The shallow water equations lab

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September 18 , 2019

The shallow water equations

- System of conservation laws:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ q \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} f_1(h,q) \\ f_2(h,q) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$f_1(h,q) = q (= vh), \qquad f_2(h,q) = q^2/h + gh^2/2$$

- The Jacobian of the flux functions is

$$\begin{pmatrix} \frac{\partial f_1}{\partial h} & \frac{\partial f_1}{\partial q} \\ \frac{\partial f_2}{\partial h} & \frac{\partial f_2}{\partial q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q^2/h^2 + gh & 2q/h \end{pmatrix}$$

As long as $h > 0$, the eigenvalues of the Jacobian,

$$\lambda = \frac{q}{h} \pm \sqrt{gh} = v \pm \sqrt{gh}$$

are **real** and **distinct**

- The Jacobian is then **diagonalizable** with **real eigenvalues** $\Rightarrow$ the system is hyperbolic

The shallow water equations

- There are **two** sets of characteristics, associated with the two eigenvalues

$$\lambda_\pm = v \pm \sqrt{gh}$$

- v is the water's velocity, $\sqrt{gh}$ is the wave speed
- Note: the shock speed is neither v nor λ_+!
- Look at `https://wikiwaves.org/Nonlinear_Shallow_Water_Waves` for more info on the shallow water equations!

1D vs 2D

1D: $\quad u_t + f(u)_x = 0,$

$$u = \begin{pmatrix} h \\ q \end{pmatrix}, \qquad f(u) = \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix}$$

Scheme: $u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$

Lax–Friedrich: $F_{i+1/2}^n = \frac{1}{2}\left(f(u_{i+1}^n) + f(u_{i-1}^n)\right) - \frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_i^n) \Rightarrow$

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right)$$

1D vs 2D

2D: $\quad u_t + f(u)_x + g(u)_y = 0,$

$$u = \begin{pmatrix} h \\ q \\ r \end{pmatrix}, \quad f(u) = \begin{pmatrix} q \\ q^2/h + gh^2/2 \\ qr/h \end{pmatrix}, \quad g(u) = \begin{pmatrix} r \\ qr/h \\ r^2/h + gh^2/2 \end{pmatrix}$$

Scheme: $u_{i,j}^{n+1} = u_{i,j}^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2,j}^n - F_{i-1/2,j}^n + G_{i,j+1/2}^n - G_{i,j-1/2}^n\right)$

Lax–Friedrich:

$$\begin{aligned} F_{i+1/2,j}^n &= \frac{1}{2}\left(f(u_{i+1,j}^n) + f(u_{i-1,j}^n)\right) - \frac{\Delta t}{4\Delta x}(u_{i+1,j}^n - u_{i,j}^n) \\ G_{i,j+1/2}^n &= \frac{1}{2}\left(g(u_{i,j+1}^n) + g(u_{i,j-1}^n)\right) - \frac{\Delta t}{4\Delta x}(u_{i,j+1}^n - u_{i,j}^n) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} u_{i,j}^{n+1} = {} & \frac{1}{4}\left(u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n\right) \\ & - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1,j}^n) - f(u_{i-1,j}^n) + g(u_{i,j+1}^n) - g(u_{i,j-1}^n)\right) \end{aligned}$$