Quantitative Methods

Human Sciences, 2020–21

Elias Nosrati

Lecture 1: 15 October 2020

General setup

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- Statistics and data science in **R**.

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- Weekly problem sheets to be submitted day before teaching.
- All materials available on Canvas and on eliasnosrati.github.io.

Syllabus

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- Descriptive inference (data discovery and pattern recognition).

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- Introduction to **R** (tutorial).

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- And much more!

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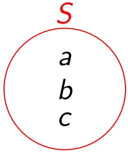

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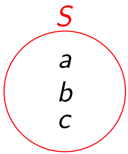

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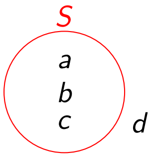

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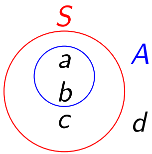

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- $a \in S$: a is a member (or element) of S.
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- If $A = \{a, b\}$, then $A \subset S$: A is a *subset* of S.

Set theory: definition of a subset

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A set A is a *subset* of a set B, or $A \subset B$, if every member of A is also a member of B:

$$(x \in A) \Rightarrow (x \in B)$$

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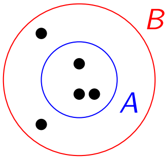

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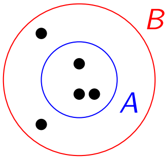

Remark

If $A \subset B$ and $A \neq B$, then A is said to be a *proper subset* of B. Otherwise, we often write $A \subseteq B$.

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- The set of real numbers $\mathbb{R}$ (think of the number line).

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Example

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Note that for any set S, $S \subseteq S$ and $\varnothing \subseteq S$. Hence $S \in \mathcal{P}(S)$ and $\varnothing \in \mathcal{P}(S)$ for all S. Note also for any set with n elements, its power set has 2^n elements (why?).

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Remark

We say that A and B are *disjoint* (mutually exclusive) is they have no elements in common, i.e., if $A \cap B = \varnothing$.

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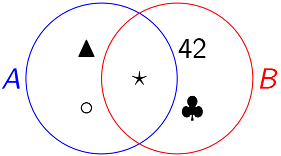

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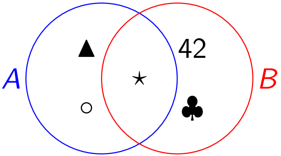

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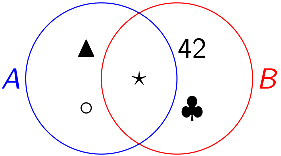

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Example

Let $A = \{\text{Denmark, Norway, Sweden}\}$ and
$B = \{\text{Botswana, Namibia, Zimbabwe}\}$. Then $A \cap B = \{\} = \varnothing$.

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Example

Let A be the set of all individuals named Henry and let B be the set of all individuals with brown hair. Then A^c is the set of all people whose name is not Henry and $A - B$ is the set of all people named Henry who do not have brown hair.

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Definition

The *sample space* S of an experiment is the set of all possible outcomes of the experiment. An *event* A is a subset of this sample space, and if the actual outcome is an element of A, we say that A *occurred*.

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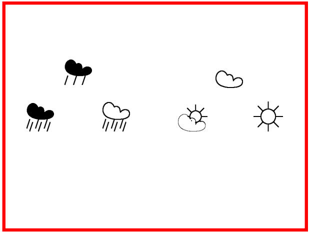

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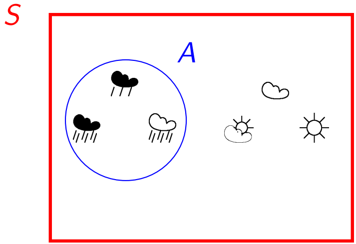

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Answer:

$$B = A_1 \cup \cdots \cup A_{10} = \bigcup_{i=1}^{10} A_i.$$

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Answer:

$$D = (A_1 \cap A_2) \cup \cdots \cup (A_9 \cap A_{10}) = \bigcup_{i=1}^{9} (A_i \cap A_{i+1}).$$

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Remark

Note that $(A \cup B)^c = A^c \cap B^c$. This is one of *De Morgan's laws*. The other is $(A \cap B)^c = A^c \cup B^c$.

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Functions

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Let X and Y be sets. A *function* f from X to Y is a mapping from the *domain* X to the *codomain* Y that assigns a value $f(x) \in Y$ to each $x \in X$:

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Equivalently:

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

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Exercise: Can your prove these properties (using only the three axioms of probability)?

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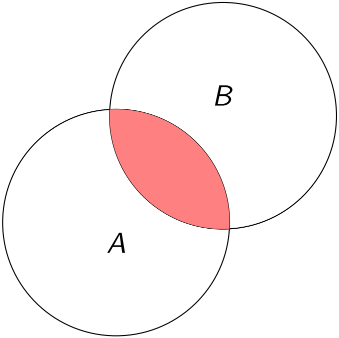

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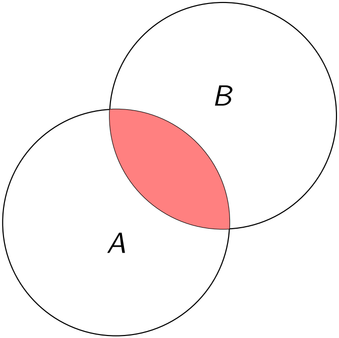

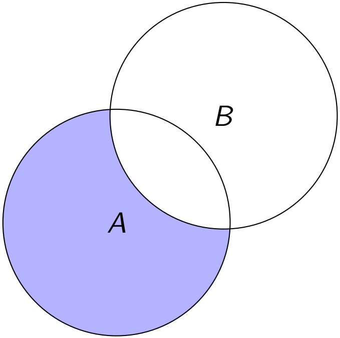

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Unlike a permutation, a combination is a way of choosing k out of n objects *without* regard to order. Formally, this is known as a binomial coefficient:

Definition

A *binomial coefficient* counts the number of subsets of size k for a set of size n. For any non-negative integers k and n, the binomial coefficient, read as "n choose k", is defined as

$$C_k^n = \binom{n}{k} = \frac{P_k^n}{k!} = \frac{n!}{k!(n-k)!}.$$

Since sets are unordered (i.e., $\{x, y\} = \{y, x\}$), this amounts to choosing k out of n objects, without replacement and without distinguishing the order in which they are chosen.

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(*Vandermonde's identity*.)