Recursive Least Squares

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1 Least squares

Solving the linear least squares

$$\min \sum \left(y_i - \varphi_i^T\,\theta\right)^2 \tag{1}$$

Then the solution is obtained from the ls-command or pseudo-inverse

$$M = \begin{bmatrix} \varphi_1^T \\ \varphi_2^T \\ \vdots \\ \varphi_N^T \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \tag{2}$$

$$\hat{\theta} = (M^T\,M)^{-1}M^T Y \tag{3}$$

Assuming that the model is

$$y_i = \varphi_i^T\,\theta + V \tag{4}$$

where $V \sim \mathcal{N}(0, \sigma_V^2)$, given $\hat{\theta}$, we can compute the error $e_i = y_i - \varphi_i^T\,\theta$ we can estimate σ_V^2 as

$$\hat{\sigma}_V^2 = \mathrm{Cov}(e) = \texttt{np.var(Y - M @ THETA)} \tag{5}$$

We then know that $Y \sim \mathcal{N}(0, I\,\sigma_V^2)$ and we can use standard rules when computing the variance for linear equations of random variables.

$$\begin{aligned} \mathrm{Cov}(\hat{\theta}) &= \mathrm{Cov}((M^T\,M)^{-1}M^T Y) \\ &= \left[(M^T\,M)^{-1}M^T\right]\,\mathrm{Cov}(Y)\,\left[(M^T\,M)^{-1}M^T\right]^T \\ &= \sigma_V^2\,(M^T\,M)^{-1}M^T\,M\left[(M^T\,M)^{-1}\right]^T \\ &= \sigma_V^2(M^T\,M)^{-1} \end{aligned} \tag{6}$$

2 Recursive Least Squares

From "Adaptive control" By K.J. Åström & Björn Wittermark, the RLS algorithm is defined as:

$$K = P_{k-1}\,\varphi_{k-1}\,(\lambda + \varphi_{k-1}^T\,P_{k-1}\,\varphi_{k-1})^{-1} \tag{7}$$

$$P_k = (I - K\,\varphi_{k-1}^T)\,P_{k-1}\,\frac{1}{\lambda} \tag{8}$$

$$\theta_k = \theta_{k-1} + K\,(y_k - \varphi_{k-1}^T\,\theta_{k-1}) \tag{9}$$

For a sample intervall of δ_T and a desired forgetting time constant T_f the choice of lambda is recomended as

$$\lambda = e^{-\delta_T/T_f} \tag{10}$$