Quantitative Methods

Human Sciences, 2020–21

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Lecture 7: 26 November 2020

Today

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- Statistical models (continued).

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- Introduction to causal inference.

Model notation

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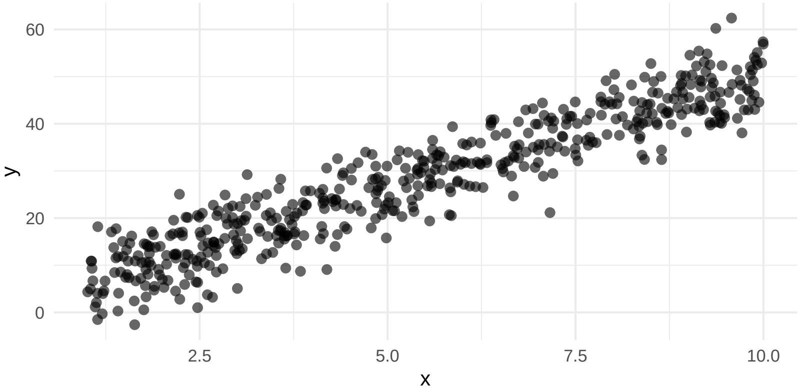

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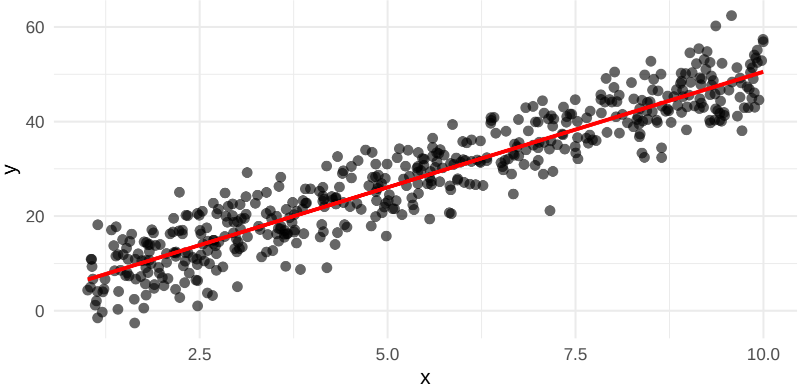

Model notation (cont.)

```
# Specify regression line
model <- lm(y ~ x, data)
model

##
## Call:
## lm(formula = y ~ x, data = data)
##
## Coefficients:
## (Intercept)            x
##       1.647        4.888
```

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##     1     2     3     4     5     6
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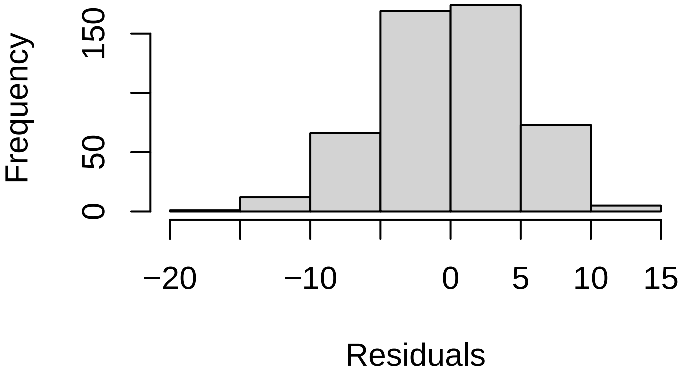
```r
summary(residuals)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  -15.49   -3.29    0.02    0.00    3.47   13.95
```

Model notation (cont.)

```r
# Plot residuals
hist(Residuals, main = "")
```



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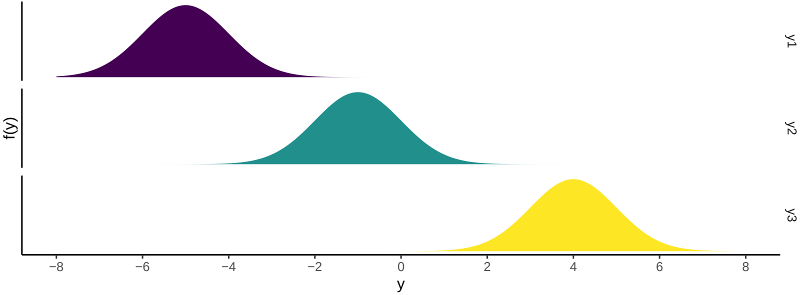

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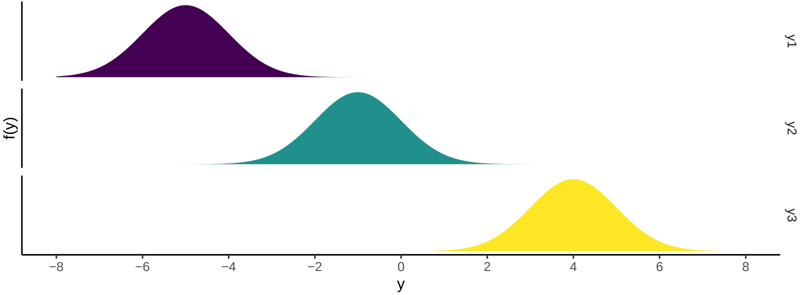

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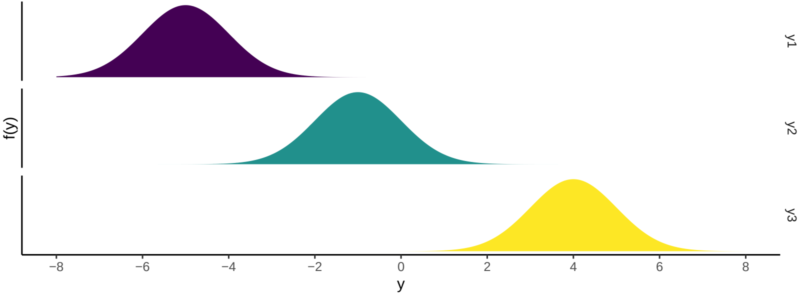
- (2) is the systematic component of the model.



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- If the CEF is not exactly linear, then the regression of Y on X gives the *best linear approximation* to this non-linear CEF (this is done by minimising the squared deviation between the values of the linear model and those of the CEF).

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- If Y and X are uncorrelated, then $\beta = 0$.
- But if $\beta \neq 0$, does that mean X *causes* Y?

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- How do we know if an association is causal?

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- The action is referred to as an *intervention*, an *exposure*, or a *treatment*.

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- The causal effect itself is defined as $Y_1 - Y_0$.

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- The *population average treatment effect* (PATE) is defined as a contrast between expected values of counterfactual outcomes:

$$\mathbb{E}(Y_1 - Y_0) = \mathbb{E}(Y_1) - \mathbb{E}(Y_0).$$

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- Have you obtained your quantity of interest?

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- Hence, *causation* is defined by a contrast in the same population (or two otherwise identical populations) under two different values of T.
- Key challenge: how do we ensure that the treatment and control groups are *exchangeable* (they have the same *pre-treatment characteristics*)?

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- **Quiz**: What is the difference between an observed outcome and a potential (counterfactual) outcome?