# Quantitative Methods

## Human Sciences, 2020–21

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Lecture 6: 19 November 2020

# Today

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- Statistical models and the problem of inference.

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- Introduction to linear regression models.

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- Model might not be "true" or entirely realistic, but does it provide us with useful information about some quantity of interest?

# Data generating process

You are given a data set called `lifexp`:

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## [1] 69 74 78 73 70 75

tail(lifexp)

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summary(lifexp)

##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
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What is the underlying data generating process? What are the key features of the underlying distribution?

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- But how do we estimate $\mu$?

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- A *relative* (not absolute) measure of uncertainty.

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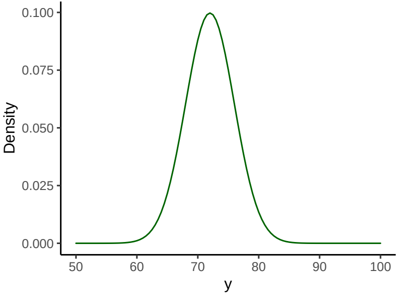


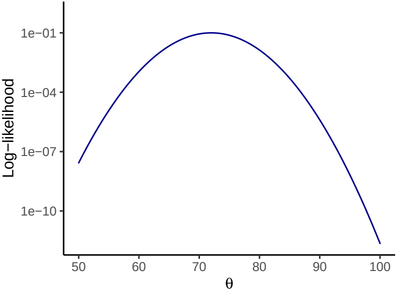

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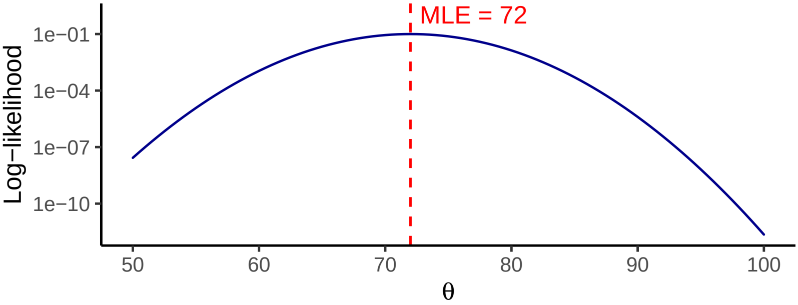

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- Practical difference: minor unless prior is actually important.

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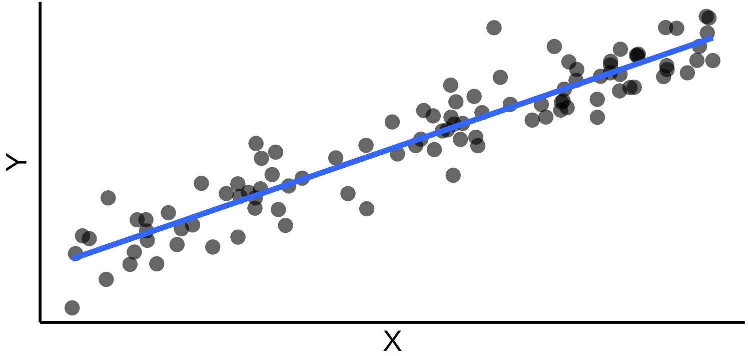
- Bayesian: use prior information about $p$ (e.g., previous poll data or evidence of rigged election).
- Challenge: what is the exact value of the prior?
- Typical solution: try a range of possible values.

# A non-theory of inference: ordinary least squares



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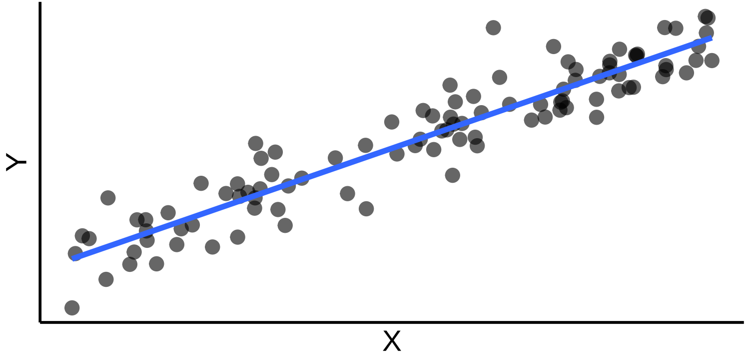


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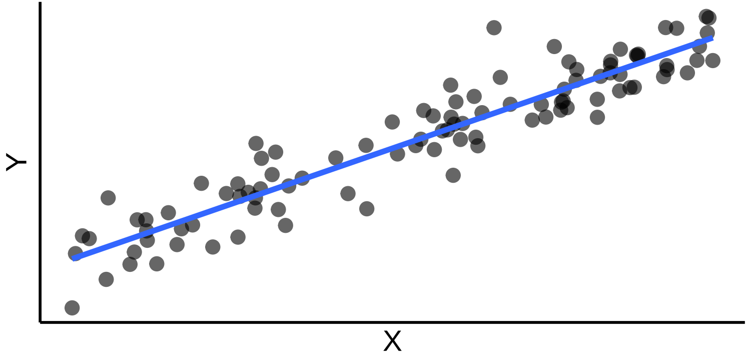


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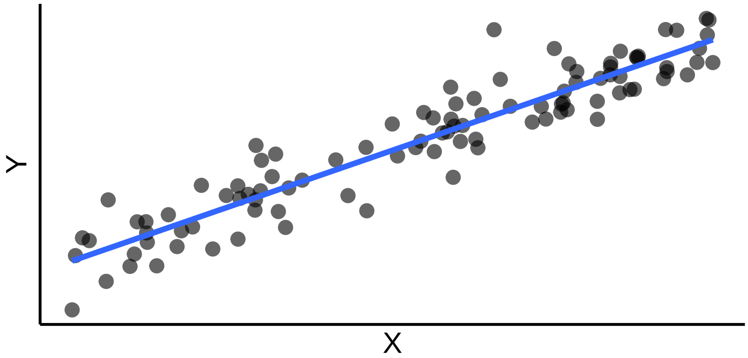


- A common approach: minimise ("*least*") the squared deviations ("*squares*") of the observed data from the estimated parameter value.
- Classical example: how to fit a line to a cloud of points.
- The corresponding parameter estimate (slope of line) is known as the *ordinary least squares* (OLS) estimate.

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- Bonus: easy to implement.

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- $\mu$: intercept (mean).
- $\epsilon$: error (disturbance) term, with $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

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```r
mean(lifexp)

## [1] 71.932
```