The shallow water equations

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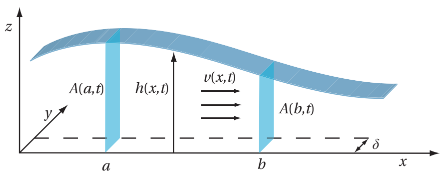
The shallow water equations

- Examples:
  - Coastal flows; effects of storm surges e. g.
  - Flows in dams and rivers
  - Tsunami modeling
- Water-flow case in which **vertical dimensions $\ll$ horizontal dimensions**
- Here: shallow and narrow **channel** of water; constant density ρ

Assumptions



(a) **Narrow, straight channel; width δ constant**. Horizontal movement only in x direction Everything constant in y direction.
Vertical cross section area:

$$A(x,t) = h(x,t)\delta \tag{1}$$

(b) **Shallow channel**: Horizontal dynamics dominates. Vertical dynamics ignored:
- Ignoring vertical velocity: $\boldsymbol{v}(\boldsymbol{x},t) = \big(v(x,t), 0, 0\big)$
- $v(x,t)$ represents a **depth-averaged** velocity
- Effects of air pressure ignored
- Hydrostatic water pressure in the vertical direction:

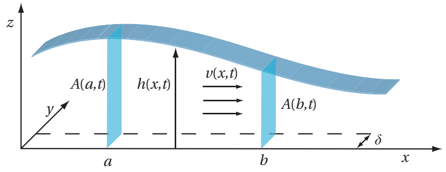
$$p(x,z,t) = \rho g\big(h(x,t) - z\big) \qquad \text{for } z \le h \tag{2}$$

(c) Friction forces at bottom and walls ignored

The shallow water equations

- The shallow water equations: a system of 2 conservation laws in $h(x,t)$ and $v(x,t)$
- If not narrow channel, 1 additional conservation law for additional horizontal velocity component
- These equations derived from conservation of mass + conservation of momentum + relation (2).

Mass conservation



- Water mass between a and b:

$$m(t) = \int_a^b \rho A(x,t)\,dx \qquad (3)$$

- Mass flux at $x = a$ in positive x direction:

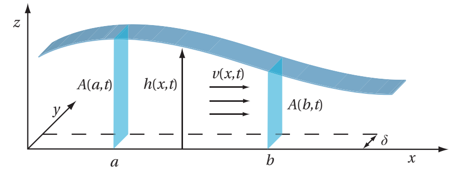
$$\rho v(a,t)A(a,t) \qquad \text{[kg/s]} \qquad (4)$$

- Mass conservation and expressions (3), (4):

$$\begin{aligned} \frac{d}{dt}m(t) &= \frac{d}{dt}\int_a^b \rho A(x,t)\,dx \\ &= \rho v(a,t)A(a,t) - \rho v(b,t)A(b,t) = -\int_a^b \frac{\partial}{\partial x}(\rho v A)\,dx \end{aligned}$$

$$\Rightarrow \qquad \int_a^b \left(\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho v A)\right) = 0$$

Mass conservation



$$\int_a^b \left(\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho v A)\right) = 0$$

Recall (1): $A(x,t) = h(x,t)\delta$ and that ρ, δ constant $\Rightarrow$

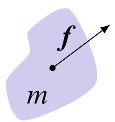
$$\int_a^b \left(\frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(vh)\right) = 0$$

$a < b$ arbitrarily $\Rightarrow$

$$\frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(vh) = 0 \qquad (5)$$

First conservation law done!

Momentum balance



Momentum balance for rigid bodies = Newton's 2nd law:

$$m\boldsymbol{a} = \boldsymbol{f}$$

Generalization for changing mass:

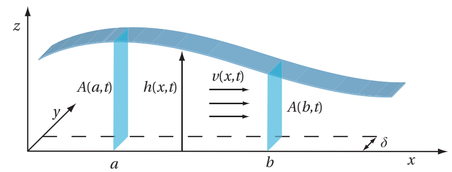
$$\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{f}$$

- Momentum $\boldsymbol{r} = m\boldsymbol{v}$ is a **vector**
- An applied force changes momentum (direction and/or magnitude)
- In 1D, the momentum r can be positive or negative
- (Compare the above with mass and energy conservation!)

Force on a body V with boundary ∂V from an exterior pressure field p:

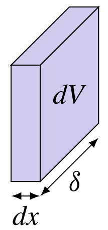
$$\boldsymbol{f} = -\int_{\partial V} p\boldsymbol{n}\,dS = -\int_{\partial V} p\,d\boldsymbol{S}$$

Momentum balance



Momentum balance law for region $a < x < b$:

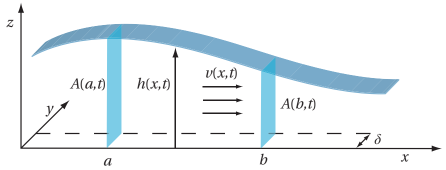
$$\frac{dr}{dt} = \text{[mom. inflow at } a\text{]} - \text{[mom. outflow at } b\text{]} + \text{[force on region]}$$



- Infinitesimal volume $dV = A(x,t)\,dx = h(x,t)\delta\,dx$
- Momentum in dV: $dr = \rho v A\,dx = \rho v h\delta\,dx$
- Momentum in region $a < x < b$:

$$r(t) = \int_a^b dr = \int_a^b \rho v A\,dx = \int_a^b \rho v h \delta\,dx \qquad (6)$$

Momentum flux



$$\textbf{Momentum flux} \text{ is } \underbrace{\textbf{momentum density}}_{\frac{[\text{momentum}]}{\text{m}^3}} \times \underbrace{\textbf{volume flux}}_{\frac{\text{m}^3}{\text{s}}}$$

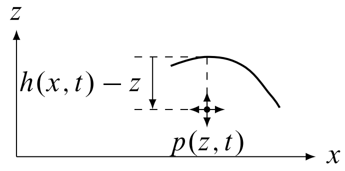
$$
\begin{aligned}
[\text{mom. inflow at } a] &= \underbrace{\rho v(a,t)}_{\frac{[\text{momentum}]}{\text{m}^3}} \underbrace{A(a,t) v(a,t)}_{\frac{\text{m}^3}{\text{s}}} = \underbrace{\rho v(a,t)^2 A(a,t)}_{\frac{[\text{momentum}]}{\text{s}}} \\
&= \rho v(a,t)^2 h(a,t) \delta && \text{(7a)} \\
[\text{mom. outflow at } b] &= \rho v(b,t)^2 A(b,t) = \rho v(b,t)^2 h(b,t) \delta && \text{(7b)}
\end{aligned}
$$

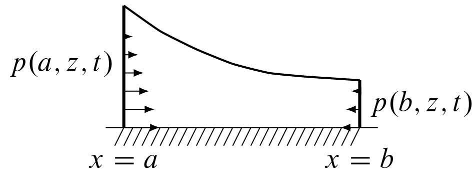
Forces

- Only forces from water pressure (ignoring friction)
- Hydrostatic water pressure in vertical direction:

$$p(x,z,t) = \rho g \big(h(x,t) - z\big)$$

- Forces on region $a < x < b$ from pressures at $x = a$ and $x = b$





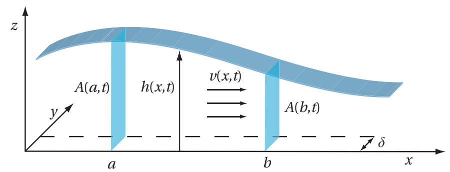
Forces

Hydrostatic water pressure: $p(x,z,t) = \rho g \big(h(x,t) - z\big)$

Integrated pressure forces on vertical surface $A(x,t) = h(x,t)\delta$:

$$
\begin{aligned}
\int_0^\delta \int_0^h p(x,z,t)\, dz\, dy &= \delta \int_0^h p(x,z,t)\, dz = \delta \int_0^h \rho g \big(h - z\big)\, dz \\
&= -\delta \rho g \frac{(h-z)^2}{2} \bigg|_{z=0}^{z=h} = \delta \rho g \frac{h(x,t)^2}{2}
\end{aligned} \tag{8}
$$

Momentum balance



Recall:

$$\frac{dr}{dt} = [\text{mom. inflow at } a] - [\text{mom. outflow at } b] + [\text{force on region}]$$

Equations (6), (7), (8) $\Rightarrow$

$$
\begin{aligned}
\frac{d}{dt} \int_a^b \rho v h \delta \, dx &= \rho v^2(a,t) h(a,t) \delta - \rho v^2(b,t) h(b,t) \delta \\
&\quad + \delta \rho g \frac{h(a,t)^2}{2} - \delta \rho g \frac{h(b,t)^2}{2} \\
&= -\int_a^b \left[\frac{\partial}{\partial x} \big(\rho v^2 h \delta\big) + \frac{\delta \rho g}{2} \frac{\partial}{\partial x} h^2 \right] dx
\end{aligned}
$$

The final system

δ, ρ constant $\Rightarrow$

$$\int_a^b \left[\frac{\partial}{\partial t} vh + \frac{\partial}{\partial x} v^2 h + \frac{g}{2} \frac{\partial}{\partial x} h^2 \right] dx = 0$$

$a < b$ arbitrary $\Rightarrow$ conservation law

$$\frac{\partial}{\partial t} vh + \frac{\partial}{\partial x} v^2 h + \frac{g}{2} \frac{\partial}{\partial x} h^2 = 0$$

Recall mass balance (5):

$$\frac{\partial}{\partial t} h + \frac{\partial}{\partial x} (vh) = 0$$

Defining $q = vh$ (δq is the volume flux), the system can be written

$$\frac{\partial}{\partial t} \begin{pmatrix} h \\ q \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Is the system hyperbolic?

$$\frac{\partial}{\partial t} \begin{pmatrix} h \\ q \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Can be written

$$\mathbf{u}_t + \mathbf{f}_x = \mathbf{0},$$

where

$$\mathbf{u} = \begin{pmatrix} h \\ q \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix}$$

Hyperbolic if the Jacobian

$$\mathbf{Df} = \begin{pmatrix} \frac{\partial f_1}{\partial h} & \frac{\partial f_1}{\partial q} \\ \frac{\partial f_2}{\partial h} & \frac{\partial f_2}{\partial q} \end{pmatrix}$$

is **diagonalizable** with **real eigenvalues**

Is the system hyperbolic?

$$f_1 = q$$

$$f_2 = \frac{q^2}{h} + \frac{g}{2} h^2$$

$$\mathbf{Df} = \begin{pmatrix} \frac{\partial f_1}{\partial h} & \frac{\partial f_1}{\partial q} \\ \frac{\partial f_2}{\partial h} & \frac{\partial f_2}{\partial q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q^2/h^2 + gh & 2q/h \end{pmatrix}$$

Characteristic equation for eigenvalues λ:

$$\left| \lambda \mathbf{I} - \mathbf{Df} \right| = \begin{vmatrix} \lambda & -1 \\ q^2/h^2 - gh & \lambda - 2q/h \end{vmatrix}$$

$$= \lambda \left(\lambda - \frac{2q}{h} \right) + \frac{q^2}{h^2} - gh = 0$$

$$\lambda = \frac{q}{h} \pm \sqrt{\frac{q^2}{h^2} - \frac{q^2}{h^2} + gh} = \frac{q}{h} \pm \sqrt{gh} \tag{9}$$

Is the system hyperbolic?

Recall $q = vh \Rightarrow$

$$\lambda = v \pm \sqrt{gh}$$

$h \geq 0 \Rightarrow$ real eigenvalues!

$h > 0 \Rightarrow$ separate eigenvalues $\Rightarrow$ two linearly independent eigenvectors $\Rightarrow$ diagonalizable $\mathbf{Df}$

Thus, system is hyperbolic for $h > 0$