Assignment 8

TTK4215

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Problem 5 Chapter 5

p 212 in PDF Consider the second order plant

$$y_p = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} u_p \tag{1}$$

where a_0, a_1, b_0 and b_1 are constants with b_0, $b_1 > 0$. The reference model is given by

$$y_m = \frac{4}{s+5} r. \tag{2}$$

a)

Assume that a_0, a_1, b_0 and b_1 are known. Design an MRC law that guarantees closed-loop stability and meets the control objective $y_p \to y_m$ as $t \to \infty$ for any bounded reference signal r.

General case

The plant and reference model can be written as

$$y_p = \underbrace{k_p \frac{Z_p(s)}{R_p(s)}}_{G_p(s)} u_p$$

$$y_m = \underbrace{k_m \frac{Z_m(s)}{R_m(s)}}_{W_m(s)} r$$

In order to meet the MRC objective, the following assumptions on the plant and reference model must be satisfied:

Plant assumtions:

- P1. $Z_p(s)$ is a monic Hurwitz polynomial.
- P2. An upper bound n of the degree n_p of $R_p(s)$ is known.
- P3. The relative degree $n^* = n_p - m_p$ of $G_p(s)$ is known, where m_p is the degree of $Z_p(s)$.
- P4. The sign of the high-frequency gain k_p is known.

Reference model assumtions:

- M1. $Z_m(s)$, $R_m(s)$ are monic Hurwitz polynomials of degree q_m, p_m, respectively, where $p_m \leq n$.
- M2. The relative degree $n_m^* = p_m - q_m$ of $W_m(s)$ is the same as that of $G_p(s)$, that is, $n_m^* = n^*$.

The feedback law is chosen as

$$u_p = \theta_1^{*T} \frac{\alpha(s)}{\Lambda(s)} u_p + \theta_2^{*T} \frac{\alpha(s)}{\Lambda(s)} y_p + \theta_3^* y_p + c_0^* r \tag{3}$$

see Figure 1, where

$$\begin{aligned} \alpha(s) &\triangleq \alpha_{n-2}(s) = [s^{n-2}, s^{n-3}, \ldots, s, 1]^T && \text{for } n \geq 2, \\ \alpha(s) &\triangleq 0 && \text{for } n = 1 \end{aligned}$$

c_0^*, $\theta_3^* \in R$; θ_1^*, $\theta_2^* \in R^{n-1}$ are constant parameters to be desinged; and $\Lambda(s)$ is an arbitrary monic Hurwitz polynomial of degree $n-1$ that contains $Z_m(s)$ as a factor, that is,

$$\Lambda(s) = \Lambda_o(s) Z_m(s), \tag{4}$$

which implies that $\Lambda_0(s)$ is monic, Hurwitz, and of degree $n_0 = n - 1 - q_m$. The controller parameter vector

$$\theta^* = [\theta_1^{*T}, \theta_2^{*T}, \theta_3^*, c_0^*]^T \in R^{2n} \tag{5}$$

is to be chosen so that the transfer function from r to y_p is equal to $W_m(s)$.

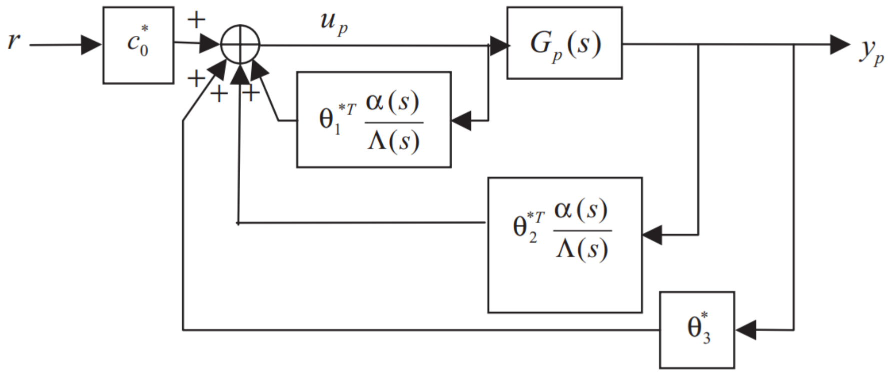

Figure 1: The MRC shceme

The closed-loop plant in Figure 1 is described by

$$y_p = G_c(s) r, \tag{6}$$

where

$$G_c(s) = \frac{c_0^* k_p Z_p(s) \Lambda^2(s)}{\Lambda(s) \left[\left(\Lambda(s) - \theta_1^{*T} \alpha(s) \right) R_p(s) - k_p Z_p(s) \left(\theta_2^{*T} \alpha(s) + \theta_3^* \Lambda(s) \right) \right]}$$

The control objective is met if we select the controller parameters θ_1^*, θ_2^*, θ_3^*, c_0^* such that the closed-loop poles are stable and the closed-loop transfer function $G_c(s) = W_m(s)$, that is

$$\frac{c_0^* k_p Z_p(s)\Lambda^2(s)}{\Lambda(s)\left[\left(\Lambda(s) - \theta_1^{*T}\alpha(s)\right)R_p(s) - k_p Z_p(s)\left(\theta_2^{*T}\alpha(s) + \theta_3^*\Lambda(s)\right)\right]} = k_m \frac{Z_m}{R_m}$$

is satisfied for all $s \in \mathcal{C}$. Choosing

$$c_0^* = \frac{k_m}{k_p} \tag{7}$$

and using $\Lambda(s) = \Lambda_o(s)Z_m(s)$, the matching equation becomes

$$\left(\Lambda - \theta_1^{*T}\alpha\right)R_p - k_p Z_p\left(\theta_2^{*T}\alpha + \theta_3^*\Lambda\right) = Z_p\Lambda_0 R_m \tag{8}$$

or

$$\theta_1^{*T}\alpha(s)R_p(s) + k_p\left(\theta_2^{*T}\alpha(s) + \theta_3^*\Lambda(s)\right)Z_p(s) = \Lambda(s)R_p(s) - Z_p(s)\Lambda_0(s)R_m(s) \tag{9}$$

Equating the coefficients of the powers of s on both side of Equation 9 we can express it in terms of the algebraic equation

$$S\bar{\theta^*} = p \tag{10}$$

where $\bar{\theta^*} = [\theta_1^{*T}, \theta_2^{*T}, \theta_3^*]^T$; S is an $(n+n_p-1)\times(2n-1)$ matrix that depends on the coefficients of R_p, k_pZ_p and Λ; and p is an $n+n_p-1$ vector with the coefficients of $\Lambda R_p - Z_p\Lambda_0 R_m$.

Our case

Equation 1 can be written as

$$y_p = b_1\frac{(s+\frac{b_0}{b_1})}{s^2 + a_1 s + a_0}u_p, \tag{11}$$

then , and . n is then $deg(R_p) = 2$,

- $k_p = b_1$
- $Z_p = s + \frac{b_0}{b_1}$
- $deg(Z_p) = n_p = 1$
- $R_p = s^2 + a_1 s + a_0$
- $deg(R_p) = m_p = 2$
- $n^* = m_p - n_p = 1$

Equation 2 can be written as

$$y_m = 4\frac{1}{s+5}r, \tag{12}$$

then

- $k_m = 4$

- $Z_m = 1$
- $deg(Z_m) = q_m = 0$
- $R_m = s + 5$
- $deg(R_m) = p_m = 1$
- $n_m^* = p_m - q_m = 1$

Then $\Lambda(s) = \Lambda_0(s)Z_m(s) = \Lambda_0 = s + \lambda_0$ and $\alpha(s) = 1$, since $n = 2$. c_0^* becomes $\frac{4}{b_1}$ which makes the matching equation

$$\begin{gathered}\left(\Lambda - \theta_1^{*T}\alpha\right) R_p - k_p Z_p \left(\theta_2^{*T}\alpha + \theta_3^*\Lambda\right) = Z_p\Lambda_0 R_m \\ \downarrow \\ \theta_1^{*T}\alpha R_p + k_p\left(\theta_2^{*T}\alpha + \theta_3^*\Lambda\right) Z_p = \Lambda R_p - Z_p\Lambda_0 R_m \\ \downarrow \\ \theta_1^{*T}\alpha R_p + k_p\theta_2^{*T}\alpha Z_p + k_p\theta_3^*\Lambda Z_p = \Lambda R_p - Z_p\Lambda_0 R_m \\ \downarrow \\ S\bar{\theta}^* = p\end{gathered}$$

Inserting our values gives:

$$\theta_1^{*T}(s^2+a_1s+a_0)+b_1\theta_2^{*T}(s+\frac{b_0}{b_1})+b_1\theta_3^*(s+\lambda_0)(s+\frac{b_0}{b_1}) = (s+\lambda_0)(s^2+a_1s+a_0)-(s+\frac{b_0}{b_1})(s+\lambda_0)(s+5)$$

Setting the coefficients in front of s^2, s^1, s^0 equal on each side of the equation and rearranging with the θ^*'s on the left hand side, we obtain

$$\begin{aligned}&\boldsymbol{s^2} : \theta_1^{*T} + \theta_3^{*T}b_1 = a_1 - 5 - \frac{b_0}{b_1} \\ &\boldsymbol{s^1} : \theta_1^*a_1 + \theta_2^*b_1 + \theta_3^*b_1\lambda_0 + \theta_3^*b_0 = a_0 + a_1\lambda_0 - 5\lambda_0 - 5\frac{b_0}{b_1} - \frac{b_0}{b_1}\lambda_0 \\ &\boldsymbol{s^0} : \theta_1^*a_0 + \theta_2^*b_0 + \theta_3^*\lambda_0 b_0 = \lambda_0 a_0 - 5\frac{b_0}{b_1}\lambda_0\end{aligned}$$

which in matrix form becomes

$$S = \begin{bmatrix} 1 & 0 & b_1 & 0 \\ a_1 & b_1 & b_1\lambda_0 + b_0 & 0 \\ a_0 & b_0 & \lambda_0 b_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \bar{\theta}^* = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ c_0^* \end{bmatrix} \tag{13}$$

$$p = \begin{bmatrix} a_1 - 5 - \frac{b_0}{b_1} \\ a_0 + a_1\lambda_0 - 5\lambda_0 - 5\frac{b_0}{b_1} - \frac{b_0}{b_1}\lambda_0 \\ \lambda_0 a_0 - 5\frac{b_0}{b_1}\lambda_0 \\ \frac{4}{b_1} \end{bmatrix} \tag{14}$$

We can set up our control law as

$$u_p = \theta^{*T}\omega$$

, where

$$\omega = \begin{bmatrix} \frac{\alpha(s)}{\Lambda(s)}[u_p] \\ \frac{\alpha(s)}{\Lambda(s)}[y_p] \\ y_p \\ r \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ y_p \\ r \end{bmatrix}$$

Further analyzing ω_i

$$\omega_1 = \frac{\alpha(s)}{\Lambda(s)}[u_p] = \frac{1}{s+\lambda_0}u_p \to s\omega_1 + \lambda_0\omega_1 = u_p$$
$$\omega_2 = \frac{\alpha(s)}{\Lambda(s)}[u_p] = \frac{1}{s+\lambda_0}y_p \to s\omega_2 + \lambda_0\omega_2 = y_p$$

Inverse Laplace-transform and we get that

$$\dot{\omega_1} = -\lambda_0\omega_1 + u_p$$
$$\dot{\omega_2} = -\lambda_0\omega_1 + y_p$$

The MRC is therefore given as

$$u_p = \theta^{*T}\omega$$

, where

$$\theta^* = S^{-1}p$$

and

$$\omega = \begin{bmatrix} \omega_1 & \omega_2 & y_p & r \end{bmatrix}^T$$
$$\dot{\omega_1} = -\lambda_0\omega_1 + u_p$$
$$\dot{\omega_2} = -\lambda_0\omega_1 + y_p$$

b)

Repeat a) when a_0, a_1, b_0, b_1 *are unknown and* b_1, $b_0 > 0$

When the plant parameters are unknown, we simply use one of the estimation schemes used in previous assignments and replace the values in S and p with the estimated values. Rewriting Equation 1 into an SPM:

$$y_p s^2 + y_p a_1 s + y_p a_0 = b_1 s u_p + b_0 u_p$$
$$\Lambda(s) = s^2 + \lambda_1 s + \lambda_0$$
$$\frac{s^2}{\Lambda(s)}[y_p] = \frac{s}{\Lambda(s)}[u_p]b_1 + \frac{1}{\Lambda(s)}[u_p]b_0 - \frac{s}{\Lambda(s)}[y_p]a_1 - \frac{1}{\Lambda(s)}[y_p]a_0$$
$$\underbrace{\frac{s^2}{\Lambda(s)}[y_p]}_{z_p} = \underbrace{[b_1, b_0, a_1, a_0]}_{\theta_p^{*T}} \underbrace{\begin{bmatrix} \frac{s}{\Lambda(s)}[u_p] \\ \frac{1}{\Lambda(s)}[u_p] \\ -\frac{s}{\Lambda(s)}[y_p] \\ -\frac{1}{\Lambda(s)}[y_p] \end{bmatrix}}_{\phi_p}$$

Using instantaneous cost

$$\varepsilon = \frac{z - \theta^{*T}\phi}{1 + \alpha\phi_p^T\phi_p}$$
$$\dot{\theta}_p = \Gamma\varepsilon\phi_p$$

The MRC parameters are obtained by

$$\theta^* = \hat{S}^{-1}\hat{p} \tag{15}$$

c)

If $a_0 = 1$, $a_1 = 0$, $b_1 = 1$, and $b_0 > 0$ are known, are known, indicate the simplification that results in the control law

The control law simplifies to

$$S = \begin{bmatrix} 1 & 0 & b_1 & 0 \\ 0 & 1 & \lambda_0 + b_0 & 0 \\ 1 & b_0 & \lambda_0 b_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \bar{\theta}^* = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ c_0^* \end{bmatrix} \tag{16}$$

$$p = \begin{bmatrix} -5 - \frac{b_0}{b_1} \\ 1 - 5\lambda_0 - 5\frac{b_0}{1} - \frac{b_0}{1}\lambda_0 \\ \lambda_0 - 5\frac{b_0}{1}\lambda_0 \\ \frac{4}{1} \end{bmatrix} \tag{17}$$

and now we only have to estimate b_0 as done i b).

Problem 5 Chapter 6

Consider the plant

$$y = \frac{s+b}{s(s+a)}u. \tag{18}$$

a)

Design an adaptive law to generate $\hat{a}$ and $\hat{b}$, the estimates of a and b, respectively, online. Rewriting Equation 18 as an SPM:

$$\begin{aligned} s^2y + sya &= su + bu \\ s^2y - su &= bu - sya \\ \Lambda(s) &= s^2 + \lambda_1 s + \lambda_0 \\ \underbrace{\frac{s^2}{\Lambda(s)}[y] - \frac{s}{\Lambda(s)}[u]}_{z} &= \underbrace{[b, a]}_{\theta^{*T}} \underbrace{\begin{bmatrix} \frac{1}{\Lambda(s)}[u] \\ -\frac{s}{\Lambda(s)}[y] \end{bmatrix}}_{\phi} \end{aligned}$$

Using instantaneous cost

$$\begin{aligned} \varepsilon &= \frac{z - \theta^{*T}\phi}{1 + \alpha\phi_p^T\phi_p} \\ \dot{\theta}_p &= \Gamma\varepsilon\phi_p \end{aligned}$$

b)

Design an APPC scheme to stabilize the plant and regulate y to zero.

The tracking error is defined as

$$e_1 = y_p - y_m = -\frac{1}{1 + CG_p}y_m, \tag{19}$$

where the controller C is given by

$$C(s) = \frac{P(s)}{L(s)}\frac{1}{Q_m(s)}, \tag{20}$$

where P, L, Q_m are polynomials to be selected. We now have

$$e_1 = -\frac{1}{1 + \frac{P}{L}\frac{1}{Q_m}\frac{Z_p}{R_p}}y_m = -\frac{LR_p}{LQ_mR_p + PZ_p}Q_my_m \tag{21}$$

Since we want to follow a constant reference singal (0), we select $Q_m = s$. Since the derivative of a constant is zero, we get that $Q_my_m = 0$:). If we then can select L, P such that

$$LQ_mR_p + PZ_p = A^* \tag{22}$$

then we have

$$e_1 = -\frac{LR_p}{A^*}[0] \tag{23}$$

so $e_1 \to 0$ asymtotically.

We have to define som degrees

- $n = deg(R_p) = 2$
- $q = deg(Z_n) = 1$
- $deg(P) = q + n - 1 = 2$
- $del(L) = n - 1 = 1$
- $deg(\Lambda) = q + n - 1 = 2$

L, Λ is monic, while L is not, giving us

$$P(s) = p_2 s^2 + p_1 s + p_0 \tag{24}$$
$$L(s) = s + l_0 \tag{25}$$
$$\Lambda(s) = s^2 + \lambda_1 s + \lambda_0 \tag{26}$$

Solving Equation 22 gives us

$$(s + l_0)s(s(s + a)) + (p_2 s^2 + p_1 s + p_0)(s + b) = A^*$$
$$s^4 + as^3 + l_0 s + l_0 as^2 + p_2 s^3 + p_2 bs^2 + p_1 s^2 + p_1 bs + p_0 s + p_0 b = s^4 + a_3^* s^3 + a_2^* s^2 + a_1^* s + a_0^*$$
$$s^4 + (a + l_0 + p_2 b)s^3 + (l_0 a + p_1 + bp_2)s^2 + (p_0 + p_1 b)s + p_0 b = s^4 + a_3^* s^3 + a_2^* s^2 + a_1^* s + a_0^*$$

which again gives us four equations:

$$s^3 : a + l_0 + p_2 b = a_3^*$$
$$s^2 : l_0 a + p_1 + bp_2 = a_2^*$$
$$s^1 : p_0 + p_1 b = a_1^*$$
$$s^0 : p_0 b = a_0^*$$

which in matrix form is

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & b & a \\ 1 & b & 0 & 0 \\ b & 0 & 0 & 0 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ l_0 \end{bmatrix}}_{\theta} = \underbrace{\begin{bmatrix} a_3^* - a \\ a_2^* \\ a_1^* \\ a_0^* \end{bmatrix}}_{p} \tag{27}$$

The parameters in p are known and a, b are estimated, the controller parameters can be found by

$$\theta = S^{-1} p \tag{28}$$

and inserted into the control law

$$u_p = \frac{\Lambda - Q_m L}{\Lambda} u_p - \frac{P}{\Lambda}(y_p - y_m) \tag{29}$$

c)

Discuss the stabilizability condition that $\hat{a}$ and $\hat{b}$ must satisfy at each time t

The book states that the existence and uniqueness of $\hat{L}(s,t), \hat{P}(s,t)$ is guaranteed, provided that $\hat{R}_p(s,t)Q_m(s), \hat{Z}_p(s,t)$ are coprime at each frozen time t.

Checking what this means

$$\hat{R}_p(s,t)Q_m(s) = s(s+\hat{a})s$$
$$\hat{Z}_p(s,t) = s+\hat{b}$$

They are noe coprime if $\hat{a} \neq \hat{b}$ or $\hat{b} \neq 0$

d)

What additional assumptions do you need to impose on the parameters a and b so that the adaptive algorithm can be modified to guarantee the stabilizability condition? Use these assumptions to propose a modified APPC scheme.

In order to make sure that the equations are coprime, one can implement logic in the adaptive laws that ensure this does not happen, e.g. projection or resetting.