Quantitative Methods

Human Sciences, 2020–2021

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Lecture 2: 22 October 2020

Today

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- Recap on probability and counting: the birthday problem.

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- Introduction to conditional probability.

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- Problem sheet 1 (tutorial).

The birthday problem

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- Assume each person's birthday is equally likely to be any of the 365 days of the year and assume people's birthdays are independent.
- What is the probability that at least one pair of people in the group have the same birthday?
- Hint: Recall that $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$.

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- $\mathbb{P}(\text{at least one birthday match}) = 1 - \mathbb{P}(\text{no birthday match})$.
- In this room, $\mathbb{P}(\text{at least one birthday match}) \approx 4\%$.

The birthday problem in R

```r
# Create a function
pmatch <- function(n) {
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```r
# Alternative method
probs <- sapply(1:70, pmatch)
```

The birthday problem in **R** (cont.)

```
save <- data.frame("n" = 1:70, "prob" = probs)
```

The birthday problem in **R** (cont.)

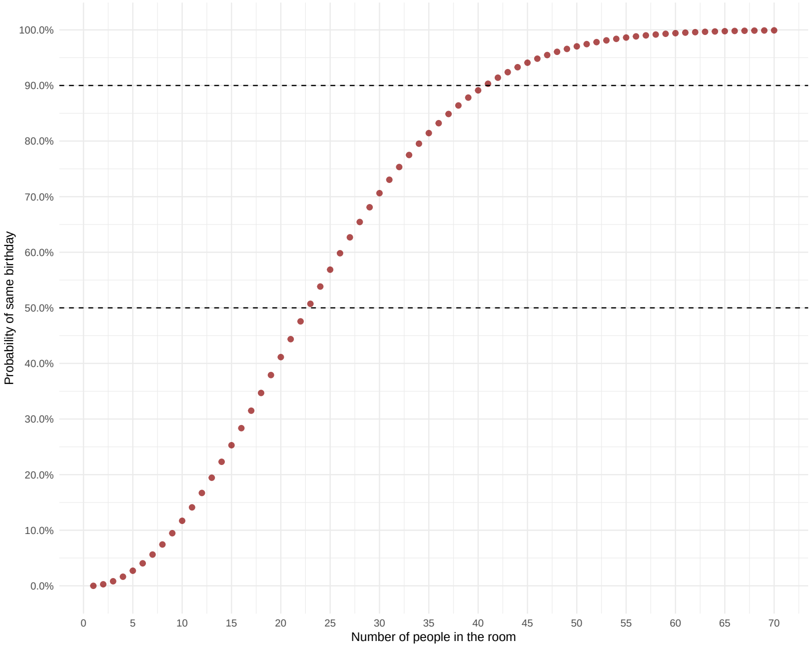
```r
save <- data.frame("n" = 1:70, "prob" = probs)
```

```r
head(save)

##   n        prob
## 1 1 0.000000000
## 2 2 0.002739726
## 3 3 0.008204166
## 4 4 0.016355912
## 5 5 0.027135574
## 6 6 0.040462484
```

The birthday problem in **R** (cont.)

```
ggplot(save, aes(n, prob)) +
  labs( # Axis labels
    x = "Number of people in the room",
    y = "Probability of same birthday") +
  scale_x_continuous( # Modify X-axis
    breaks = seq(0, 70, 5)) +
  scale_y_continuous( # Modify Y-axis
    breaks = seq(0, 1, 0.1),
    label = scales::percent) +
  geom_point( # Modify points
    size = 2,
    colour = "darkred",
    alpha = 0.7) +
  geom_hline( # When is P(match) > 0.5 or 0.9?
    yintercept = c(0.5, 0.9), linetype = "dashed") +
  theme_minimal() # Remove redundant lines
```


Probability of same birthday
100.0%
90.0%
80.0%
70.0%
60.0%
50.0%
40.0%
30.0%
20.0%
10.0%
0.0%
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
Number of people in the room

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- Whenever new evidence is observed, we acquire information that may affect out uncertainties.
- Conditional probability allows us to update our beliefs in light of new evidence.
- "Conditioning is the soul of statistics" (Blitzstein and Hwang, 2019: 46).

Defining conditional probability

Definition

If A and B are events, with $\mathbb{P}(B) > 0$, then the *conditional probability* of A given B is

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

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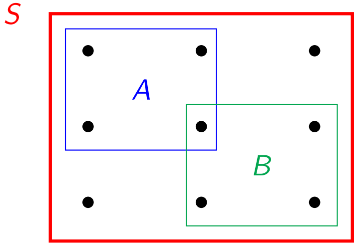
- Note that $\mathbb{P}(A \mid A) = 1$.

Visualising a conditional probability

Suppose we have a sample space S and two events A and B:



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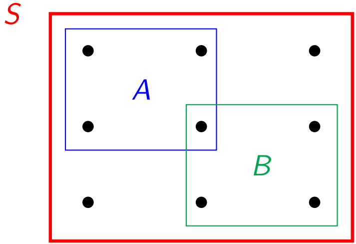

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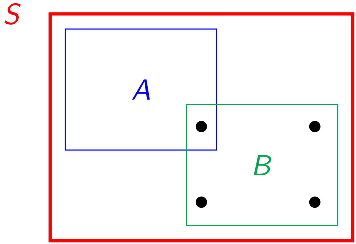

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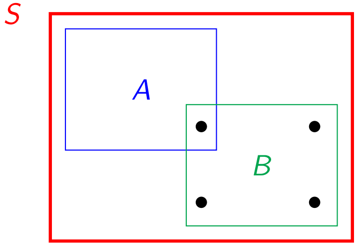

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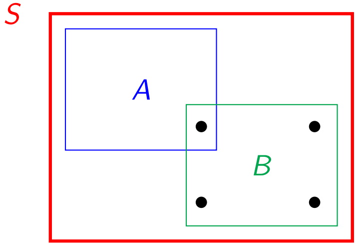

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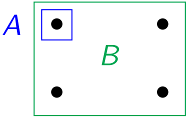
- Divide by $\mathbb{P}(B)$, the total mass of the outcomes in B.

Visualising a conditional probability (cont.)

The updated probability measure assigned to the event A is the conditional probability

$$\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B)/\mathbb{P}(B).$$



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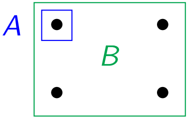

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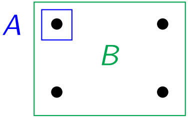

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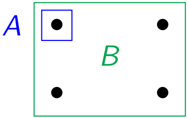

- We have updated our probabilities in accordance with observed evidence.
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- Relative measures of uncertainty are redistributed amongst remaining possible outcomes.

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- Thus

$$\mathbb{P}(A \mid B) = n_{AB}/n_B = (n_{AB}/n)/(n_B/n) = \mathbb{P}(A \cap B)/\mathbb{P}(B).$$

Joint probability and conditional probability

Theorem

For any events A and B with positive probabilities,

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This can be generalised to the intersection of n events:

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2) \cdots \mathbb{P}(A_n|A_1 \cap \cdots \cap A_n).$$

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This follows immediately from the previous Theorem (which in turn follows immediately from the definition of conditional probability).

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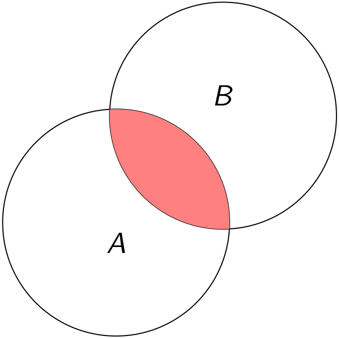
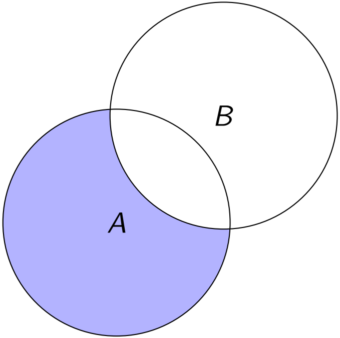
Recall that, for any two events A and B, the Law of Total Probability states that

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c).$$





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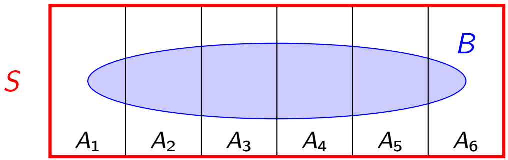

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&\approx 16\%.
\end{aligned}$$

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Conclusion:

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- *All probabilities are conditional probabilities.*

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$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \quad \text{and} \quad \mathbb{P}(B \mid A) = \mathbb{P}(B).$$

- Independence is a symmetric relation: if A is independent of B, B is independent of A.

Independence

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- Independence is a symmetric relation: if A is independent of B, B is independent of A.
- Warning: independence $\neq$ disjointness. In fact, disjoint events can only be independent if $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$. (Why?)

Conditional independence

Definition

Events A and B are said to be *conditionally independent* given a third event E if

$$\mathbb{P}(A \cap B \mid E) = \mathbb{P}(A \mid E)\mathbb{P}(B \mid E).$$

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Health warning: independence does *not* imply conditional independence and vice versa.

Conditional independence and complements

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- Suppose there are two types of teachers: those who give grades that reflect student effort (E), and those who randomly assign grades, regardless of student effort (E^c).

Conditional independence and complements

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- Let W be the event that you work hard and let G be the event that you receive a good grade.

Conditional independence and complements

- Suppose there are two types of teachers: those who give grades that reflect student effort (E), and those who randomly assign grades, regardless of student effort (E^c).
- Let W be the event that you work hard and let G be the event that you receive a good grade.
- Then W and G are conditionally independent given E^c, but they are not conditionally independent given E.

Conditional independence $\not\Rightarrow$ independence

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- You pick one of the coins at random, without knowing which one you've chosen, and flip it several times.
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- However, the coin flips are not unconditionally independent: without knowing which coin we've chosen, each flip gives us new data from which we can predict outcomes of future tosses.

Conditional independence $\not\Rightarrow$ independence

- You have one fair coin and one biased coin which lands Heads with probability 3/4.
- You pick one of the coins at random, without knowing which one you've chosen, and flip it several times.
- Conditional on choosing either the fair or the biased coin, the coin flips are independent.
- However, the coin flips are not unconditionally independent: without knowing which coin we've chosen, each flip gives us new data from which we can predict outcomes of future tosses.
- (Think about the definition of independence.)

Independence $\not\Rightarrow$ conditional independence

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- Each day, they decide independently whether to call me that day.
- Let A be the event that Alice calls me tomorrow and let B be the event that Ben calls me tomorrow.

Independence $\not\Rightarrow$ conditional independence

- Suppose my friends Alice and Ben are the only two people who call me on my mobile phone.
- Each day, they decide independently whether to call me that day.
- Let A be the event that Alice calls me tomorrow and let B be the event that Ben calls me tomorrow.
- Then A and B are unconditionally independent, with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$.

Independence $\not\Rightarrow$ conditional independence

- Suppose my friends Alice and Ben are the only two people who call me on my mobile phone.
- Each day, they decide independently whether to call me that day.
- Let A be the event that Alice calls me tomorrow and let B be the event that Ben calls me tomorrow.
- Then A and B are unconditionally independent, with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$.
- However, given that I receive exactly one call tomorrow (C), A and B are no longer independent:

$$\mathbb{P}(A \mid C) > 0, \quad \text{but} \quad \mathbb{P}(A \mid C \cap B) = 0.$$

The Monty Hall problem

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- Choose one of three doors: two with goats, one with car.

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The Monty Hall problem

- Choose one of three doors: two with goats, one with car.
- After your choice, Monty Hall opens one door with a goat and asks if you want to switch to the remaining door. Should you switch?
- Solve analytically or by simulation.

The Monty Hall problem: solving by simulation

```
sims <- 10 ^ 5 # Simulations
doors <- 1:3 # Label doors
win_no_switch <- 0 # Win count without switch
win_switch <- 0 # Win count with switch
```

The Monty Hall problem: solving by simulation

```r
sims <- 10 ^ 5 # Simulations
doors <- 1:3 # Label doors
win_no_switch <- 0 # Win count without switch
win_switch <- 0 # Win count with switch
```

```r
for (i in 1:sims) {
  # Pick a door for the car
  car_door <- sample(doors, 1)
  # Pick a door for the contestant
  chosen_door <- sample(doors, 1)
  # If they match, add to "no switch" count
  if (car_door == chosen_door)
    win_no_switch <- win_no_switch + 1
  # Otherwise, add to switch count
  else
    win_switch <- win_switch + 1
}
```

The Monty Hall problem: solving by simulation (cont.)

```
cat("P(car | no switch) =", win_no_switch / sims)

## P(car | no switch) = 0.33504
```

The Monty Hall problem: solving by simulation (cont.)

```r
cat("P(car | no switch) =", win_no_switch / sims)

## P(car | no switch) = 0.33504
```

```r
cat("P(car | switch) =", win_switch / sims)

## P(car | switch) = 0.66496
```

How is this possible?

The Monty Hall problem: analytic solution

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- Let C_i be the event that the car is behind door i, for $1 \leq i \leq 3$, and let W be the event that you have chosen the winning door.

The Monty Hall problem: analytic solution

- Conditioning as wishful thinking: condition on what you wish you knew.
- Let C_i be the event that the car is behind door i, for $1 \leq i \leq 3$, and let W be the event that you have chosen the winning door.
- Since the C_i partition the sample space, we can use the Law of Total Probability:

$$\mathbb{P}(W) = \mathbb{P}(W \mid C_1) \times \frac{1}{3} + \mathbb{P}(W \mid C_2) \times \frac{1}{3} + \mathbb{P}(W \mid C_3) \times \frac{1}{3}.$$

The Monty Hall problem: analytic solution

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- Suppose, without loss of generality, that you chose door 1.

The Monty Hall problem: analytic solution

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- Suppose, without loss of generality, that you chose door 1.
- If C_1, the switching strategy will fail:

$$\mathbb{P}(W \mid C_1) = 0.$$

The Monty Hall problem: analytic solution

- Conditioning as wishful thinking: condition on what you wish you knew.
- Let C_i be the event that the car is behind door i, for $1 \leq i \leq 3$, and let W be the event that you have chosen the winning door.
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- Suppose, without loss of generality, that you chose door 1.
- If C_1, the switching strategy will fail:

$$\mathbb{P}(W \mid C_1) = 0.$$

- If the car is behind doors 2 or 3, switching will succeed:

$$\mathbb{P}(W \mid C_2) = \mathbb{P}(W \mid C_3) = 1.$$

The Monty Hall problem: analytic solution

- Conditioning as wishful thinking: condition on what you wish you knew.
- Let C_i be the event that the car is behind door i, for $1 \leq i \leq 3$, and let W be the event that you have chosen the winning door.
- Since the C_i partition the sample space, we can use the Law of Total Probability:

$$\mathbb{P}(W) = \mathbb{P}(W \mid C_1) \times \frac{1}{3} + \mathbb{P}(W \mid C_2) \times \frac{1}{3} + \mathbb{P}(W \mid C_3) \times \frac{1}{3}.$$

- Suppose, without loss of generality, that you chose door 1.
- If C_1, the switching strategy will fail:

$$\mathbb{P}(W \mid C_1) = 0.$$

- If the car is behind doors 2 or 3, switching will succeed:

$$\mathbb{P}(W \mid C_2) = \mathbb{P}(W \mid C_3) = 1.$$

- Hence, $\mathbb{P}(W) = 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{2}{3}$.

Problem sheet 1: preamble

```r
# Clear environment, set working directory
rm(list = ls())
setwd("/Users/Elias/Documents/.../QM/Problem sheets")
```

```r
# Load tidyverse packages
library(tidyverse)
```

Question A1

```r
# Construct die and sample space
die <- 1:6
S <- expand.grid(die, die, die)
```

Question A2

```r
# Create new variable
S <- S %>%
  mutate(Value = Var1 + Var2 + Var3)
```

Question A3

```r
# P(A) = # elements in A / # elements in S
sum(S$Value == 12) / nrow(S)

## [1] 0.1157407
```

Question A4

```r
# What if dice are biased?
Prob <- c("1" = 1/8, "2" = 1/8, "3" = 1/8,
          "4" = 1/8, "5" = 1/8, "6" = 3/8)
```

Question A4

```r
# What if dice are biased?
Prob <- c("1" = 1/8, "2" = 1/8, "3" = 1/8,
          "4" = 1/8, "5" = 1/8, "6" = 3/8)
```

```r
# Assign individual and joint probabilities
S$Prob1 <- Prob[S$Var1]
S$Prob2 <- Prob[S$Var2]
S$Prob3 <- Prob[S$Var3]
S$Prob_joint <- S$Prob1 * S$Prob2 * S$Prob3
```

Question A5

```r
# Extract event A and calculate probability
A <- subset(S, Value == 12)
sum(A$Prob_joint)

## [1] 0.1074219
```

Question B1

```r
#  Load data sets
kenya <- read_csv("kenya.csv")
sweden <- read_csv("sweden.csv")
world <- read_csv("world.csv")
```

Question B2

```r
#  Inspect
summary(kenya)
summary(sweden)
summary(world)

glimpse(kenya)
glimpse(sweden)
glimpse(world)

head(kenya)
head(sweden)
head(world)

print(kenya, n = 30)
print(sweden, n = 30)
print(world, n = 30)
```

Question B3

```r
# Calculate age-specific fertility rate
asfr <- function(data) {
  data %>%
    mutate(
      asfr = births / py.women) %>%
    select(period, age, asfr) %>%
    data.frame() # Convert tibble to data frame
}
```

Question B3 (cont.)

```
asfr(kenya)[c(4:10, 19:25), ]

##         period   age       asfr
## 4  1950-1955 15-19 0.16884585
## 5  1950-1955 20-24 0.35596942
## 6  1950-1955 25-29 0.34657814
## 7  1950-1955 30-34 0.28946367
## 8  1950-1955 35-39 0.20644016
## 9  1950-1955 40-44 0.11193267
## 10 1950-1955 45-49 0.03905205
## 19 2005-2010 15-19 0.10057087
## 20 2005-2010 20-24 0.23583536
## 21 2005-2010 25-29 0.23294721
## 22 2005-2010 30-34 0.18087964
## 23 2005-2010 35-39 0.13126805
## 24 2005-2010 40-44 0.05626214
## 25 2005-2010 45-49 0.03815044
```

Question B3 (cont.)

```
asfr(sweden)[c(4:10, 19:25), ]

##        period   age         asfr
## 4  1950-1955 15-19 0.0389089519
## 5  1950-1955 20-24 0.1277108826
## 6  1950-1955 25-29 0.1252436647
## 7  1950-1955 30-34 0.0873641591
## 8  1950-1955 35-39 0.0486037714
## 9  1950-1955 40-44 0.0162101857
## 10 1950-1955 45-49 0.0013418290
## 19 2005-2010 15-19 0.0059709097
## 20 2005-2010 20-24 0.0507320271
## 21 2005-2010 25-29 0.1162085625
## 22 2005-2010 30-34 0.1322744621
## 23 2005-2010 35-39 0.0625923991
## 24 2005-2010 40-44 0.0121600765
## 25 2005-2010 45-49 0.0006143942
```

Question B3 (cont.)

```
asfr(world)[c(4:10, 19:25), ]

##       period   age       asfr
## 4  1950-1955 15-19 0.090295213
## 5  1950-1955 20-24 0.237633702
## 6  1950-1955 25-29 0.252452289
## 7  1950-1955 30-34 0.204164096
## 8  1950-1955 35-39 0.138105344
## 9  1950-1955 40-44 0.063608319
## 10 1950-1955 45-49 0.015190644
## 19 2005-2010 15-19 0.048489719
## 20 2005-2010 20-24 0.151971307
## 21 2005-2010 25-29 0.146980966
## 22 2005-2010 30-34 0.093813813
## 23 2005-2010 35-39 0.046689639
## 24 2005-2010 40-44 0.016268995
## 25 2005-2010 45-49 0.004510245
```

Question B4

```r
# Calculate total fertility rate
tfr <- function(data) {
  out <- asfr(data)
  out %>%
    group_by(period) %>%
    summarise(
      tfr = 5 * sum(asfr))
}
```

Question B4 (cont.)

```
tfr(kenya)

## # A tibble: 2 x 2
##   period      tfr
##   <chr>     <dbl>
## 1 1950-1955  7.59
## 2 2005-2010  4.88

tfr(sweden)

## # A tibble: 2 x 2
##   period      tfr
##   <chr>     <dbl>
## 1 1950-1955  2.23
## 2 2005-2010  1.90
```

Question B4 (cont.)

```
tfr(world)

## # A tibble: 2 x 2
##   period      tfr
##   <chr>     <dbl>
## 1 1950-1955  5.01
## 2 2005-2010  2.54
```

Question B5

```r
# Calculate age-specific death rate
asdr <- function(data) {
  data %>%
    mutate(
      # Convert rates to per 1000 population
      asdr = 1000 * deaths / (py.men + py.women)) %>%
    select(period, age, asdr) %>%
    data.frame() # Convert tibble to data frame
}
```

Question B5 (cont.)

```
sample_n(asdr(kenya), 10)

##       period   age      asdr
## 1  1950-1955 15-19  5.869582
## 2  2005-2010 60-69 25.395531
## 3  1950-1955 10-14  5.972093
## 4  1950-1955 55-59 24.433007
## 5  2005-2010 30-34 10.603913
## 6  1950-1955 60-69 41.996801
## 7  1950-1955 20-24  7.651103
## 8  1950-1955 40-44 12.633744
## 9  1950-1955   0-4 66.826532
## 10 2005-2010 20-24  3.885368
```

Question B5 (cont.)

```
sample_n(asdr(sweden), 10)

##       period   age         asdr
## 1  1950-1955   80+ 167.81702554
## 2  2005-2010   5-9   0.08138094
## 3  2005-2010 40-44   1.03925622
## 4  2005-2010 25-29   0.49414399
## 5  2005-2010 60-69   9.82877193
## 6  1950-1955 10-14   0.48964064
## 7  1950-1955 25-29   1.11409103
## 8  1950-1955 40-44   2.50955411
## 9  1950-1955 70-79  59.98230926
## 10 1950-1955   5-9   0.43205374
```

Question B5 (cont.)

```
sample_n(asdr(world), 10)

##        period   age       asdr
## 1  2005-2010 30-34   2.623982
## 2  1950-1955   80+ 184.364978
## 3  1950-1955 45-49  13.459846
## 4  1950-1955 30-34   7.132501
## 5  2005-2010   80+ 120.679385
## 6  2005-2010 70-79  47.457519
## 7  1950-1955 55-59  24.265320
## 8  2005-2010 20-24   1.832602
## 9  2005-2010 35-39   3.031563
## 10 2005-2010 40-44   3.753402
```

Question B6

```r
# Collect ASFR and ASDR for each country
ken <- left_join(asfr(kenya), asdr(kenya))
swe <- left_join(asfr(sweden), asdr(sweden))
wor <- left_join(asfr(world), asdr(world))
```

Question B6

```r
# Collect ASFR and ASDR for each country
ken <- left_join(asfr(kenya), asdr(kenya))
swe <- left_join(asfr(sweden), asdr(sweden))
wor <- left_join(asfr(world), asdr(world))
```

```r
# Create one data frame with all results
df <- rbind(ken, swe, wor)
df$country <- c(rep("Kenya", 30),
                rep("Sweden", 30), rep("World", 30))
```

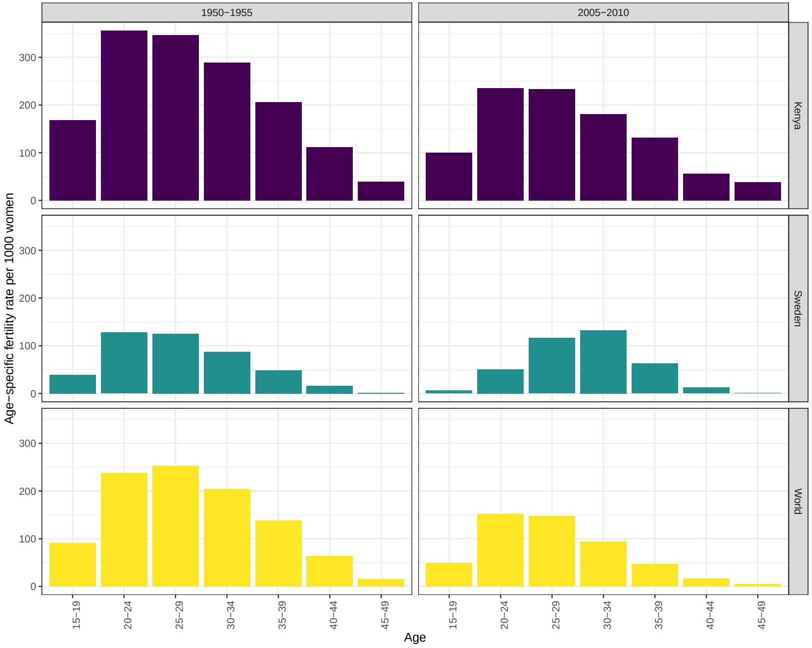
Question B6 (cont.)

```r
# Transform age groups to ordered factor
df$age <- factor(df$age,
                 levels = c("0-4", "5-9", "10-14",
                            "15-19", "20-24", "25-29",
                            "30-34", "35-39", "40-44",
                            "45-49", "50-54", "55-59",
                            "60-69", "70-79", "80+"))

# Age groups for reproductive age range
age_groups <- c("15-19", "20-24", "25-29",
                "30-34", "35-39", "40-44", "45-49")
```

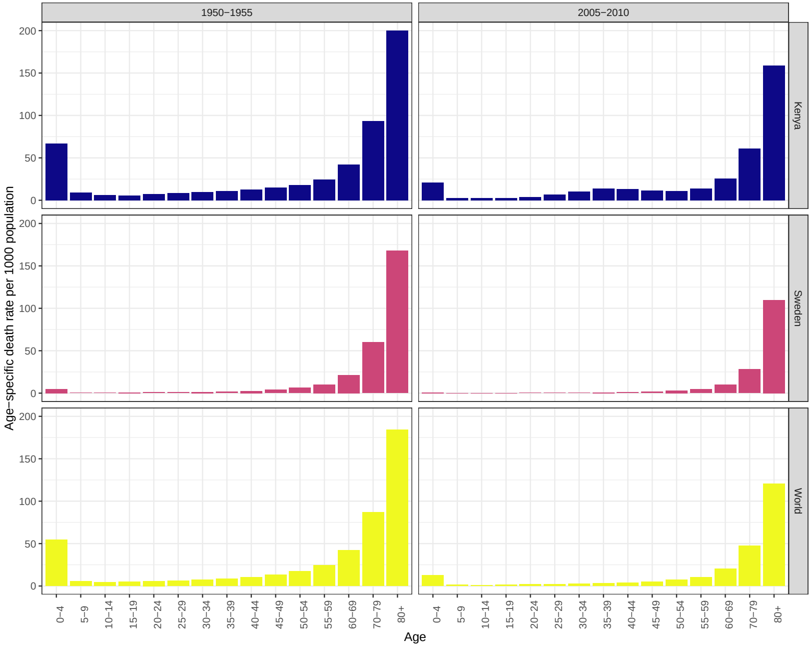
Question B6 (cont.)

```r
# Visualise ASFR
g1 <- ggplot(subset(df, age %in% age_groups), # Age range
             aes(age, 1000 * asfr, # Modify rate
                 fill = country)) + # Colour code
  geom_col() + # Show as columns
  labs( # Axis labels
    x = "Age",
    y = "Age-specific fertility rate per 1000 women") +
  scale_fill_viridis_d() + # Choose a nice colour palette
  facet_grid(country ~ period) + # Stratify
  theme_bw() + # Remove redundant lines
  theme( # Avoid cluttering
    legend.position = "none",
    axis.text.x = element_text(angle = 90))
```


1950−1955
2005−2010
Kenya
Sweden
World
Age−specific fertility rate per 1000 women
0
100
200
300
15−19
20−24
25−29
30−34
35−39
40−44
45−49
Age

Question B6 (cont.)

```
# Visualise ASDR
g2 <- ggplot(df, aes(age, asdr,
                     fill = country)) + # Colour code
  geom_col() + # Show as columns
  scale_y_continuous(breaks = seq(0, 200, 50)) + # Y-axis
  labs( # Axis labels
    x = "Age",
    y = "Age-specific death rate per 1000 population") +
  scale_fill_viridis_d(option = "plasma") + # Colour
  facet_grid(country ~ period) + # Stratify
  theme_bw() + # Remove redundant lines
  theme( # Avoid cluttering
    legend.position = "none",
    axis.text.x = element_text(angle = 90))
```


1950−1955
2005−2010
Kenya
Sweden
World
Age−specific death rate per 1000 population
0
50
100
150
200
0−4
5−9
10−14
15−19
20−24
25−29
30−34
35−39
40−44
45−49
50−54
55−59
60−69
70−79
80+
Age