Let $S = \{s_1, ..., s_n\}$ be a set of $n$ standards, with concentrations $X = \{x_1, ..., x_n\}$ and measurements $Y = \{y_1, ..., y_n\}$.

The Four parameter logistic curve we will try to fit is defined as

$$f(x) = d + \frac{a-d}{1 + \left(\frac{x}{c}\right)^b}.$$

Each of the parameters serve a purpose:

- $a$ is the lower asymptotic bound of $f$
- $d$ is the upper asymptotic bound of $f$
- $c$ determines the point of inflection of the curve
- $d$ determines the slope at the point of inflection

We will try to find a local minimum for $a, b, c$ and $d$ using gradient descent.

When plotted on a graph with a logarithmic scaling x-axis, it has a sigmoidal shape, so we will substitute $x$ with $\hat{x} = \ln x \Leftrightarrow x = e^{\hat{x}}$. Moreover we want $c$ to scale proportionally with the point of inflection, so we will substitute it with $\hat{c} = \ln c \Leftrightarrow c = e^{\hat{c}}$. This gives us

$$f(x) = d + \frac{a-d}{1 + \left(\frac{e^{\hat{x}}}{e^{\hat{c}}}\right)^b} = d + \frac{a-d}{1 + e^{b(\hat{x}-\hat{c})}}$$

Our cost function shall be the sum of squares.

$$C(Y, X, a, b, c, d) = \frac{1}{n}\sum_{i=1}^{n} \left(y_i - f(x_i)\right)^2$$

First we compute mutual parts of the derivatives

$$C(v) = \frac{1}{n}\sum_{i=1}^{n} v^2, \quad v(y, u) = y - u, \quad u(\hat{x}, a, b, \hat{c}, d) = d + \frac{a-d}{1 + e^{b(\hat{x}-\hat{c})}}$$

$$\frac{\partial C(v)}{\partial v} = \frac{\partial}{\partial v}\frac{1}{n}\sum_{i=1}^{n} v^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial v}v^2 = \frac{1}{n}\sum_{i=1}^{n} 2v = \frac{2}{n}\sum_{i=1}^{n} v$$

$$\frac{\partial v}{\partial u} = \frac{\partial}{\partial u}(y - u) = -1$$

We take the partial derivative of $u$ with respect to $a, b, \hat{c}$ and $d$.

$$\frac{\partial u}{\partial a} = \frac{1}{1 + e^{b(\hat{x}-\hat{c})}}$$

$$\frac{\partial u}{\partial b} = -\frac{(a-d)(\hat{x}-\hat{c})e^{b(\hat{x}-\hat{c})}}{\left(1 + e^{b(\hat{x}-\hat{c})}\right)^2}$$

$$\frac{\partial u}{\partial \hat{c}} = \frac{b(a-d)e^{b(\hat{x}-\hat{c})}}{\left(1 + e^{b(\hat{x}-\hat{c})}\right)^2}$$

$$\frac{\partial u}{\partial d} = \frac{1}{1 + e^{b(\hat{x}-\hat{c})}}$$

Now, putting it all together

$$\begin{aligned}
\frac{\partial C(v)}{\partial a} &= \frac{\partial C(v)}{\partial u}\frac{\partial u}{\partial a} = -\frac{2}{n}\sum_{i=1}^{n}\left(y_i - d - \frac{a-d}{1+e^{b(\hat{x}_i-\hat{c})}}\right)\frac{1}{1+e^{b(\hat{x}_i-\hat{c})}} \\
\frac{\partial C(v)}{\partial b} &= \frac{\partial C(v)}{\partial u}\frac{\partial u}{\partial b} = -\frac{2}{n}\sum_{i=1}^{n}\left(y_i - d - \frac{a-d}{1+e^{b(\hat{x}_i-\hat{c})}}\right)\cdot\left(-\frac{(a-d)(\hat{x}_i-\hat{c})e^{b(\hat{x}_i-\hat{c})}}{\left(1+e^{b(\hat{x}_i-\hat{c})}\right)^2}\right) \\
&= \frac{2}{n}(a-d)\sum_{i=1}^{n}\left(y_i - d - \frac{a-d}{1+e^{b(\hat{x}_i-\hat{c})}}\right)\frac{(\hat{x}_i-\hat{c})e^{b(\hat{x}_i-\hat{c})}}{\left(1+e^{b(\hat{x}_i-\hat{c})}\right)^2} \\
\frac{\partial C(v)}{\partial \hat{c}} &= \frac{\partial C(v)}{\partial u}\frac{\partial u}{\partial \hat{c}} = -\frac{2}{n}\sum_{i=1}^{n}\left(y_i - d - \frac{a-d}{1+e^{b(\hat{x}_i-\hat{c})}}\right)\frac{b(a-d)e^{b(\hat{x}_i-\hat{c})}}{\left(1+e^{b(\hat{x}_i-\hat{c})}\right)^2} \\
&= -\frac{2}{n}b(a-d)\sum_{i=1}^{n}\left(y_i - d - \frac{a-d}{1+e^{b(\hat{x}_i-\hat{c})}}\right)\frac{e^{b(\hat{x}_i-\hat{c})}}{\left(1+e^{b(\hat{x}_i-\hat{c})}\right)^2} \\
\frac{\partial C(v)}{\partial d} &= \frac{\partial C(v)}{\partial u}\frac{\partial u}{\partial d} = -\frac{2}{n}\sum_{i=1}^{n}\left(y_i - d - \frac{a-d}{1+e^{b(\hat{x}_i-\hat{c})}}\right)\frac{1}{1+e^{b(\hat{x}_i-\hat{c})}}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{i=1}^{n}\left(y_i - d - \frac{a-d}{1+e^{b(\hat{x}_i-\hat{c})}}\right)\cdot g(x_i) \\
&= \sum_{i=1}^{n} y_i\cdot g(x_i) - d\sum_{i=1}^{n} g(x_i) - (a-d)\sum_{i=1}^{n}\frac{g(x_i)}{1+e^{b(\hat{x}_i-\hat{c})}}
\end{aligned}$$

so the function can be split up further, reducing total operations per iteration