Systems Theory Project Report

Nonlinear Model Predictive Control Design

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Abstract

1 Introduction

The goal of the project is to combine the methods of Sum of Squares (SOS) Programming and nonlinear Model Predictive Control (MPC). Based on an exemplary nonlinear dynamical system an MPC is designed in such a way that it acts outside a specified region in the state space and drives it towards this region. The region is called *terminal region*. Inside the terminal region a linear controller takes over, this concept is called *Dual Mode Control*. SOS methods are used to estimate the terminal region which is a *region of attraction* of the closed loop dynamics of the linearly controlled system. The report first introduces the basics of MPC in 2. Continuing with the concept of dual mode controller design focussing on SOS methods in 3. Lastly, the methods presented are applied to the ball beam system. The results are shown and discussed in 4.

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2 Model Predictive Control

Model Predicitve Control is a flexible and powerful control strategy. It naturally handles nonlinearities constraints as well as multiple inputs and outputs. The underlying idea is based on optimal control theory. MPC is characterized by two key features. First, an iterative online optimization is performed to compute the next control input. Second, at each instance of time an Optimal Control Problem (OCP) is solved, therefore the time horizon of the OCP moves forward. [1]

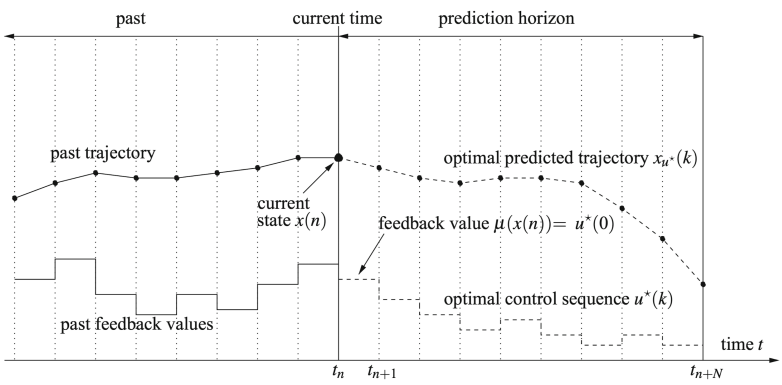

Figure 1: Moving horizon and optimization of an MPC in discrete time [1]

Figure 1 illustrates the two key feastures of MPC in discrete time. At time instance t_n the OCP for the next N time steps is solved. Given that the OCP is feasible, the first optimal control input u_0^* is applied to the system. The system evolves to the next time instance t_{n+1} and the process repeats.

2.1 Problem Formulation

The discrete time finite horizon optimal control problem can be formulated as follows:

$$
\begin{aligned}
\min_{\nu} \quad & E\left(x_{k+N|k}\right) + \textstyle\sum_{i=k}^{k+N-1} F\left(x_{i|k}, u_{i|k}\right) \\
\text{s.t.} \quad & x_{i+1|k} = f\left(x_{i|k}, u_{i|k}\right), \quad x_{k|k} = x_k \\
& u_{i|k} \in \mathcal{U}, \quad i \in [k, k+N-1] \\
& x_{i|k} \in \mathcal{X}, \quad i \in [k, k+N-1] \\
& x_{k+N|k} \in \mathcal{E}
\end{aligned} \tag{1}
$$

where

$$
J(\nu, x_k) = \underbrace{E\left(x_{k+N|k}\right)}_{\text{terminal costs}} + \underbrace{\sum_{i=k}^{k+N-1} F\left(x_{i|k}, u_{i|k}\right)}_{\text{stage costs}} \tag{2}
$$

with $\mathcal{U}$ input constraints, $\mathcal{X}$ state constraints and $\mathcal{E}$ terminal region/set as the constraining sets. The OCP is solved for the prediction horizon N, the initial condition x_k at the current time step t_k, the dynamics of the system given by $f\left(x_{i|k}, u_{i|k}\right)$ and the terminal state $x_{k+N|k}$. The cost function (2) consists of the terminal costs at last time instance t_N of the optimization and the stage costs for all other time instances starting from the initial condition. The optimal control sequence is defined as

$$
\nu^* = [u^*_{k|k}, \ldots, u^*_{k+N-1|k}]^T. \tag{3}
$$

The control input μ at each discrete time step t_k is the first element of the optimal control sequence ν^*

$$
\mu(x(t_k)) := \nu^*(0) \tag{4}
$$

See [1] and [2] for details on the formulation of the OCP resp. the MPC formulation.

2.2 Terminal Conditions

The shown MPC includes terminal conditions. These conditions are part of the design of the MPC and are used to guarantee certain stability properties of the closed loop system. The followong Theorem connects terminal conditions and stability of the closed loop system. [2]

Theorem 1

The closed loop system is asymptotically stable if the optimal control problem is feasible at the first time instant and the following assumptions are satisfied for a terminal cost E, a terminal region $\mathcal{E}$, and a locally stabilizing control law $u_k = \varphi(x_k)$: [A1] $E(x_k) > 0, \forall x_k \in \mathbb{R}^n \backslash \{0\}$
[A2] $\mathcal{E} \subseteq \mathcal{X}, 0 \in \mathcal{E}$
[A3] $\varphi(x_k) \in \mathcal{U}, \forall x_k \in \mathcal{E}$
[A4] $f(x_k, \varphi(x_k)) \in \mathcal{E}, \forall x_k \in \mathcal{E}$
[A5] $E(f(x_k, \varphi(x_k))) - E(x_k) \leq -F(x_k, \varphi(x_k)), \forall x_k \in \mathcal{E}$.

Several approaches based on Theorem 1 exist to ensure closed loop stability. Those investigated in this work are the following:

Zero Terminal State Constraint
The idea is to use $E(x_k) = 0$, $\mathcal{E} = \{0\}$ and $\varphi(x_k) = 0$. No linear controller is used. The stability of the MPC is guaranteed through $x_{k+N|k} = 0$. When assuming an equlibrium at the origin of the system, this ensures stability at the last step of the OCP.

Terminal Region
No terminal costs are used, but a terminal region $\mathcal{E} = \{x_k \in \mathbb{R}^n \mid x_k^T P x_k \leq \alpha\}$ is

choosen as well as a locally stabilizing linear controller $\varphi(x_k) = Kx_k$. This approach is also *dual mode control*.

Terminal Region and Terminal Cost
The so-called quasi-infinite horizon MPC uses a terminal region $\mathcal{E} = \{x_k \in \mathbb{R}^n \mid x_k^T P x_k \leq \alpha\}$ and a terminal cost $E(x_k) = x_k^T P x_k$ as well as locally stabilizing linear controller $\varphi(x_k) = Kx_k$. The terminal cost E, terminal region $\mathcal{E}$ and locally stabilizing linear controller $\varphi(x_k)$ are calculated off-line. The procedure is described in section 3.

2.3 Variations of MPC

There are several variations of the MPC formulation. Two important distinctions have to be made regarding the length of the prediction horiton. One differentiate between *finite* and *infinite* horizon optimal control. presented above is the finite horizon MPC. The infinite horizon MPC, as the name implies, has an infinite prediction horizon $N \to \infty$. It can be shown that the inifinte horizon MPC asymptotically stabilizes the system, without need of any terminal conditions. Hence the name *quasi-infinite horizon* is motivated by this property. Furthermore, it should be differentiated between *stabilizing* and *tracking* MPC. The stabilizing MPC is used to stabilize the system around an equilibrium point. The tracking MPC is used to follow a reference trajectory as good as possible. For the stabilizing MPC the reference can assumed by constant zero, when the system is stable around the origin. Given the definitions above, it is possible to introduce the tracking MPC by utilizing the difference, or error, of the current state to the reference $e_k = x_k - x_{\text{ref},k}$ as a new state. In other words, the tracking MPC is a stabilizing MPC for the error dynamics of the system. Tracking MPC won't be considered in the follwing. For more details see [1].

3 Synthesis of Terminal Conditions

4 Exemplary Application

5 Conclusion & Outlook

References

[1] Jürgen Pannek Lars Grüne. *Nonlinear Model Predicitve Control*. Springer Cham, 2 edition, 2016.

[2] T. Raff, C. Ebenbauer, R. Findeisen, and F. Allgöwer. *Nonlinear Model Predictive Control and Sum of Squares Techniques*, volume 10, pages 325–344. 07 2007.

Todo list

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