Quantitative Methods

Human Sciences, 2020–21

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Lecture 1: 15 October 2020

General setup

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- Statistics and data science in **R**.

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- Weekly problem sheets to be submitted day before teaching.
- All materials available on Canvas and on eliasnosrati.github.io.

Syllabus

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- Descriptive inference (data discovery and pattern recognition).

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- Introduction to **R** (tutorial).

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- And much more!

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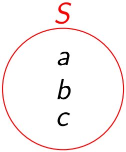

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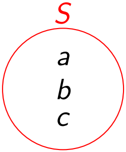

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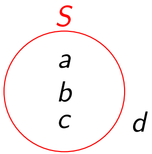

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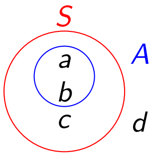

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- $a \in S$: a is a member (or element) of S.
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- If $A = \{a, b\}$, then $A \subset S$: A is a *subset* of S.

Set theory: definition of a subset

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A set A is a *subset* of a set B, or $A \subset B$, if every member of A is also a member of B:

$$(x \in A) \Rightarrow (x \in B)$$

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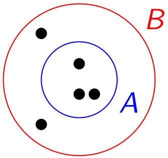

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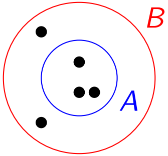

Remark

If $A \subset B$ and $A \neq B$, then A is said to be a *proper subset* of B. Otherwise, we often write $A \subseteq B$.

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- The set of real numbers $\mathbb{R}$ (think of the number line).

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Example

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Note that for any set S, $S \subseteq S$ and $\varnothing \subseteq S$. Hence $S \in \mathcal{P}(S)$ and $\varnothing \in \mathcal{P}(S)$ for all S. Note also for any set with n elements, its power set has 2^n elements (why?).

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Remark

We say that A and B are *disjoint* (mutually exclusive) is they have no elements in common, i.e., if $A \cap B = \varnothing$.

Set theory: examples of union and intersection

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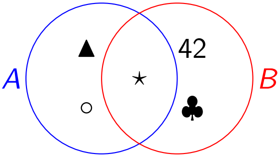

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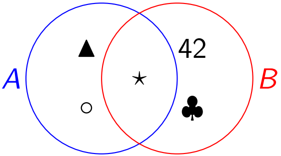

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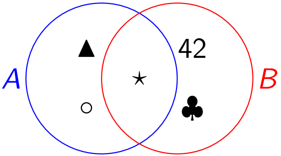

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Example

Let $A = \{\text{Denmark, Norway, Sweden}\}$ and
$B = \{\text{Botswana, Namibia, Zimbabwe}\}$. Then $A \cap B = \{\} = \varnothing$.

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Example

Let A be the set of all individuals named Henry and let B be the set of all individuals with brown hair. Then A^c is the set of all people whose name is not Henry and $A - B$ is the set of all people named Henry who do not have brown hair.

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Definition

The *sample space* S of an experiment is the set of all possible outcomes of the experiment. An *event* A is a subset of this sample space, and if the actual outcome is an element of A, we say that A *occurred*.

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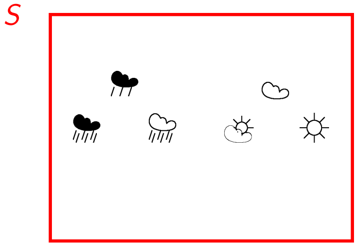

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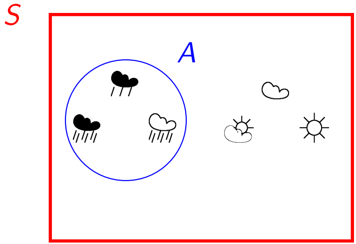

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Answer:

$$B = A_1 \cup \dots \cup A_{10} = \bigcup_{i=1}^{10} A_i.$$

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Answer:

$$D = (A_1 \cap A_2) \cup \cdots \cup (A_9 \cap A_{10}) = \bigcup_{i=1}^{9} (A_i \cap A_{i+1}).$$

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Remark

Note that $(A \cup B)^c = A^c \cap B^c$. This is one of *De Morgan's laws*. The other is $(A \cap B)^c = A^c \cup B^c$.

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Functions

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Let X and Y be sets. A *function* f from X to Y is a mapping from the *domain* X to the *codomain* Y that assigns a value $f(x) \in Y$ to each $x \in X$:

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A *probability space* consists of a sample space S, a class of events defined on S, and a *probability function*

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   Equivalently:

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Exercise: Can your prove these properties (using only the three axioms of probability)?

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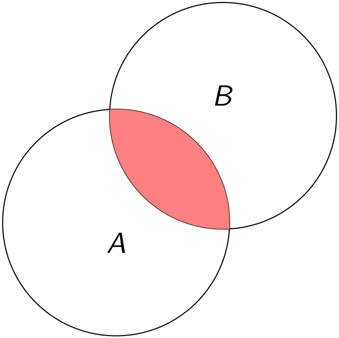

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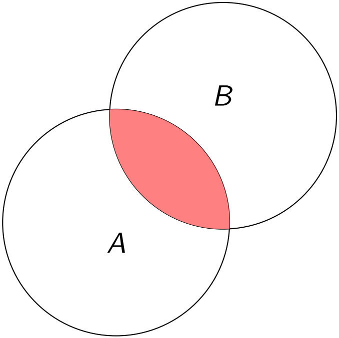

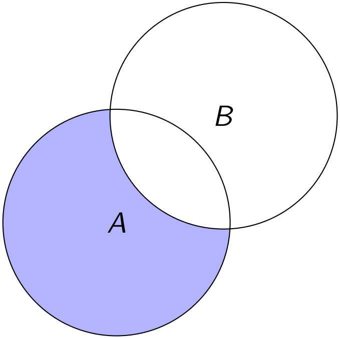

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Unlike a permutation, a combination is a way of choosing k out of n objects *without* regard to order. Formally, this is known as a binomial coefficient:

Definition

A *binomial coefficient* counts the number of subsets of size k for a set of size n. For any non-negative integers k and n, the binomial coefficient, read as "n choose k", is defined as

$$C_k^n = \binom{n}{k} = \frac{P_k^n}{k!} = \frac{n!}{k!(n-k)!}.$$

Since sets are unordered (i.e., $\{x, y\} = \{y, x\}$), this amounts to choosing k out of n objects, without replacement and without distinguishing the order in which they are chosen.

Combinations: committee example

Consider a group of four people. How many ways are there to choose a two-person committee?

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No, because this overcounts by a factor of 2 (picking A and B to be a team is equivalent to picking C and D to be a team)!

$$\frac{\binom{4}{2}}{2} = 3.$$

Some important identities (with story proofs)

- Choosing the complement:

$$\binom{n}{k} = \binom{n}{n-k}.$$

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(*Vandermonde's identity*.)

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- **R** or **RStudio**?

Basic operations

```r
4 + 2
```

```
## [1] 6
```

Basic operations

```
4 + 2

## [1] 6
```

```
4 - 2

## [1] 2
```

Basic operations

```
4 + 2

## [1] 6
```

```
4 - 2

## [1] 2
```

```
4 * 2

## [1] 8
```

Basic operations (cont.)

```r
4 * (5 - 3)

## [1] 8
```

Basic operations (cont.)

```
4 * (5 - 3)

## [1] 8
```

```
4 / 2

## [1] 2
```

Basic operations (cont.)

```
4 * (5 - 3)

## [1] 8
```

```
4 / 2

## [1] 2
```

```
4 ^ 2

## [1] 16
```

Basic operations (cont.)

```r
4 * (5 - 3)

## [1] 8
```

```r
4 / 2

## [1] 2
```

```r
4 ^ 2

## [1] 16
```

```r
sqrt(4)

## [1] 2
```

Creating objects

Create an object `x` that saves information:

```
x <- 4 * 2
```

Creating objects

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```
x <- 4 * 2
```

View object:

```
x

## [1] 8

(x <- 4 * 2)

## [1] 8
```

Numeric objects: double and integer

```
(x1 <- 2.0)

## [1] 2

typeof(x1)

## [1] "double"
```

Numeric objects: double and integer

```r
(x1 <- 2.0)

## [1] 2

typeof(x1)

## [1] "double"
```

```r
(x2 <- 2L)

## [1] 2

typeof(x2)

## [1] "integer"
```

Character objects

```r
(instructor <- "Elias")

## [1] "Elias"
```

Character objects

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(instructor <- "Elias")

## [1] "Elias"
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```r
(instructor <- "Elias Nosrati")

## [1] "Elias Nosrati"
```

Character objects

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(instructor <- "Elias")

## [1] "Elias"
```

```r
(instructor <- "Elias Nosrati")

## [1] "Elias Nosrati"
```

```r
typeof(instructor)

## [1] "character"
```

Character objects

```
(instructor <- "Elias")

## [1] "Elias"
```

```
(instructor <- "Elias Nosrati")

## [1] "Elias Nosrati"
```

```
typeof(instructor)

## [1] "character"
```

```
number_as_character <- "10"
number_as_character / 2

## Error in number_as_character/2:  non-numeric
argument to binary operator
```

Logical objects

```r
5 > 2
```

```
## [1] TRUE
```

Logical objects

```
5 > 2

## [1] TRUE
```

```
x <- 5
x > 2

## [1] TRUE
```

Logical objects

```r
5 > 2

## [1] TRUE
```

```r
x <- 5
x > 2

## [1] TRUE
```

```r
y <- x > 2
typeof(y)

## [1] "logical"
```

Data structures: vectors

```r
x <- c(1, 2, 3)
x

## [1] 1 2 3
```

Data structures: vectors

```r
x <- c(1, 2, 3)
x

## [1] 1 2 3
```

```r
a <- c("a", 2, FALSE)
b <- c("z", 47L)
c <- c(a, b)
c

## [1] "a"     "2"     "FALSE" "z"     "47"
```

Data structures: vectors (cont.)

```
(x <- c(1, 2, 3, 4, 5))

## [1] 1 2 3 4 5
```

Data structures: vectors (cont.)

```r
(x <- c(1, 2, 3, 4, 5))

## [1] 1 2 3 4 5
```

```r
(x <- 1:5)

## [1] 1 2 3 4 5
```

Data structures: vectors (cont.)

```
(x <- c(1, 2, 3, 4, 5))

## [1] 1 2 3 4 5
```

```
(x <- 1:5)

## [1] 1 2 3 4 5
```

```
(x <- seq(1, 5))

## [1] 1 2 3 4 5
```

Data structures: vectors (cont.)

```r
(x <- c(1, 2, 3, 4, 5))

## [1] 1 2 3 4 5
```

```r
(x <- 1:5)

## [1] 1 2 3 4 5
```

```r
(x <- seq(1, 5))

## [1] 1 2 3 4 5
```

```r
(x <- seq(10, 50, by = 10))

## [1] 10 20 30 40 50
```

Data structures: matrices

```
matrix(1:4, nrow = 2)

##      [,1] [,2]
## [1,]    1    3
## [2,]    2    4
```

Data structures: matrices

```
matrix(1:4, nrow = 2)

##      [,1] [,2]
## [1,]    1    3
## [2,]    2    4
```

```
matrix(1:4, nrow = 2, byrow = TRUE)

##      [,1] [,2]
## [1,]    1    2
## [2,]    3    4
```

Data structures: lists

```r
list(1:3, c("a", "b"), TRUE, 44.7, " ")

## [[1]]
## [1] 1 2 3
##
## [[2]]
## [1] "a" "b"
##
## [[3]]
## [1] TRUE
##
## [[4]]
## [1] 44.7
##
## [[5]]
## [1] " "
```

Data structures: data frame

```
data.frame("Var_1" = 1:3, "Var_2" = 4:6)

##   Var_1 Var_2
## 1     1     4
## 2     2     5
## 3     3     6
```

Data structures: data frame

```
data.frame("Var_1" = 1:3, "Var_2" = 4:6)

##   Var_1 Var_2
## 1     1     4
## 2     2     5
## 3     3     6
```

```
df <- data.frame("Var_1" = 1:3, "Var_2" = 4:6)
names(df) <- c("Variable_1", "Variable_2")
df

##   Variable_1 Variable_2
## 1          1          4
## 2          2          5
## 3          3          6
```

Converting between data structures

```
m <- matrix(1:100, nrow = 10)
as.data.frame(m)

##    V1 V2 V3 V4 V5 V6 V7 V8 V9 V10
## 1   1 11 21 31 41 51 61 71 81  91
## 2   2 12 22 32 42 52 62 72 82  92
## 3   3 13 23 33 43 53 63 73 83  93
## 4   4 14 24 34 44 54 64 74 84  94
## 5   5 15 25 35 45 55 65 75 85  95
## 6   6 16 26 36 46 56 66 76 86  96
## 7   7 17 27 37 47 57 67 77 87  97
## 8   8 18 28 38 48 58 68 78 88  98
## 9   9 19 29 39 49 59 69 79 89  99
## 10 10 20 30 40 50 60 70 80 90 100
```

Indexing

```
x <- 1:10
x[5]

## [1] 5
```

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```
x <- 1:10
x[5]

## [1] 5
```

```
x[2:7]

## [1] 2 3 4 5 6 7
```

Indexing

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x[5]

## [1] 5
```

```
x[2:7]

## [1] 2 3 4 5 6 7
```

```
x[c(2, 7)]

## [1] 2 7
```

Indexing

```
x <- 1:10
x[5]

## [1] 5
```

```
x[2:7]

## [1] 2 3 4 5 6 7
```

```
x[c(2, 7)]

## [1] 2 7
```

```
x[-c(2, 7)]

## [1]  1  3  4  5  6  8  9 10
```

Indexing (cont.)

```
df <- data.frame("V1" = 1:3, "V2" = 4:6)
df[2]

##   V2
## 1  4
## 2  5
## 3  6
```

Indexing (cont.)

```r
df <- data.frame("V1" = 1:3, "V2" = 4:6)
df[2]

##   V2
## 1  4
## 2  5
## 3  6
```

```r
df$V2

## [1] 4 5 6
```

Indexing (cont.)

```
df[3, 2]
```

```
## [1] 6
```

Indexing (cont.)

```
df[3, 2]

## [1] 6
```

```
subset(df, V1 == 1 & V2 == 4)

##   V1 V2
## 1  1  4
```

Indexing (cont.)

```
df[3, 2]

## [1] 6
```

```
subset(df, V1 == 1 & V2 == 4)

##   V1 V2
## 1  1  4
```

```
subset(df, V1 == 2 | V2 != 4)

##   V1 V2
## 2  2  5
## 3  3  6
```

Functions

We have already seen several functions: `c()`, `class()`, `data.frame()`, etc.

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```
my_function <- function(input) {

  # Define output using input here

  return(output)
}
```

Functions: constructing $f(x) = x^2 + 4$

```r
my_function <- function(x) { # Function takes input x
  y <- x^2 + 4 # Expression for f(x)
  return(y) # Output
}
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```r
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  return(y) # Output
}
```

```r
my_function(2)

## [1] 8
```

Functions: creating a summary function

```r
my_summary <- function(x) { # Input
  s_out <- sum(x) # Sum
  l_out <- length(x) # Length
  m_out <- s_out / l_out # Mean
  out <- c(s_out, l_out, m_out) # Define output
  names(out) <- c("Sum", "Length", "Mean") # Labels
  return(out) # End function by calling output
}
```

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input <- 1:10
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  names(out) <- c("Sum", "Length", "Mean") # Labels
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}
```

```
input <- 1:10
```

```
my_summary(input)

##    Sum Length   Mean
##   55.0   10.0    5.5
```

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- You load them into **R** by typing `library(name_of_package)`.
- To get help on usage, type `package?name_of_package` and `help(package = "name_of_package")`.

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- Write and execute code with editor and save text file with .R file extension.

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Some hands-on exercises

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- Then type `install_course_github("kosukeimai", "qss-swirl")`.
- Type `swirl()` to start the first exercise.

Homework

- Chapters 2 and 3 of Grolemund's *Hands-On Programming with R* (available online).

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- Complete Problem Sheet 1 and submit your **R** scripts by email at least 24h before the next lecture.