Quantitative Methods

Human Sciences, 2020–21

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Lecture 2: 22 October 2020

Today

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- Recap on probability and counting: the birthday problem.

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- Introduction to conditional probability.

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- Problem sheet 1 (tutorial).

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- Assume each person's birthday is equally likely to be any of the 365 days of the year and assume people's birthdays are independent.
- What is the probability that at least one pair of people in the group have the same birthday?
- Hint: Recall that $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$.

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- In this room, $\mathbb{P}(\text{at least one birthday match}) \approx 4\%$.

The birthday problem in **R**

```
# Create a function
pmatch <- function(n) {
        1 - prod(365:(365 - n + 1)) / (365 ^ n)
}
```

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# Placeholder
probs <- NULL

# For loop
for (i in 1:70) {
  probs[i] <- pmatch(i)
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```r
# Alternative method
probs <- sapply(1:70, pmatch)
```

The birthday problem in **R** (cont.)

```
save <- data.frame("n" = 1:70, "prob" = probs)
```

The birthday problem in **R** (cont.)

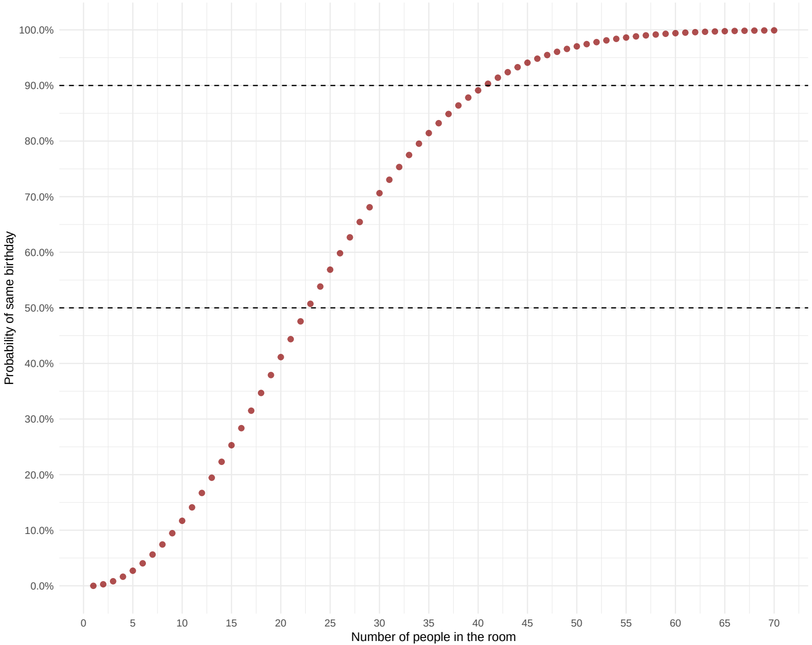
```r
save <- data.frame("n" = 1:70, "prob" = probs)
```

```r
head(save)

##   n        prob
## 1 1 0.000000000
## 2 2 0.002739726
## 3 3 0.008204166
## 4 4 0.016355912
## 5 5 0.027135574
## 6 6 0.040462484
```

The birthday problem in **R** (cont.)

```r
ggplot(save, aes(n, prob)) +
  geom_point( # Modify points
    size = 2,
    colour = "darkred",
    alpha = 0.7) +
  labs( # Axis labels
    x = "Number of people in the room",
    y = "Probability of same birthday") +
  scale_x_continuous( # Modify X-axis
    breaks = seq(0, 70, 5)) +
  scale_y_continuous( # Modify Y-axis
    breaks = seq(0, 1, 0.1),
    label = scales::percent) +
  geom_hline( # When is P(match) > 0.5 or 0.9?
    yintercept = c(0.5, 0.9), linetype = "dashed") +
  theme_minimal() # Remove redundant lines
```



Probability of same birthday
100.0%
90.0%
80.0%
70.0%
60.0%
50.0%
40.0%
30.0%
20.0%
10.0%
0.0%
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
Number of people in the room

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- Whenever new evidence is observed, we acquire information that may affect out uncertainties.
- Conditional probability allows us to update our beliefs in light of new evidence.
- "Conditioning is the soul of statistics" (Blitzstein and Hwang, 2019: 46).

Defining conditional probability

Definition

If A and B are events, with $\mathbb{P}(B) > 0$, then the *conditional probability* of A given B is

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

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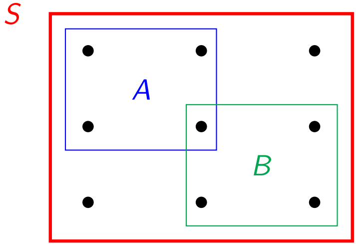
- Note that $\mathbb{P}(A \mid A) = 1$.

Visualising a conditional probability

Suppose we have a sample space S and two events A and B:



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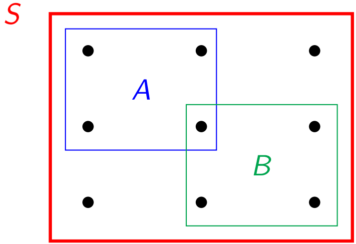

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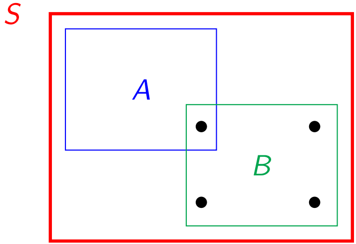

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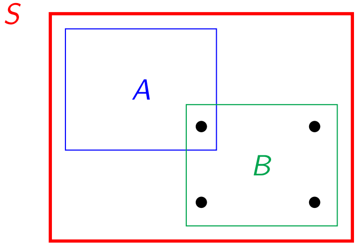

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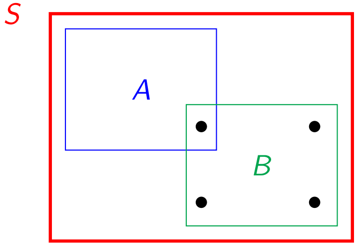

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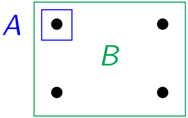
- Divide by $\mathbb{P}(B)$, the total mass of the outcomes in B.

Visualising a conditional probability (cont.)

The updated probability measure assigned to the event A is the conditional probability

$$\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B)/\mathbb{P}(B).$$



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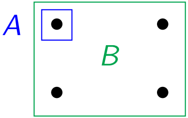

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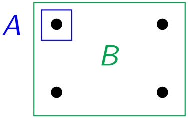

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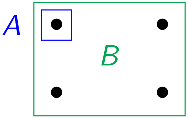

- We have updated our probabilities in accordance with observed evidence.
- Outcomes that contradict the evidence are discarded.
- Relative measures of uncertainty are redistributed amongst remaining possible outcomes.

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- Thus

$$\mathbb{P}(A \mid B) = n_{AB}/n_B = (n_{AB}/n)/(n_B/n) = \mathbb{P}(A \cap B)/\mathbb{P}(B).$$

Joint probability and conditional probability

Theorem

For any events A and B with positive probabilities,

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A \mid B) = \mathbb{P}(A)\mathbb{P}(B \mid A).$$

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This can be generalised to the intersection of n events:

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2) \cdots \mathbb{P}(A_n|A_1 \cap \cdots \cap A_n).$$

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This follows immediately from the previous Theorem (which in turn follows immediately from the definition of conditional probability).

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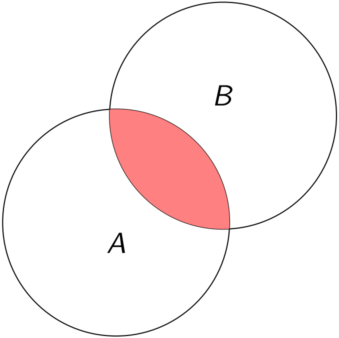
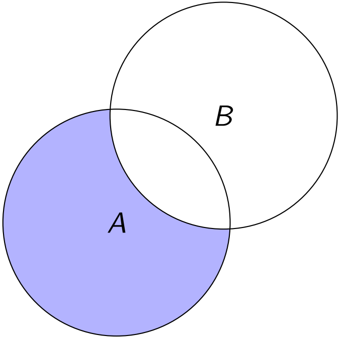
Recall that, for any two events A and B, the Law of Total Probability states that

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c).$$





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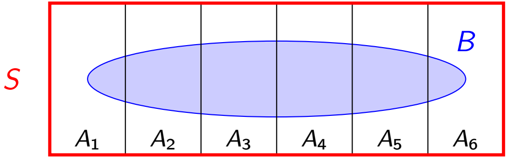

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&\approx 16\%.
\end{aligned}$$

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Conclusion:

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- *All probabilities are conditional probabilities.*

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$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \quad \text{and} \quad \mathbb{P}(B \mid A) = \mathbb{P}(B).$$

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- Independence is a symmetric relation: if A is independent of B, B is independent of A.
- Warning: independence $\neq$ disjointness. In fact, disjoint events can only be independent if $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$. (Why?)

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Health warning: independence does *not* imply conditional independence and vice versa.

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- Let W be the event that you work hard and let G be the event that you receive a good grade.
- Then W and G are conditionally independent given E^c, but they are not conditionally independent given E.

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- (Think about the definition of independence.)

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- Then A and B are unconditionally independent, with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$.
- However, given that I receive exactly one call tomorrow (C), A and B are no longer independent:

$$\mathbb{P}(A \mid C) > 0, \quad \text{but} \quad \mathbb{P}(A \mid C \cap B) = 0.$$