Assignment 7 TTK4130 Elias Olsen Almenningen

Problem 1:

$$\dot{x}_1 = x_1 + x_2 + z \qquad (1a)$$

$$\dot{x}_2 = z + u \qquad (1b)$$

$$0 = \frac{1}{2}\left(x_1^2 + x_2^2 - 1\right) \qquad (1c) \qquad x_1 + x_2 \neq 0$$

a) (1) is a DAE because z does not appear as time differentiated.

Alternatively:

(1) can be written as $F(\dot{n}, n, u) = \begin{bmatrix} \dot{x}_1 - x_1 - x_2 - z \\ \dot{x}_2 - z - u \\ \frac{1}{2}(x_1^2 + x_2^2 - 1) \end{bmatrix}$, $n = \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}$

if $\left|\frac{\partial F}{\partial \dot{n}}\right| = 0$, (1) is a DAE

$$\frac{\partial F}{\partial \dot{n}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{\det}{\Rightarrow} 0 \Rightarrow \text{(1) is a DAE}$$

b) By differentiating (1c) until $\dot{z}$ appears, we get the differential index of (1)

$$c(x, z) = \frac{1}{2}\left(x_1^2 + x_2^2 - 1\right)$$

$$\dot{c} = x_1 \dot{x}_1 + x_2 \dot{x}_2$$

$$= x_1(x_1 + x_2 + z) + x_2(z + u)$$

$$\ddot{c} = \dot{x}_1(x_1 + x_2 + z) + \dot{x}_2(z + u) + x_1(\dot{x}_1 + \dot{x}_2 + \dot{z}) + x_2(\dot{z} + \dot{u})$$

Then solve $\ddot{c} = 0$ for z

$$\dot{z}(x_1 + x_2) = -\dot{x}_1(x_1 + x_2 + z) - \dot{x}_2(z+u) - x_1(\dot{x}_1 + \dot{x}_2) - x_2\dot{u}$$

$$\dot{z} = \frac{-\dot{x}_1(x_1 + x_2 + z) - \dot{x}_2(z+u) - x_1(\dot{x}_1 + \dot{x}_2) - x_2\dot{u}}{x_1 + x_2}$$

(1) had to be differentiated 2 times for $\dot{z}$ to appear
$\Rightarrow$ (1)'s differential index is $\underline{\underline{2}}$

c) Index-reduction "form" is when we are "one step away" from becoming an ODE, which is

$$\dot{x}_1 = x_1 + x_2 + z$$
$$\dot{x}_2 = z + u$$
$$0 = x_1(x_1 + x_2 + z) + x_2(z+u)$$
$$= x_1^2 + x_1x_2 + x_1z + x_2z + x_2u$$

Problem 2:

$$\dot{x} = -\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x - z \qquad (3a)$$

$$\varepsilon \dot{z} = \frac{1}{10} x - Az \qquad (3b) \qquad\qquad A = \begin{bmatrix} x_1^2 & x_2 \\ 0 & x_2^2 \end{bmatrix} + \alpha I \qquad (4)$$

$$= \begin{bmatrix} x_1^2 + \alpha & x_2 \\ 0 & x_2^2 + \alpha \end{bmatrix}$$

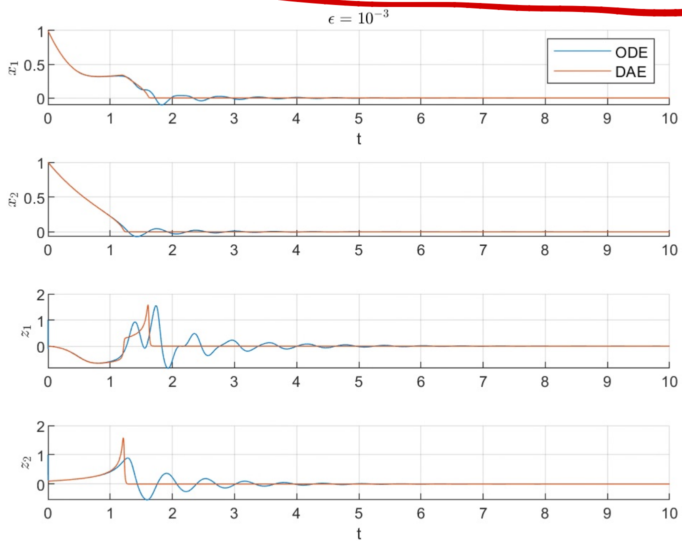
$$\varepsilon, \alpha \geq 0$$

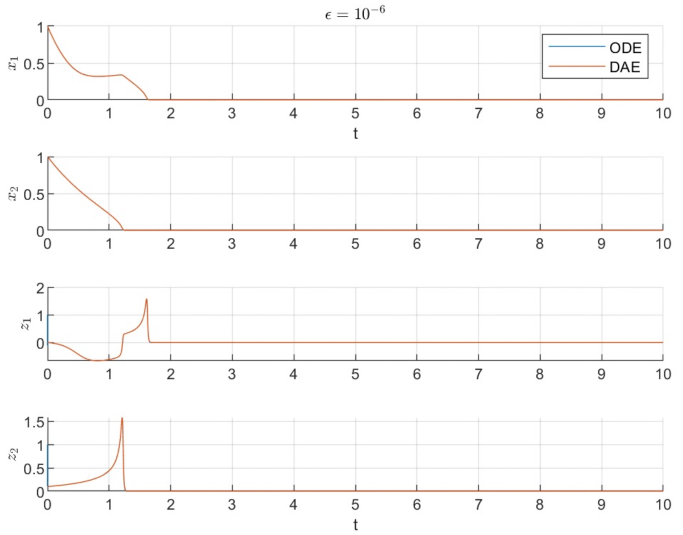
a) (3) is an ODE $\forall \varepsilon > 0$, and an DAE when $\varepsilon = 0$

b) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{matrix} -x_1 - x_2 - z_1 \\ -x_2 - z_2 \end{matrix}$

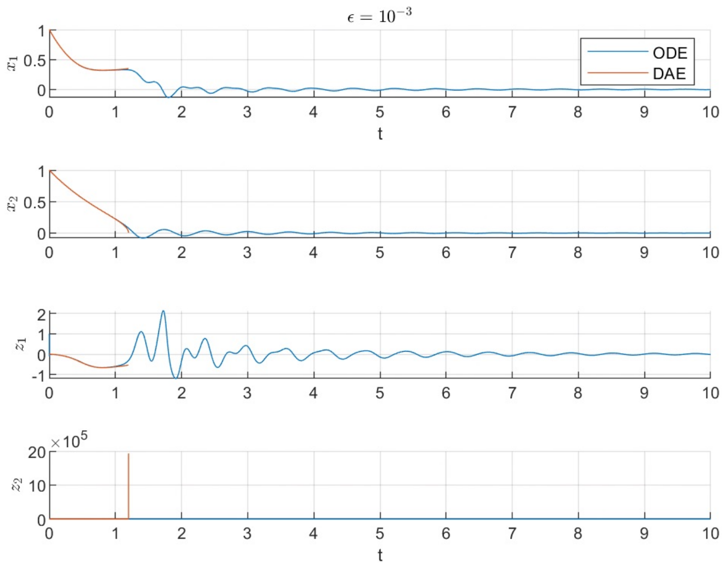
$$\varepsilon \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{matrix} \frac{1}{10} x_1 - \left((x_1^2 + \alpha) z_1 + x_2 z_2 \right) \\ \frac{1}{10} x_2 - \left((x_2^2 + \alpha) z_2 \right) \end{matrix}$$

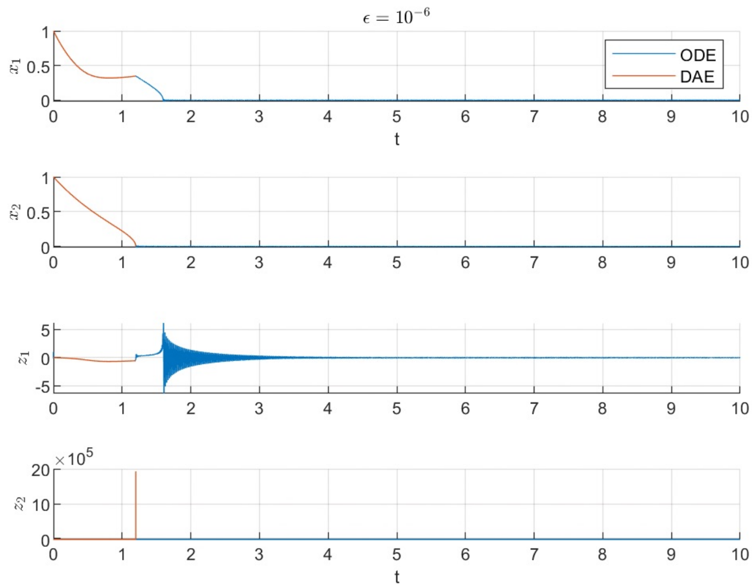
$= 0$ og finn z som tilfredsstiller





When $\varepsilon = 10^{-3}$, the ODE and DAE behaves somewhat different, but when $\varepsilon = 10^{-6}$ they seem to be identical c)



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Command Window
Warning: Failure at t=1.198640e+00.  Unable to meet integration tolerances without reducing the step size below the smallest value allowed (3.552714e-15)
at time t.
> In ode15s (line 723)
In Problem_2c (line 18)
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The DAE simulation fails at $t = 1.2$, but up to that the behaviour is similar to b)

d) Tikhonovs Theorem supposes

- the dynamics $\dot{z} = g(x, z)$ are stable $\forall x$
- matrix $\frac{\partial g}{\partial z}$ is full rank (i.e.) invertible everywhere

then $\Rightarrow \lim_{\varepsilon \to 0} x_\varepsilon(t), z_\varepsilon(t) = x_0(t), z_0(t)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{matrix} -x_1 - x_2 - z_1 \\ -x_2 - z_2 \end{matrix}$$

$$\varepsilon \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{matrix} \frac{1}{10} x_1 - \left(\left(x_1^2 + \alpha \right) z_1 + x_2 z_2 \right) = \frac{1}{\varepsilon}\left(\frac{1}{10} x_1 - x_1^2 z_1 - \alpha z_1 - x_2 z_2 \right) \\ \frac{1}{10} x_2 - \left(\left(x_2^2 + \alpha \right) z_2 \right) = \frac{1}{\varepsilon}\left(\frac{1}{10} x_2 - x_2^2 z_2 - \alpha z_2 \right) \end{matrix}$$

$$\frac{\partial g}{\partial z} = \frac{1}{\varepsilon} \begin{bmatrix} -x_1^2 - \alpha & -x_2 \\ 0 & -x_2^2 - \alpha \end{bmatrix}$$

$$\det\left(\frac{\partial g}{\partial z}\right) = (-x_1^2-\alpha)(-x_2^2-\alpha)$$

This is not singular if $\alpha > 0$, its allways invertible

If $\alpha = 0$, as in c), $\det\left(\frac{\partial g}{\partial z}\right) = 0$, if either

x_1 or $x_2 = 0$

Problem 3:

a) $\dot{x}_1 + u + x_1 + x_2 = 0$

$u + x_2 + \dot{x}_2\dot{x}_1 + \dot{x}_2 u + \dot{x}_2 x_1 + \dot{x}_2 x_2 + u^2 = 0$

$$\dot{x}_2(\underbrace{\dot{x}_1 + u + x_1 + x_2}_{=0}) = -u - u^2 - x_2$$

$$0 = -u - u^2 - x_2$$

$$x_2 = -u - u^2$$

Giving us:
$$\begin{aligned} \dot{x}_1 + u + x_1 + x_2 &= 0 \\ x_2 + u + u^2 &= 0 \end{aligned} \iff \begin{aligned} \dot{x}_1 + u + x_1 + z &= 0 \\ z + u + u^2 &= 0 \end{aligned}$$

This is a DAE

OR

$$F(\dot{x}, x, u) = \begin{bmatrix} \dot{x}_1 + u + x_1 + x_2 \\ \dot{x}_2(\dot{x}_1 + u + x_1 + x_2) + u + u^2 + x_2 \end{bmatrix} = 0$$

$$\frac{\partial F}{\partial \dot{x}} = \begin{bmatrix} 1 & 0 \\ \dot{x}_2 & \underbrace{\dot{x}_1 + u + x_1 + x_2}_{0} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ \dot{x}_2 & 0 \end{bmatrix} \Rightarrow \text{rank deficient}$$

$\Rightarrow$ DAE

b) $u + \dot{x}_1 x_1 + \dot{x}_2 x_2 = 0$ $\qquad$ $\dot{x}_1 x_1 = -u - \dot{x}_2 x_2$

$u \dot{x}_1 x_1 + \dot{x}_2 u x_2 = 0$ $\qquad$ $-u^2 - \dot{x}_2 u x_2 + \dot{x}_2 u x_2 = 0$

$$\Rightarrow u^2 = 0 \quad ?$$

?

$$F(\dot{x}, x, u) = \begin{bmatrix} u + \dot{x}_1 x_1 + \dot{x}_2 x_2 \\ u \dot{x}_1 x_1 + \dot{x}_2 u x_2 \end{bmatrix} = 0$$

$$\frac{\partial F}{\partial \dot{x}} = \begin{bmatrix} x_1 & x_2 \\ u x_1 & u x_2 \end{bmatrix} \overset{\det}{\Longrightarrow} (u x_1 x_2) - (u x_1 x_2) = 0$$

Problem 4:

$$\dot{x} + u + \tanh(\dot{x}) + x z = 0 \qquad (10a)$$

$$\tanh(2u - z) = 0 \qquad (10b)$$

$x, z, u \in \mathbb{R}$

a)

- **Semi-explicit DAEs** split explicitly the differential and algebraic equations, and can generally be written in the form:

$$\dot{x} = f(x, z, u) \qquad (5.24a)$$

$$0 = g(x, z, u) \qquad (5.24b)$$

A semi-explicit DAE can be trivially written as a fully-implicit DAE by defining:

$$F(\dot{x}, x, z, u) = \begin{bmatrix} \dot{x} - f(x, z, u) \\ g(x, z, u) \end{bmatrix} = 0 \qquad (5.25)$$

Conversely, a fully implicit can be trivially written as a semi-explicit DAE by introducing some "helper variables" (labelled v here), i.e.

$$\dot{x} = v \qquad (5.26a)$$

$$0 = F(v, x, z, u) \qquad (5.26b)$$

$$\dot{x} = v$$

$$0 = v + u + \tanh(v) + x z$$

$$0 = \tanh(2u - z)$$

b) $\cancel{F(v,x,z,u) = \begin{bmatrix} v + u + \tanh(v) + xz \\ \tanh(2u - z) \end{bmatrix} = 0}$ $\quad \dot{x} = \begin{bmatrix} x \\ v \\ z \end{bmatrix}$ **?**

$$\cancel{\frac{\partial F}{\partial \dot{x}} = \begin{bmatrix} z & 1 + \operatorname{sech}^2(v) & x \\ 0 & 0 & -\operatorname{sech}^2(-z-2u) \end{bmatrix} ?}$$

For which parameters of F, do we need to "solve for" $\dot{x}$: v, z

$1 - \tanh^2(2u - z)$

$$F(v,x,z,u) = \begin{bmatrix} v + u + \tanh(v) + xz \\ \tanh(2u - z) \end{bmatrix} = 0$$

$$M = \begin{bmatrix} \frac{\partial F}{\partial v} & \frac{\partial F}{\partial z} \end{bmatrix} = \begin{bmatrix} 2 - \tanh^2(v)) & x \\ 0 & -\operatorname{sech}^2(z-2u) \end{bmatrix}$$

$$\det(M) = (2 - \tanh^2 v)(-\operatorname{sech}^2(z-2u)) = 0$$
$$(2 - \tanh^2(v))(\tanh^2(z-2u) - 1) = 0$$
$$> 0 \qquad > 0$$

$\tanh(x) \in (-1, 1)$ $\qquad \det(M) \neq 0$

Since $\det(M) \neq 0$, IFT guarantees that we can solve for $\dot{x}$ and z, and the trajectory is therefore well defined

c) To find the differential index, we must differentiate

$$\tanh(2u-z) = 0$$

$$\frac{\partial}{\partial z} \tanh(2u-z) = 1-\tanh^2(2u-z) \neq 0$$

2 ?