

Quantitative Methods

Human Sciences, 2020–2021

Elias Nosrati

Lecture 1: 15 October 2020

General setup

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- ▶ Statistics and data science in **R**.

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- ▶ Weekly problem sheets to be submitted day before teaching.
- ▶ All materials available on Canvas and on eliasnosrati.github.io.

Syllabus

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- ▶ Descriptive inference (data discovery and pattern recognition).

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- ▶ Some mathematical preliminaries.

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- ▶ Probability theory.
- ▶ Introduction to **R** (tutorial).

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- ▶ And much more!

Set theory: definition of a set

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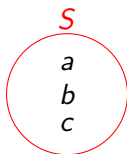
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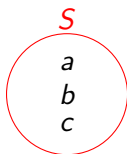
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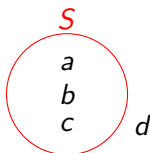
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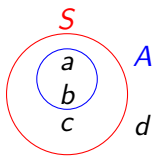
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- ▶ $a \in S$: a is a member (or element) of S .
- ▶ $d \notin S$: d is not a member (or element) of S .
- ▶ If $A = \{a, b\}$, then $A \subset S$: A is a *subset* of S .

Set theory: definition of a subset

Definition

A set A is a *subset* of a set B , or $A \subset B$, if every member of A is also a member of B :

$$(x \in A) \Rightarrow (x \in B)$$

for all elements x .

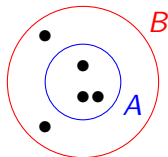
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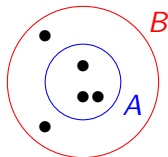
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Remark

If $A \subset B$ and $A \neq B$, then A is said to be a *proper subset* of B . Otherwise, we often write $A \subseteq B$.

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- ▶ The set of real numbers \mathbb{R} (think of the number line).

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Note that for any set S , $S \subseteq S$ and $\emptyset \subseteq S$. Hence $S \in \mathcal{P}(S)$ and $\emptyset \in \mathcal{P}(S)$ for all S . Note also for any set with n elements, its power set has 2^n elements (why?).

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Remark

We say that A and B are *disjoint* (mutually exclusive) if they have no elements in common, i.e., if $A \cap B = \emptyset$.

Set theory: examples of union and intersection

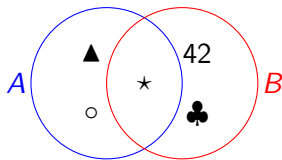
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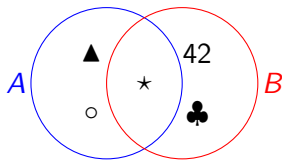
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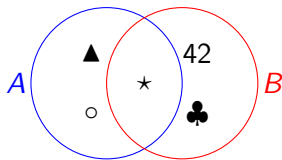


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Then $A \cup B = \{\star, \circ, \blacktriangle, 42, \clubsuit\}$ and $A \cap B = \{\star\}$.

Example

Let $A = \{\text{Denmark, Norway, Sweden}\}$ and $B = \{\text{Botswana, Namibia, Zimbabwe}\}$. Then $A \cap B = \{\} = \emptyset$.

Set theory: complement and set difference

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Example

Let A be the set of all individuals named Henry and let B be the set of all individuals with brown hair. Then A^c is the set of all people whose name is not Henry and $A - B$ is the set of all people named Henry who do not have brown hair.

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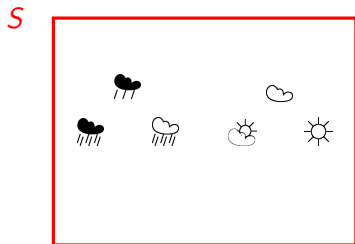
The *sample space* S of an experiment is the set of all possible outcomes of the experiment. An *event* A is a subset of this sample space, and if the actual outcome is an element of A , we say that A *occurred*.

Sample space: weather example

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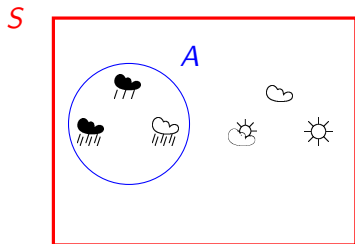
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Now let $A_1 \subset S$ be the event that the outcome of the first flip of the coin is T , i.e., $s_1 = T$:

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Answer:

$$B = A_1 \cup \dots \cup A_{10} = \bigcup_{i=1}^{10} A_i.$$

Sample space: coin toss example (cont.)

Another quiz: Let D be the event that we obtain at least two consecutive T 's. How can we express this in set-theoretical terms?

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Answer:

$$D = (A_1 \cap A_2) \cup \cdots \cup (A_9 \cap A_{10}) = \bigcup_{i=1}^9 (A_i \cap A_{i+1}).$$

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Remark

Note that $(A \cup B)^c = A^c \cap B^c$. This is one of *De Morgan's laws*. The other is $(A \cap B)^c = A^c \cup B^c$.

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Equivalently:

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i).$$

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Exercise: Can you prove these properties (using only the three axioms of probability)?

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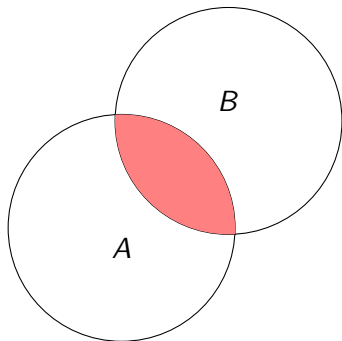
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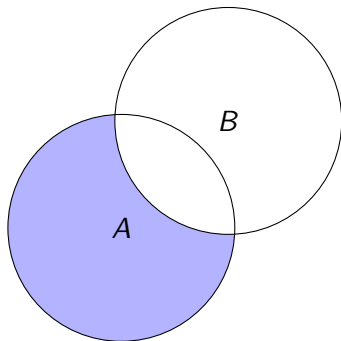
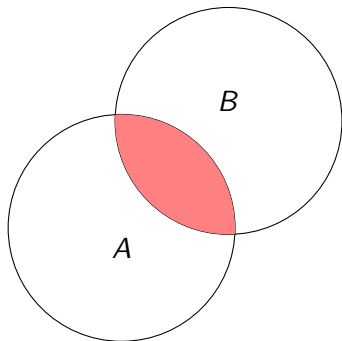
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The *permutation* of n objects is an arrangement of these objects in a specific order. The number of ways of permuting k objects out of n unique objects is given by

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Since sets are unordered (i.e., $\{x, y\} = \{y, x\}$), this amounts to choosing k out of n objects, without replacement and without distinguishing the order in which they are chosen.

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$$\frac{\binom{4}{2}}{2} = 3.$$

Some important identities (with story proofs)

- ▶ Choosing the complement:

$$\binom{n}{k} = \binom{n}{n-k}.$$

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(*Vandermonde's identity.*)

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- ▶ R or RStudio?

Basic operations

```
4 + 2
```

```
## [1] 6
```

Basic operations

```
4 + 2
```

```
## [1] 6
```

```
4 - 2
```

```
## [1] 2
```

Basic operations

4 + 2

```
## [1] 6
```

4 - 2

```
## [1] 2
```

4 * 2

```
## [1] 8
```

Basic operations (cont.)

```
4 * (5 - 3)
```

```
## [1] 8
```

Basic operations (cont.)

```
4 * (5 - 3)
```

```
## [1] 8
```

```
4 / 2
```

```
## [1] 2
```

Basic operations (cont.)

```
4 * (5 - 3)
```

```
## [1] 8
```

```
4 / 2
```

```
## [1] 2
```

```
4 ^ 2
```

```
## [1] 16
```

Basic operations (cont.)

```
4 * (5 - 3)
```

```
## [1] 8
```

```
4 / 2
```

```
## [1] 2
```

```
4 ^ 2
```

```
## [1] 16
```

```
sqrt(4)
```

```
## [1] 2
```


Creating objects

Create an object `x` that saves information:

```
x <- 4 * 2
```

Creating objects

Create an object `x` that saves information:

```
x <- 4 * 2
```

View object:

```
x
```

```
## [1] 8
```

```
(x <- 4 * 2)
```

```
## [1] 8
```

Numeric objects: double and integer

```
(x1 <- 2.0)
```

```
## [1] 2
```

```
typeof(x1)
```

```
## [1] "double"
```

Numeric objects: double and integer

```
(x1 <- 2.0)
```

```
## [1] 2
```

```
typeof(x1)
```

```
## [1] "double"
```

```
(x2 <- 2L)
```

```
## [1] 2
```

```
typeof(x2)
```

```
## [1] "integer"
```

Character objects

```
(instructor <- "Elias")
```

```
## [1] "Elias"
```

Character objects

```
(instructor <- "Elias")
```

```
## [1] "Elias"
```

```
(instructor <- "Elias Nosrati")
```

```
## [1] "Elias Nosrati"
```

Character objects

```
(instructor <- "Elias")
```

```
## [1] "Elias"
```

```
(instructor <- "Elias Nosrati")
```

```
## [1] "Elias Nosrati"
```

```
typeof(instructor)
```

```
## [1] "character"
```

Character objects

```
(instructor <- "Elias")
```

```
## [1] "Elias"
```

```
(instructor <- "Elias Nosrati")
```

```
## [1] "Elias Nosrati"
```

```
typeof(instructor)
```

```
## [1] "character"
```

```
number_as_character <- "10"
```

```
number_as_character / 2
```

```
## Error in number_as_character/2: non-numeric  
argument to binary operator
```


Logical objects

```
5 > 2
```

```
## [1] TRUE
```

Logical objects

```
5 > 2
```

```
## [1] TRUE
```

```
x <- 5
```

```
x > 2
```

```
## [1] TRUE
```

Logical objects

```
5 > 2
```

```
## [1] TRUE
```

```
x <- 5
```

```
x > 2
```

```
## [1] TRUE
```

```
y <- x > 2
```

```
typeof(y)
```

```
## [1] "logical"
```

Data structures: vectors

```
x <- c(1, 2, 3)
```

```
x
```

```
## [1] 1 2 3
```

Data structures: vectors

```
x <- c(1, 2, 3)
```

```
x
```

```
## [1] 1 2 3
```

```
a <- c("a", 2, FALSE)
```

```
b <- c("z", 47L)
```

```
c <- c(a, b)
```

```
c
```

```
## [1] "a"      "2"      "FALSE"  "z"      "47"
```

Data structures: vectors (cont.)

```
(x <- c(1, 2, 3, 4, 5))
```

```
## [1] 1 2 3 4 5
```

Data structures: vectors (cont.)

```
(x <- c(1, 2, 3, 4, 5))
```

```
## [1] 1 2 3 4 5
```

```
(x <- 1:5)
```

```
## [1] 1 2 3 4 5
```

Data structures: vectors (cont.)

```
(x <- c(1, 2, 3, 4, 5))
```

```
## [1] 1 2 3 4 5
```

```
(x <- 1:5)
```

```
## [1] 1 2 3 4 5
```

```
(x <- seq(1, 5))
```

```
## [1] 1 2 3 4 5
```


Data structures: vectors (cont.)

```
(x <- c(1, 2, 3, 4, 5))
```

```
## [1] 1 2 3 4 5
```

```
(x <- 1:5)
```

```
## [1] 1 2 3 4 5
```

```
(x <- seq(1, 5))
```

```
## [1] 1 2 3 4 5
```

```
(x <- seq(10, 50, by = 10))
```

```
## [1] 10 20 30 40 50
```

Data structures: matrices

```
matrix(1:4, nrow = 2)
```

```
##      [,1] [,2]  
## [1,]    1    3  
## [2,]    2    4
```

Data structures: matrices

```
matrix(1:4, nrow = 2)
```

```
##      [,1] [,2]  
## [1,]    1    3  
## [2,]    2    4
```

```
matrix(1:4, nrow = 2, byrow = TRUE)
```

```
##      [,1] [,2]  
## [1,]    1    2  
## [2,]    3    4
```

Data structures: lists

```
list(1:3, c("a", "b"), TRUE, 44.7, " ")
```

```
## [[1]]
```

```
## [1] 1 2 3
```

```
##
```

```
## [[2]]
```

```
## [1] "a" "b"
```

```
##
```

```
## [[3]]
```

```
## [1] TRUE
```

```
##
```

```
## [[4]]
```

```
## [1] 44.7
```

```
##
```

```
## [[5]]
```

```
## [1] " "
```

Data structures: data frame

```
data.frame("Var_1" = 1:3, "Var_2" = 4:6)
```

```
##   Var_1 Var_2
## 1     1     4
## 2     2     5
## 3     3     6
```

Data structures: data frame

```
data.frame("Var_1" = 1:3, "Var_2" = 4:6)
```

```
##   Var_1 Var_2
## 1     1     4
## 2     2     5
## 3     3     6
```

```
df <- data.frame("Var_1" = 1:3, "Var_2" = 4:6)
names(df) <- c("Variable_1", "Variable_2")
df
```

```
##   Variable_1 Variable_2
## 1           1           4
## 2           2           5
## 3           3           6
```

Converting between data structures

```
m <- matrix(1:100, nrow = 10)
as.data.frame(m)
```

##		V1	V2	V3	V4	V5	V6	V7	V8	V9	V10
##	1	1	11	21	31	41	51	61	71	81	91
##	2	2	12	22	32	42	52	62	72	82	92
##	3	3	13	23	33	43	53	63	73	83	93
##	4	4	14	24	34	44	54	64	74	84	94
##	5	5	15	25	35	45	55	65	75	85	95
##	6	6	16	26	36	46	56	66	76	86	96
##	7	7	17	27	37	47	57	67	77	87	97
##	8	8	18	28	38	48	58	68	78	88	98
##	9	9	19	29	39	49	59	69	79	89	99
##	10	10	20	30	40	50	60	70	80	90	100

Indexing

```
x <- 1:10
```

```
x[5]
```

```
## [1] 5
```


Indexing

```
x <- 1:10
```

```
x[5]
```

```
## [1] 5
```

```
x[2:7]
```

```
## [1] 2 3 4 5 6 7
```

Indexing

```
x <- 1:10
```

```
x[5]
```

```
## [1] 5
```

```
x[2:7]
```

```
## [1] 2 3 4 5 6 7
```

```
x[c(2, 7)]
```

```
## [1] 2 7
```

Indexing

```
x <- 1:10
```

```
x[5]
```

```
## [1] 5
```

```
x[2:7]
```

```
## [1] 2 3 4 5 6 7
```

```
x[c(2, 7)]
```

```
## [1] 2 7
```

```
x[-c(2, 7)]
```

```
## [1] 1 3 4 5 6 8 9 10
```

Indexing (cont.)

```
df <- data.frame("V1" = 1:3, "V2" = 4:6)  
df[2]
```

```
##      V2  
## 1     4  
## 2     5  
## 3     6
```

Indexing (cont.)

```
df <- data.frame("V1" = 1:3, "V2" = 4:6)  
df[2]
```

```
##    V2  
## 1   4  
## 2   5  
## 3   6
```

```
df$V2
```

```
## [1] 4 5 6
```

Indexing (cont.)

```
df[3, 2]
```

```
## [1] 6
```

Indexing (cont.)

```
df[3, 2]
```

```
## [1] 6
```

```
subset(df, V1 == 1 & V2 == 4)
```

```
##    V1 V2
```

```
## 1  1  4
```

Indexing (cont.)

```
df[3, 2]
```

```
## [1] 6
```

```
subset(df, V1 == 1 & V2 == 4)
```

```
##      V1 V2
```

```
## 1    1  4
```

```
subset(df, V1 == 2 | V2 != 4)
```

```
##      V1 V2
```

```
## 2    2  5
```

```
## 3    3  6
```


Functions

We have already seen several functions: `c()`, `class()`, `data.frame()`, etc.

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```
my_function <- function(input) {  
  
  # Define output using input here  
  
  return(output)  
}
```

Functions: constructing $f(x) = x^2 + 4$

```
my_function <- function(x) { # Function takes input x
  y <- x^2 + 4 # Expression for f(x)
  return(y) # Output
}
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Functions: constructing $f(x) = x^2 + 4$

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}
```

```
my_function(2)
```

```
## [1] 8
```

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}
```

```
my_function(2)
```

```
## [1] 8
```

```
my_function(2)
```

```
## [1] 8
```

Functions: creating a summary function

```
my_summary <- function(x) { # Input
  s_out <- sum(x) # Sum
  l_out <- length(x) # Length
  m_out <- s_out / l_out # Mean
  out <- c(s_out, l_out, m_out) # Define output
  names(out) <- c("Sum", "Length", "Mean") # Labels
  return(out) # End function by calling output
}
```

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```

```
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```

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  names(out) <- c("Sum", "Length", "Mean") # Labels
  return(out) # End function by calling output
}
```

```
input <- 1:10
```

```
my_summary(input)
```

##	Sum	Length	Mean
##	55.0	10.0	5.5

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- ▶ To get help on usage, type `package?name_of_package` and `help(package = "name_of_package")`.

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- ▶ Change directory: `setwd("new_directory")`.

Some hands-on exercises

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- ▶ Type `swirl()` to start the first exercise.

Homework

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- ▶ Complete Problem Sheet 1 and submit your **R** scripts by email at least 24h before the next lecture.