Quantitative Methods

Human Sciences, 2020–21

Elias Nosrati

Lecture 4: 5 November 2020

Today

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- Recap on random variables and probability distributions

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- Key features of probability distributions

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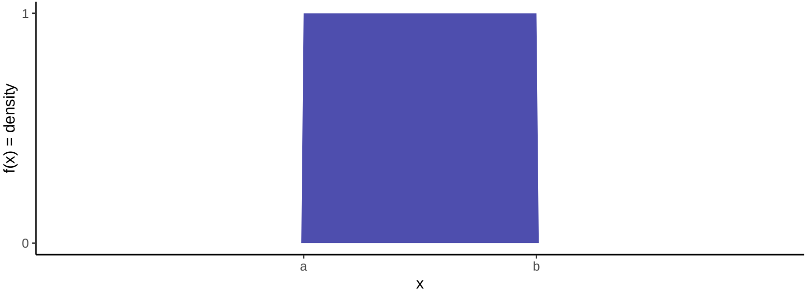
- Key features of probability distributions
- Problem sheet 3 (tutorial)

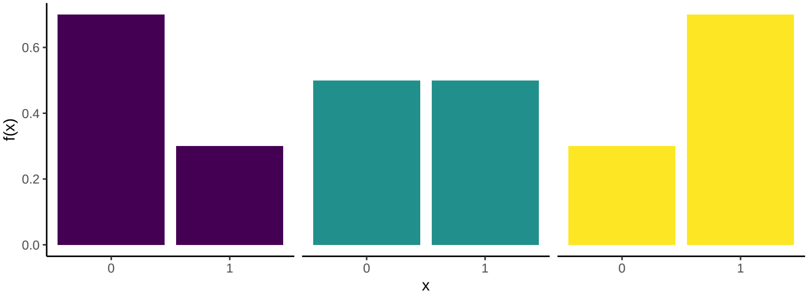
Uniform distribution

Uniform distribution



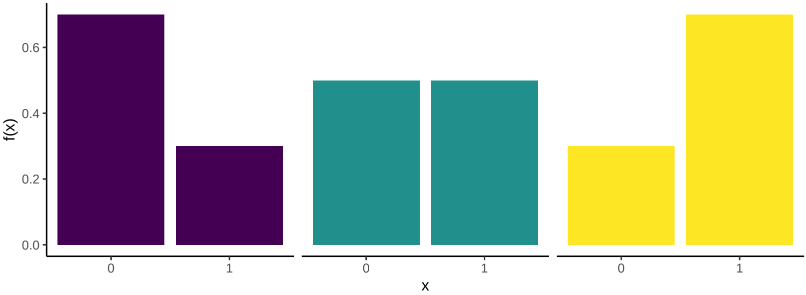
A random variable X has a *Uniform distribution* on the interval (a, b) if it takes every value within the interval (a, b) with equal likelihood and any value outside this interval with zero likelihood.

Bernoulli distribution



A random variable X has a *Bernoulli distribution* if it takes only two distinct and mutually exclusive values.

Bernoulli distribution



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Example

X is the number of Tails obtained when flipping a coin.

Binomial distribution

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X is the number of Tails obtained when flipping a coin n times.

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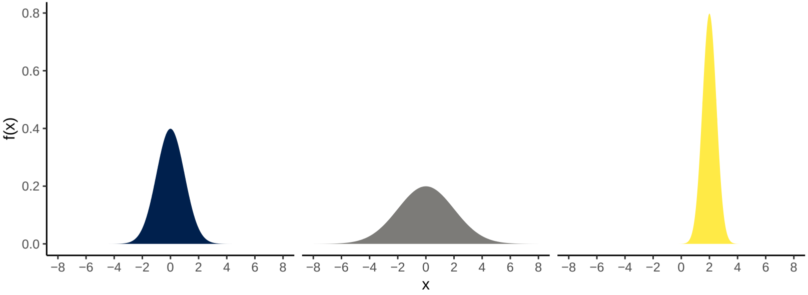
Remark

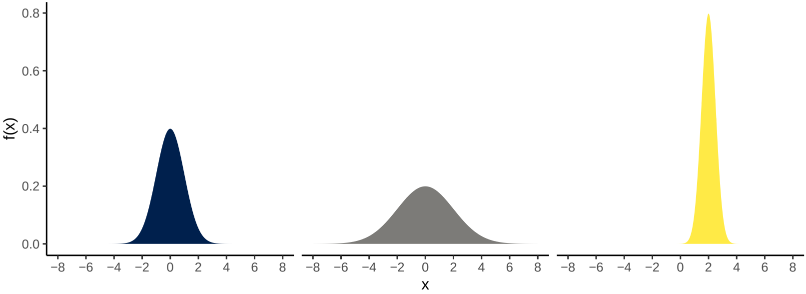
Independence assumption: the number of global pandemics per decade may not follow a Poisson distribution if the first wave of the pandemic increases the probability of follow-up waves of similar magnitude.

Normal distribution



A continuous random variable X has a *Normal distribution* if its probability density function is shaped like a bell curve.

Normal distribution



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What are the differences between these three Normal distributions?

Expected value

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Definition

The *expected value* (or *population mean*) of a random variable X is a weighted average of the possible values that X can take, weighted by their probability densities:

$$\mathbb{E}(X) = \begin{cases} \sum x\, f(x) & \text{if } X \text{ is discrete,} \\ \int x\, f(x)\, dx & \text{if } X \text{ is continuous.} \end{cases}$$

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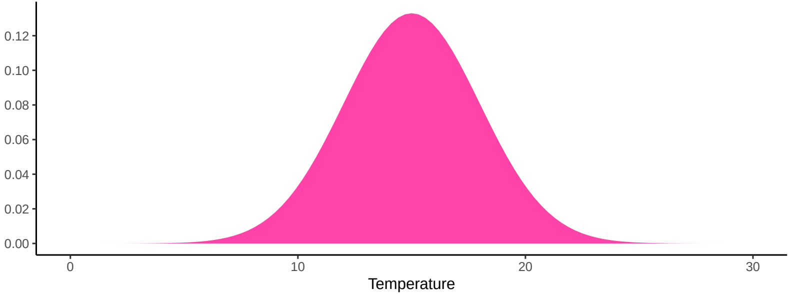
Note that, in this example, X never equals its expected value (you can never roll 3.5!).

Expected value: continuous case



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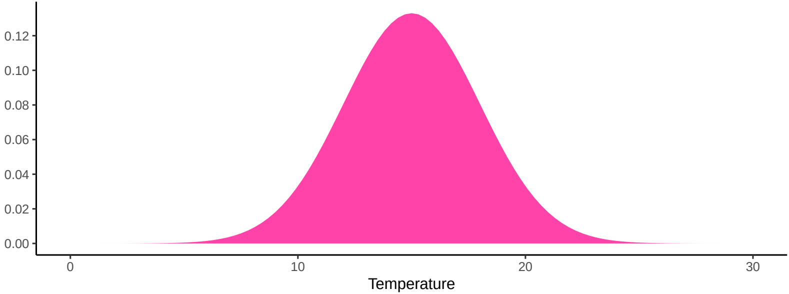

What to do when X takes on an infinite number of possible values?

- Solve analytically.

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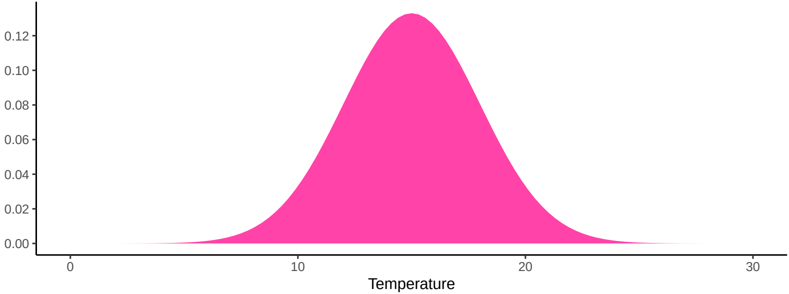

What to do when X takes on an infinite number of possible values?

- Solve analytically.
- Use the Law of Large Numbers: as the sample size increases, the sample mean converges to the population mean or expected value.

Law of Large Numbers: motivating example

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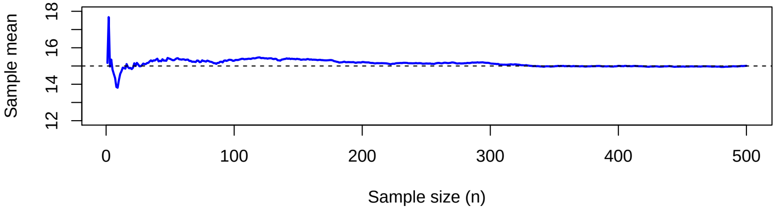

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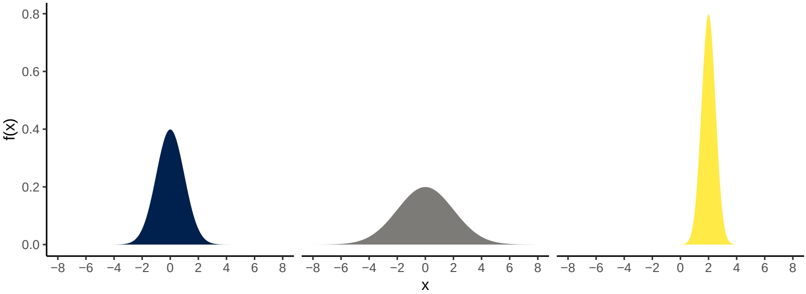
$$\overline{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n} \rightsquigarrow \mathbb{E}(X).$$

Normal distribution



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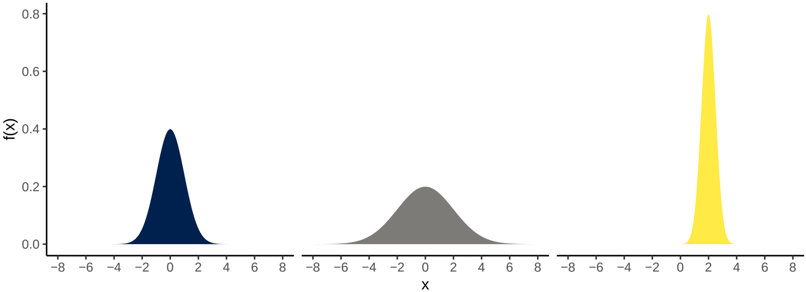
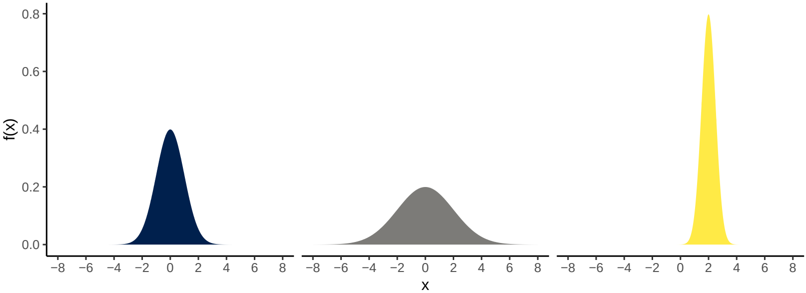


- Expectation as a measure of central tendency.

Normal distribution



- Expectation as a measure of central tendency.
- Variance as a measure of variability or spread.

Variance and standard deviation

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The square root of $\mathbb{V}(X)$ is known as the *standard deviation* of X.

Expectation and variance in **R**

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X <- rnorm(1000, 15, 3)
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```r
sd(X)
```

```
## [1] 3.015831
```

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- Equivalently, suppose we gather a random sample of observations $X_1, \ldots, X_n$ and calculate the sample mean $\overline{X} = \frac{1}{n} \sum_i X_i$.
- If this procedure is performed many times, the Central Limit Theorem states that the probability distribution of $\overline{X}$ will be Normally distributed.

The Central Limit Theorem in **R**

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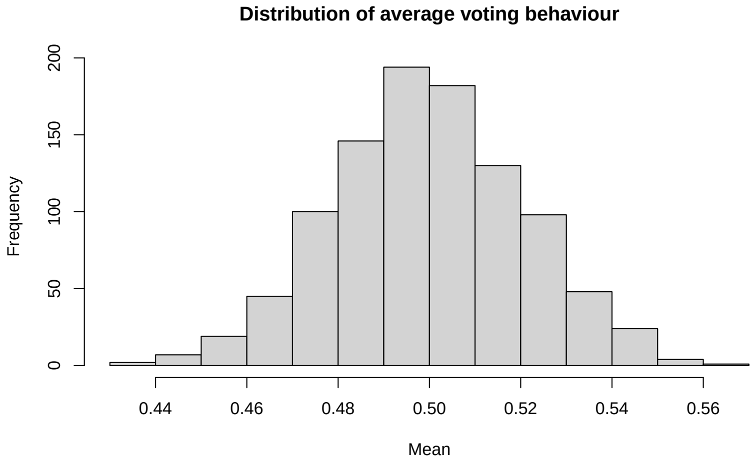
```r
# To be filled
prop_vote <- rep(NA, 1000)

# Repeat experiment 1000 times
for (i in 1:1000) {
  # Sample 600 people and ask about voting
  # Yes or no: Bernoulli random variable
  sample <- rbinom(600, 1, 0.5)
  # Save proportion of voters
  prop_vote[i] <- mean(sample)
}
```

The Central Limit Theorem in **R** (cont.)

```
hist(prop_vote,
     main = "Distribution of average voting behaviour",
     xlab = "Mean")
```



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Note that $\mathrm{Cov}(X, X) = \mathbb{V}(X)$ and that $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$.

Measures of association (cont.)

Two random variables that have zero covariance are *uncorrelated*:

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Definition

The *correlation* between two random variables X and Y is

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathbb{V}(X)\mathbb{V}(Y)}}.$$

Correlation and dependence