 'information' traverses the domain, which is equivalent to the isothermal sound speed of the medium.

If we do a Fourier decomposition of the density perturbation, we can handle each wave mode independently, since any arbitrary perturbation can be attained from a linear superposition of individual modes. It is thus sufficient to only consider a single mode of the type

$$\delta\rho(x,t) = Ae^{i(kx-\omega t)}, \tag{14}$$

which we can can substitute into Eq.(13) to obtain

$$\left[[-i\omega + u_0 ik]^2 + c_s^2 k^2 \right] Ae^{i(kx-\omega t)} = 0. \tag{15}$$

Since this equation has to hold true for all x and t, for a nontrivial solution where $A \neq 0$, the large parenthesis has to equal zero. Simplifying this expression further and solving for ω gives the dispersion relation

$$\omega = k(u_0 \pm c_s), \tag{16}$$

which is linear in k, as expected for planar waves.

Furthermore, we can write the system of equations of Eq.(1-2) in vector notation as per

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{q})}{\partial x} = 0, \tag{17}$$

where we define the state vector $\boldsymbol{q}$ as

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \tag{18}$$

and the corresponding flux functions $\boldsymbol{F}$ as

$$\boldsymbol{F} = \begin{bmatrix} q_2 \\ \frac{q_2^2}{q_1} + q_1 c_s^2 \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + P \end{bmatrix}. \tag{19}$$

Taking the PDE in Eq.(17), which is written in conservative form, and transforming it into primitive form as per

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{q})}{\partial \boldsymbol{q}} \frac{\partial \boldsymbol{q}}{\partial x} = 0, \tag{20}$$

we can notice that the derivative of $\boldsymbol{F}$ with respect to $\boldsymbol{q}$ corresponds to the Jacobian $\mathbf{J}$ of the dynamical system as per

$$\mathbf{J} = \frac{\partial \boldsymbol{F}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{bmatrix} 0 & 1 \\ \left(c_s^2 - \frac{q_2^2}{q_1^2}\right) & 2\frac{q_2}{q_1} \end{bmatrix}. \tag{21}$$

The characteristic polynomial of the matrix $\mathbf{J}$ is

$$\det[\mathbf{J} - \lambda \mathbb{1}] = \lambda^2 - [\operatorname{tr} \mathbf{J}]\lambda + \det \mathbf{J} = 0, \tag{22}$$

which can be solved as

$$\lambda_\pm = \frac{1}{2} \operatorname{tr} \mathbf{J} \pm \frac{1}{2} \sqrt{[\operatorname{tr} \mathbf{J}]^2 - 4 \det \mathbf{J}}, \tag{23}$$

to obtain the eigenvalues

$$\lambda_\pm = \frac{q_2}{q_1} \pm c_s = u \pm c_s, \tag{24}$$

which can be substituted back into the original eigenvalue problem

$$\mathbf{J}\boldsymbol{v} = \lambda \boldsymbol{v}, \tag{25}$$

to obtain the corresponding eigenvectors

$$\boldsymbol{v}_\pm = \begin{bmatrix} 1 \\ \lambda_\pm \end{bmatrix} = \begin{bmatrix} 1 \\ u \pm c_s \end{bmatrix}. \tag{26}$$