Quantitative Methods

Human Sciences, 2020–21

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Lecture 3: 29 October 2020

Today

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- Recap on conditional probability: prosecutor's fallacy and the Monty Hall problem.

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- Random variables and their distributions.
- Problem sheet 2 (tutorial).

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- The concluding argument: the probability of Clark's innocence was 1 in 73 million.
- What (if anything) is wrong with this line of reasoning?

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- Re-write $\mathbb{P}(\text{evidence})$ using Law of Total Probability:

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- Note that $\mathbb{P}(\text{guilt}) \approx 0$ and hence

$$\mathbb{P}(\text{innocence} \mid \text{evidence}) \approx \frac{\mathbb{P}(\text{evidence} \mid \text{innocence})\mathbb{P}(\text{innocence})}{\mathbb{P}(\text{evidence} \mid \text{innocence})\mathbb{P}(\text{innocence})} = 1.$$

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- After your choice, Monty Hall opens one door with a goat and asks if you want to switch to the remaining door. Should you switch?
- Solve analytically or by simulation.

The Monty Hall problem: solving by simulation

```
sims <- 10 ^ 5 # Simulations
doors <- 1:3 # Label doors
win_no_switch <- 0 # Win count without switch
win_switch <- 0 # Win count with switch
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```r
for (i in 1:sims) {
  # Pick a door for the car
  car_door <- sample(doors, 1)
  # Pick a door for the contestant
  chosen_door <- sample(doors, 1)
  # If they match, add to "no switch" count
  if (car_door == chosen_door)
    win_no_switch <- win_no_switch + 1
  # Otherwise, add to switch count
  else
    win_switch <- win_switch + 1
}
```

The Monty Hall problem: solving by simulation (cont.)

```
cat("P(car | no switch) =", win_no_switch / sims)

## P(car | no switch) = 0.33102
```

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```
cat("P(car | switch) =", win_switch / sims)

## P(car | switch) = 0.66898
```

How is this possible?

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- Let C_i be the event that the car is behind door i, for $1 \leq i \leq 3$, and let W be the event that you win the car.
- Since the C_i partition the sample space, we can use the Law of Total Probability:

$$\mathbb{P}(W) = \mathbb{P}(W \mid C_1) \times \frac{1}{3} + \mathbb{P}(W \mid C_2) \times \frac{1}{3} + \mathbb{P}(W \mid C_3) \times \frac{1}{3}.$$

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- If C_1, the switching strategy will fail:

$$\mathbb{P}(W \mid C_1) = 0.$$

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- Suppose, without loss of generality, that you have chosen door 1 but decide to switch.
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- Hence, $\mathbb{P}(W) = 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{2}{3}$.

Random variables

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Given an experiment with sample space S, a *random variable* is a function from the sample space S to the set of real numbers $\mathbb{R}$. Thus a random variable X assigns a numerical value $X(s)$ to each possible outcome s of the experiment:

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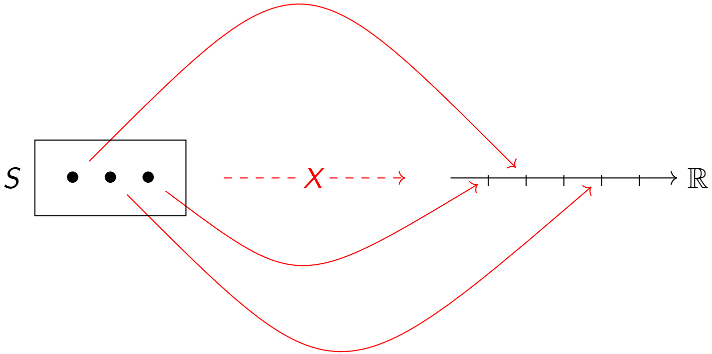
- The function itself is deterministic.
- The randomness derives from the experiment whose outcomes have probabilities described by the probability function $\mathbb{P}(\cdot)$.

Random variables (cont.)



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- Note that two random variables can be defined on the same sample space: the outcomes are the same, but the numerical values assigned to the outcomes are different.

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Example

The number of Heads resulting from a single coin flip is a *binary* random variable: it takes on only two values (0 or 1). The sum of the number of eyes on two dice or the number of votes won by a presidential candidate are also discrete random variables.

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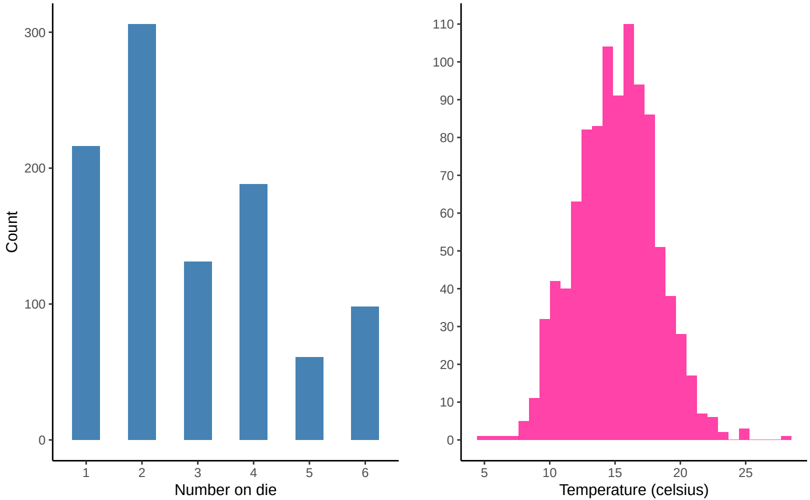
Example

The total amount of rainfall over the next year is a continuous random variable. So is the temperature on a randomly chosen day, or the height of a randomly selected person.

The distribution of random variables



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  2. $\sum f(x) = 1$ (discrete) or $\int f(x) = 1$ (continuous).

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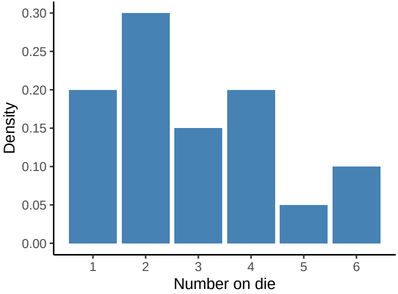
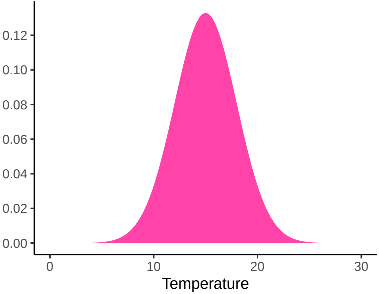
- To visualise the distribution of X, we 'discretise' the variable by creating bins or intervals along the variable of interest and count the number of observations that fall within each bin.
- The *density* for each bin is then defined as

$$\text{density} = \frac{\text{proportion of observations in the bin}}{\text{width of the bin}}.$$

What exactly is a density? (cont.)





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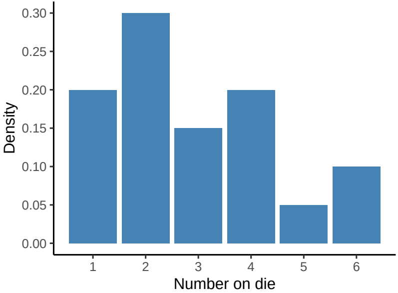

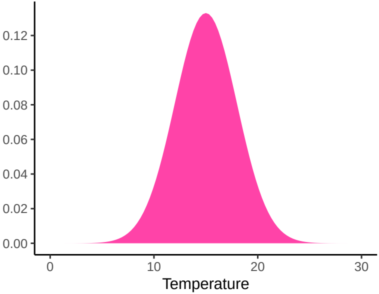

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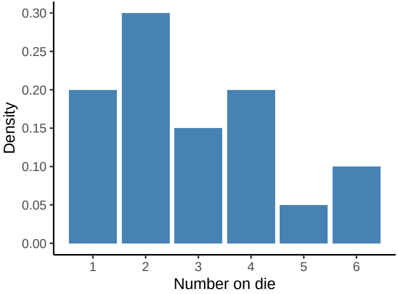

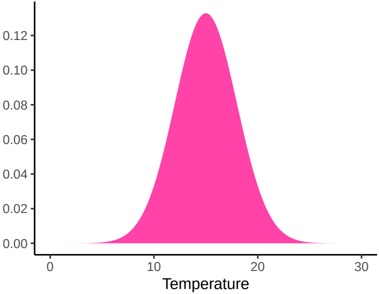

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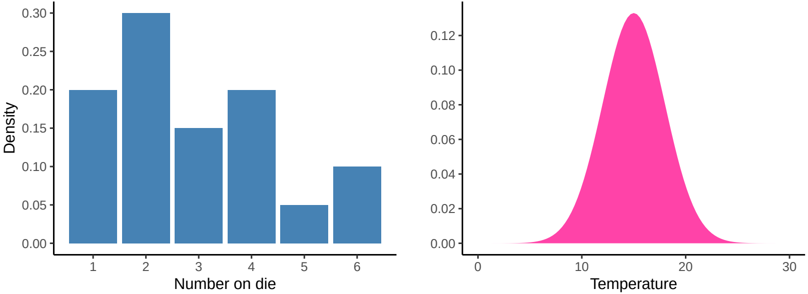

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- For a discrete random variable, probability = density.
- For a continuous random variable, probability = area under curve.