

Quantitative Methods

Human Sciences, 2020–21

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Lecture 2: 22 October 2020

Today

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- ▶ Recap on probability and counting: the birthday problem.

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- ▶ Introduction to conditional probability.

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- ▶ Introduction to conditional probability.
- ▶ Problem sheet 1 (tutorial).

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- ▶ What is the probability that at least one pair of people in the group have the same birthday?
- ▶ Hint: Recall that $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$.

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- ▶ $\mathbb{P}(\text{at least one birthday match}) = 1 - \mathbb{P}(\text{no birthday match})$.
- ▶ In this room, $\mathbb{P}(\text{at least one birthday match}) \approx 4\%$.

The birthday problem in R

```
# Create a function  
pmatch <- function(n) {  
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```
# Alternative method  
probs <- sapply(1:70, pmatch)
```

The birthday problem in **R** (cont.)

```
save <- data.frame("n" = 1:70, "prob" = probs)
```

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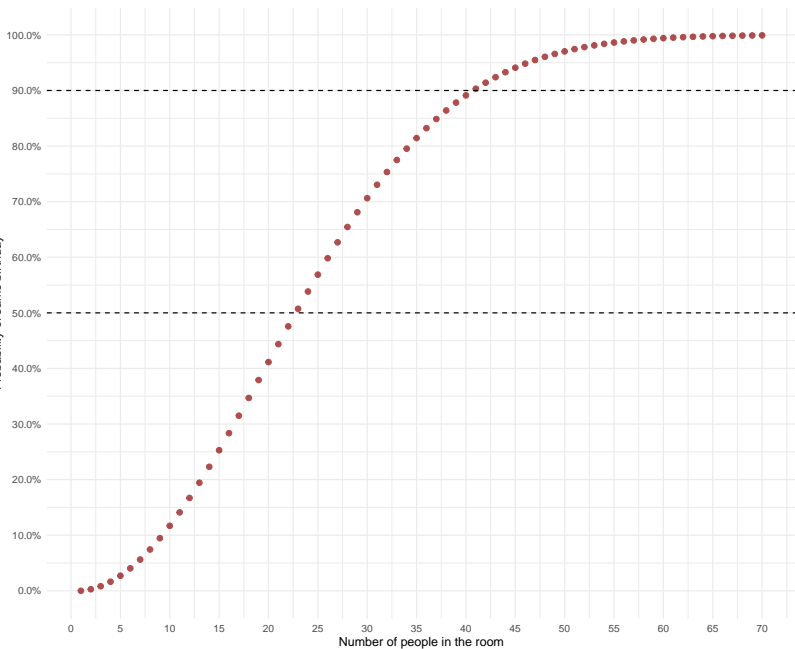
```
head(save)
```

##	n	prob
## 1	1	0.000000000
## 2	2	0.002739726
## 3	3	0.008204166
## 4	4	0.016355912
## 5	5	0.027135574
## 6	6	0.040462484

The birthday problem in **R** (cont.)

```
ggplot(save, aes(n, prob)) +  
  labs( # Axis labels  
    x = "Number of people in the room",  
    y = "Probability of same birthday") +  
  scale_x_continuous( # Modify X-axis  
    breaks = seq(0, 70, 5)) +  
  scale_y_continuous( # Modify Y-axis  
    breaks = seq(0, 1, 0.1),  
    label = scales::percent) +  
  geom_point( # Modify points  
    size = 2,  
    colour = "darkred",  
    alpha = 0.7) +  
  geom_hline( # When is P(match) > 0.5 or 0.9?  
    yintercept = c(0.5, 0.9), linetype = "dashed") +  
  theme_minimal() # Remove redundant lines
```

Probability of same birthday



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- ▶ Whenever new evidence is observed, we acquire information that may affect our uncertainties.
- ▶ Conditional probability allows us to update our beliefs in light of new evidence.
- ▶ “Conditioning is the soul of statistics” (Blitzstein and Hwang, 2019: 46).

Defining conditional probability

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If A and B are events, with $\mathbb{P}(B) > 0$, then the *conditional probability* of A given B is

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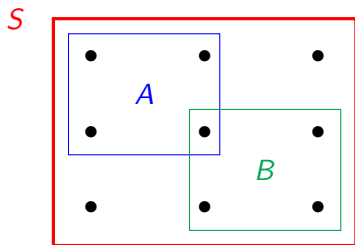
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- ▶ Note that $\mathbb{P}(A \mid A) = 1$.

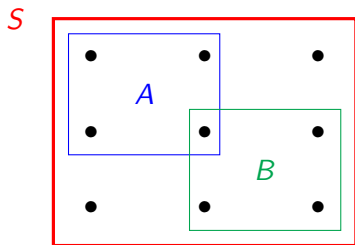
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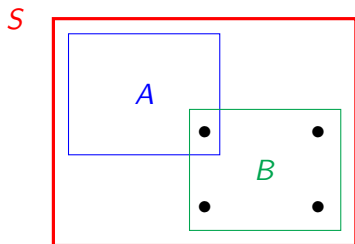
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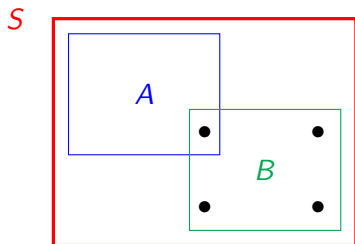
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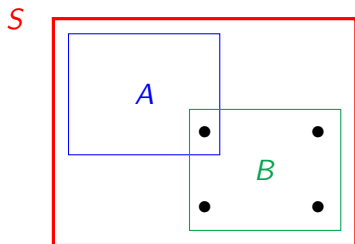
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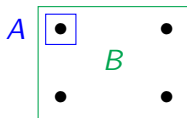


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- ▶ Divide by $\mathbb{P}(B)$, the total mass of the outcomes in B .

Visualising a conditional probability (cont.)

The updated probability measure assigned to the event A is the conditional probability

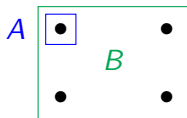
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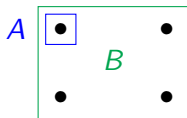


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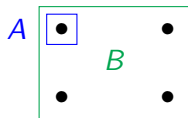


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- ▶ Relative measures of uncertainty are redistributed amongst remaining possible outcomes.

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- ▶ Thus

$$\mathbb{P}(A \mid B) = n_{AB}/n_B = (n_{AB}/n)/(n_B/n) = \mathbb{P}(A \cap B)/\mathbb{P}(B).$$

Joint probability and conditional probability

Theorem

For any events A and B with positive probabilities,

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This can be generalised to the intersection of n events:

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2) \cdots \mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1}).$$

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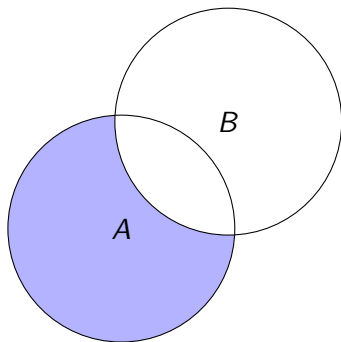
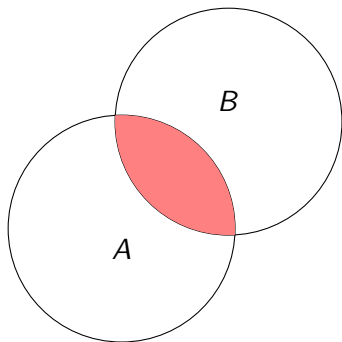
This follows immediately from the previous Theorem (which in turn follows immediately from the definition of conditional probability).

The Law of Total Probability (revisited)

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Recall that, for any two events A and B , the Law of Total Probability states that

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c).$$



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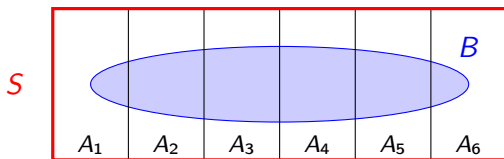
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Conclusion:

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- ▶ *All probabilities are conditional probabilities.*

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- ▶ Independence is a symmetric relation: if A is independent of B , B is independent of A .

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Events A and B are *independent* if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

- ▶ If $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, this is equivalent to

$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \quad \text{and} \quad \mathbb{P}(B \mid A) = \mathbb{P}(B).$$

- ▶ Independence is a symmetric relation: if A is independent of B , B is independent of A .
- ▶ Warning: independence \neq disjointness. In fact, disjoint events can only be independent if $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$. (Why?)

Conditional independence

Definition

Events A and B are said to be *conditionally independent* given a third event E if

$$\mathbb{P}(A \cap B \mid E) = \mathbb{P}(A \mid E)\mathbb{P}(B \mid E).$$

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Health warning: independence does *not* imply conditional independence and vice versa.

Conditional independence and complements

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Conditional independence and complements

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- ▶ Let W be the event that you work hard and let G be the event that you receive a good grade.

Conditional independence and complements

- ▶ Suppose there are two types of teachers: those who give grades that reflect student effort (E), and those who randomly assign grades, regardless of student effort (E^c).
- ▶ Let W be the event that you work hard and let G be the event that you receive a good grade.
- ▶ Then W and G are conditionally independent given E^c , but they are not conditionally independent given E .

Conditional independence $\not\Rightarrow$ independence

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- ▶ You have one fair coin and one biased coin which lands Heads with probability $3/4$.

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- ▶ You pick one of the coins at random, without knowing which one you've chosen, and flip it several times.
- ▶ Conditional on choosing either the fair or the biased coin, the coin flips are independent.
- ▶ However, the coin flips are not unconditionally independent: without knowing which coin we've chosen, each flip gives us new data from which we can predict outcomes of future tosses.
- ▶ (Think about the definition of independence.)

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- ▶ Suppose my friends Alice and Ben are the only two people who call me on my mobile phone.
- ▶ Each day, they decide independently whether to call me that day.
- ▶ Let A be the event that Alice calls me tomorrow and let B be the event that Ben calls me tomorrow.
- ▶ Then A and B are unconditionally independent, with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$.

Independence \nRightarrow conditional independence

- ▶ Suppose my friends Alice and Ben are the only two people who call me on my mobile phone.
- ▶ Each day, they decide independently whether to call me that day.
- ▶ Let A be the event that Alice calls me tomorrow and let B be the event that Ben calls me tomorrow.
- ▶ Then A and B are unconditionally independent, with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$.
- ▶ However, given that I receive exactly one call tomorrow (C), A and B are no longer independent:

$$\mathbb{P}(A \mid C) > 0, \quad \text{but} \quad \mathbb{P}(A \mid C \cap B) = 0.$$

Problem sheet 1: preamble

```
# Clear environment, set working directory  
rm(list = ls())  
setwd("/Users/Elias/Documents/.../QM/Problem sheets")
```

```
# Load tidyverse packages  
library(tidyverse)
```

Question A1

```
# Construct die and sample space  
die <- 1:6  
S <- expand.grid(die, die, die)
```

Question A2

```
# Create new variable  
S <- S %>%  
  mutate(Value = Var1 + Var2 + Var3)
```

Question A3

```
# P(A) = # elements in A / # elements in S  
sum(S$Value == 12) / nrow(S)  
  
## [1] 0.1157407
```

Question A4

```
# What if dice are biased?
```

```
Prob <- c("1" = 1/8, "2" = 1/8, "3" = 1/8,  
          "4" = 1/8, "5" = 1/8, "6" = 3/8)
```

Question A4

```
# What if dice are biased?
```

```
Prob <- c("1" = 1/8, "2" = 1/8, "3" = 1/8,  
          "4" = 1/8, "5" = 1/8, "6" = 3/8)
```

```
# Assign individual and joint probabilities
```

```
S$Prob1 <- Prob[S$Var1]  
S$Prob2 <- Prob[S$Var2]  
S$Prob3 <- Prob[S$Var3]  
S$Prob_joint <- S$Prob1 * S$Prob2 * S$Prob3
```

Question A5

```
# Extract event A and calculate probability  
A <- subset(S, Value == 12)  
sum(A$Prob_joint)  
  
## [1] 0.1074219
```


Question B1

```
# Load data sets  
kenya <- read_csv("kenya.csv")  
sweden <- read_csv("sweden.csv")  
world <- read_csv("world.csv")
```

Question B2

```
# Inspect
summary(kenya)
summary(sweden)
summary(world)

glimpse(kenya)
glimpse(sweden)
glimpse(world)

head(kenya)
head(sweden)
head(world)

print(kenya, n = 30)
print(sweden, n = 30)
print(world, n = 30)
```

Question B3

```
# Calculate age-specific fertility rate
asfr <- function(data) {
  data %>%
    mutate(
      asfr = births / py.women) %>%
    select(period, age, asfr) %>%
    data.frame() # Convert tibble to data frame
}
```

Question B3 (cont.)

```
asfr(kenya)[c(4:10, 19:25), ]
```

##	period	age	asfr
## 4	1950-1955	15-19	0.16884585
## 5	1950-1955	20-24	0.35596942
## 6	1950-1955	25-29	0.34657814
## 7	1950-1955	30-34	0.28946367
## 8	1950-1955	35-39	0.20644016
## 9	1950-1955	40-44	0.11193267
## 10	1950-1955	45-49	0.03905205
## 19	2005-2010	15-19	0.10057087
## 20	2005-2010	20-24	0.23583536
## 21	2005-2010	25-29	0.23294721
## 22	2005-2010	30-34	0.18087964
## 23	2005-2010	35-39	0.13126805
## 24	2005-2010	40-44	0.05626214
## 25	2005-2010	45-49	0.03815044

Question B3 (cont.)

```
asfr(sweden)[c(4:10, 19:25), ]
```

##		period	age	asfr
## 4	1950-1955	15-19	0.0389089519	
## 5	1950-1955	20-24	0.1277108826	
## 6	1950-1955	25-29	0.1252436647	
## 7	1950-1955	30-34	0.0873641591	
## 8	1950-1955	35-39	0.0486037714	
## 9	1950-1955	40-44	0.0162101857	
## 10	1950-1955	45-49	0.0013418290	
## 19	2005-2010	15-19	0.0059709097	
## 20	2005-2010	20-24	0.0507320271	
## 21	2005-2010	25-29	0.1162085625	
## 22	2005-2010	30-34	0.1322744621	
## 23	2005-2010	35-39	0.0625923991	
## 24	2005-2010	40-44	0.0121600765	
## 25	2005-2010	45-49	0.0006143942	

Question B3 (cont.)

```
asfr(world)[c(4:10, 19:25), ]
```

##		period	age	asfr
## 4	1950-1955	15-19	0.090295213	
## 5	1950-1955	20-24	0.237633702	
## 6	1950-1955	25-29	0.252452289	
## 7	1950-1955	30-34	0.204164096	
## 8	1950-1955	35-39	0.138105344	
## 9	1950-1955	40-44	0.063608319	
## 10	1950-1955	45-49	0.015190644	
## 19	2005-2010	15-19	0.048489719	
## 20	2005-2010	20-24	0.151971307	
## 21	2005-2010	25-29	0.146980966	
## 22	2005-2010	30-34	0.093813813	
## 23	2005-2010	35-39	0.046689639	
## 24	2005-2010	40-44	0.016268995	
## 25	2005-2010	45-49	0.004510245	

Question B4

```
# Calculate total fertility rate
tfr <- function(data) {
  out <- asfr(data)
  out %>%
    group_by(period) %>%
    summarise(
      tfr = 5 * sum(asfr))
}
```

Question B4 (cont.)

```
tfr(kenya)
```

```
## # A tibble: 2 x 2
##   period      tfr
##   <chr>      <dbl>
## 1 1950-1955  7.59
## 2 2005-2010  4.88
```

```
tfr(sweden)
```

```
## # A tibble: 2 x 2
##   period      tfr
##   <chr>      <dbl>
## 1 1950-1955  2.23
## 2 2005-2010  1.90
```


Question B4 (cont.)

```
tfr(world)
```

```
## # A tibble: 2 x 2
```

```
##   period      tfr
```

```
##   <chr>      <dbl>
```

```
## 1 1950-1955  5.01
```

```
## 2 2005-2010  2.54
```

Question B5

```
# Calculate age-specific death rate
asdr <- function(data) {
  data %>%
    mutate(
      # Convert rates to per 1000 population
      asdr = 1000 * deaths / (py.men + py.women)) %>%
    select(period, age, asdr) %>%
    data.frame() # Convert tibble to data frame
}
```

Question B5 (cont.)

```
sample_n(asdr(kenya), 10)
```

##		period	age	asdr
## 1		2005-2010	70-79	61.261551
## 2		1950-1955	25-29	8.838750
## 3		1950-1955	40-44	12.633744
## 4		1950-1955	10-14	5.972093
## 5		1950-1955	80+	200.016381
## 6		2005-2010	15-19	2.942986
## 7		2005-2010	10-14	2.918895
## 8		2005-2010	80+	158.620510
## 9		1950-1955	70-79	93.683927
## 10		2005-2010	5-9	2.911301

Question B5 (cont.)

```
sample_n(asdr(sweden), 10)
```

##		period	age	asdr
## 1		2005-2010	40-44	1.0392562
## 2		1950-1955	40-44	2.5095541
## 3		2005-2010	50-54	2.9887149
## 4		1950-1955	20-24	1.0177339
## 5		2005-2010	10-14	0.1135496
## 6		1950-1955	80+	167.8170255
## 7		2005-2010	30-34	0.5057066
## 8		2005-2010	55-59	4.7099135
## 9		1950-1955	70-79	59.9823093
## 10		1950-1955	25-29	1.1140910

Question B5 (cont.)

```
sample_n(asdr(world), 10)
```

##		period	age	asdr
## 1		1950-1955	10-14	4.261869
## 2		1950-1955	20-24	5.891020
## 3		1950-1955	55-59	24.265320
## 4		1950-1955	30-34	7.132501
## 5		2005-2010	55-59	10.477192
## 6		2005-2010	30-34	2.623982
## 7		2005-2010	70-79	47.457519
## 8		1950-1955	80+	184.364978
## 9		1950-1955	15-19	4.752908
## 10		2005-2010	35-39	3.031563

Question B6

```
# Collect ASFR and ASDR for each country  
ken <- left_join(asfr(kenya), asdr(kenya))  
swe <- left_join(asfr(sweden), asdr(sweden))  
wor <- left_join(asfr(world), asdr(world))
```

Question B6

```
# Collect ASFR and ASDR for each country  
ken <- left_join(asfr(kenya), asdr(kenya))  
swe <- left_join(asfr(sweden), asdr(sweden))  
wor <- left_join(asfr(world), asdr(world))
```

```
# Create one data frame with all results  
df <- rbind(ken, swe, wor)  
df$country <- c(rep("Kenya", 30),  
                rep("Sweden", 30), rep("World", 30))
```

Question B6 (cont.)

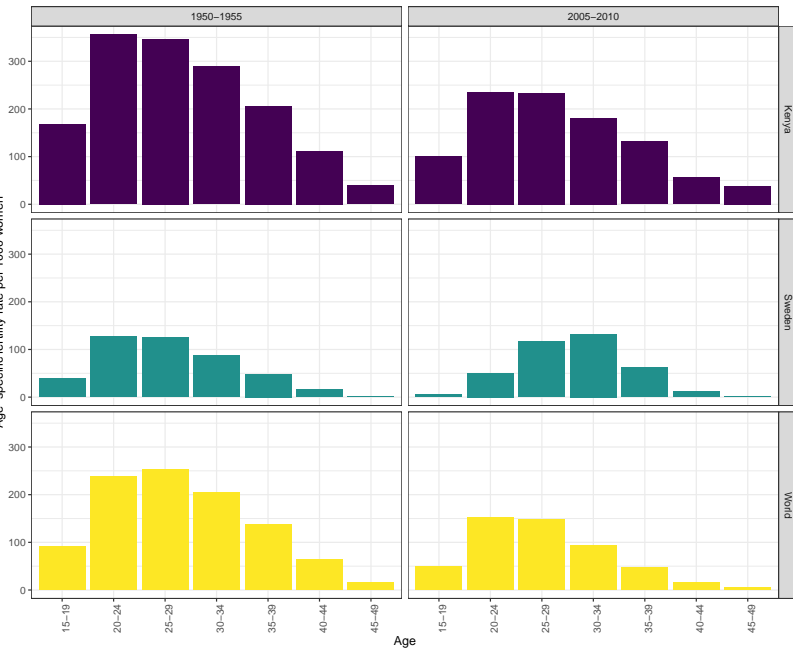
```
# Transform age groups to ordered factor
df$age <- factor(df$age,
                 levels = c("0-4", "5-9", "10-14",
                             "15-19", "20-24", "25-29",
                             "30-34", "35-39", "40-44",
                             "45-49", "50-54", "55-59",
                             "60-69", "70-79", "80+"))

# Age groups for reproductive age range
age_groups <- c("15-19", "20-24", "25-29",
                "30-34", "35-39", "40-44", "45-49")
```


Question B6 (cont.)

```
# Visualise ASFR
g1 <- ggplot(subset(df, age %in% age_groups), # Age range
             aes(age, 1000 * asfr, # Modify rate
                 fill = country)) + # Colour code
  geom_col() + # Show as columns
  labs( # Axis labels
    x = "Age",
    y = "Age-specific fertility rate per 1000 women") +
  scale_fill_viridis_d() + # Choose a nice colour palette
  facet_grid(country ~ period) + # Stratify
  theme_bw() + # Remove redundant lines
  theme( # Avoid cluttering
    legend.position = "none",
    axis.text.x = element_text(angle = 90))
```

Age-specific fertility rate per 1000 women



Question B6 (cont.)

```
# Visualise ASDR
g2 <- ggplot(df, aes(age, asdr,
                      fill = country)) + # Colour code
  geom_col() + # Show as columns
  scale_y_continuous(breaks = seq(0, 200, 50)) + # Y-axis
  labs( # Axis labels
    x = "Age",
    y = "Age-specific death rate per 1000 population") +
  scale_fill_viridis_d(option = "plasma") + # Colour
  facet_grid(country ~ period) + # Stratify
  theme_bw() + # Remove redundant lines
  theme( # Avoid cluttering
    legend.position = "none",
    axis.text.x = element_text(angle = 90))
```

