Assignment 5

TTK4215	
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Version: 1.0	Date: October 16, 2023

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Problem 1

The nonlinear dynamics of a damped simple pendulum is given by

$$ml^2\ddot{q} + \beta\dot{q} + mglsin(q) = \tau \tag{1}$$

where q is the angle in radians, measured clockwise from the downward hanging equilibrium, m is the mass, β is a damping coefficient, l is the length, g is the gravitational acceleration and τ is a control torque.

a)

Given measurements of q and τ , design an adaptive law to estimate m, l and β. You can assume that g is known.

Rewriting Equation 1 and taking the laplace transform gives

$$\begin{aligned} \ddot{q} &= \frac{1}{ml^2}\tau - \frac{\beta}{ml^2}\dot{q} - \frac{g}{l}sin(q) \\ \mathcal{L}(\ddot{q}) &\downarrow \\ s^2 q &= \frac{1}{ml^2}\tau - \frac{\beta}{ml^2}sq - \frac{g}{l}sin(q) \end{aligned} \tag{2}$$

To obtain the SPM form of Equation 2, we filter this equation by $\frac{1}{\Lambda(s)}$, where $\Lambda(s) = s^2 + \lambda_1 s + \lambda_0$ and obtain the SPM

$$z = \theta^{*\top}\phi \tag{3}$$

where

$$\underbrace{\frac{s^2}{\Lambda(s)}[q]}_{z} = \underbrace{\begin{bmatrix} \frac{1}{ml^2} & \frac{\beta}{ml^2} & \frac{1}{l} \end{bmatrix}}_{\theta^{*T}} \underbrace{\begin{bmatrix} \frac{1}{\Lambda(s)}[\tau] \\ \frac{-s}{\Lambda(s)}[q] \\ \frac{-g}{\Lambda(s)}[sin(q)] \end{bmatrix}}_{\phi}$$

Choosing the adaptive law as instantaneous cost

$$\dot{\theta} = \Gamma\varepsilon\phi \tag{4}$$

where

$$\Gamma = \begin{bmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{bmatrix}$$

$$\varepsilon = \frac{z - \theta^{*\top}\phi}{m_s^2}$$

where $m = 1 + n_s^2$, where n_s is chosen s.t. $\frac{\phi}{m} \in \mathcal{L}_\infty$. The normal value for this is often $n_s^2 = \alpha\phi^\top\phi$, $\alpha > 0$.

After obtaining an estimate $\theta(t) = [\theta_1(t)\,, \theta_2(t)\,, \theta_3(t)]^\top$, estimates of $m\,, l$ and θ can be obtained by

$$\hat{m}(t) = \frac{\theta_3(t)^2}{\theta_1(t)},$$
$$\hat{l} = \frac{1}{\theta_3(t)},$$
$$\hat{\beta} = \frac{\theta_2(t)}{\theta_1(t)}.$$

b)

From Lecture notes 3: Rule of thumb: We need one distinct frequency for every second unknown parameter, provided that H(s) does not lose its linear independence. In our case, H (s) = ϕ.

Therefore, $\tau = \tau_1 + \tau_2$, where $\tau_i = A_i sin(\omega_i t)$. Testing showed that

```
%% Estimator paramters
% Gains
g11         = 5000;             % 1 / ml^2
g12         = 0;                % cross gain?
g13         = 0;                % cross gain?
g21         = 0;                % cross gain?
g22         = 5;                % beta / ml^2
g23         = 0;                % cross gain?
g31         = 0;                % cross gain?
g32         = 0;                % cross gain?
g33         = 50;               % 1 / l
Gamma       = [ g11 g12 g13;
                g21 g22 g23;
                g31 g32 g33];

m_0 = .1;
l_0 = 1.2;
beta_0 = 0.006;
% Initial estimates
theta_0 = [1/(m_0*l_0^2) beta_0/(m_0*l_0^2) 1/l_0]';          % 1 / ml^2 ...
    , beta / ml^2, 1 / l

% Normalization
alpha = 1;

% Input signals
A1 = 1;
A2 = .1;
w1 = pi;
w2 = pi*.3;
```

gave a satisfactory convergence, see Figure 1.

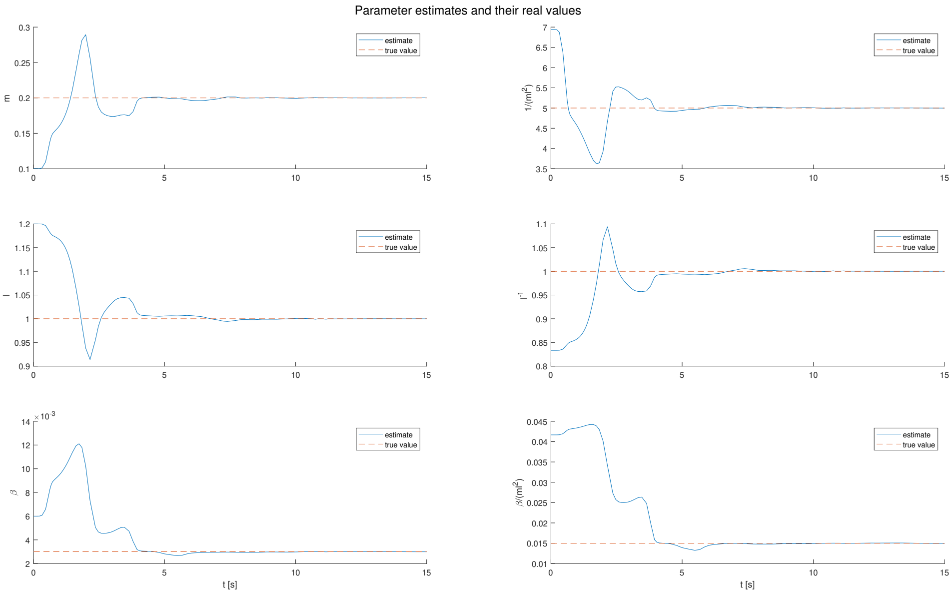


Figure 1: Estimates

Note that the gains used here are identical to the gains used in the solution. They were given to me after spending 1.5 hours with a student assistant trying to figure out why things did not converge. It seems that the cause is that τ was "incorrect".

c)

If in addition to q and τ we are given measurements of the angular velocity $\dot{q}$, how would you change the design from section ? Will this increase performance?

If the measurment of $\dot{q}$ is available, $\ddot{q}$ can be filtered down to $\dot{q}$ and the filter will be simpler (and maybe faster???)

d)

Linearize the dynamics around the upright equilibrium ($q = \pi$) to obtain a linear state-space model of the form $\dot{x} = Ax + Bu$. Let $u = \tau$ and $x = [\Delta q\,,\, \dot{q}]^\top$, where $\Delta q = q - \pi$ is the deviation from equilibrium. Using this model and a method of your choice (e.g. pole placement or LQR), design a linear state-feedback controller $u = -Kx$ to stabilize the upright equilibrium of Equation 1. Assume that both $q(t)$ and $\dot{q}(t)$ can be measured.

We want to linearize about the upright equlibrium point $x_0 = \pi$. Defining the peturbation as $\Delta q = q - \pi$ and taking the time derivative of it gives

$$\Delta\dot{q} = \dot{q} - \dot{\pi}$$
$$\Delta\ddot{q} = \ddot{q} - \ddot{\pi}$$

Inserting these into Equation 1 gives

$$ml^2\Delta\ddot{q} + \beta\Delta\dot{q} + mglsin(\Delta q) = \tau \tag{5}$$

Defining the state as $x = [\Delta q\,,\,\dot{q}]^\top$ gives us the system

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{ml^2}(\tau - \beta x_2 - mlgx_1) \end{bmatrix} \tag{6}$$

To get it on a state space form, the matrices **A** and **B** has to be calculated:

$$\boldsymbol{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} |_{x=x_0} = \begin{bmatrix} 0 & 1 \\ \\ -\frac{g}{l}cos(x_1+\pi) & -\frac{\beta}{ml^2} \end{bmatrix} |_{x=x_0} = \begin{bmatrix} 0 & 1 \\ \\ \frac{g}{l} & -\frac{\beta}{ml^2} \end{bmatrix} \tag{7}$$

$$\boldsymbol{B} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \\ \frac{\partial f_1}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \frac{1}{ml^2} \end{bmatrix} \tag{8}$$

which finally gives us

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \\ \frac{g}{l} & -\frac{\beta}{ml^2} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ \\ \frac{1}{ml^2} \end{bmatrix} u \tag{9}$$

The linear state-feedback controller $u = -Kx$ is derived using LQR in matlab

```
%% Model parameters
m           = 0.2;      % kg
l           = 1;        % m
beta        = 0.003;    % kg m^2 / s
g           = 9.81;     % m / s^2

%% State space model
A = [0 1;g/l -beta/(m*l^2)];
B = [0; 1/(m*l^2)];
%% LQR
Q = diag([1 1]);
R = 1;
[K, ¬, ¬] = lqr(A,B,Q,R);
```

e)

The state-feedback is added to the simulink model, as seen i Figure 2, with state feedback subsystem shown in Figure 3 and the pendulum trajectory can be seen in Figure 4.

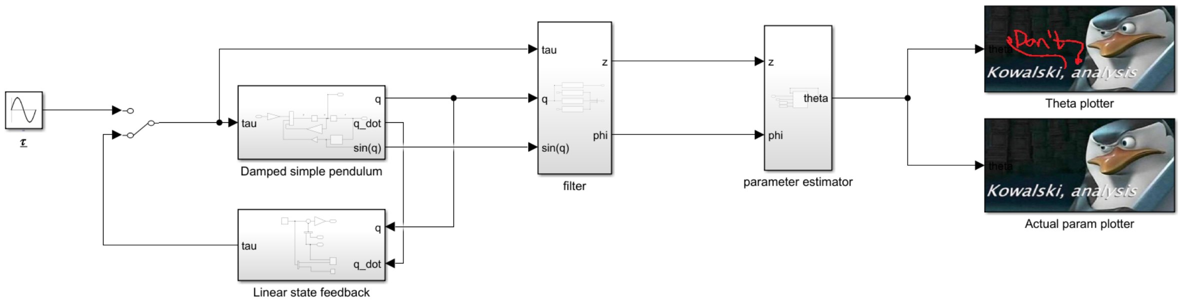


Figure 2: The simulink model with state-feedback

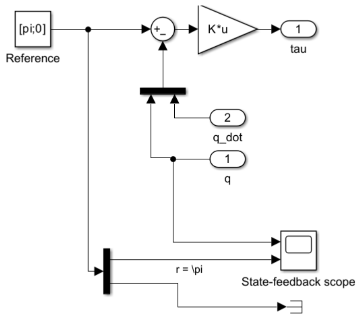


Figure 3: The state-feedback subsystem

Problem 2

Consider the first order plant

$$y = \frac{b}{s-1}u \tag{10}$$

where $b > 0$ the only unknown parameter. Design and analyze a direct MRAC scheme that can stabilize the plant and force y to follow the output y_m of the reference model

$$y_m = \frac{2}{s+2}r \tag{11}$$

for any bounded and continuous reference signal r. Simulate the system in closed loop with your model reference adaptive controller and investigate the conditions for convergence of k and l to k^* and l^* , respectively.

In general we have a system on the form

$$\dot{x} = ax + bu \tag{12}$$

where a and b are unknown, but the sign of b is known. The goal of direct MRAC is to choose a control law u such that all signals in the closed loop are bounded and x tracks the state of x_m of the reference model given by

$$\dot{x}_m = -a_m x + b_m r \tag{13}$$

The control law is chosen as

$$u = -k^* x + l^* r \tag{14}$$

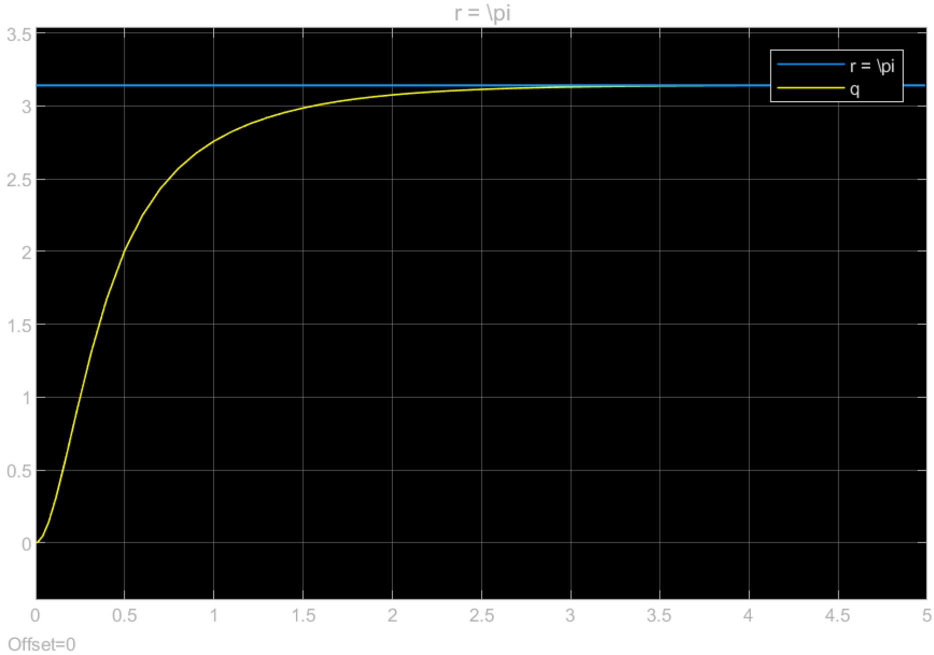


Figure 4: The simulation

where k^*, l^* are calculated so that the closed-loop transfer function from r to x is equal to that of the reference model, that is

$$\frac{x}{r} = \frac{bl^*}{s - a + bk^*} = \frac{b_m}{s + a_m} = \frac{x_m}{r}.$$

For this to be the case

$$k^* = \frac{a_m + a}{b} \tag{15}$$

$$l^* = \frac{b_m}{b} \tag{16}$$

where $b \neq 0$.

The error is defined as $e(t) = x(t) - x_m(t)$ and the dynamics can be manipulated as

$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{x_m} \\
&= ax + bu - (-a_m x_m + b_m r) \\
&= {\color{red}-a_m x + a_m x} + ax + bu + a_m x_m - b_m r \\
&= -a_m e + a_m x + ax - b_m r + bu \\
&\text{inserting for } a_m + a \text{ and } b_m \\
&= -a_m e + k^* bx - bl^* r + bu \\
&= -a_m e + b(k^* x - l^* r + u)
\end{aligned} \tag{17}$$

It is not possible to insert Equation 14 for u in Equation 17, since k^* and l^* are uncertain. Therefore, we define the estimates of k and l as $k(t)\,, l(t)$ and substitue $u = -kx + lr$ for u in

Equation 17:

$$
\begin{aligned}
\dot{e} &= -a_m e + b(k^* x - l^* r - kx + lr) \\
&= -a_m e + b(-\tilde{k}x + \tilde{l}r)
\end{aligned}
\tag{18}
$$

where $\tilde{k} = k - k^*$ and $\tilde{l} = l - l^*$. The error dynamics are now man into an asymptotically stable system driven by terms that are proportional to the parameter estimation errors. We can then select the Lyapunov function candidate

$$
V = \frac{e^2}{2} + \frac{1}{2\gamma_1}\tilde{k}^2 + \frac{1}{2\gamma_2}\tilde{l}^2 \tag{19}
$$

where $\gamma_1 , \gamma_2 > 0$. Taking the time derivative of V gives

$$
\begin{aligned}
\dot{V} &= e\dot{e} + \overbrace{\frac{1}{\gamma_1}\tilde{k}\dot{\tilde{k}} + \frac{1}{\gamma_2}\tilde{l}\dot{\tilde{l}}}^{k^*, l^* x = \text{constant}} \\
&= -a_m e^2 - eb\tilde{k}x + \frac{1}{\gamma_1}\tilde{k}\dot{k} + eb\tilde{l}r + \frac{1}{\gamma_2}\tilde{l}\dot{l} \\
&= -a_m e^2 + \frac{\tilde{k}}{\gamma_1}\Big(\underbrace{\dot{k} - \gamma_1 ebx}_{=0 \text{gives law for } k} \Big) + \frac{\tilde{l}}{\gamma_2}\Big(\underbrace{\dot{l} + \gamma_2 ebr}_{=0 \text{gives law for } l} \Big) \\
\dot{V} &= -a_m e^2
\end{aligned}
\tag{20}
$$

The adaption laws are then

$$
\dot{k} = \gamma_1 ebx \tag{21}
$$

$$
\dot{l} = -\gamma_2 ebr \tag{22}
$$

Since we only know the sign of b, we can do a dirty trick and divide by $|b|$ and since $\frac{b}{|b|} = sign(b)$ we finally get

$$
\dot{k}(t) = \gamma_1 e(t)\text{sign}(b)x(t) \tag{23}
$$

$$
\dot{l}(t) = -\gamma_2 e(t)\text{sign}(b)r(t) \tag{24}
$$

Implementing the adaptive laws in simulink as shown in Figure 5 By tuning the following gains were found

```
%% Model parameters
% Reference model
am      = 2;
bm      = 2;

% Real model
a       = 1;
b       = 5;        % Unknown

```

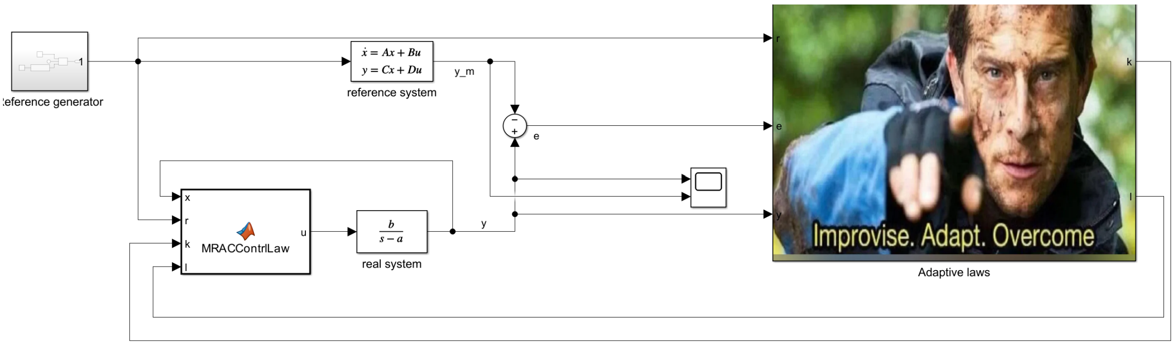


Figure 5: The simulink model for problem 2

```
%% Parameter estimator
gamma1   = 1000;      %k
gamma2   = 1000;     %l

%% Other
ref      = 1;
Ts       = 5;         % Time constant reference signal
a        = 1;         % Amplitude time varying reference signal
w        = 1;         % Frequency --||--

%% Run simulation
t        = 50;        % Simulation time

sim("Problem2.slx");
```

and the control law manages to follow the model reference rather quickly, as seen in Figure 6

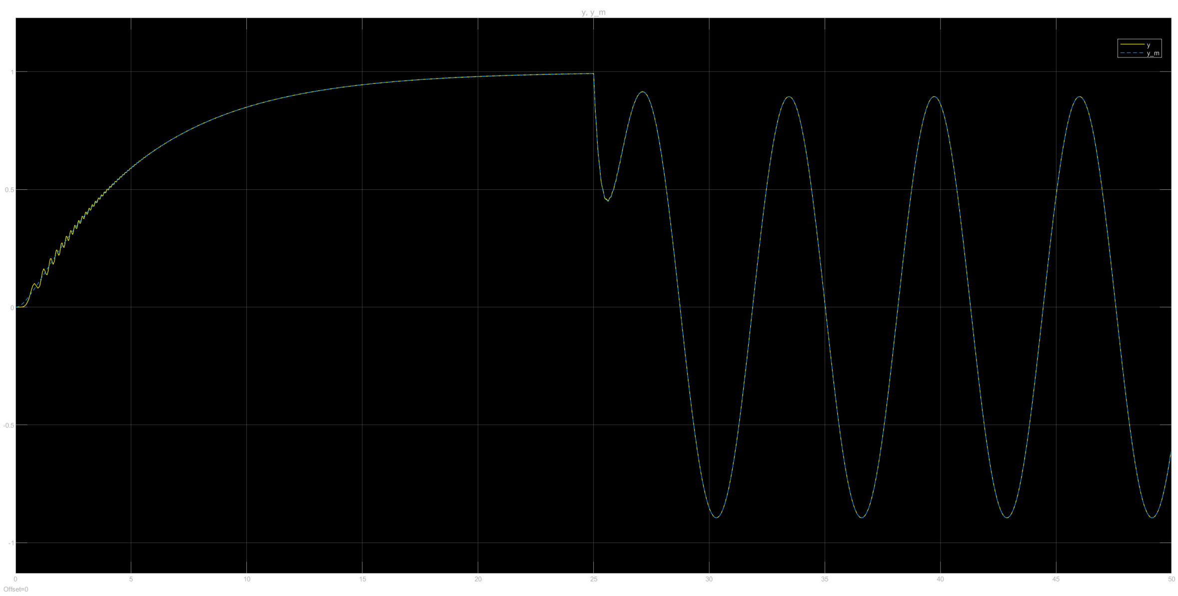
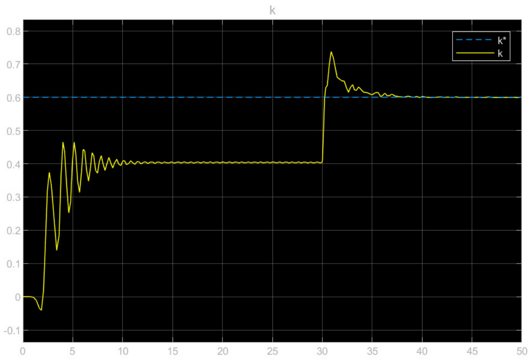
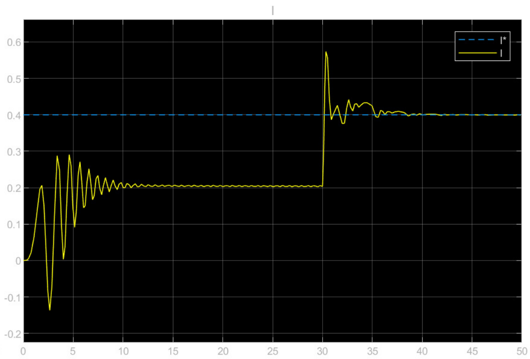


Figure 6: The real system following the reference model

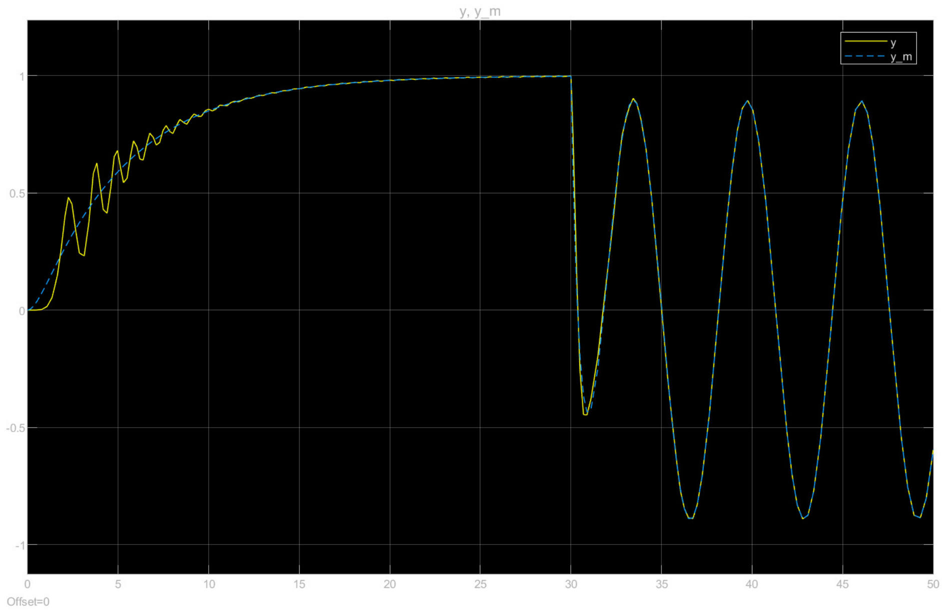
In order for k and l to converge, it seems that the reference signal has to be time-varying. In Figure 7, the reference signal is a step response and then a sine wave. When the step response reaches its final value, the estimates of k, l stop converging, reaching an incorrect value. When the reference flips to a sine wave, the estimates start converging again, and this time they reach the correct value. It seems even though the estimates are wrong, the real system manages to follow the reference system, from $t = [10\,30]$ the estimated values are consant, but wrong. After $t > 30$ the estimates converge to the true values.



(a) K and k^*



(b) l and l^*



(c) Reference following

Figure 7: Three subfigures