Truth without Contra(di)ction[*]

Elia Zardini

Instituto de Investigaciones Filosóficas

Universidad Nacional Autónoma de México

`elia.zardini@filosoficas.unam.mx`

March 19, 2011

1   Introduction and Overview

If P, doesn't it follow that 'P' is true? And if 'P' is true, doesn't it follow that P? Well, that arguably depends. If I am hungry, it does not follow that your *utterance* of the sentence 'I am hungry' is true—you might well not be hungry, in which case your utterance would intuitively be false. It is not even clear whether and in what sense it follows that the *sentence* 'I am hungry' is true—if I am hungry and you are not hungry why should the sentence 'I am hungry' give more weight to my hunger than to your satiety (see Zardini [2008], pp. 545–561; [2010a])? Moreover, it might seem possible (albeit not actual) that there are no linguistic entities. But, if there were no linguistic entities, it is not very clear whether and in what sense it would follow that 'There are no linguistic entities' is true.

The two above kinds of problematic cases should certainly not be seen as forcing a wholesale rejection of every type of *correlation laws or rules*[1] with which this paper opened—laws or rules correlating its being the case that P with 'P''s being true—but

---

[1]Throughout, by '*law*' I mean a proposition accepted or rejected by a certain logical or truth-theoretic system (for example, the law of excluded middle, or the law of T-necessity, see below in the text), by '*rule*' an argument accepted by such system (for example, the rule of simplification, or the rule of T-introduction, see below in the text) and by '*metarule*' a conditional claim about rules accepted by such system (for example, the metarule of universal generalisation, or the metarule according to which logical consequence requires truth preservation, see below in the text). (In effect, I will follow common lore and treat laws as a special (zero-premise or zero-conclusion) case of rules—although from a more general point of view I'd be very suspicious of such identification, it will serve well for the purposes of this paper.) I'll use '*principle*' as a catch-all phrase for laws, rules and metarules (and, of course, for all the stuff to which that phrase is usually applied but which lies beyond the scope of logical and truth-theoretic systems). Finally, I'll use '*rule of inference*' for the principles governing deductive systems (see section 4).

they do call for significant refinements of the understanding of the notions of a *truth bearer* (as implicitly presupposed by the use of quotation marks) and of *following-from* which should be employed in stating an acceptable correlation law or rule. Thus, as far as the first kind of problematic case is concerned, one could take the truth bearers to be utterances, and restrict the correlation law or rule to utterances made in one's own context, or one could take the truth bearers to be sentences, and either restrict the correlation law or rule to sentences that do not exhibit a problematic kind of semantic context dependence[2] or understand the law or rule as governing an implicitly relativised notion of sentential truth (truth of a sentence relative to one's own context).[3] As far as the second kind of problematic case is concerned, one should use a notion of following-from[4] that only cares about situations where the underlying metaphysics of language (and, more generally, of representations) that is presupposed by the correlation law or rule is held fixed.

Mainly for simplicity, I will henceforth take sentences to be the truth bearers (and be what the usual quotation environments refer to) while also assuming that the sentences we'll be interested in do not exhibit a problematic kind of semantic context dependence.[5] Moreover, I will use notions of logical consequence and implication that only care about situations where our ordinary metaphysics of language (and, more generally, of representations) is held fixed. With these (and possibly other) refinements in place, the fully unrestricted correlation rules:

T-introduction: 'P' entails ' 'P' is true'

T-elimination: ' 'P' is true' entails 'P'

assume an extremely high plausibility in virtue of what seems to be a fundamental feature of our notion of truth. It lies beyond the scope of this paper to offer an in-depth investigation of such feature, but let me at least cursorily indicate that I regard it as grounded in a

---

[2]What is a "problematic kind of context dependence"? For the purposes of this paper, we don't need to go into the details of this issue; without further explanation, suffice it to say that, assuming for simplicity the two-stage semantic framework made familiar by Kaplan [1989], it is context dependence due to variation in features of the *context of utterance* rather than in features of the *circumstance of evaluation*.

[3]One thing that one *cannot* do is to get away with the problem by shifting to an unrestricted correlation law or rule along the lines of one stating that, if an utterance u says that P, if P, it follows that u is true, and if u is true, it follows that P, for such a law or rule as well would fall afoul of more subtle forms of semantic context dependence arguably exhibited by natural languages (see Zardini [2008], pp. 545–561).

[4]Throughout, I use 'follow from' and its relatives to denote the relation of logical consequence (broadly understood so as to encompass also the "logic of truth"), 'entail' and its relatives to denote the converse relation and 'implication' and its relatives to denote the operation expressed by the conditional. 'Equivalence' and its relatives denote two-way entailment.

[5]Given that the focus of this paper are the semantic paradoxes, many of the sentences we'll be interested in contain occurrences of 'true' and its likes. Hence, the last assumption made in the text requires 'true' and its likes not to exhibit any problematic kind of semantic context dependence that does not derive from a problematic kind of semantic context dependence of sentences to which such expressions meaningfully apply. This is notoriously denied by *contextualist* solutions to the semantic paradoxes (going back at least as far as Parsons [1974]). I'm thus setting aside these theories, reserving a discussion of their pros and cons for another occasion.

conception of truth as a property of *objective representational correctness*—more specifically, the property that a sentence has just in case *it describes things the way they are*.[6,7] That seems an eminently interesting property for a sentence to have, and analogous properties applying to other kinds of truth bearers such as propositions, beliefs and assertions seem indeed to be central to our self-understanding as inquirers seeking to represent an objective world.

Let us call a theory of truth validating fully unrestricted (suitable formalisations of) T-introduction and T-elimination over a *sufficiently rich language* (i.e. roughly, a language rich enough as to allow for the formulation of semantic paradoxes) 'a *naive theory of truth*'. In fact, the considerations just given support not only the fully unrestricted entailment claims constituting T-introduction and T-elimination, but also the two fully unrestricted conditional claims which are got by setting as antecedent the premise and as consequent the conclusion of those two rules and which constitute the correlation laws:

T-necessity: If P, 'P' is true

T-sufficiency: If 'P' is true, P

This is so because the considerations just given not only justify, say, that ''P' is true' is *assertable iff* 'P' is *assertable* (which, at least for some theorists, would be enough to support an entailment claim but not enough to support a conditional claim, see Field [2008], pp. 162–164 for a brief discussion), but also justify that its being the case that 'P' is true is indeed a *necessary and sufficient condition* for its being the case that P—as I said, it seems to be a fundamental feature of our notion of truth that truth is that specific property of objective representational correctness that a sentence has *just in case* it describes things the way they are. And, more generally, that truth is a property of *objective* representational correctness seems to require just that: if it is the case that P, then, no matter whether it is assertable or not that P, that is sufficient for its being the case that 'P' is true, and if it is the case that 'P' is true, then, no matter whether it is assertable or not that 'P' is true, that is sufficient for its being the case that P. Let us call a theory of truth validating fully unrestricted (suitable formalisations of) T-necessity and T-sufficiency over a sufficiently rich language 'an *ingenuous theory of truth*'.

As I've defined it, a naive (ingenuous) theory of truth endorses *all* the instances of T-introduction and T-elimination (T-necessity and T-sufficiency) over a sufficiently rich language. However, in spite of the apparent centrality of unrestricted properties of

---

[6]There would be much to say about the best interpretation of the locution 'the way things are' and of its likes, but that is better left for another occasion. For the purposes of this paper, an intuitive understanding of that locution will suffice.

[7]The fact that truth is such a property can in turn be given very different explanations according to different theories of what the ultimate *nature* of truth is. Many standard *correspondence* and *deflationist* theories are at least in the ballpark for being able to give such an explanation. For what is worth, my own preference would go for an explanation of the fact that truth is a property of objective representational correctness in terms of a non-deflationary theory about the ultimate nature of truth, but I will not try to argue the point in this paper.

objective representational correctness to our self-understanding as inquirers seeking to represent an objective world, such properties cannot possibly exist in a world governed by the laws of (full) *classical* logic.[8] Such is the lesson of the *semantic paradoxes*, the most famous of which, the Liar paradox, originally seems to go back to the Megarian Eubulides of Miletus (see Diogenes Laertius' *De vitis*, 18.02; I'll give concrete examples of the Liar and some other such paradoxes in sections 2.1 and 2.2). This is the problem we'll be concerned with. Without further argument, the philosophical point of view presented in this paper will assume that the culprit is classical logic[9] and will focus on offering a new solution to the semantic paradoxes based on a principled weakening of the logic. It will do so by, after briefly motivating philosophically such weakening, developing a formal naive (and ingenuous) theory of truth and proving its consistency.

The weakening in question consists in the failure, in the background logic of the theory, of the metarule of *contraction*, which very roughly says that if the premises $A_0, A_1, A_2 \ldots B, B \ldots$ entail the conclusion C, then $A_0, A_1, A_2 \ldots B \ldots$ already entail C (note that 'contraction' is often used in the literature on the semantic paradoxes in a related but importantly different sense, which I clarify in fn 45).[10] Contraction is valid not only in classical logic, but in most logics proposed to deal with the semantic paradoxes and, as I'll indicate in sections 2.1 and 2.2, it is in fact implicitly used in the semantic paradoxes. I think that it is in particular this implicit use of contraction that is the culprit for the semantic paradoxes and I will show that ridding truth of contraction suffices for ridding it of contradiction.[11]

---

[8]There are consistent naive theories of truth that, in a sense, validate all classical laws and rules but not all classical metarules (for example, the supervaluationist naive theory of McGee [1991], which validates all classical laws and rules but does not validate reasoning by cases and some other related classical metarules). For these theories, there is a largely terminological question concerning their "classicality". For the purposes of this paper, it'll be more convenient to leave them out of the extension of 'classical' and its relatives.

[9]This has been extensively and variously argued e.g. by Priest [2006a] and Field [2008]. I'm sympathetic to the broad outlines of their treatments of this issue, but don't endorse many of the details. A vindication of the claim that classical theories are unacceptable requires a case-by-case consideration of the best classical theories, a task that is better left for other occasions (see López de Sa and Zardini [2007]; [2010]; Zardini [2008]; [2010a] for a start). I will say much more in section 3 about some of the substantial advantages of the naive theory of truth presented in this paper over its (non-classical) naive rivals, but there again the main purpose of the paper will be not so much to criticise these other theories as to develop a new theory with interesting distinctive features.

[10]Let me straight at the outset enter a *caveat* about restricting contraction. Obviously, any non-contractive logic cannot maintain that logical consequence can be (adequately represented by) a relation holding between *sets* of premises and conclusions (given that the set $\{A_0, A_1, A_2 \ldots B, B \ldots\}$ is identical with the set $\{A_0, A_1, A_2 \ldots B \ldots\}$). This raises very interesting and unfortunately badly underinvestigated issues in the philosophical interpretation of non-contractive logics. Although I'll have something general to say about such interpretation in section 2.3, in this paper I won't inquire further into this particular issue, and will rest content with the idea that the consequence relation does not hold between premises and conclusion *simpliciter*, but only between premises and conclusion *taken a certain (countable) number of times* (see section 3.2 for a generalisation of this point to a multiple-conclusion framework and for a standard way of representing all this mathematically using the theory of multisets).

[11]I don't know of anywhere in the literature where restriction of contraction is the key component in a philosophically motivated approach specifically focussed on the semantic paradoxes. There are though

At the end of this introduction, let me say a word on the scope and limits of the work in this paper. The aim, as I said, is to offer a new solution to the semantic paradoxes by, after briefly motivating philosophically the failure of contraction, developing a formal non-contractive naive theory of truth and proving its consistency. The focus will thus be on demonstrating the *logical and truth-theoretic strength and coherence of the theory*, especially in the surprisingly many respects of philosophically interesting strength in which it outperforms its naive rivals. As the reader will certainly notice, this focus will require a number of simplifications leaving many open problems, which I will list and briefly

a couple of logical and computer-science (rather than philosophical) traditions that have worked on the technical details of certain non-contractive logics, and have applied these to the set-theoretic (rather than semantic) paradoxes. The most relevant tradition is probably that represented by BCK *set theories* (whose study has been initiated by Grišin [1974]). Very regrettably, it would appear that that paper is only in Russian with no translation available into English, with even the Russian version not being easily accessible. Because of these circumstances, a more informed study of the technical points of similarity and dissimilarity between the two theories will have to wait. However, so much can already be said. Grišin's theory includes both "*multiplicative*" connectives and "*additive*" ones (see section 3.2), without specifying which (if any) are supposed to express the notions of conjunction and disjunction (this is probably due to the more general lack of philosophical underpinnings of the paper; as will become clear in section 3, it is precisely the focus on the "multiplicative" operators that gives my theory much of its philosophical interest). Relatedly, it only includes "additive" quantifiers, which, as I'll argue in section 3.2, yield a wrong theory of quantification. The latter difference leads to his theory remaining *finitary*, while mine goes *infinitary*; it also leads to major differences in the *deductive systems* codifying the respective theories and in the consistency proofs based on those. As I said, all antecedent non-contractive traditions tend to be concerned with sets rather than truth, and Grišin's is no exception. That tendency is something this paper aims to correct, as, *modulo* the (substantial) changes that I've mentioned I'll make to the background logic of BCK set theories, a non-contractive logic fits truth in a surprisingly optimal fashion, getting us what we want about logic and truth and getting us rid of what we don't want about logic and truth (as I'll proceed to show in detail in sections 3 and 4). The same cannot be said of the application of a non-contractive logic to sets, where arguably more radical changes than the mere restriction of contraction envisaged by BCK set theories are required to get a satisfactory theory. Grišin's own theory is proof of this, since, in merely restricting contraction, while it manages to uphold the full principle of *comprehension*, it still is deeply unsatisfactory as it does not manage to retain consistency with the full principle of *extensionality* (as is realised in Grišin [1981]), which is arguably essential to sets. Similar points apply to the couple of theorists that have followed Grišin on all these questionable choices (White [1987]; [1993]; Petersen [2000]; [2002], all of which actually develop theories that are even weaker than Grišin's in some philosophically crucial respect), although, contrary to all other authors mentioned in this footnote, Petersen's work has rich philosophical foundations that I hope in future work to be able to discuss and compare with the approach I sketch in section 2.3. The second, more loosely connected tradition I want to mention is that represented by *linear logics* (whose study has been initiated by Girard [1987]). These logics have all the drawbacks I've mentioned about the background logic of BCK set theories. Additionally, they are substantially weaker in that they do not validate the metarule of *monotonicity* (see the metarules K-L and K-R in section 3.2 for a precise formulation of monotonicity), which, as we'll see in sections 3.2 and 3.3, plays a crucial role in securing the strength of my theory. Much of the work done in linear logics is actually focussed on certain 1ary modal connectives, called '*exponentials*', which, while unexceptionable (and indeed very interesting and useful) from a purely logical and technical point of view, have no space in my theory (I hope to be able to discuss this issue in future work). Some authors have studied whether and to what extent linear logics can be used as background logics for set theories based on unrestricted comprehension (see e.g. Girard [1998]). However, exactly because of the presence of exponentials, while shedding light on important *computational issues*, the results have been rather disappointing for those in seek of a viable *theory of sets*.

comment on in section 5. I do plan to overcome most of the limitations of this paper in future work, but even so restricted I hope that the present aim will appear to the reader to be worth the effort. Achieving it will certainly prove not a trivial task (it will indeed keep us occupied for the rest of this long paper), and it is the necessary first step along a path leading to a new view on these millenary problems.

The rest of this paper is organised as follows. Section 2 starts by reviewing some of the most venerable semantic paradoxes; analysing the crucial role played by contraction in various pieces of standard paradoxical reasoning, it introduces the idea of solving the paradoxes by restricting contraction and offers a sketch of a metaphysical picture that would make sense of such a restriction. Section 3 contains the development of a formal non-contractive naive theory of truth: it lays down the syntax of a language capable of a *minimum* of self-reference, proceeds to develop the non-contractive background logic of the theory (highlighting its various strengths against the background logics of its naive rivals) and concludes by developing the theory of truth proper (again, highlighting its various strengths against its naive rivals). Section 4 introduces a way of looking at the system developed so far as a deductive system; after some stage-setting, this enables the proof of a cut-elimination theorem from which several consistency properties follow as straightforward applications. Section 5 concludes by listing and briefly commenting on some open problems lying ahead for future research.

2 Paradox and Contraction

2.1 Lies

Let us see some indicative examples of how contraction is actually crucial in paradoxical reasoning. For the time being, I wish to leave things at a fairly informal level, with the familiar bits of standard notation plus the use of T[12] as a truth predicate and the use of lower-case Gothic letters (possibly with numerical subscripts) as names of sentences (I'll make the notation more precise and slightly revise it in section 3). We start by considering the standard paradoxical reasoning using the instance of the law of excluded middle for a strengthened Liar sentence. Very roughly, a *strengthened Liar sentence* is a sentence saying of itself that it is not true: let us take as an example $\neg T\mathfrak{l}_0$, where $\mathfrak{l}_0$ is a name of $\neg T\mathfrak{l}_0$. The standard paradoxical reasoning using the instance of the law of *excluded middle* for $T\mathfrak{l}_0$ (either $\mathfrak{l}_0$ is true or $\mathfrak{l}_0$ is not true) involves two subarguments respectively deriving a violation of the law of *non-contradiction* for $T\mathfrak{l}_0$ (rejection that $\mathfrak{l}_0$ is true and $\mathfrak{l}_0$ is not true) from $T\mathfrak{l}_0$ and the same violation of the law of non-contradiction from $\neg T\mathfrak{l}_0$. *Reasoning by cases*, paradox would then ensue. Each subargument, however, crucially involves contraction. Let us look for example at the subargument deriving a violation of the law of non-contradiction from $T\mathfrak{l}_0$:[13]

---

[12]Throughout, formal and semi-formal symbols are understood autonymously to refer to themselves.

[13]Throughout, I use ⊢ to refer to the contextually relevant consequence relation. Also, I'll present the relevant pieces of reasoning in a *sequent*-style format rather than in a (philosophically more familiar)

6

$$\frac{\overline{T\mathfrak{l}_0 \ \vdash \ T\mathfrak{l}_0} \ \text{reflexivity} \quad \overline{T\mathfrak{l}_0 \ \vdash \ \neg T\mathfrak{l}_0} \ T\text{-elimination}}{\dfrac{T\mathfrak{l}_0, T\mathfrak{l}_0 \ \vdash \ T\mathfrak{l}_0 \wedge \neg T\mathfrak{l}_0}{T\mathfrak{l}_0 \ \vdash \ T\mathfrak{l}_0 \wedge \neg T\mathfrak{l}_0} \ \text{contraction}} \ \wedge\text{-introduction}$$

Even given both $T\mathfrak{l}_0 \vdash T\mathfrak{l}_0$ and $T\mathfrak{l}_0 \vdash \neg T\mathfrak{l}_0$, the intuitive metarule of $\wedge$-introduction requires to take the premises in both arguments as many times as they occur as premises in both arguments, which only yields $T\mathfrak{l}_0, T\mathfrak{l}_0 \vdash T\mathfrak{l}_0 \wedge \neg T\mathfrak{l}_0$.[14] *It is contraction* that allows us to go from that to $T\mathfrak{l}_0 \vdash T\mathfrak{l}_0 \wedge \neg T\mathfrak{l}_0$. In the absence of contraction, it is only $T\mathfrak{l}_0$ *taken twice* that entails a contradiction—$T\mathfrak{l}_0$ in itself does not, and so the paradoxical reasoning is blocked.

Things do not substantially change if we consider instead the standard paradoxical reasoning using the instance of the law of bivalence for a simple Liar sentence. Very roughly, a *simple Liar sentence* is a sentence saying of itself that it is false: using F as a falsity predicate, let us take as an example $F\mathfrak{l}_1$, where $\mathfrak{l}_1$ is a name of $F\mathfrak{l}_1$. The standard paradoxical reasoning using the instance of the law of *bivalence* for $\mathfrak{l}_1$ (either $\mathfrak{l}_1$ is true or $\mathfrak{l}_1$ is false) involves two subarguments respectively deriving a violation of the law of *contravalence* for $\mathfrak{l}_1$ (rejection that $\mathfrak{l}_1$ is true and $\mathfrak{l}_1$ is false) from $T\mathfrak{l}_1$ and the same violation of the law of contravalence from $F\mathfrak{l}_1$. *Reasoning by cases*, paradox would then ensue. Each subargument, however, crucially involves contraction. Let us look for example at the subargument deriving a violation of the law of contravalence from $T\mathfrak{l}_1$:

$$\frac{\overline{T\mathfrak{l}_1 \ \vdash \ T\mathfrak{l}_1} \ \text{reflexivity} \quad \overline{T\mathfrak{l}_1 \ \vdash \ F\mathfrak{l}_1} \ T\text{-elimination}}{\dfrac{T\mathfrak{l}_1, T\mathfrak{l}_1 \ \vdash \ T\mathfrak{l}_1 \wedge F\mathfrak{l}_1}{T\mathfrak{l}_1 \ \vdash \ T\mathfrak{l}_1 \wedge F\mathfrak{l}_1} \ \text{contraction}} \ \wedge\text{-introduction}$$

Comments analogous to those made about the strengthened Liar sentence apply. For comparative purposes, notice that, in both the strengthened-Liar and simple-Liar case, most naive theories of truth would rather block the paradoxical reasoning either at the step assuming the law of excluded middle or the law of bivalence (as happens in *analetheic*[15]

---

natural-deduction-style format. The former brings out the implicit use of contraction involved in paradoxical reasoning in a more immediate and dramatic fashion than the latter does. As is well known, if one were to analyse such pieces of reasoning using instead a natural-deduction-style format, the implicit use of contraction would be manifested in the multiple simultaneous discharge of different assumptions of the same sentence.

[14]The point in the text can be buttressed by observing that it should be uncontroversial that the general metarule of $\wedge$-introduction is to the effect that, if $A \vdash B$ holds and $C \vdash D$ holds, $A, C \vdash B \wedge D$ holds. But then, in the specific case where $A \vdash B$ holds, $C \vdash D$ holds and A is identical with C (so that in fact $A \vdash B$ holds and $A \vdash D$ holds), that metarule *by itself* only yields $A, A \vdash B \wedge D$. No matter how obvious the move might seem, a further non-trivial assumption would thus be needed to go from that to $A \vdash B \wedge D$.

[15]Some theorists have recently started calling theories that reject the law of excluded middle '*paracomplete*' (see for example Field [2008]). Although this is a purely terminological issue, I think this tendency should actively be discouraged in the interest of avoiding an unnecessarily unsystematic and confusing terminology. For 'paracomplete' should clearly be used to refer to the position which is the dual of *paraconsistency*. Given that a consequence relation $\vdash$ is *strongly* (*weakly*) *paraconsistent* iff $A \wedge \neg A \vdash B$

theories, see e.g. Brady [2006]; Field [2008]), or at the step assuming the relevant version of the metarule of reasoning by cases (as happens in *supervaluationist* or *revision* theories, see e.g. McGee [1991]; Gupta and Belnap [1993]), or at the step assuming the law of non-contradiction or the law of contravalence (as happens in *dialetheic* theories, see e.g. Priest [2006a]; Beall [2009]). I find all these laws and the relevant version of the metarule very compelling, and I regard it as a major virtue of the naive theory of truth presented in this paper that, by restricting contraction instead, it validates all of them (see section 3.2 for the details).

Of course, one needs neither the law of excluded middle (plus the law of non-contradiction) nor the law of bivalence (plus the law of contravalence) nor the metarule of reasoning by cases to generate paradox. One could for example assume $T\mathfrak{l}_0$, derive by T-elimination $\neg T\mathfrak{l}_0$ and infer from this derivation $\neg T\mathfrak{l}_0$ under no assumptions. The last inference would be licensed by the (intuitionistically acceptable version of the) metarule of *reductio ad absurdum.* (This is incidentally why intuitionist logic has never been an option for a weakening of the logic able to deal with the semantic paradoxes.) But what is the argument for justifying this metarule? Very interestingly, the standard one crucially involves contraction. Let us consider it. We assume that a formula A entails its own negation and reason as follows:

$$\cfrac{\cfrac{\cfrac{\overline{A \;\vdash\; A}\;\text{reflexivity} \qquad A \;\vdash\; \neg A}{A, A \;\vdash\; A \wedge \neg A}\;\wedge\text{-introduction}}{A \;\vdash\; A \wedge \neg A}\;\text{contraction}}{\vdash\; A \to A \wedge \neg A}\;\to\text{-introduction}$$

where the formula on the right-hand side of the last line is in many systems equivalent and fully intersubstitutable with $\neg A$ (including the system I'll present and develop in section 3). Once again, even given both $A \vdash A$ and $A \vdash \neg A$, the intuitive metarule of $\wedge$-introduction requires to take the premises in both arguments as many times as they occur as premises in both arguments, which only yields $A, A \vdash A \wedge \neg A$. *It is contraction* that allows us to go from that to $A \vdash A \wedge \neg A$. In the absence of contraction, it is only A *taken twice* that entails a contradiction—A in itself does not, and so the reasoning in favour

---

($A, \neg A \vdash B$) does not hold, $\vdash$ should be *strongly (weakly) paracomplete* iff $B \vdash A \vee \neg A$ ($B \vdash A, \neg A$) does not hold. $\vdash$'s being strongly paracomplete in this sense is perfectly compatible with $\vdash$'s validating the law of excluded middle, i.e. with $\oslash \vdash A \vee \neg A$ holding (where '$\oslash$' indicates that no sentence is taken a positive number of times (in the sense of fn 10); see definition 5 for a way of making this precise by letting $\oslash$ be the empty multiset), since failure of strong paracompleteness of $\vdash$ only follows from that *together with monotonicity in the premises*, which need not be valid in $\vdash$—more informally, one could accept $A \vee \neg A$ as a logical truth without thinking that it is entailed by everything. (This is point is dual to the one to the effect that dialetheic theories are only misleadingly called 'paraconsistent', as $\vdash$'s being strongly paraconsistent is perfectly compatible with $\vdash$'s validating the law of non-contradiction, i.e. with $A \wedge \neg A \vdash \oslash$ holding, since failure of strong paraconsistency of $\vdash$ only follows from that *together with monotonicity in the conclusions*, which need not be valid in $\vdash$—more informally, one could reject $A \wedge \neg A$ as a logical absurdity without thinking that it entails everything.) It is for this reason that I find the label 'analetheic' much more appropriate for theories that reject the law of excluded middle.

of *reductio ad absurdum* is blocked. Notice that although A in itself does not *entail a contradiction*, it does *entail its own negation*: while these two things are lumped together in most logics, a non-contractive framework allows us to keep them nicely distinct (I will delve a bit more into this distinction in section 3.2). For comparative purposes, notice also that most naive theories of truth would rather block the argument at the $\rightarrow$-introduction step. I find that metarule very compelling, and I regard it as a major virtue of the naive theory of truth presented in this paper that, by restricting contraction instead, it validates it (see section 3.2 for the details).

2.2   Curries

Consideration of *reductio ad absurdum* has naturally led to consideration of conditionals, and consideration of conditionals naturally leads to consideration of a second kind of semantic paradox: Curry's (see Curry [1942]). Let us consider the standard paradoxical reasoning with a Curry sentence for B. A *Curry sentence* for B is a sentence saying of itself that, if it is true, B: let us take as an example $T\mathfrak{c}_0 \rightarrow B$, where $\mathfrak{c}_0$ is a name for $T\mathfrak{c}_0 \rightarrow B$. The standard paradoxical reasoning with $\mathfrak{c}_0$ involves a subargument deriving B from $T\mathfrak{c}_0$, inferring from this, by $\rightarrow$-introduction, $T\mathfrak{c}_0 \rightarrow B$ under no assumptions, then inferring from the latter $T\mathfrak{c}_0$ and finally inferring from all this B. The subargument, however, crucially involves contraction. Let us look at it:

$$\dfrac{\dfrac{\dfrac{T\mathfrak{c}_0 \;\vdash\; T\mathfrak{c}_0}{} \text{ reflexivity} \quad \dfrac{B \;\vdash\; B}{} \text{ reflexivity}}{\dfrac{T\mathfrak{c}_0, T\mathfrak{c}_0 \rightarrow B \;\vdash\; B}{} \rightarrow\text{-elimination} \qquad \dfrac{T\mathfrak{c}_0 \;\vdash\; T\mathfrak{c}_0 \rightarrow B}{} T\text{-elimination}}{\dfrac{T\mathfrak{c}_0, T\mathfrak{c}_0 \;\vdash\; B}{T\mathfrak{c}_0 \;\vdash\; B} \text{ contraction}} \text{ transitivity}$$

Even given $T\mathfrak{c}_0, T\mathfrak{c}_0 \rightarrow B \vdash B$, T-elimination and transitivity only yield $T\mathfrak{c}_0, T\mathfrak{c}_0 \vdash B$.[16] *It is contraction* that allows us to go from that to $T\mathfrak{c}_0 \vdash B$. In the absence of contraction, it is only $T\mathfrak{c}_0$ *taken twice* that entails B—$T\mathfrak{c}_0$ in itself does not, and so the paradoxical reasoning is blocked.

---

[16]Again, the point in the text can be buttressed by observing that it should be uncontroversial that the general metarule of transitivity which is here relevant is to the effect that, if $A, B \vdash C$ holds and $D \vdash B$ holds, $A, D \vdash C$ holds. But then, in the specific case where $A, B \vdash C$ holds, $D \vdash B$ holds and A is identical with D (so that in fact $A, B \vdash C$ holds and $A \vdash B$ holds), that metarule *by itself* only yields $A, A \vdash C$. No matter how obvious the move might seem, a further non-trivial assumption would thus be needed to go from that to $A \vdash C$. (More precisely, what the general metarule of transitivity which is here relevant becomes in the case where A is identical with D is a version of what is known as '*cumulative transitivity*', a metarule which is particularly useful in the study of non-monotonic logics (see Gabbay [1985]). The metarule of cumulative transitivity is usually stated in the literature on those logics with the stronger consequent to the effect that $A \vdash C$ holds, but that's exactly because, in that literature, contraction is usually implicitly assumed.)

2.3 The Idea of Solving the Paradoxes by Restricting Contraction

There are, of course, many other semantic paradoxes involving the notion of truth, some of which are interestingly different from the Liar paradox and Curry's paradox. I think that all of these crucially involve contraction, at one step or another (theorem 20 and its corollaries can be taken as a proof of this claim, at least for those paradoxes expressible in the system developed in section 3). I also think that it is actually a very worthwhile and instructive enterprise, for some of these paradoxes, to analyse exactly where contraction is involved in them, as I just did for the Liar paradox and Curry's paradox. However, those further analyses are better left for another occasion.

I now rather want to pursue the thought—suggested by this brief survey—that, as contraction seems to be such a crucial ingredient in the generation of the semantic paradoxes, there is some hope to tame these by restricting that principle. The bulk of this paper, in sections 3 and 4, is in effect devoted to vindicating that hope. But what is the intuitive rationale for restricting contraction? What is it about the state-of-affairs expressed by a sentence that explains its failure to contract? Although the answers to these and related questions deserve an extended treatment that lies beyond the scope of this paper (which is focussed instead on demonstrating the logical and truth-theoretic coherence and strength of a non-contractive naive theory of truth), it seems in order that, before the start of the more formal development in section 3, I at least give a rough idea of where I think such answers lie. As the tone of my second question should already suggest, I believe that in attempting at finding these answers one has to step out of the abstract realm of *formal* theories of truth and engage in some concrete *metaphysics* of truth (more generally, over and above the specifics of the project undertaken in this paper, I believe that the interaction between formal theories of truth and more traditional theories concerning themselves with the metaphysics of truth is a very welcome and productive trend in the contemporary debate, see e.g. Maudlin [2004] for a paradigmatic case where a certain formal theory of truth is ultimately justified by a certain substantial metaphysics of truth).

As far as the naive theory of truth presented in this paper is concerned, I think that the key to understanding what it is about the state-of-affairs expressed by a sentence that explains its failure to contract is given by thinking of that state-of-affairs as distinctively *unstable*. I conceive of instability as the *metaphysical* property attaching to states-of-affairs exemplification of which *causes* the exemplification of the *logical* property of failing to contract attaching to the corresponding sentences. Let me thus explain how I understand this property and its exemplification by states-of-affairs by taking $\mathfrak{l}_0$ as a paradigmatic example. By T-introduction, if the state-of-affairs expressed by $\neg T\mathfrak{l}_0$ obtained, it would lead to the state-of-affairs expressed by $T\mathfrak{l}_0$, but the instability of the former state-of-affairs consists in the fact that *it would do so only by ceasing to obtain* (conversely, by T-elimination, if the state-of-affairs expressed by $T\mathfrak{l}_0$ obtained, it would lead to the state-of-affairs expressed by $\neg T\mathfrak{l}_0$, but the instability of the former state-of-affairs consists in the fact that *it would do so only by ceasing to obtain*). Unstable states-of-affairs behave a bit like states in physics: they have a potential for leading to

other states, but they do so only by ceasing to obtain (for example, the state in which one moving object is about to enter into collision with another moving object typically leads to a state where the direction and velocity of the two objects are different from the original ones: these are not preserved in the transition from the former state to the latter state, and so the former state leads to the latter state only by ceasing to obtain).[17]

I assume that the states-of-affairs expressed by paradoxical sentences are indeed unstable in the way I've just indicated. I think that this assumption is independently plausible and, once it has been made, it can be claimed that it is exactly such instability that causes failures of contraction: for, given that $\neg T\mathfrak{l}_0, \neg T\mathfrak{l}_0 \vdash T\mathfrak{l}_0 \wedge \neg T\mathfrak{l}_0$ holds, contraction implies that $\neg T\mathfrak{l}_0 \vdash T\mathfrak{l}_0 \wedge \neg T\mathfrak{l}_0$ holds, but that in turn requires that, if the state-of-affairs expressed by $\neg T\mathfrak{l}_0$ obtained, it would lead to the state-of-affairs expressed by $T\mathfrak{l}_0$ *without ceasing to obtain*, which cannot be given the instability of the state-of-affairs expressed by $\neg T\mathfrak{l}_0$. (A *stable* state-of-affairs, on the contrary, is exactly one that leads to its consequences without ceasing to obtain, so that, if A expresses such a state-of-affairs and $A \vdash B$ holds, one would expect $A \vdash B \wedge A$ to hold as well.) Failure of contraction is the logical symptom of an underlying unstable metaphysical reality.

Of course, by the law of excluded middle, we know that either the state-of-affairs expressed by $T\mathfrak{l}_0$ obtains or the state-of-affairs expressed by $\neg T\mathfrak{l}_0$ obtains. But, whichever obtains, *it only obtains unstably*—it immediately leads to the opposite state-of-affairs and ceases to obtain, and this in turn immediately leads to the original state-of-affairs and ceases to obtain, thus giving rise to an endless atemporal cycle. Hence, in a sense, it does not make sense to ask which one "obtains", if one means which one *stably* obtains (which is what one would usually mean by 'obtain' and its likes), for, in that sense, neither does: either way, the truth consists instead in the endless atemporal self-removal of one state-of-affairs in favour of its opposite and *vice versa*.

I think that this underlying metaphysical picture is in itself very appealing and holds potential for capturing the strong albeit vague intuition of "dynamicity" that is naturally associated with paradoxical sentences. Of course, the picture raises in turn several pressing questions, for example whether and how it can be generalised beyond the case of $\mathfrak{l}_0$ so as to cover *all* cases of failure of contraction required by the naive theory of truth presented in this paper, and how best one can understand a *dynamic* process that is also supposed to be *atemporal*. These and many other questions do need to be addressed, but clearly their treatment lies beyond the scope of this paper. It was here sufficient to give at least a rough idea of my favoured way of making philosophical sense of failure of contraction.

---

[17]In a roughly similar fashion, the written title of this paper, when this was first presented as a talk, led the chair (after some hesitation!) to a pronunciation that included the material within brackets, but it did so only by losing its potential for leading him to the incompatible pronunciation with the material within brackets left unpronounced.

3 A Non-Contractive Naive Theory of Truth

3.1 Syntax of and in the Language

It is time to develop a formal non-contractive naive theory of truth. For reasons that will become apparent in sections 3.2 and 3.3, I will call such theory '$\mathbf{IKT}^\omega$' and its background logic '$\mathbf{IK}^\omega$'. We'll work with a standard first-order language $\mathscr{L}^1$.[18]

Definition 1. The set $AS_{\mathscr{L}^1}$ of the *atomic symbols* of $\mathscr{L}^1$ is defined by enumeration as follows (where ':=' is a metalinguistic symbol expressing that what flanks it on the right-hand side is the *definiens* of what flanks it on the left-hand side):

- The denumerable set $CONST_{\mathscr{L}^1}$ of individual constants $a_0, a_1, a_2 \ldots,$[19] $b_0, b_1, b_2 \ldots, c_0, c_1, c_2 \ldots$ is a subset of $AS_{\mathscr{L}^1}$;

- The denumerable set $VAR_{\mathscr{L}^1}$ of individual variables $x_0, x_1, x_2 \ldots, y_0, y_1, y_2 \ldots, z_0, z_1, z_2 \ldots$ is a subset of $AS_{\mathscr{L}^1}$;

- For every i, the denumerable set $FUNCT^i_{\mathscr{L}^1}$ of $i$ary functors $f^i_0, f^i_1, f^i_2 \ldots, g^i_0, g^i_1, g^i_2 \ldots, h^i_0, h^i_1, h^i_2 \ldots$ is a subset of $AS_{\mathscr{L}^1}$. $FUNCT_{\mathscr{L}^1} := \bigcup\limits_{i \in \omega} (FUNCT^i_{\mathscr{L}^1})$;

- For every i, the denumerable set $PRED^i_{\mathscr{L}^1}$ of $i$ary predicate constants $P^i_0, P^i_1, P^i_2 \ldots, Q^i_0, Q^i_1, Q^i_2 \ldots, R^i_0, R^i_1, R^i_2 \ldots$ is a subset of $AS_{\mathscr{L}^1}$. $PRED_{\mathscr{L}^1} := \bigcup\limits_{i \in \omega} (PRED^i_{\mathscr{L}^1})$;

---

[18]I emphasise that the *metalanguage* within which we'll conduct our study of $\mathscr{L}^1$ will be assumed to be classical (in particular, assumed to be such that its consequence relation is contractive), and that the *metatheory* used in the metalanguage will be the classical set theory ZFC. The use of a classical metalanguage in the explanation of a non-classical object language is of course one of the *cruces* of any proposal of deviation from classical logic. I cannot hope to address here the philosophical issues related to this asymmetry nor the crucial philosophical and technical question as to whether and how much of an explanation of a non-classical object language can be provided using a metalanguage with the same logic, neither in general nor for the particular case at hand. Let me only (all too briefly) remark that, *insofar* as the mathematical metatheory, being simply supposed to provide a useful model for the systematic investigation of $\mathbf{IK}^\omega$ and $\mathbf{IKT}^\omega$, can be ultimately formulated without employing a notion of naive truth, and *insofar* as the background logic $\mathbf{IK}^\omega$ is only proposed for languages that express a notion of naive truth, there should be no objection to the way of proceeding adopted in this paper (contrast with other proposed deviations from classical logic, e.g. intuitionist logic, where what motivates the deviation is exactly a perceived inadequacy of classical logic for the relevant mathematical language).

[19]Henceforth, I'll often use the infamous three dots—when immediately preceded by reference to three progressively indexed items and when not immediately followed by reference to any other finitely indexed item—to refer to the ω-sequence obviously suggested by the first three items referred to immediately before the dots (for example, in this case, the ω-sequence whose elements are all and only the individual constants of the form 'a_i'). For the purposes of this paper, I trust that the immediacy, conciseness and familiarity of this kind of informal notation warrants its use over more precise but also more clumsy alternatives.

- The 1ary connective $\neg$ belongs to $AS_{\mathscr{L}^1}$;

- The 2ary connectives $\wedge$, $\vee$ and $\rightarrow$ belong to $AS_{\mathscr{L}^1}$;

- The first-order quantifiers $\forall$ and $\exists$ (universal and particular respectively) belong to $AS_{\mathscr{L}^1}$;

- , belongs to $AS_{\mathscr{L}^1}$;

- (and) belong to $AS_{\mathscr{L}^1}$;

- Nothing else belongs to $AS_{\mathscr{L}^1}$.

As we'll see in section 3.2, given the logical strength of $\mathbf{IK}^\omega$ this vocabulary contains many redundancies. Still, it will prove useful to proceed with all the standard primitives in order to appreciate with ease some of the fine workings of the theory. We pick designated individual constants (say, $a_0, a_1, a_2 \ldots$) to serve as canonical names of all sentences in the language (see definition 4); if an individual constant is the canonical name of a sentence A, we'll refer to that individual constant by '$\ulcorner A \urcorner$' or by a lower-case Gothic letter (possibly with numerical subscripts). We assume to have paradoxical sentences in the language, such as, for example, those denoted by $\mathfrak{l}_0$, $\mathfrak{l}_1$ and $\mathfrak{c}_0$ in section 2. We also pick a designated 1ary predicate constant in $PRED^1_{\mathscr{L}^1}$ (say, P^1_{100}) to serve as a truth predicate, and denote it with 'T' (as I've announced in section 2.1, I'm now making the notation more precise and slightly revising it).

Definition 2. The set $TERM_{\mathscr{L}^1}$ of *terms* of $\mathscr{L}^1$ can be defined by recursion in the usual way:[20]

- If $\tau \in CONST_{\mathscr{L}^1}, VAR_{\mathscr{L}^1}$, then $\tau \in TERM_{\mathscr{L}^1}$;

- For every i, if $\tau_0, \tau_1, \tau_2 \ldots \tau_{i-1} \in TERM_{\mathscr{L}^1}$ and $\rho^i \in FUNCT^i_{\mathscr{L}^1}$, $\rho^i(\tau_0, \tau_1, \tau_2 \ldots \tau_{i-1}) \in TERM_{\mathscr{L}^1}$.

Definition 3. The set $WFF_{\mathscr{L}^1}$ of *well-formed formulae* (wffs) of $\mathscr{L}^1$ can be defined by recursion in the usual way:[21]

- For every i, if $\tau_0, \tau_1, \tau_2 \ldots \tau_{i-1} \in TERM_{\mathscr{L}^1}$ and $\Phi^i \in PRED^i_{\mathscr{L}^1}$, $\Phi^i(\tau_0, \tau_1, \tau_2 \ldots \tau_{i-1}) \in WFF_{\mathscr{L}^1}$;

- If $\varphi \in WFF_{\mathscr{L}^1}$, $(\neg\varphi) \in WFF_{\mathscr{L}^1}$;

---

[20]Throughout, 'τ' and 'υ' (possibly with numerical subscripts) are used as metalinguistic variables ranging over $TERM_{\mathscr{L}^1}$; 'ρ' (possibly with numerical subscripts and superscripts (to indicate arity)) is used as a metalinguistic variable ranging over $FUNCT_{\mathscr{L}^1}$.

[21]Throughout, 'Φ' (possibly with numerical subscripts and superscripts (to indicate arity)) is used as a metalinguistic variable ranging over $PRED_{\mathscr{L}^1}$; 'φ', 'ψ' and 'χ' (possibly with numerical subscripts) are used as metalinguistic variables ranging over $WFF_{\mathscr{L}^1}$; 'ξ' (possibly with numerical subscripts) is used as a metalinguistic variable ranging over $VAR_{\mathscr{L}^1}$.

- If $\varphi, \psi \in WFF_{\mathscr{L}^1}$, $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi) \in WFF_{\mathscr{L}^1}$;

- If $\varphi \in WFF_{\mathscr{L}^1}$, $\xi \in VAR_{\mathscr{L}^1}$ and neither $\forall \xi$ nor $\exists \xi$ occur in φ, then $(\forall \xi (\varphi))$, $(\exists \xi (\varphi)) \in WFF_{\mathscr{L}^1}$;

- Nothing else belongs to $WFF_{\mathscr{L}^1}$.

Henceforth, to save on brackets, I will presuppose the following conventions. I will assume the usual scope hierarchy among the connectives (with $\neg$ binding more strongly than $\wedge$ and $\vee$, and with these in turn binding more strongly than $\rightarrow$) and left associativity for each 2ary connective (so that $(\varphi_0 \star \varphi_1 \star \varphi_2 \ldots \star \varphi_i)$ reads $(((\ldots (\varphi_0 \star \varphi_1) \star \varphi_2) \ldots \star \varphi_i)$, with $\star$ being a 2ary connective). I will also assume that quantifiers bind as strong as $\neg$ and that right associativity holds among 1ary operators[22] (so that $(\star_0 \star_1 \star_2 \ldots \star_i \varphi)$ reads $(\star_0 (\star_1 (\star_2 \ldots \star_i (\varphi) \ldots)))$, with each $\star_j$ being a 1ary operator). I will finally drop the brackets of functional and predicative application, the outermost brackets of a self-standing wff and the commas of argument composition.

Definition 4. The set $SENT_{\mathscr{L}^1}$ of *sentences* of $\mathscr{L}^1$ can then be defined as usual by setting $SENT_{\mathscr{L}^1} := WFF_{\mathscr{L}^1} \setminus \{\varphi : \text{for some } \xi, \xi \text{ occurs free in } \varphi\}$.

3.2 The Background Logic

Given some crucial features of $\mathbf{IK}^{\omega}$, a *multiple-conclusion* framework will be necessary in order to give an adequate treatment of disjunction. Moreover, given the failures of contraction and as already intimated in fn 10, we'll have to think of logical consequence as (being adequately represented by)[23] a relation holding not between *sets* of sentences (i.e. between a set of premises and a set of conclusions), but between *multisets* of sentences (i.e. between a multiset of premises and a multiset of conclusions). In turn, multisets are informally like sets except that they are sensitive to the number of times that a member occurs in them. For the purposes of this paper, it will suffice to focus attention on multisets that discriminate number of occurrences only within the domain of the *countable*.[24] We

---

[22]Following standard usage, I've been calling $\neg$, $\wedge$, $\vee$, $\rightarrow$ '*connectives*', while $\forall$ and $\exists$ (possibly together with the immediately following variable) '*quantifiers*'. I'll use '*operator*' for both kinds of logical expressions.

[23]The parenthetical qualification hints at the fact that the *informal* notion of logical consequence would rather seem to be a notion of a relation holding among (certain finely structured) *pluralities*, just like the relation of being more numerous than informally holds between Andy, Bill and Charlie on the one side and Dave and Emmie on the other side, although it is usually adequately represented by a relation holding among the sets {Andy, Bill, Charlie} and {Dave, Emmie} (as I've hinted at in fn 10, in the case of $\mathbf{IK}^{\omega}$ the pluralities should be so finely structured as to discriminate how many times something is one of them). I will ignore such niceties in what follows.

[24]While crucial for the development of $\mathbf{IK}^{\omega}$, this inclusion of the countably infinite is actually non-standard for investigations of multisets, which are typically restricted to multisets that discriminate number of occurrences only within the domain of the *finite* (see e.g. Blizard [1989], which nevertheless offers a useful survey of the literature on and systematic development of the (standard) theory of multisets).

make this informal notion mathematically precise with the following definitions:[25]

Definition 5. A *multiset* of wffs of $\mathscr{L}^1$ is a function whose domain is $WFF_{\mathscr{L}^1}$ and whose range is $\omega + 1$.[26] We write *pairwise* and *countable combinations* of multisets as follows:[27]

- $\Gamma, \Delta(\varphi) := \Gamma(\varphi) + \Delta(\varphi)$;

- $\bigsqcup_{0 \leq i < \omega} (\Gamma_i)(\varphi) := \sum_{0 \leq i < \omega} (\Gamma_i(\varphi))$.

We also write a list of sentences enclosed by square brackets for the multiset containing, for every φ, as many occurrences[28] of φ as there are in the list and set $\Gamma, \varphi := \Gamma, [\varphi]$ (including the case where Γ is the *empty multiset*, i.e. the multiset with no positive occurrences, which we denote with '$\oslash$'). We also understand talk of multiset membership and its likes to be restricted to positive occurrences.

With the notion of a multiset in place, we can proceed to define also the notion of a logic in a mathematically precise way:

Definition 6. A *logic* **L** for $\mathscr{L}^1$ is any subset of $\omega + 1^{WFF_{\mathscr{L}^1}} \times \omega + 1^{WFF_{\mathscr{L}^1}}$.

We can then proceed to specify the logic $\mathbf{IK}^\omega$ as the smallest logic containing as axiom the structural rule:[29]

$$\frac{}{\varphi \ \vdash_{\mathbf{IK}^\omega} \ \varphi} \text{ I}$$

closed under the structural metarules:

$$\frac{\Gamma_0 \ \vdash_{\mathbf{IK}^\omega} \ \Delta}{\Gamma_0, \Gamma_1 \ \vdash_{\mathbf{IK}^\omega} \ \Delta} \text{ K-L} \qquad\qquad \frac{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0}{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0, \Delta_1} \text{ K-R}$$

---

[25]Comments analogous to those in fn 23 apply to the relation between the informal notion of a multiset as just explained in the text and the mathematically precise notion to be introduced in definition 5. So there are in effect *two* levels of modelling: we model logical consequence—informally, a relation holding among (certain finely structured) pluralities—as a relation holding between multisets, and we in turn model multisets—informally, collections sensitive to the number of times that a member occurs in them—as standard set-theoretic functions with the properties described in definition 5.

[26]Every function is assumed to be total unless otherwise specified.

[27]Throughout, 'Γ', 'Δ', 'Θ', 'Λ', 'Ξ' (possibly with numerical subscripts) are used as metalinguistic variables ranging over the set of multisets of wffs of $\mathscr{L}^1$.

[28]This notion of occurrence extends naturally the standard notion of occurrence as a relation among *linguistic types* which is also used in this paper (as when it is said, for example, that a variable occurs twice in a wff). I trust that no confusion will arise as to which of these two notions is meant in a particular context.

[29]I'll take a rule or metarule to be *structural* if it does not refer to any particular member of $AS_{\mathscr{L}^1}$. Otherwise, I'll say that the rule or metarule is *operational*.

$$\frac{\Gamma_0 \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0, \varphi \qquad \Gamma_1, \varphi \ \vdash_{\mathbf{IK}^\omega} \ \Delta_1}{\Gamma_0, \Gamma_1 \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0, \Delta_1} \ \text{S}$$

and under the operational metarules:

$$\frac{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \varphi}{\Gamma, \neg\varphi \ \vdash_{\mathbf{IK}^\omega} \ \Delta} \ \neg\text{-L} \qquad\qquad\qquad \frac{\Gamma, \varphi \ \vdash_{\mathbf{IK}^\omega} \ \Delta}{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \neg\varphi} \ \neg\text{-R}$$

$$\frac{\Gamma, \varphi, \psi \ \vdash_{\mathbf{IK}^\omega} \ \Delta}{\Gamma, \varphi \wedge \psi \ \vdash_{\mathbf{IK}^\omega} \ \Delta} \ \wedge\text{-L} \qquad\qquad \frac{\Gamma_0 \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0, \varphi \qquad \Gamma_1 \ \vdash_{\mathbf{IK}^\omega} \ \Delta_1, \psi}{\Gamma_0, \Gamma_1 \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0, \Delta_1, \varphi \wedge \psi} \ \wedge\text{-R}$$

$$\frac{\Gamma_0, \varphi \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0 \qquad \Gamma_1, \psi \ \vdash_{\mathbf{IK}^\omega} \ \Delta_1}{\Gamma_0, \Gamma_1, \varphi \vee \psi \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0, \Delta_1} \ \vee\text{-L} \qquad\qquad \frac{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \varphi, \psi}{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \varphi \vee \psi} \ \vee\text{-R}$$

$$\frac{\Gamma_0 \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0, \varphi \qquad \Gamma_1, \psi \ \vdash_{\mathbf{IK}^\omega} \ \Delta_1}{\Gamma_0, \Gamma_1, \varphi \rightarrow \psi \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0, \Delta_1} \ \rightarrow\text{-L} \qquad\qquad \frac{\Gamma, \varphi \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \psi}{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \varphi \rightarrow \psi} \ \rightarrow\text{-R}$$

$$\frac{\Gamma, \varphi_{v_0/\xi}, \varphi_{v_1/\xi}, \varphi_{v_2/\xi} \ldots \ \vdash_{\mathbf{IK}^\omega} \ \Delta}{\Gamma, \forall\xi\varphi \ \vdash_{\mathbf{IK}^\omega} \ \Delta} \ \forall\text{-L}$$

$$\frac{\Gamma_0 \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0, \varphi_{v_0/\xi} \qquad \Gamma_1 \ \vdash_{\mathbf{IK}^\omega} \ \Delta_1, \varphi_{v_1/\xi} \qquad \Gamma_2 \ \vdash_{\mathbf{IK}^\omega} \ \Delta_2, \varphi_{v_2/\xi} \ldots}{\bigsqcup_{0 \leq i < \omega} (\Gamma_i) \ \vdash_{\mathbf{IK}^\omega} \ \bigsqcup_{0 \leq i < \omega} (\Delta_i), \forall\xi\varphi} \ \forall\text{-R}$$

$$\frac{\Gamma_0, \varphi_{v_0/\xi} \ \vdash_{\mathbf{IK}^\omega} \ \Delta_0 \qquad \Gamma_1, \varphi_{v_1/\xi} \ \vdash_{\mathbf{IK}^\omega} \ \Delta_1 \qquad \Gamma_2, \varphi_{v_2/\xi} \ \vdash_{\mathbf{IK}^\omega} \ \Delta_2 \ldots}{\bigsqcup_{0 \leq i < \omega} (\Gamma_i), \exists\xi\varphi \ \vdash_{\mathbf{IK}^\omega} \ \bigsqcup_{0 \leq i < \omega} (\Delta_i)} \ \exists\text{-L}$$

$$\frac{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \varphi_{v_0/\xi}, \varphi_{v_1/\xi}, \varphi_{v_2/\xi} \ldots}{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \exists\xi\varphi} \ \exists\text{-R}$$

(where φ_{τ_0/τ_1} is the result of substituting τ_0 for all free occurrences of τ_1 in φ, with τ_0 being free for τ_1 in φ, and where '$v_0, v_1, v_2 \ldots$' and its likes denote a designated complete enumeration of $TERM_{\mathscr{L}^1}$ and its likes).

Emphatically, we do *not* include the structural metarules of contraction in $\mathbf{IK}^\omega$:

$$\frac{\Gamma, \varphi, \varphi \ \vdash_{\mathbf{IK}^\omega} \ \Delta}{\Gamma, \varphi \ \vdash_{\mathbf{IK}^\omega} \ \Delta} \ \text{W-L} \qquad\qquad\qquad \frac{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \varphi, \varphi}{\Gamma \ \vdash_{\mathbf{IK}^\omega} \ \Delta, \varphi} \ \text{W-R}$$

In fact, as corollary 11 to theorem 20 will show, not only are W-L and W-R *absent from the defining principles* of $\mathbf{IK}^\omega$ (and $\mathbf{IKT}^\omega$); they are simply not *admissible* for $\mathbf{IK}^\omega$ (or $\mathbf{IKT}^\omega$), i.e. neither system is closed under them.

Literature on substructural logics, and in particular on linear logics (see fn 11), typically considers also operators governed by *additive* (or *context-sharing*) rather than *multiplicative* (or *context-free*) metarules, where the latter are pairs of metarules in which (in the multipremise*[30] metarule) the wffs that explicitly occur in the premises* are allowed to be in *different* contexts and these are *not* contracted in the conclusion*, while the former are pairs of metarules in which (in the multipremise* metarule) the wffs that explicitly occur in the premises* are required to be in the *same* context and these are *contracted* in the conclusion*.[31] Thus, for example, also the following metarules for a conjunctive connective $\overset{A}{\wedge}$ and for a disjunctive connective $\overset{A}{\vee}$ are considered:

$$\frac{\Gamma, \varphi \vdash \Delta}{\Gamma, \varphi \overset{A}{\wedge} \psi \vdash \Delta} \overset{A}{\wedge}\text{-L}^0 \qquad \frac{\Gamma, \psi \vdash \Delta}{\Gamma, \varphi \overset{A}{\wedge} \psi \vdash \Delta} \overset{A}{\wedge}\text{-L}^1 \qquad\qquad \frac{\Gamma \vdash \Delta, \varphi \qquad \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \varphi \overset{A}{\wedge} \psi} \overset{A}{\wedge}\text{-R}$$

$$\frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \overset{A}{\vee} \psi \vdash \Delta} \overset{A}{\vee}\text{-L} \qquad\qquad \frac{\Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \varphi \overset{A}{\vee} \psi} \overset{A}{\vee}\text{-R}^0 \quad \frac{\Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \varphi \overset{A}{\vee} \psi} \overset{A}{\vee}\text{-R}^1$$

Moreover, to the best of my knowledge, when it comes to quantifiers the above-mentioned literature exclusively considers something along the lines of the following rules for a universal quantifier $\overset{A}{\forall}$ and for a particular quantifier $\overset{A}{\exists}$:

$$\frac{\Gamma, \varphi_{\tau/\xi} \vdash \Delta}{\Gamma, \overset{A}{\forall} \xi\varphi \vdash \Delta} \overset{A}{\forall}\text{-L} \qquad\qquad\qquad\qquad \frac{\Gamma \vdash \Delta, \varphi_{\tau/\xi}}{\Gamma \vdash \Delta, \overset{A}{\forall} \xi\varphi} \overset{A}{\forall}\text{-R}$$

$$\frac{\Gamma, \varphi_{\tau/\xi} \vdash \Delta}{\Gamma, \overset{A}{\exists} \xi\varphi \vdash \Delta} \overset{A}{\exists}\text{-L} \qquad\qquad\qquad\qquad \frac{\Gamma \vdash \Delta, \varphi_{\tau/\xi}}{\Gamma \vdash \Delta, \overset{A}{\exists} \xi\varphi} \overset{A}{\exists}\text{-R}$$

(where in $\overset{A}{\forall}$-R and $\overset{A}{\exists}$-L τ does not occur free in either Γ or Δ).

---

[30]To avoid confusion, I reserve unstarred 'premise', 'conclusion' and their likes for the things multisets of which stand in the relation of logical consequence, while I use 'premise*', 'conclusion*' and their likes for the inputs and outputs of the metarules of the systems in question.

[31]Of course, given monotonicity but not contraction (as is the case for $\mathbf{IK}^\omega$), it is the second conjunct that really matters, while, given contraction but not monotonicity (as would be the case for example for certain presentations of certain relevant logics), it is the first conjunct that really matters.

While such metarules and the operators governed by them are unexceptionable from a purely logical and technical point of view, and have indeed proven to be very interesting and useful additions to a logic, they are multiply unacceptable from the philosophical point of view presented in this paper. From this point of view, they are in general unacceptable because premise and conclusion combination in general should go by the pairwise- and countable-sum operations given in definition 5. This is in turn so because of at least two reasons. First (and foremost), focussing without loss of generality on the case of premise combination, one of the guiding ideas of the approach pursued in this paper is that the *joint logical strength* of Γ and Γ can be different from the logical strength of Γ alone. That is what failure of contraction requires. Hence, assuming that premise combination is supposed to give us the joint logical strength of the multisets of premises to be combined,[32] it should result in a combining multiset that preserves the distinctness of the premises' occurrences in the combined multisets, so that the logical strength of each such occurrence is preserved in the combining multiset. And this is exactly what is achieved by summing the premises' distinct occurrences. Second (and less important), short of appealing to monotonicity (which is present in $\mathbf{IK}^\omega$ but may not be present in similarly motivated logics), premise and conclusion combination going by the pairwise- and countable-sum operations given in definition 5 seems to be the only natural option in the very common case where *different* premises or conclusions occur in different multisets that have to be combined. But then it would be arbitrary to deviate from that general method of combination in the very special case where it is exactly the *same* premises or conclusions that occur in each multiset that has to be combined.

From the philosophical point of view presented in this paper, the metarules for the additive operators are also unacceptable on more specific grounds. For it would follow from an easy extension of the cut-elimination theorem 20 to a deductive system including the metarules for the additive operators that, for example, $\oslash \vdash \varphi \overset{A}{\vee} \psi$ holds iff either $\oslash \vdash \varphi$ holds or $\oslash \vdash \psi$ holds (and that $\oslash \vdash \overset{A}{\exists} \xi\varphi$ holds iff, for some τ, $\oslash \vdash \varphi_{\tau/\xi}$ holds). This is so because of the more general fact that, in a standard substructural deductive (sequent-

---

[32]I think this assumption is hardly questionable with respect to what premise combination is informally supposed to do in reasoning: pooling together different pieces of information. But I'm very open to investigating, for certain purposes, alternative notions of premise combination. According to one such notion, for example, premise combination is supposed to give us, rather than the *joint* logical strength of the multisets of premises to be combined, the logical strength of *any* of them. (This is like the contrast between "You can have both English and continental breakfast" and the less generous but more sensible "You can have either English or continental breakfast".) Indeed, that is arguably the notion underlying the metarules for the additive operators, and in fact we can see the main feature of that notion reflected in the logical behaviour of those very same operators. Thus, for example, speaking a bit roughly, while from $\varphi \overset{A}{\wedge} \psi$ one can infer φ and one can infer ψ, one cannot infer both φ and ψ. Speaking a bit less roughly, even if $\varphi, \psi \vdash \chi$, it does not follow (even in the presence of S) that $\varphi \overset{A}{\wedge} \psi \vdash \chi$ (while of course, by $\wedge$-L, it does follow that $\varphi \wedge \psi \vdash \chi$). As I have said, I'm very open to investigating, for certain purposes, this alternative notion of premise combination (and others as well). But I think that, insofar as one is concerned with developing a logic for the purpose of using it reasoning, the relevant notion of premise combination is the one according to which it gives us the joint logical strength of the multisets of premises to be combined.

calculus) system $\Rightarrow$ with only I, K-L and K-R as structural rules and metarules (with S possibly being admissible), the sequent $\oslash \Rightarrow \varphi \overset{A}{\vee} \psi$ must be got from either $\overset{A}{\vee}$-R^0 or $\overset{A}{\vee}$-R^1, which clearly can only happen if either $\oslash \Rightarrow \varphi$ holds or $\oslash \Rightarrow \psi$ holds (and the sequent $\oslash \Rightarrow \overset{A}{\exists} \xi\varphi$ must be got from $\overset{A}{\exists}$-R, which clearly can only happen if, for some τ, $\oslash \Rightarrow \varphi_{\tau/\xi}$ holds).[33],[34] That imposes on additive operators strong *constructivist* features that few of us would expect to be present in our informal notions of conjunction, disjunction and quantification. Of course the issues arising at this juncture go far beyond the scope of this paper. Let me just register the non-constructivist character of the philosophical point of view presented in this paper and its consequent rejection of additive operators as giving adequate expression to our informal notions of conjunction, disjunction and quantification.[35],[36]

Conversely, no doubt the shift to multiplicative quantifiers has here been achieved at what are usually regarded as significant costs. Firstly, $\forall$-L and $\exists$-R are *infinitary* in the sense of having infinitely many explicitly occurring premises or conclusions, while $\forall$-R and $\exists$-L are infinitary in the sense of having infinitely many premises* or conclusions* (with overall infinitely many explicitly occurring premises or conclusions). Secondly, $\forall$-L and $\exists$-R clearly only make sense with respect to the intended meaning of $\forall$ and $\exists$ as *objectual* quantifiers if every object is denoted by a term in $TERM_{\mathscr{L}^1}$, which in turn implies that the objects these quantifiers range over not only form a set, but are indeed

---

[33]Note that an analogous fact does not hold for $\vee$, since $\oslash \vdash_{\mathbf{IK}^\omega} \varphi \vee \neg\varphi$ holds for every φ (see theorem 33), while it is a consequence of corollary 8 to theorem 20 that, for example, neither $\oslash \vdash_{\mathbf{IK}^\omega} P_0^0$ nor $\oslash \vdash_{\mathbf{IK}^\omega} \neg P_0^0$ hold. Note also that, since $\oslash \vdash_{\mathbf{IK}^\omega} \varphi, \neg\varphi$ also holds (see again theorem 33), in a suitable extension of $\mathbf{IK}^\omega$, by $\overset{A}{\vee}$-R^0 and $\overset{A}{\vee}$-R^1, $\oslash \vdash_{\mathbf{IK}^\omega} \varphi \overset{A}{\vee} \neg\varphi, \varphi \overset{A}{\vee} \neg\varphi$ would also hold, but of course we could not from this infer $\oslash \vdash_{\mathbf{IK}^\omega} \varphi \overset{A}{\vee} \neg\varphi$ because of the absence of W-R.

[34]Indeed, even stronger results encompassing arguments with premises would be available: for example, if the only members of Γ are *Rasiowa-Harrop* wffs (i.e. wffs where no disjunction or particular quantification has a strictly positive occurrence), $\Gamma \vdash \varphi \overset{A}{\vee} \psi$ holds iff either $\Gamma \vdash \varphi$ holds or $\Gamma \vdash \psi$ holds (and $\Gamma \vdash \overset{A}{\exists} \xi\varphi$ holds iff, for some τ, $\Gamma \vdash \varphi_{\tau/\xi}$ holds).

[35]Again, the last point is compatible with the recognition of the fact that additive operators are unexceptionable from a purely logical and technical point of view, and that they have indeed proven to be very interesting and useful additions to a logic. Even more strongly, it is compatible with such operators expressing interesting notions (different from our informal notions of conjunction, disjunction and quantification). I take no stand on this specific issue, while recording a certain scepticism as to whether rules are ever sufficient to determine a meaning. Be that as it may, the issue is to a large extent tangential to the purposes of this paper: as I've already observed in the text, theorem 20 is easily extendable to a deductive system including the metarules for the additive operators, if one wishes to include them (however, in order not to raise complexity beyond necessity, I won't spell out the details of such extension).

[36]Having said all this, I believe that the metarules for the additive operators do capture some important features of our informal notions of conjunction, disjunction and quantification: as can easily be seen from theorems 7, 8, 9 and 10, the theory of conjunction, disjunction and quantification developed in this paper validates analogues of the unipremise* metarules $\overset{A}{\wedge}$-L^0, $\overset{A}{\wedge}$-L^1, $\overset{A}{\vee}$-R^0, $\overset{A}{\vee}$-R^1, $\overset{A}{\forall}$-L and $\overset{A}{\exists}$-R *unrestrictedly*, and also validates analogues of the multipremise* (or unipremise* but with the usual no-free-occurrence restriction) metarules $\overset{A}{\wedge}$-R, $\overset{A}{\vee}$-L, $\overset{A}{\forall}$-R and $\overset{A}{\exists}$-L *for the special case in which* $\Gamma = \Delta = \oslash$.

countably many. While I must confess to not being particularly moved by the first worry (at least without further elaboration and detail), and will say no more about it, the second worry does point to a very severe limitation of the systems developed in this paper. It is however, I think, a limitation worth incurring in order to keep complexity within necessity, and I trust that it will be easy to see that and how the systems developed in this paper can be so generalised as to lift the restriction to the countable. Moreover, overall, the naturalness and coherence of the systems developed in this paper does show that there are at least some intuitive and well-behaved notions of multiplicative quantifiers, as opposed to a certain scepticism to be found in the literature ("no good context-free versions are known", Troelstra and Schwichtenberg [2000], p. 295). This is even more important if I am right in my contention that additive quantifiers have little or no place in the development of the kind of non-contractive solution to the semantic paradoxes presented in this paper.

Whereas, in comparison with other deviations from classical logic proposed in order to deal with the semantic paradoxes while preserving the naive theory of truth, $\mathbf{IK}^\omega$ is obviously weak in the specific respect constituted by the absence from it of W-L and W-R, it is surprisingly strong in many other respects, approximating the simplicity and symmetry of classical logic to an extent unmatched by its naive rivals. I will now go through an indicative survey of principles valid in $\mathbf{IK}^\omega$, for most of the time letting the self-evidence of such principles speak for itself, but commenting a bit on certain consequences and comparisons.

To begin with, in $\mathbf{IK}^\omega$ $\neg\varphi$ is *complete* over φ:

Theorem 1. *The law of* excluded middle *and the attendant rule of* exhaustion *hold in* $\mathbf{IK}^\omega$, *that is:*

(LEM) $\oslash \vdash_{\mathbf{IK}^\omega} \varphi \vee \neg\varphi$

(EXH) $\oslash \vdash_{\mathbf{IK}^\omega} \varphi, \neg\varphi$

hold.[37]

Proof. By I, $\varphi \vdash_{\mathbf{IK}^\omega} \varphi$ holds, and hence, by $\neg$-R, $\oslash \vdash_{\mathbf{IK}^\omega} \varphi, \neg\varphi$ holds, and so, by $\vee$-R, $\oslash \vdash_{\mathbf{IK}^\omega} \varphi \vee \neg\varphi$ holds.

$\square$

This is in sharp contrast to analetheic naive theories of truth (see some references in section 2.1), in which both the law of excluded middle and the rule of exhaustion fail.

Given the dualities of the logic, it is no surprise that in $\mathbf{IK}^\omega$ $\neg\varphi$ is also *inconsistent* with φ:

---

[37]Throughout, for clarity I use more or less traditional names (like 'the law of excluded middle') for principles seen not as specific to a certain system, but as able to be valid or invalid relative to different systems (as when we say for example that the law of excluded middle is valid in classical logic but invalid in intuitionist logic), while I use the corresponding acronyms (like '(LEM)') for the $\mathbf{IK}^\omega$- or $\mathbf{IKT}^\omega$-specific principles, which can only either absolutely hold or absolutely fail to hold, depending on the properties of $\mathbf{IK}^\omega$ and $\mathbf{IKT}^\omega$ (as when I say that (LEM) holds).

Theorem 2. *The law of* non-contradiction *and the attendant rule of* explosion *hold in* $\mathbf{IK}^\omega$*, that is:*

(LNC) $\varphi \wedge \neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$

(EXP) $\varphi, \neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$

hold.

Proof. By I, $\varphi \vdash_{\mathbf{IK}^\omega} \varphi$ holds, and hence, by ¬-L, $\varphi, \neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$ holds, and so, by ∧-L, $\varphi \wedge \neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$ holds.

□

This is in sharp contrast to dialetheic naive theories of truth (see some references in section 2.1), in which both the law of non-contradiction and the rule of explosion fail. In fact, by ¬-R, (LNC) yields $\oslash \vdash_{\mathbf{IK}^\omega} \neg(\varphi \wedge \neg\varphi)$. This latter, extremely plausible law is upheld in the best dialetheic theories (such as those referenced in section 2.1)[38] but disappointingly fails even in the best analetheic theories (such as those referenced in section 2.1), which are thus not in a position to assert extremely plausible claims like the one that that it is not the case that [the strengthened Liar sentence is true and the strengthened Liar sentence is not true].[39,40]

Indeed, the joint strength of ¬-L and ¬-R ensures that negation enjoys many of its habitual interactions with *suppositional reasoning*:

Theorem 3. *The metarules of* weak reductio ad absurdum *hold in in* $\mathbf{IK}^\omega$*, that is:*

(WRAA$_0$) *If* $\Gamma \vdash_{\mathbf{IK}^\omega} \varphi$ *holds,* $\Gamma, \neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$ *holds.*

(WRAA$_1$) *If* $\Gamma, \varphi \vdash_{\mathbf{IK}^\omega} \oslash$ *holds,* $\Gamma \vdash_{\mathbf{IK}^\omega} \neg\varphi$ *holds.*

hold.

Proof. I give the argument for (WRAA$_0$). Suppose that $\Gamma \vdash_{\mathbf{IK}^\omega} \varphi$ holds. Then, by ¬-L, $\Gamma, \neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$ holds. A dual argument holds for (WRAA$_1$).

□

---

[38]Whence the impropriety of calling that principle 'the law of non-contradiction', since it is acceptable even for those theorists that accept contradictions. Although the point ultimately depends on subtleties concerning the interpretation of arguments whose conclusion is $\oslash$ which I cannot go into here, I think it can reasonably be argued that, on the contrary, it is the validity of the analogues of (LNC) in a logic $\vdash$ that adequately represents the fact that $\vdash$ "rules out" contradictions.

[39]Henceforth, I use square brackets to disambiguate constituent structure in English.

[40]It'll probably be rejoined that the alleged extremely plausible law is equivalent, given the relevant De Morgan principles, to the law of excluded middle, and hence that, since the latter is resistible, so is the former. But this way of reasoning strikes me as deeply misguided: given that the alleged extremely plausible law is indeed such, it is not resistible, and hence one should either also accept the law of excluded middle or else reject the relevant De Morgan principles (as intuitionists do).

Thus, although (the intuitionistically acceptable version of) full *reductio ad absurdum* has to fail in the background logic of the naive theory of truth presented in this paper,[41] we still preserve the core of the connections between negation and inconsistency in suppositional reasoning, contrary to most naive theories of truth (certainly all those referenced in this paper), where either of the metarules fails. (Indeed, it is very much arguable that, once the distinction has been drawn between (WRAA$_1$) and the metarule standardly called '*reductio ad absurdum*' formulated in fn 41, the name '*reductio ad absurdum*' for the latter is an egregious misnomer. For a metarule properly so-called should have an antecedent saying that certain premises lead to an absurdity (i.e. are inconsistent, i.e. entail $\oslash$), which is clearly the case for (WRAA$_1$) and clearly not the case for the metarule standardly called '*reductio ad absurdum*' formulated in fn 41, which would more properly be called '*reductio ad ipsius negationem*'.)

Negation in $\mathbf{IK}^\omega$ has thus all the hallmarks of *Boolean* negation (see Priest [2006b] for a good critical discussion of this notion).[42] Of course, it also has all the hallmarks of *De Morgan* negation:

Theorem 4. *The* sentential De Morgan *rules hold in* $\mathbf{IK}^\omega$*, that is:*

(SDM$_0$) $\varphi \wedge \psi \vdash_{\mathbf{IK}^\omega} \neg(\neg\varphi \vee \neg\psi)$

(SDM$_1$) $\varphi \vee \psi \vdash_{\mathbf{IK}^\omega} \neg(\neg\varphi \wedge \neg\psi)$

(SDM$_2$) $\neg(\varphi \wedge \psi) \vdash_{\mathbf{IK}^\omega} \neg\varphi \vee \neg\psi$

(SDM$_3$) $\neg(\varphi \vee \psi) \vdash_{\mathbf{IK}^\omega} \neg\varphi \wedge \neg\psi$

hold.

---

[41](The intuitionistically acceptable version of) full *reductio ad absurdum* says that, if $\Gamma, \varphi \vdash \Delta, \neg\varphi$ holds, $\Gamma \vdash \Delta, \neg\varphi$ holds. This metarule has to fail in the background logic of the naive theory of truth presented in this paper because, as I've already observed in section 2.1, it would reinstate paradox for example with the strengthened Liar sentence $\mathfrak{l}_0$ (for then, since $T\mathfrak{l}_0$ entails $\neg T\mathfrak{l}_0$, $\neg T\mathfrak{l}_0$ would be a law, which is easily seen to lead to inconsistency in the naive theory of truth presented in this paper). Notice that, applying $\neg$-R to $\Gamma, \varphi \vdash_{\mathbf{IK}^\omega} \Delta, \neg\varphi$, we do get $\Gamma \vdash_{\mathbf{IK}^\omega} \Delta, \neg\varphi, \neg\varphi$, but, given the absence of W-R, we cannot go from that to $\Gamma \vdash_{\mathbf{IK}^\omega} \Delta, \neg\varphi$.

[42]Unfortunately, that book completely ignores the kind of proposal presented in this paper, and in so doing overstates its case when it says (p. 94, where contraction is blatantly presupposed in the "proof" of the claim): "[...] if we have Boolean negation and the truth predicate (together with self-reference), triviality ensues" and when it says (p. 99, where again the only way envisaged of keeping both Boolean negation and naive truth is to give up self-reference, which, as Priest says, is "clearly no ground for smugness either"): "[n]or has a classical logician any reason to feel smug about this. As we have seen, if Boolean negation is meaningful, then a predicate satisfying the unrestricted T-schema cannot be". The naive theory of truth presented in this paper has both Boolean negation and naive truth (indeed, as corollary 2 to theorem 18 will show, has naive truth in a much stronger form than the one Priest envisages and endorses), but, as corollary 9 to theorem 20 will show, is not trivial. This does give me some grounds for smugness.

Proof. I give the argument for (SDM$_0$). By I, $\varphi \vdash_{\mathbf{IK}^\omega} \varphi$ holds, and hence, by $\neg$-L, $\varphi, \neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$ holds. Analogously, $\psi, \neg\psi \vdash_{\mathbf{IK}^\omega} \oslash$ holds. Thus, by $\vee$-L, $\varphi, \psi, \neg\varphi \vee \neg\psi \vdash_{\mathbf{IK}^\omega} \oslash$ holds, and hence, by $\wedge$-L, $\varphi \wedge \psi, \neg\varphi \vee \neg\psi \vdash_{\mathbf{IK}^\omega} \oslash$ holds, and so, by $\neg$-R, $\varphi \wedge \psi \vdash_{\mathbf{IK}^\omega} \neg(\neg\varphi \vee \neg\psi)$ holds. The arguments for (SDM$_1$), (SDM$_2$) and (SDM$_3$) are similar or dual.

$\square$

Theorem 5. *The* quantificational De Morgan *rules hold in* $\mathbf{IK}^\omega$, *that is:*

(QDM$_0$) $\forall\xi\varphi \vdash_{\mathbf{IK}^\omega} \neg\exists\xi\neg\varphi$

(QDM$_1$) $\exists\xi\varphi \vdash_{\mathbf{IK}^\omega} \neg\forall\xi\neg\varphi$

(QDM$_2$) $\neg\forall\xi\varphi \vdash_{\mathbf{IK}^\omega} \exists\xi\neg\varphi$

(QDM$_3$) $\neg\exists\xi\varphi \vdash_{\mathbf{IK}^\omega} \forall\xi\neg\varphi$

hold.

Proof. I give the argument for (QDM$_0$). For every i, by I, $\varphi_{v_i/\xi} \vdash_{\mathbf{IK}^\omega} \varphi_{v_i/\xi}$ holds, and hence, by $\neg$-L, $\varphi_{v_i/\xi}, \neg\varphi_{v_i/\xi} \vdash_{\mathbf{IK}^\omega} \oslash$ holds. Thus, by $\exists$-L, $\varphi_{v_0/\xi}, \varphi_{v_1/\xi}, \varphi_{v_2/\xi} \ldots \exists\xi\neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$ holds, and hence, by $\forall$-L, $\forall\xi\varphi, \exists\xi\neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$ holds, and so, by $\neg$-R, $\forall\xi\varphi \vdash_{\mathbf{IK}^\omega} \neg\exists\xi\neg\varphi$ holds. The arguments for (QDM$_1$), (QDM$_2$) and (QDM$_3$) are similar or dual.

$\square$

Theorem 6. *The rules of* double-negation introduction *and* double-negation elimination *hold in* $\mathbf{IK}^\omega$, *that is:*

(DNI) $\varphi \vdash_{\mathbf{IK}^\omega} \neg\neg\varphi$

(DNE) $\neg\neg\varphi \vdash_{\mathbf{IK}^\omega} \varphi$

hold.

Proof. I give the argument for (DNI). By I, $\varphi \vdash_{\mathbf{IK}^\omega} \varphi$ holds, and hence, by $\neg$-L, $\varphi, \neg\varphi \vdash_{\mathbf{IK}^\omega} \oslash$ holds, and so, by $\neg$-R, $\varphi \vdash_{\mathbf{IK}^\omega} \neg\neg\varphi$ holds. A dual argument holds for (DNE).

$\square$

Given the other standard features of $\mathbf{IK}^\omega$, theorems 4, 5 and 6 clearly suffice to establish the classical equivalences, in the presence of negation, between conjunction and disjunction on the one hand and between universal quantification and particular quantification on the other hand, so that, in the presence of negation, each of these two pairs consists of interdefinable operations.[43]

We can also verify that these operators obey some other rules that are typically uncontroversial for naive theories of truth:

---

[43]Strictly speaking, equivalence guarantees definability in the usual sense (which requires full intersubstitutability) only in the presence of theorem 18.

Theorem 7. *The rules of* simplification *and* addition *hold in* **IK**$^\omega$*, that is:*

(SIMP$_0$) $\varphi \wedge \psi \vdash_{\mathbf{IK}^\omega} \varphi$

(SIMP$_1$) $\varphi \wedge \psi \vdash_{\mathbf{IK}^\omega} \psi$

(ADD$_0$) $\varphi \vdash_{\mathbf{IK}^\omega} \varphi \vee \psi$

(ADD$_1$) $\psi \vdash_{\mathbf{IK}^\omega} \varphi \vee \psi$

hold.

Proof. I give the argument for (SIMP$_0$). By I, $\varphi \vdash_{\mathbf{IK}^\omega} \varphi$ and hence, by K-L, $\varphi, \psi \vdash_{\mathbf{IK}^\omega} \varphi$, and so, by $\wedge$-L, $\varphi \wedge \psi \vdash_{\mathbf{IK}^\omega} \varphi$. Similar or dual arguments hold for (SIMP$_1$), (ADD$_0$) and (ADD$_1$).

$\square$

Theorem 8. *The rules of* universal instantiation *and* particular generalisation *hold in* **IK**$^\omega$*, that is:*

(UI) $\forall \xi \varphi \vdash_{\mathbf{IK}^\omega} \varphi_{\tau/\xi}$

(PG) $\varphi_{\tau/\xi} \vdash_{\mathbf{IK}^\omega} \exists \xi \varphi$

hold.

Proof. I give the argument for (UI). Let $\tau = v_i$. By I, $\varphi_{v_i/\xi} \vdash_{\mathbf{IK}^\omega} \varphi_{v_i/\xi}$ holds, and hence, by K-L, $\varphi_{v_0/\xi}, \varphi_{v_1/\xi}, \varphi_{v_2/\xi} \ldots \varphi_{v_i/\xi} \ldots \vdash_{\mathbf{IK}^\omega} \varphi_{v_i/\xi}$ holds, and so, by $\forall$-L, $\forall \xi \varphi \vdash_{\mathbf{IK}^\omega} \varphi_{\tau/\xi}$ holds. A dual argument holds for (PG).

$\square$

Indeed, even more straightforwardly, **IK**$^\omega$ validates the converse rules:

Theorem 9. *The rules of* finite adjunction *and* finite abjunction *hold in* **IK**$^\omega$*, that is:*

(ADJ^f) $\varphi, \psi \vdash_{\mathbf{IK}^\omega} \varphi \wedge \psi$

(ABJ^f) $\varphi \vee \psi \vdash_{\mathbf{IK}^\omega} \varphi, \psi$

hold.

Proof. I give the argument for (ADJ^f). By I, $\varphi \vdash_{\mathbf{IK}^\omega} \varphi$ and $\psi \vdash_{\mathbf{IK}^\omega} \psi$ hold, and hence, by $\wedge$-R, $\varphi, \psi \vdash_{\mathbf{IK}^\omega} \varphi \wedge \psi$ holds. A dual argument holds for (ABJ^f).

$\square$

Theorem 10. *The rules of* denumerable adjunction *and* denumerable abjunction *hold in* $\mathbf{IK}^\omega$*, that is:*

$(\mathrm{ADJ}^d)\ \ \varphi_{v_0/\xi}, \varphi_{v_1/\xi}, \varphi_{v_2/\xi} \ldots \vdash_{\mathbf{IK}^\omega} \forall \xi \varphi$

$(\mathrm{ABJ}^d)\ \ \exists \xi \varphi \vdash_{\mathbf{IK}^\omega} \varphi_{v_0/\xi}, \varphi_{v_1/\xi}, \varphi_{v_2/\xi} \ldots$

hold.

Proof. I give the argument for (ADJ^d). For every i, by I, $\varphi_{v_i/\xi} \vdash_{\mathbf{IK}^\omega} \varphi_{v_i/\xi}$ holds, and hence, by $\forall$-R, $\varphi_{v_0/\xi}, \varphi_{v_1/\xi}, \varphi_{v_2/\xi} \ldots \vdash_{\mathbf{IK}^\omega} \forall \xi \varphi$ holds. A dual argument holds for (ABJ^d).
$\square$

Together with $\wedge$-L and $\vee$-R, theorem 9 ensures that the operation of conjunction denoted by $\wedge$ and the operation of disjunction denoted by $\vee$ perfectly capture the operation of finite combination of premises and finite combination of conclusions respectively. And together with $\forall$-L and $\exists$-R, theorem 10 ensures that the operation of universal quantification denoted by $\forall$ and the operation of particular quantification denoted by $\exists$ perfectly capture the operation of denumerable combination of premises (all of the form $\varphi_{v_i/\xi}$) and denumerable combination of conclusions (all of the form $\varphi_{v_i/\xi}$) respectively.

Indeed, the joint strength of $\vee$-L and $\vee$-R ensures that disjunction enjoys many of its habitual interactions with *suppositional reasoning*:

Theorem 11. *The metarule of* weak reasoning by cases *holds in* $\mathbf{IK}^\omega$*, that is:*

(WRBC) *If* $\Gamma_0, \varphi_0 \vdash_{\mathbf{IK}^\omega} \Delta_0, \psi_0$ *and* $\Gamma_1, \varphi_1 \vdash_{\mathbf{IK}^\omega} \Delta_1, \psi_1$ *hold,* $\Gamma_0, \Gamma_1, \varphi_0 \vee \varphi_1 \vdash_{\mathbf{IK}^\omega}$
$\Delta_0, \Delta_1, \psi_0 \vee \psi_1$ *holds.*

holds.

Proof. Suppose that $\Gamma_0, \varphi_0 \vdash_{\mathbf{IK}^\omega} \Delta_0, \psi_0$ and $\Gamma_1, \varphi_1 \vdash_{\mathbf{IK}^\omega} \Delta_1, \psi_1$ hold. Then, by $\vee$-L, $\Gamma_0, \Gamma_1, \varphi_0 \vee \varphi_1 \vdash_{\mathbf{IK}^\omega} \Delta_0, \Delta_1, \psi_0, \psi_1$ holds, and hence, by $\vee$-R, $\Gamma_0, \Gamma_1, \varphi_0 \vee \varphi_1 \vdash_{\mathbf{IK}^\omega}$ $\Delta_0, \Delta_1, \psi_0 \vee \psi_1$ holds.
$\square$

Theorem 12. *The metarule of* inconsistency transmission from disjuncts to disjunction *holds in* $\mathbf{IK}^\omega$*, that is:*

(ITDD) *If* $\Gamma_0, \varphi_0 \vdash_{\mathbf{IK}^\omega} \oslash$ *and* $\Gamma_1, \varphi_1 \vdash_{\mathbf{IK}^\omega} \oslash$ *hold,* $\Gamma_0, \Gamma_1, \varphi_0 \vee \varphi_1 \vdash_{\mathbf{IK}^\omega} \oslash$ *holds.*

holds.

Proof. An instance of $\vee$-L.
$\square$

This is in sharp contrast to supervaluationist and revision naive theories of truth (see some references in section 2.1), in (possibly an appropriate extension to a multiple-conclusion framework of) which the rules of finite and denumerable abjunction and the metarules of weak reasoning by cases and inconsistency transmission from disjuncts to disjunction all fail. Thus, although full reasoning by cases has to fail in the background logic of the naive theory of truth presented in this paper,[44] we still preserve the core of the function of disjunction in suppositional reasoning and the core of the connections between disjunction and inconsistency in suppositional reasoning. Analogous considerations apply for particular quantification.

Turning to implication, it exhibits a pair of mutually converse features traditionally thought to be of the essence of this operation. On the one hand:

Theorem 13. *The rule of* modus ponens *holds in* $\mathbf{IK}^\omega$*, that is:*

(MP) $\varphi, \varphi \to \psi \vdash_{\mathbf{IK}^\omega} \psi$

holds.

Proof. By I, $\varphi \vdash_{\mathbf{IK}^\omega} \varphi$ and $\psi \vdash_{\mathbf{IK}^\omega} \psi$ hold, and hence, by $\to$-L, $\varphi, \varphi \to \psi \vdash_{\mathbf{IK}^\omega} \psi$ holds.

$\square$

On the other hand, a strong, *side-premise* version of the metarule represented by the *deduction theorem* is of course nothing more than $\to$-R. This is in sharp contrast to most naive theories of truth (certainly to all those referenced in this paper), as in such theories no connective that satisfies *modus ponens* also satisfies even the *single-premise* version of the deduction theorem. For consider again the Curry sentence denoted by $\mathfrak{c}_0$ and the paradoxical argument presented in section 2.2. That argument relies only on the naive theory of truth, on the structural rules and metarules of reflexivity, transitivity and contraction, on *modus ponens* and on the single-premise version of the deduction theorem. As most naive theories of truth (certainly all those referenced in this paper) validate the first four, most naive theories of truth (certainly all those referenced in this paper) have to give up the fifth.

---

[44]Full reasoning by cases for a disjunctive operation is in effect the analogue for that operation of what $\overset{A}{\vee}$-L is for the operation denoted by $\overset{A}{\vee}$. This metarule has to fail in the background logic of the naive theory of truth presented in this paper because it would reinstate W-R (for then, since $\varphi \vdash_{\mathbf{IK}^\omega} \varphi$ holds, $\varphi \vee \varphi \vdash_{\mathbf{IK}^\omega} \varphi$ would hold, which, by $\vee$-R and S, allows one to derive W-R in $\mathbf{IK}^\omega$). In turn, W-R would reinstate paradox for example with the strengthened Liar sentence $\mathfrak{l}_0$ (for then, since according to the naive theory of truth presented in this paper $\oslash \vdash T\mathfrak{l}_0, T\mathfrak{l}_0$ holds, $\oslash \vdash T\mathfrak{l}_0$ would hold, which is easily seen to lead to inconsistency in the naive theory of truth presented in this paper). Notice that all this does not imply that $\overset{A}{\vee}$ cannot consistently be added to $\mathscr{L}_1$, for the operation it denotes does not obey the analogue for it of what $\vee$-R is for the operation denoted by $\vee$.

Although this is not the place to elaborate on the deep issues involved here, I do want to emphasise that this is a *prima facie* very serious (and strangely underestimated)[45] problem for such theories: for, if φ entails ψ, it would seem eminently reasonable to infer 'If φ, ψ'. If that eminently reasonable inference is blocked, an explanation should certainly be given of what it is about logical consequence, or what it is about any *modus-ponens*-satisfying kind of implication, that blocks the inference—an explanation should certainly be given of how it can be that φ *entails* ψ while failing to be in any interesting sense a *sufficient condition* for it (although one should be open to the idea that there are kinds of *modus-ponens*-satisfying implication for which that inference fails, we would seem to have the notion of at least one such kind for which that inference is valid). I myself don't know of any remotely persuasive explanation for this up-to-date, and, in the absence of such an explanation, the question must arise as to why that treatment of this version of Curry's paradox is anything more than a piece of adhocery.

We should note that, in the presence of K-L and K-R, the deduction theorem makes implication very similar to classical material implication:

Theorem 14. *The rules of* positive sufficiency *and* negative sufficiency *hold in* $\mathbf{IK}^{\omega}$, *that is:*

(PS) $\varphi \vdash_{\mathbf{IK}^{\omega}} \psi \to \varphi$

(NP) $\neg\varphi \vdash_{\mathbf{IK}^{\omega}} \varphi \to \psi$

hold.

Proof. I give the argument for (PS). By I, $\varphi \vdash_{\mathbf{IK}^{\omega}} \varphi$ holds, and hence, by K-L, $\varphi, \psi \vdash_{\mathbf{IK}^{\omega}} \varphi$ holds, and so, by $\to$-R, $\varphi \vdash_{\mathbf{IK}^{\omega}} \psi \to \varphi$ holds. The argument for (NP) is dual.

$\square$

Notice that, although the similarity is very high indeed, we still do not have the collapse of implication on classical material implication, as for example the classically valid contraction law mentioned in fn 45 is not valid in $\mathbf{IK}^{\omega}$. What we have is nevertheless enough to establish the classical equivalences, in the presence of negation, between implication and conjunction on the one hand and implication and disjunction on the other hand, so that,

---

[45]For what is worth, I conjecture that at least part of the explanation for this underestimation of the problem is due to the fact that, traditionally, people have worked on the semantic paradoxes in an *axiomatic* framework. In such a framework, Curry's paradox is most naturally presented in a version that, instead of appealing to the *structural metarule* of contraction, appeals to the *law* $\varphi \to (\varphi \to \psi) \to (\varphi \to \psi)$ (which is sometimes also confusingly called 'contraction' and which, using I, $\to$-L and $\to$-R, one could derive in $\mathbf{IK}^{\omega}$ iff one could contract on φ). Such a version does *not* appeal to the deduction theorem (unsurprisingly so, given that the deduction theorem is hardly ever a metarule in an axiomatic framework, since standard axiomatic systems are formulated in such a way that the deduction theorem is rather merely admissible in them). This is certainly another respect in which the shift of focus from the axioms of the Fregean and Hilbertian tradition to the consequence relations of the Tarskian tradition helps to improve our understanding of certain issues in logic and its philosophy.

in the presence of negation, each of these two pairs consists of interdefinable operations (with, as we've already seen, conjunction and disjunction also being interdefinable in the presence of negation, see fn 43 and the text to which it is appended for further details):

Theorem 15. *The rules of* no-counterexample *and of* false-antecedent-or-true-consequent *hold in* $\mathbf{IK}^{\omega}$*, that is:*

(NC) $\varphi \to \psi \vdash_{\mathbf{IK}^{\omega}} \neg(\varphi \land \neg\psi)$ *and* $\neg(\varphi \land \neg\psi) \vdash_{\mathbf{IK}^{\omega}} \varphi \to \psi$ *hold.*

(FAOTC) $\varphi \to \psi \vdash_{\mathbf{IK}^{\omega}} \neg\varphi \lor \psi$ *and* $\neg\varphi \lor \psi \vdash_{\mathbf{IK}^{\omega}} \varphi \to \psi$ *hold.*

hold.

Proof. I give the argument for (NC). First, by (MP), $\varphi, \varphi \to \psi \vdash_{\mathbf{IK}^{\omega}} \psi$ holds, and hence, by ¬-L, $\varphi, \neg\psi, \varphi \to \psi \vdash_{\mathbf{IK}^{\omega}} \oslash$ holds, and so, by ∧-L, $\varphi \land \neg\psi, \varphi \to \psi \vdash_{\mathbf{IK}^{\omega}} \oslash$ holds, whence, by ¬-R, $\varphi \to \psi \vdash_{\mathbf{IK}^{\omega}} \neg(\varphi \land \neg\psi)$ holds. Second, by I, $\varphi \vdash_{\mathbf{IK}^{\omega}} \varphi$ holds. But, by (EXH), $\oslash \vdash_{\mathbf{IK}^{\omega}} \psi, \neg\psi$ holds, and hence, by ∧-R, $\varphi \vdash_{\mathbf{IK}^{\omega}} \psi, \varphi \land \neg\psi$ holds, and so, by ¬-L, $\varphi, \neg(\varphi \land \neg\psi) \vdash_{\mathbf{IK}^{\omega}} \psi$ holds, whence, by →-R, $\neg(\varphi \land \neg\psi) \vdash_{\mathbf{IK}^{\omega}} \varphi \to \psi$ holds. The argument for (FAOTC) follows by the equivalence between $\neg(\varphi \land \neg\psi)$ and $\neg\varphi \lor \psi$, which, given the other standard features of $\mathbf{IK}^{\omega}$, is ensured by theorems 4 and 6.

□

I hope I've said enough to give a good and concrete sense of the strength of $\mathbf{IK}^{\omega}$, especially when compared with other deviations from classical logic proposed in order to deal with the semantic paradoxes while preserving the naive theory of truth. Of course, $\mathbf{IK}^{\omega}$ is obviously weak in the specific respect constituted by the absence from it of contraction in the premises and in the conclusions, which is on the contrary valid in those other logics. Notice again that it is not only that W-L and W-R are absent from the defining principles of $\mathbf{IK}^{\omega}$ (and $\mathbf{IKT}^{\omega}$): as should be expected and as will be shown in corollary 11 to theorem 20, W-L and W-R are simply not admissible in $\mathbf{IK}^{\omega}$ (or in $\mathbf{IKT}^{\omega}$)—they *fail*.

However, quite generally, the fact that a certain principle is not valid in a certain logic in a certain sense does not mean that much even from the standpoint of an adherent of that logic. For that the principle is not valid simply means that it is not *formally valid*, and for that it suffices that *at least one instance* of the principle be unacceptable either from the point of view of the logic itself or from the more general outlook (philosophical or otherwise) informing the logic (of course, if the logic is at least sound with respect to the outlook, the latter property entails the former). And, clearly, that sufficient condition is perfectly compatible with *other instances* of the principle being acceptable either from the point of view of the logic itself or from the more general outlook (philosophical or otherwise) informing the logic (of course, if the logic is at least sound with respect to the outlook, the former property entails the latter). The trite example of the status of the law of excluded middle in intuitionism illustrates well both cases. Although the law is formally invalid in intuitionist logic, some instances of it are acceptable from the point of view of the

28

logic itself and are indeed deemed to be logical truths—for example, 'Either, if Goldbach's Conjecture is true, Goldbach's Conjecture is true, or it is not the case that, if Goldbach's Conjecture is true, Goldbach's Conjecture is true'. And although the law is formally invalid in intuitionist logic, some instances of the law are acceptable from the more general philosophical outlook informing the logic (at least the usual one) and are deemed to be true—for example, 'Either $341785601361360 + 703217609131275 = 104341785492635$ or it is not the case that $341785601361360 + 703217609131275 = 104341785492635$'. The first case simply reflects the familiar point about the importance of the right level of fineness of grain in logical form; the second case is more interesting and demonstrates how a particular instance of a principle can fail to be acceptable in a logic (by failing to be an instance of a principle valid in the logic, which is the only kind of acceptability envisioned in standard, formal logics) while being acceptable from the more general outlook (philosophical or otherwise) informing the logic.

There are of course various ways in which one could fill the second gap: mainly, one could envision a non-standard logic where the relevant instances of the law of excluded middle do get counted as logical truths, or, since one is anyways in the business of using the logic as a background for a theory, one could simply add those instances to the relevant theory. Setting aside the interesting question of what the substantial differences are between these two courses (and others), the latter is the one usually taken. Now, it is a very beautiful fact about intuitionist logic that, very roughly speaking, the addition to a theory of the instance of the law of excluded middle for φ has the consequence that φ behaves in effect as it would behave in classical logic, and that, more generally, adding all instances of the law of excluded middle to every set of premises yields in effect classical logic.

It is a very beautiful fact about $\mathbf{IK}^\omega$ that it enjoys a similar property, where the relevant instances are not instances of the law of excluded middle but instances of the law of *superidempotency of self-conjunction* $\varphi \rightarrow \varphi \wedge \varphi$ and of the law of *subidempotency of self-disjunction* $\varphi \vee \varphi \rightarrow \varphi$:

Definition 7. Let $\mathbf{IK}^\omega\mathbf{W}^X$ be the system got by adding to $\mathbf{IK}^\omega$ the instances of the laws of superidempotency of self-conjunction and of subidempotency of self-disjunction for every $\varphi \in X$:

(SPSC) $\oslash \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \varphi \rightarrow \varphi \wedge \varphi$

(SBSD) $\oslash \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \varphi \vee \varphi \rightarrow \varphi$

Theorem 16. *For every $\varphi \in X$, contraction on φ in the premises holds in $\mathbf{IK}^\omega\mathbf{W}^X$: if $\Gamma, \varphi, \varphi \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \Delta$ holds, so does $\Gamma, \varphi \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \Delta$. For every $\varphi \in X$, contraction on φ in the conclusions also holds in $\mathbf{IK}^\omega\mathbf{W}^X$: if $\Gamma \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \Delta, \varphi, \varphi$ holds, so does $\Gamma \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \Delta, \varphi$.*

Proof. I give the argument for contraction on φ in the premises. Suppose that $\Gamma, \varphi, \varphi \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \Delta$ holds. Then, by $\wedge$-L, $\Gamma, \varphi \wedge \varphi \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \Delta$ holds. But, by (SPSC), $\oslash \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \varphi \rightarrow \varphi \wedge \varphi$ holds, and hence, by (MP) and S, $\varphi \vdash_{\mathbf{IK}^\omega\mathbf{W}^x} \varphi \wedge \varphi$ holds, and

so, by S, $\Gamma, \varphi \vdash_{\mathbf{IK}^\omega \mathbf{W}^x} \Delta$ holds. A dual argument holds for contraction on φ in the conclusions.

□

Corollary 1. $\mathbf{IK}^\omega \mathbf{W}^{WFF_{\mathscr{L}^1}}$ *is classical logic with the ω-rule over* $\mathscr{L}^1$.

Proof. By theorem 16, $\mathbf{IK}^\omega \mathbf{W}^{WFF_{\mathscr{L}^1}}$ has all the principles of $\mathbf{IK}^\omega$ plus contraction. That clearly yields classical logic with the ω-rule over $\mathscr{L}^1$.

□

3.3   The Theory of Truth

So much for the background logic $\mathbf{IK}^\omega$. We now add to it the following metarules for T:

$$\frac{\Gamma, \varphi \vdash_{\mathbf{IKT}^\omega} \Delta}{\Gamma, T\ulcorner\varphi\urcorner \vdash_{\mathbf{IKT}^\omega} \Delta} \; T\text{-L} \qquad\qquad\qquad \frac{\Gamma \vdash_{\mathbf{IKT}^\omega} \Delta, \varphi}{\Gamma \vdash_{\mathbf{IKT}^\omega} \Delta, T\ulcorner\varphi\urcorner} \; T\text{-R}$$

thereby obtaining the theory of truth $\mathbf{IKT}^\omega$.

As for $\mathbf{IK}^\omega$, I will now go through an indicative survey of principles valid in $\mathbf{IKT}^\omega$, for most of the time letting the self-evidence of such principles speak for itself, but commenting a bit on certain consequences and comparisons (given the properties of $\mathbf{IKT}^\omega$, some of the points to be made will of course be analogous to points made about $\mathbf{IK}^\omega$). We start by noting that $\mathbf{IKT}^\omega$ has the required minimal strength:

Theorem 17. $\mathbf{IKT}^\omega$ *is a naive theory of truth.*

Proof.

- $\varphi \vdash_{\mathbf{IKT}^\omega} T\ulcorner\varphi\urcorner$. By I, $\varphi \vdash_{\mathbf{IKT}^\omega} \varphi$ holds, and hence, by, T-R, $\varphi \vdash_{\mathbf{IKT}^\omega} T\ulcorner\varphi\urcorner$ holds.

- $T\ulcorner\varphi\urcorner \vdash_{\mathbf{IKT}^\omega} \varphi$. By I, $\varphi \vdash_{\mathbf{IKT}^\omega} \varphi$ holds, and hence, by T-L, $T\ulcorner\varphi\urcorner \vdash_{\mathbf{IKT}^\omega} \varphi$ holds.

□

But $\mathbf{IKT}^\omega$ is much stronger than any old naive theory of truth. To appreciate this, the following general intersubstitutability fact will be useful:

Theorem 18. *If φ_1 is the result of replacing ψ by χ in φ_0, with $\psi \vdash_{\mathbf{IKT}^\omega} \chi$ and $\chi \vdash_{\mathbf{IKT}^\omega} \psi$ holding, then $[\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds iff $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds] and $[\Gamma \vdash_{\mathbf{IKT}^\omega} \Delta, \varphi_0$ holds iff $\Gamma \vdash_{\mathbf{IKT}^\omega} \Delta, \varphi_1$ holds].*

Proof. It clearly suffices to prove the result for the following cases:

- $\varphi_0 = \psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.

- $\varphi_0 = \neg\psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\psi \vdash_{\mathbf{IKT}^\omega} \chi$ holds, by $\neg$-L, $\psi, \neg\chi \vdash_{\mathbf{IKT}^\omega} \oslash$ holds, and hence, by $\neg$-R, $\neg\chi \vdash_{\mathbf{IKT}^\omega} \neg\psi$ holds, and so, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.

- $\varphi_0 = \psi_0 \wedge \psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds and since, by (ADJ^f), $\psi_0, \psi \vdash_{\mathbf{IKT}^\omega} \psi_0 \wedge \psi$ holds, by S, $\psi_0, \chi \vdash_{\mathbf{IKT}^\omega} \psi_0 \wedge \psi$ holds, and hence, by $\wedge$-L, $\psi_0 \wedge \chi \vdash_{\mathbf{IKT}^\omega} \psi_0 \wedge \psi$ holds, and so, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.

- $\varphi_0 = \psi_0 \vee \psi$. Dual arguments hold.

- $\varphi_0 = \psi_0 \rightarrow \psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds and since, by (MP), $\psi_0, \psi_0 \rightarrow \chi \vdash_{\mathbf{IKT}^\omega} \chi$ holds, by S, $\psi_0, \psi_0 \rightarrow \chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds, and hence, by $\rightarrow$-R, $\psi_0 \rightarrow \chi \vdash_{\mathbf{IKT}^\omega} \psi_0 \rightarrow \psi$ holds, and so, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.

- $\varphi_0 = \psi \rightarrow \psi_0$. Dual arguments hold.

- $\varphi_0 = \forall \xi \psi$. I give the argument for the claim that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds only if $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Suppose that $\Gamma, \varphi_0 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. Then, since $\chi \vdash_{\mathbf{IKT}^\omega} \psi$ holds, clearly, for every i, $\chi_{v_i/\xi} \vdash_{\mathbf{IKT}^\omega} \psi_{v_i/\xi}$ holds, and hence, by $\forall$-R, $\chi_{v_0/\xi}, \chi_{v_1/\xi}, \chi_{v_2/\xi} \ldots \vdash_{\mathbf{IKT}^\omega} \forall \xi \psi$ holds, and so, by $\forall$-L, $\forall \xi \chi \vdash_{\mathbf{IKT}^\omega} \forall \xi \psi$ holds, whence, by S, $\Gamma, \varphi_1 \vdash_{\mathbf{IKT}^\omega} \Delta$ holds. The arguments for the other claims are similar.

- $\varphi_0 = \exists \xi \psi$. Dual arguments hold.

$\square$

The extreme strength of $\mathbf{IKT}^\omega$ should now be manifest:

Definition 8. A theory of truth $\vdash$ is *transparent* iff, if φ_1 is the result of replacing ψ by $T\ulcorner\psi\urcorner$ in φ_0, then $[\Gamma, \varphi_0 \vdash \Delta$ holds iff $\Gamma, \varphi_1 \vdash \Delta$ holds$]$ and $[\Gamma \vdash \Delta, \varphi_0$ holds iff $\Gamma \vdash \Delta, \varphi_1$ holds$]$.

Corollary 2. $\mathbf{IKT}^\omega$ *is transparent.*

Proof. By theorems 17 and 18.

$\square$

Coupled with the previous background logic, transparency yields very intuitive, simple and elegant principles governing truth. To begin with, in addition to naivety, we also have ingenuousness:

Corollary 3. IKT$^\omega$ *is ingenuous.*

Proof. By I, $\varphi \vdash_{\mathbf{IKT}^\omega} \varphi$ holds, and hence, by $\rightarrow$-R, $\oslash \vdash_{\mathbf{IKT}^\omega} \varphi \rightarrow \varphi$ holds, and so, by corollary 2 to theorem 18, $\oslash \vdash_{\mathbf{IKT}^\omega} T^\ulcorner\varphi\urcorner \rightarrow \varphi$ and $\oslash \vdash_{\mathbf{IKT}^\omega} \varphi \rightarrow T^\ulcorner\varphi\urcorner$ hold.

□

Bearing in mind that the *falsity* of φ is very plausibly understood to consist in the truth of $\neg\varphi$, in addition to the logical law of excluded middle, we also have its semantic counterpart saying that falsity is *complete* over truth:

Corollary 4. *The law of* bivalence *holds in* **IKT$^\omega$**, *that is:*

(BIV) $\oslash \vdash_{\mathbf{IKT}^\omega} T^\ulcorner\varphi\urcorner \vee T^\ulcorner\neg\varphi\urcorner$

holds.

Proof. From (LEM) by corollary 2 to theorem 18.

□

This is in sharp contrast to analetheic naive theories of truth (see some references in section 2.1), in which the law of bivalence fails.

Given the dualities of the theory, it is no surprise that, in addition to the logical law of non-contradiction, we also have its semantic counterpart saying that falsity is *inconsistent* with truth:

Corollary 5. *The law of* contravalence *holds in* **IKT$^\omega$**, *that is:*

(CONTRAV) $T^\ulcorner\varphi\urcorner \wedge T^\ulcorner\neg\varphi\urcorner \vdash_{\mathbf{IKT}^\omega} \oslash$

holds.

Proof. From (LNC) by corollary 2 to theorem 18.

□

In fact, by $\neg$-R, (CONTRAV) yields $\oslash \vdash_{\mathbf{IKT}^\omega} \neg(T^\ulcorner\varphi\urcorner \wedge T^\ulcorner\neg\varphi\urcorner)$. This latter, extremely plausible law is upheld in the best dialetheic theories (such as those referenced in section 2.1) but disappointingly fails even in the best analetheic theories (such as those referenced in section 2.1), which are thus not in a position to assert extremely plausible claims like the one that that it is not the case that [the simple Liar sentence is true and the simple

Liar sentence is false]. Thus, the traditional conception according to which there are neither *gaps* nor *gluts* between the true and the false is fully upheld in $\mathbf{IKT}^\omega$.

Other traditional semantic laws familiar from truth-functional semantics are also valid in $\mathbf{IKT}^\omega$:

Corollary 6. *The* truth-functional *laws:*

(NEG$^\Rightarrow$) $\oslash \vdash_{\mathbf{IKT}^\omega} T^\ulcorner \neg\varphi \urcorner \rightarrow \neg T^\ulcorner \varphi \urcorner$

(NEG$^\Leftarrow$) $\oslash \vdash_{\mathbf{IKT}^\omega} \neg T^\ulcorner \varphi \urcorner \rightarrow T^\ulcorner \neg\varphi \urcorner$

(CONJ$^\Rightarrow$) $\oslash \vdash_{\mathbf{IKT}^\omega} T^\ulcorner \varphi \wedge \psi \urcorner \rightarrow T^\ulcorner \varphi \urcorner \wedge T^\ulcorner \psi \urcorner$

(CONJ$^\Leftarrow$) $\oslash \vdash_{\mathbf{IKT}^\omega} T^\ulcorner \varphi \urcorner \wedge T^\ulcorner \psi \urcorner \rightarrow T^\ulcorner \varphi \wedge \psi \urcorner$

(DISJ$^\Rightarrow$) $\oslash \vdash_{\mathbf{IKT}^\omega} T^\ulcorner \varphi \vee \psi \urcorner \rightarrow T^\ulcorner \varphi \urcorner \vee T^\ulcorner \psi \urcorner$

(DISJ$^\Leftarrow$) $\oslash \vdash_{\mathbf{IKT}^\omega} T^\ulcorner \varphi \urcorner \vee T^\ulcorner \psi \urcorner \rightarrow T^\ulcorner \varphi \vee \psi \urcorner$

(COND$^\Rightarrow$) $\oslash \vdash_{\mathbf{IKT}^\omega} T^\ulcorner \varphi \rightarrow \psi \urcorner \rightarrow (T^\ulcorner \varphi \urcorner \rightarrow T^\ulcorner \psi \urcorner)$

(COND$^\Leftarrow$) $\oslash \vdash_{\mathbf{IKT}^\omega} T^\ulcorner \varphi \urcorner \rightarrow T^\ulcorner \psi \urcorner \rightarrow T^\ulcorner \varphi \rightarrow \psi \urcorner$

hold.

Proof. By corollary 2 to theorem 18.

□

Corresponding laws could easily be proven for the quantifiers, but doing so would at least require building more syntax into $\mathbf{IKT}^\omega$ (and, for certain statements of such laws, extending the theory of truth to a theory of true-of), which would go beyond the purposes of this paper.

I would like to stress the presence in $\mathbf{IKT}^\omega$ of (CONJ$^\Leftarrow$) and (DISJ$^\Rightarrow$) on the one hand and of (COND$^\Rightarrow$) on the other hand. One is easily tempted to understand failure of contraction, especially if one focusses on the ensuing failures of (SPSC) and (SBSD) (see theorem 16), as a failure of the truth of certain conjuncts to imply the truth of their conjunction, and as a failure of the truth of a certain disjunction to imply the truth of either of its disjuncts. As made manifest by the presence in $\mathbf{IKT}^\omega$ of (CONJ$^\Leftarrow$) and (DISJ$^\Rightarrow$), that temptation should be resisted: failure of contraction is perfectly compatible with the very plausible semantic law that the truth of certain conjuncts implies the truth of their conjunction, and perfectly compatible with the dual very plausible semantic law that the truth of a certain disjunction implies the truth of either of its disjuncts. What, for example, generates or at least is connected with the failures of (SPSC) are not the alleged failures of the former law, which on the contrary continues to hold in $\mathbf{IKT}^\omega$; it is rather the fact that φ's being true cannot always be assumed to imply that [φ is true and φ is

true] (which alone is the proper statement of the fact that *both* conjuncts of $\varphi \wedge \varphi$ are true, which, by (CONJ$^{\Leftarrow}$), would then imply the truth of $\varphi \wedge \varphi$; dual considerations hold for the failures of (SBSD)). Granted, at first blush, that may seem completely baffling. But if it is, it should be no *additional* bafflement to the original one generated by the failure of (SPSC), as the failure of the implication from 'φ is true' to 'φ is true and φ is true' just is the failure of (yet another) instance of (SPSC). Moreover, I think it should start to look less baffling once one realises that that implication would give one the license to infer from φ's truth not just what intuitively follows from it, but also what intuitively follows from it *together with φ's truth being kept fixed* (for one could use one conjunct of 'φ is true and φ is true' to infer what intuitively follows from φ's truth and use the other conjunct to keep φ's truth fixed). Once one pays heed to the fact that paradoxical sentences arguably exhibit, under naive-truth conditions, a certain dynamicity, that license should start to seem problematic (this insight is clearly present in early revision theorists such as Herzberger [1982]; I've developed my favoured way of making sense of such dynamicity using a non-contractive framework in section 2.3).

The presence in **IKT**$^{\omega}$ of (COND$^{\Rightarrow}$) is also very welcome. Not only is that law very intuitive in itself, but it is also easily seen to imply, given the properties of **IKT**$^{\omega}$, that the law of *truth preservingness of* modus ponens is valid in **IKT**$^{\omega}$, that is that:

(TP$^{\mathrm{MP}}$) $\oslash \vdash_{\mathbf{IKT}^{\omega}} T\ulcorner \varphi \rightarrow \psi \urcorner \wedge T\ulcorner \varphi \urcorner \rightarrow T\ulcorner \psi \urcorner$

holds. I submit that (TP$^{\mathrm{MP}}$) is very plausible for the same reasons as (COND$^{\Rightarrow}$) is. Very interestingly, it is a surprising feature of most transparent theories of truth (certainly of all those referenced in this paper) that they cannot vindicate the law of truth preservingness of *modus ponens*, in the sense that adding that law to them generates inconsistency (the best of them do vindicate *modus ponens* and the analogues of (COND$^{\Rightarrow}$), but I think that validating *modus ponens* and the analogues of (COND$^{\Rightarrow}$) while being inconsistent with the law of truth preservingness of *modus ponens* leaves a rather bitter taste in mouth). The reason for this is relatively straightforward and, in its essence, has first been pointed out by Meyer et al. [1979]. Take a Curry sentence $T\mathfrak{c}_1 \rightarrow Q_0^0$, with $\mathfrak{c}_1$ being the designated individual constant for that sentence, and instantiate the law of truth preservingness of *modus ponens* with $T\mathfrak{c}_1$ for the antecedent and Q_0^0 for the consequent. By definition of $\mathfrak{c}_1$ and transparency, that instance is equivalent with $T\mathfrak{c}_1 \wedge T\mathfrak{c}_1 \rightarrow T\ulcorner Q_0^0 \urcorner$,[46] which, on most transparent theories of truth, is in turn equivalent with $T\mathfrak{c}_1 \rightarrow T\ulcorner Q_0^0 \urcorner$ (*since, on most transparent theories of truth, φ is equivalent and fully intersubstitutable with $\varphi \wedge \varphi$). And that in turn entails Q_0^0 on most transparent theories of truth.* I think that such inconsistency with the law of truth preservingness of *modus ponens* is extremely problematic, especially when coupled with acceptance of unrestricted *modus ponens*.

The previous argument against the consistency of the law of truth preservingness of *modus ponens* fails for the naive theory of truth presented in this paper, for the simple

---

[46]Notice that transparency is sufficient but not necessary for validating the relevant entailment here, which would equally hold, for example, in the *naive but non-transparent* theory of truth advocated by Priest [2006a]. It is on these grounds that also that theory cannot vindicate the law of truth preservingness of *modus ponens*.

reason that, in $\mathbf{IKT}^\omega$, φ is not equivalent with $\varphi \wedge \varphi$, as the former does not entail the latter (which is the crucial direction employed in the previous argument; the converse direction is indeed valid in $\mathbf{IKT}^\omega$, as it is just a particular case of (SIMP$_0$) or (SIMP$_1$)). That entailment does not hold because otherwise, by →-R, (SPSC) would hold, which contradicts theorem 16 taken together with corollary 11 to theorem 20. I also note that, very interestingly, even once one somehow manages to get up to $T\mathfrak{c}_1 \to T^\ulcorner Q_0^0 \urcorner$, in $\mathbf{IKT}^\omega$ that does not entail Q_0^0. That entailment does not hold because otherwise, by (EXH), transparency, (NS) and S, $\oslash \vdash_{\mathbf{IKT}^\omega} Q_0^0, Q_0^0$ would hold, which contradicts corollary 8 to theorem 20. While this last feature of $\mathbf{IKT}^\omega$ is certainly very interesting and I hope to expand on that in future work, it does not however constitute an additional reason for why the previous argument against the consistency of the law of truth preservingness of *modus ponens* fails in $\mathbf{IKT}^\omega$. For, while, as I've just shown, $T\mathfrak{c}_1 \to T^\ulcorner Q_0^0 \urcorner \vdash_{\mathbf{IKT}^\omega} Q_0^0$ does not hold, $T\mathfrak{c}_1 \to T^\ulcorner Q_0^0 \urcorner, T\mathfrak{c}_1 \to T^\ulcorner Q_0^0 \urcorner \vdash_{\mathbf{IKT}^\omega} Q_0^0$ clearly holds. But, if $T\mathfrak{c}_1 \wedge T\mathfrak{c}_1 \to T^\ulcorner Q_0^0 \urcorner \vdash_{\mathbf{IKT}^\omega} T\mathfrak{c}_1 \to T^\ulcorner Q_0^0 \urcorner$ held, since, by (TP$^{\mathrm{MP}}$), $\oslash \vdash_{\mathbf{IKT}^\omega} T\mathfrak{c}_1 \wedge T\mathfrak{c}_1 \to T^\ulcorner Q_0^0 \urcorner$ holds, (TP$^{\mathrm{MP}}$) would suffice to yield the problematic premise multiset $[T\mathfrak{c}_1 \to T^\ulcorner Q_0^0 \urcorner, T\mathfrak{c}_1 \to T^\ulcorner Q_0^0 \urcorner]$.

The point generalises beyond the particular case of *modus ponens* and the particular case of transparent theories of truth: in many other cases too, it is a surprising feature of most naive theories of truth (certainly of all those referenced in this paper), whether transparent or not, that they validate some rules while being inconsistent with the corresponding semantic laws stating that such rules are truth preserving (see Field [2006] for an interesting survey and for a penetrating discussion of the problem).[47,48] Now, I myself would actually be most wary of the postulation of any simple and straightforward connection between validity and truth preservation: I would be especially wary of any non-trivial principle stating that some version of truth preservation is *sufficient* for validity, but, because of various considerations relating to semantic context dependence and to higher-order indeterminacy which would lead us too far afield to rehearse here, I would also be wary of principles stating that certain versions of truth preservation are *necessary* for validity (see Zardini [2010b]; [2010c]), which is the direction of implication that fails on such naive theories of truth. Still, none of the considerations underlying that caution

---

[47]Unfortunately, that paper completely ignores the kind of proposal presented in this paper, and is so doing overstates its findings when it says (pp. 601–602): "Curry's Paradox [...] shows that any logic that accepts the standard introduction and elimination rules for the conditional and the introduction and elimination rules for truth is completely trivial: it implies anything whatsoever. Thus however compelling the argument that validity coincides with necessary truth-preservation may have seemed, it relies on assumptions that cannot be jointly accepted [...] The divergence between the rules one employs and the rules one regards as unrestrictedly truth preserving is virtually inescapable". The naive theory of truth presented in this paper validates all of →-L,→-R, T-L and T-R, but, as corollary 9 to theorem 20 will show, is not trivial and, as theorem 19 will show, entails that truth preservation is necessary for validity (which, of the various connections between validity and truth preservation, is the real target in this passage).

[48]The problem thus affects also those *naive but non-transparent* theories of truth (such as certain strong supervaluationist and revision theories) that do manage to vindicate the law of truth preservingness of *modus ponens*. These theories too validate some rules (involving the truth predicate rather than the conditional, like T-introduction and T-elimination) while being inconsistent with the corresponding semantic laws stating that such rules are truth preserving.

target the necessity for the validity of such a most basic rule as *modus ponens* of such a minimal version of truth preservation as the law of truth preservingness of *modus ponens* (or an analogous necessary condition for the rules mentioned in fn 48), so that I think that the above-mentioned theories' inability to vindicate the truth preservingness of the rules they validate remains extremely problematic.

In the presence of this important failure of most naive theories of truth, the interesting question arises whether the naive theory of truth presented in this paper can do any better. Well, we've already seen that it can do better *to a certain interesting extent* not only in *being consistent with* ($\mathrm{TP^{MP}}$), but indeed in *entailing* it on the sole strength of the background logic represented by $\mathbf{IK}^\omega$ and the naive theory of truth encapsulated in $\mathbf{IKT}^\omega$. I will now show, with a beautifully simple argument, that the theory does *much* better than even that—roughly, that whenever, according to $\mathbf{IKT}^\omega$, certain finitely-many conclusions follow from certain finitely-many premises, it is also the case that, according to $\mathbf{IKT}^\omega$, if all the premises are true, so is some of the conclusions. In fact, this follows from a yet stronger result that allows for countably-many side premises and conclusions:

Theorem 19. *If* $\Gamma, \varphi_0, \varphi_1, \varphi_2 \dots, \varphi_i \vdash_{\mathbf{IKT}^\omega} \Delta, \psi_0, \psi_1, \psi_2 \dots, \psi_j$ *holds,* $\Gamma \vdash_{\mathbf{IKT}^\omega} \Delta, T^\ulcorner\varphi_0\urcorner \wedge T^\ulcorner\varphi_1\urcorner \wedge T^\ulcorner\varphi_2\urcorner \dots \wedge T^\ulcorner\varphi_i\urcorner \to T^\ulcorner\psi_0\urcorner \vee T^\ulcorner\psi_1\urcorner \vee T^\ulcorner\psi_2\urcorner \dots \vee T^\ulcorner\psi_j\urcorner$ *holds.*

Proof. Suppose that $\Gamma, \varphi_0, \varphi_1, \varphi_2 \dots, \varphi_i \vdash_{\mathbf{IKT}^\omega} \Delta, \psi_0, \psi_1, \psi_2 \dots, \psi_j$ holds. Then, by (i applications of) T-L, $\Gamma, T^\ulcorner\varphi_0\urcorner, T^\ulcorner\varphi_1\urcorner, T^\ulcorner\varphi_2\urcorner \dots, T^\ulcorner\varphi_i\urcorner \vdash_{\mathbf{IKT}^\omega} \Delta, \psi_0, \psi_1, \psi_2 \dots, \psi_j$ holds, and so, by (j applications of) T-R, $\Gamma, T^\ulcorner\varphi_0\urcorner, T^\ulcorner\varphi_1\urcorner, T^\ulcorner\varphi_2\urcorner \dots, T^\ulcorner\varphi_i\urcorner \vdash_{\mathbf{IKT}^\omega} \Delta, T^\ulcorner\psi_0\urcorner, T^\ulcorner\psi_1\urcorner, T^\ulcorner\psi_2\urcorner \dots, T^\ulcorner\psi_j\urcorner$ holds. Thus, by (i applications of) $\wedge$-L, $\Gamma, T^\ulcorner\varphi_0\urcorner \wedge T^\ulcorner\varphi_1\urcorner \wedge T^\ulcorner\varphi_2\urcorner \dots \wedge T^\ulcorner\varphi_i\urcorner \vdash_{\mathbf{IKT}^\omega} \Delta, T^\ulcorner\psi_0\urcorner, T^\ulcorner\psi_1\urcorner, T^\ulcorner\psi_2\urcorner \dots, T^\ulcorner\psi_j\urcorner$ holds, and so, by (j applications of) $\vee$-R, $\Gamma, T^\ulcorner\varphi_0\urcorner \wedge T^\ulcorner\varphi_1\urcorner \wedge T^\ulcorner\varphi_2\urcorner \dots \wedge T^\ulcorner\varphi_i\urcorner \vdash_{\mathbf{IKT}^\omega} \Delta, T^\ulcorner\psi_0\urcorner \vee T^\ulcorner\psi_1\urcorner \vee T^\ulcorner\psi_2\urcorner \dots \vee T^\ulcorner\psi_j\urcorner$ holds, whence, by $\to$-R, $\Gamma \vdash_{\mathbf{IKT}^\omega} \Delta, T^\ulcorner\varphi_0\urcorner \wedge T^\ulcorner\varphi_1\urcorner \wedge T^\ulcorner\varphi_2\urcorner \dots \wedge T^\ulcorner\varphi_i\urcorner \to T^\ulcorner\psi_0\urcorner \vee T^\ulcorner\psi_1\urcorner \vee T^\ulcorner\psi_2\urcorner \dots \vee T^\ulcorner\psi_j\urcorner$ holds.

$\square$

A corresponding theorem could easily be proven for the case of denumerably many premises or conclusions, but, again, doing so would require building more syntax into $\mathbf{IKT}^\omega$, which would go beyond the purposes of this paper. Clearly, as the above proof demonstrates, the unique availability of theorem 19 for $\mathbf{IKT}^\omega$ is tightly connected with the unique availability of $\to$-R for $\mathbf{IKT}^\omega$. $\to$-R is a highly intuitive and compelling principle about implication, which in turn yields with theorem 19 a highly intuitive and compelling principle about validity and truth preservation. I regard the availability of these highly intuitive and compelling principles as one of the main advantages of the naive theory of truth presented in this paper over its naive rivals, which would seem to sacrifice much of their innocent simplicity to the sophistries involved in trying to uphold their rejection of these two principles.

4 The Consistency of a Non-Contractive Naive Theory of Truth

4.1 *Hauptsatz*

The informal notion of *consistency* comes in different interesting strengths, ranging from mere *non-triviality* to *conservativeness* with respect to various background theories (of course, the informal notion of consistency does not require conservativeness with respect to any background theory whatsoever—while I guess we want a theory of truth to be conservative over fundamental physics, I suppose we don't want to require that it be conservative over pure logic). For the purposes of this paper, we'll rest content here with proving a central result of intermediate strength: taking a *deductive system* $\Rightarrow_{\mathbf{I^K T}^\omega}$ that is sound and complete with respect to $\mathbf{IKT}^\omega$, we'll prove that the rule of inference corresponding to S (and commonly known as '*cut*') is *eliminable* in $\Rightarrow_{\mathbf{I^K T}^\omega}$.

More specifically:

Definition 9. $\Rightarrow_{\mathbf{I^K T}^\omega}$ is the deductive (sequent-calculus) system obtained by taking:

$$\frac{}{\Gamma, \varphi \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta, \varphi} \ \mathrm{I^K}$$

as its axiom and the closure principles in sections 3.2 and 3.3 (minus K-L and K-R) as its rules of inference.

Definition 10. A $\Rightarrow_{\mathbf{I^K T}^\omega}$-*deduction* is any upwards-branching tree with countably many branches of finite length, where each node n is a sequent of the form $\Gamma \Rightarrow_{\mathbf{I^K T}^\omega} \Delta$ such that:

- If n is a leaf, n is the conclusion* of an instance of $\mathrm{I^K}$;

- If n is immediately below all and only the Xs, n is the conclusion* and the Xs are the premises* of one of the rules of inference of $\Rightarrow_{\mathbf{I^K T}^\omega}$.

Clearly, $\Gamma \vdash_{\mathbf{IKT}^\omega} \Delta$ iff $\Gamma \Rightarrow_{\mathbf{I^K T}^\omega} \Delta$.[49]

Some additional standard definitions will be useful:

---

[49]Why did I then introduce the background logic of the naive theory of truth presented in this paper using I, K-L and K-R rather than, more simply, $\mathrm{I^K}$? Because I on the one side and K-L and K-R on the other side clearly seem to concern two quite different, logically natural properties that a consequence relation might have or fail to have, and which in a *philosophical* discussion is of the utmost importance to keep distinct. $\mathrm{I^K}$, on the contrary, clearly seems to do nothing more than rather unilluminatingly conflating these two properties. That being said, it is also the case that the simplicity of a deductive system based on $\mathrm{I^K}$ makes it a far better tool for the *proof-theoretic* investigation of this section.

Definition 11. In an instance of S:[50]

$$\frac{\mathcal{D}_0 \qquad\qquad\qquad \mathcal{D}_1}{\Gamma_0 \ \Rightarrow_{\mathbf{I\kappa T}^\omega} \ \Delta_0, \varphi \qquad \Gamma_1, \varphi \ \Rightarrow_{\mathbf{I\kappa T}^\omega} \ \Delta_1}{\Gamma_0, \Gamma_1 \ \Rightarrow_{\mathbf{I\kappa T}^\omega} \ \Delta_0, \Delta_1} \ \text{S}$$

φ is the *cut formula*.

Definition 12. For every operator $\star$, the wff with an explicit occurrence of $\star$ in $\star$-L ($\star$-R) is the *principal* wff of $\star$-L ($\star$-R).

The specifics of our proof-theoretical framework will also require some non-standard definitions:

Definition 13. Given a $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction $\mathcal{D}$, a *pertinent* subtree of $\mathcal{D}$ is any subtree T of $\mathcal{D}$ such that:

- T has the same root as $\mathcal{D}$;

- If n is a node of T and is immediately below all and only the Xs in an instance of a rule of inference other than $\forall$-R and $\exists$-L, then all of the Xs are nodes of T;

- If n is a node of T, is immediately below all and only the Xs in an instance of either $\forall$-R or $\exists$-L and is above i many instances of $\wedge$-L/R, $\vee$-L/R and $\rightarrow$-L/R,[51] then, for some $j > i$, j Xs are nodes of T.

Definition 14. Given a non-empty sequence S of $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deductions $\{\mathcal{D}_i : i < j\}$, a *route* through S is a sequence of trees $\{\mathcal{T}_i : i < j\}$ such that, for every $i < j$, T_i (which we'll also write as '$R(\mathcal{D}_i)$') is a pertinent subtree of $\mathcal{D}_i$.

Definition 15. If a $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction $\mathcal{D}$ has a unique bottommost instance of S whose premises* are given by the $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deductions $\mathcal{D}_0$ and $\mathcal{D}_1$, the *cut depth* of $\mathcal{D}$ relative to a route R $\{\mathcal{T}_i : i \leq j\}$ through a sequence to which $\mathcal{D}$ (or a $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction of which $\mathcal{D}$ is a subdeduction) belongs as its kth element ($\mathrm{cd}_R(\mathcal{D})$) is the number of nodes in the maximal subtree of $\mathcal{T}_k$ whose root is the conclusion* of $\mathcal{D}$'s bottommost instance of S. If, on the contrary, $\mathcal{D}$ is S-free, the *cut depth* of $\mathcal{D}$ relative to a route through a sequence to which $\mathcal{D}$ (or a $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction of which $\mathcal{D}$ is a subdeduction) belongs is 0.

Definition 16. A route R $\{\mathcal{T}_i : i < j\}$ through a sequence S is *monotonic* over a non-empty subsequence of S of $k \leq j$ $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deductions $\{\mathcal{D}_i : i < k\}$ that have a unique bottommost instance of S iff, for every $i < k-1$, every node of the maximal subtree of $\mathcal{T}_i$ whose root is the conclusion* of the bottommost instance of S (minus the premises* and conclusion* of such instance) is also a node of the maximal subtree of $\mathcal{T}_{i+1}$ whose root is the conclusion* of the bottommost instance of S.

---

[50]Throughout, '$\mathcal{D}_0$' and its likes as they occur in proof-display mode are understood to denote the relevant $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction that ends with the node immediately below '$\mathcal{D}_0$' (*including* such a node).

[51]Throughout, I use '$\star$-L/R' (with $\star$ being an operator) when I mean to talk indiscriminatingly about $\star$-L and $\star$-R (with context disambiguating between the 'both'-reading and the 'either'-reading).

Theorem 20. S *is eliminable in* $\mathbf{I^K T}^\omega$.

Proof. We follow the broad outlines of the standard strategy essentially going back to Gentzen [1934]. It clearly suffices to prove that topmost instances of S in a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction can be eliminated. In turn, for that it clearly suffices to prove that a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ containing an instance of S only at its last step can be transformed into a S-free $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction with the same root as $\mathcal{D}$. And in turn, for that it clearly suffices to prove that, given a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ containing an instance of S only at its last step, there are both a $\mathcal{D}$-initial sequence S of $\Rightarrow_{\mathbf{I^K T}^\omega}$-deductions with the same root as $\mathcal{D}$ and a route R through S such that, for each element $\mathcal{D}_i$ of S for which $\mathrm{cd}_R(\mathcal{D}_i) > 0$, $\mathrm{cd}_R(\mathcal{D}_{i+1}) < \mathrm{cd}_R(\mathcal{D}_i)$. We'll prove this last claim by first proving that, for every $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ containing an instance of S only at its last step, it can be transformed into a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$ with the same root as $\mathcal{D}$ for which there is a route R such that $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$. To complete the proof, it will then suffice to observe that the defined transformations have the property that, for any sequence S obtained by repeatedly applying them, there is at least one route R through S that can be used as a constant witness for the claim that, for each element $\mathcal{D}_i$ of S for which $\mathrm{cd}_R(\mathcal{D}_i) > 0$, $\mathrm{cd}_R(\mathcal{D}_{i+1}) < \mathrm{cd}_R(\mathcal{D}_i)$. We distinguish main cases, subcases, subsubcases and subsubsubcases, and presuppose the general structure and notation displayed in definition 11 for the instance of S.

Main case 1. Either premise* is the conclusion* of an instance of $\mathrm{I^K}$.

Subcase 1a. The left premise* is the conclusion* of an instance of $\mathrm{I^K}$.

Subsubcase 1aa. The cut formula only occurs as a conclusion. Then the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ has the form:

$$
\cfrac{\cfrac{}{\Gamma_0 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0, \varphi}\ \mathrm{I^K} \qquad \cfrac{\mathcal{D}_1}{\Gamma_1, \varphi \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_1}}{\Gamma_0, \Gamma_1 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0, \Delta_1}\ \mathrm{S}
$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$
\cfrac{}{\Gamma_0, \Gamma_1 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0, \Delta_1}\ \mathrm{I^K}
$$

where, for every R, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

Subsubcase 1ab. The cut formula occurs both as a premise and as a conclusion. Then the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ has the form:

$$
\cfrac{\cfrac{}{\varphi, \Gamma_0' \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0, \varphi}\ \mathrm{I^K} \qquad \cfrac{\mathcal{D}_1}{\Gamma_1, \varphi \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_1}}{\varphi, \Gamma_0', \Gamma_1 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0, \Delta_1}\ \mathrm{S}
$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$\mathcal{D}_1^{\$}$$
$$\Gamma_0', \Gamma_1, \varphi \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \Delta_1$$

where $\mathcal{D}_1^{\$}$ is a suitable variant of $\mathcal{D}_1$ (obtained by fiddling with $\mathrm{I^K}$) and where, for every R, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

Subcase 1b. A symmetrical argument holds if the right premise* is an instance of $\mathrm{I^K}$.

Main case 2. The cut formula is not principal in at least one of the premises*.

Subcase 2a. The cut formula is not principal in the left premise*. Then the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ has the form:

$$
\begin{array}{c}
\cfrac{
\cfrac{
\begin{array}{ccc}
\mathcal{D}_0' & \mathcal{D}_0'' & \mathcal{D}_0''' \\
\Gamma_0' \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0', \varphi \quad & \Gamma_0'' \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0'' \quad & \Gamma_0''' \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0''' \dots
\end{array}
}{\Gamma_0 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \varphi} \star\text{-L/R} \qquad
\begin{array}{c}
\mathcal{D}_1 \\
\Gamma_1, \varphi \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1
\end{array}
}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \Delta_1} \mathrm{S}
\end{array}
$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$
\cfrac{
\cfrac{
\begin{array}{cc}
\mathcal{D}_0' & \mathcal{D}_1 \\
\Gamma_0' \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0', \varphi \quad & \Gamma_1, \varphi \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1
\end{array}
}{\Gamma_0', \Gamma_1 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0', \Delta_1} \mathrm{S} \qquad
\begin{array}{cc}
\mathcal{D}_0'' & \mathcal{D}_0''' \\
\Gamma_0'' \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0'' \quad & \Gamma_0''' \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0''' \dots
\end{array}
}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \Delta_1} \star\text{-L/R}
$$

where, for every R monotonic over $\langle \mathcal{D}, \mathcal{E} \rangle$ and such that $R(\mathcal{D})$ includes a branch of $\mathcal{D}_0'$, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

Subcase 2b. A symmetrical argument holds if the cut formula is not principal in the right premise*.

Main case 3. The cut formula is principal in both premises*.

Subcase 3a. The cut formula is of the form $\neg\psi$. Then the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ has the form:

$$
\cfrac{
\cfrac{
\begin{array}{c}
\mathcal{D}_0' \\
\Gamma_0, \psi \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0
\end{array}
}{\Gamma_0 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \neg\psi} \neg\text{-R} \qquad
\cfrac{
\begin{array}{c}
\mathcal{D}_1' \\
\Gamma_1 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1, \psi
\end{array}
}{\Gamma_1, \neg\psi \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1} \neg\text{-L}
}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0, \Delta_1} \mathrm{S}
$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$
\cfrac{
\begin{array}{cc}
\mathcal{D}_1' & \mathcal{D}_0' \\
\Gamma_1 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1, \psi \quad & \Gamma_0, \psi \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_0
\end{array}
}{\Gamma_1, \Gamma_0 \Rightarrow_{\mathbf{I^K T}^\omega} \Delta_1, \Delta_0} \mathrm{S}
$$

40

where, for every R monotonic over $\langle \mathcal{D}, \mathcal{E}\rangle$, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

Subcase 3b. The cut formula is of the form $\psi \wedge \chi$. Then the $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction $\mathcal{D}$ has the form:

$$
\cfrac{
\cfrac{
\begin{array}{c}\mathcal{D}'_0\\ \Gamma'_0 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta'_0, \psi\end{array}
\quad
\begin{array}{c}\mathcal{D}''_0\\ \Gamma''_0 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta''_0, \chi\end{array}
}{\Gamma_0 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_0, \psi \wedge \chi} \ \wedge\text{-R}
\qquad
\cfrac{
\begin{array}{c}\mathcal{D}'_1\\ \Gamma_1, \psi, \chi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_1\end{array}
}{\Gamma_1, \psi \wedge \chi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_1} \ \wedge\text{-L}
}{\Gamma_0, \Gamma_1 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_0, \Delta_1} \ \text{S}
$$

which we can transform into the $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction $\mathcal{E}$:

$$
\cfrac{
\begin{array}{c}\mathcal{D}'_0\\ \Gamma'_0 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta'_0, \psi\end{array}
\qquad
\cfrac{
\begin{array}{c}\mathcal{D}''_0\\ \Gamma''_0 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta''_0, \chi\end{array}
\quad
\begin{array}{c}\mathcal{D}'_1\\ \Gamma_1, \psi, \chi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_1\end{array}
}{\Gamma''_0, \Gamma_1, \psi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta''_0, \Delta_1} \ \text{S}
}{\Gamma_0, \Gamma_1 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_0, \Delta_1} \ \text{S}
$$

where, for every R monotonic over $\langle \mathcal{D}, \mathcal{E}\rangle$, $\mathrm{cd}_R(\mathcal{E}') < \mathrm{cd}_R(\mathcal{D})$ (with $\mathcal{E}'$ being the maximal subtree of $\mathcal{E}$ whose root is the displayed occurrence of $\Gamma''_0, \Gamma_1, \psi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta''_0, \Delta_1$). Thus, by the induction hypothesis, the upper instance of S in $\mathcal{E}$ can be eliminated producing a S-free $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction $\mathcal{E}'^*$ of $\Gamma''_0, \Gamma_1, \psi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta''_0, \Delta_1$ and a corresponding $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction $\mathcal{E}^*$ of $\Gamma_0, \Gamma_1 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_0, \Delta_1$ having only one instance of S at its last step. Now, let $\mathcal{E}'^{*-1}$ be the $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction from which $\mathcal{E}'^*$ has immediately been obtained (clearly, $\mathcal{E}'^*$ must have been obtained from $\mathcal{E}'^{*-1}$ by one of the transformations defined in main case 1 or subsubsubcase 3eba). Take any R suitable for the elimination of the upper instance of S and such that $R(\mathcal{E}'^*)$ coincides with $R(\mathcal{E}'^{*-1})$ wherever possible (identifying a node in $\mathcal{E}'^{*-1}$ with its \$-variant in $\mathcal{E}'^*$). It is then easy to verify that the transformations defined in this proof have the property that, for any such R, $|\mathrm{fld}(\mathcal{E}'^*) \cap \mathrm{fld}(R(\mathcal{E}^*))| \leq |\mathrm{fld}(\mathcal{D}') \cap \mathrm{fld}(R(\mathcal{D}))| + |\mathrm{fld}(\mathcal{D}''_0) \cap \mathrm{fld}(R(\mathcal{D}))|$ (with $\mathrm{fld}(X)$ being the field of the relation X, with $\mathcal{E}^{*'}$ being the maximal subtree of $R(\mathcal{E}^*)$ whose root is the right premise* of the instance of S in $\mathcal{E}^*$ and with $\mathcal{D}'$ being the maximal subtree of $\mathcal{D}$ whose root is the displayed occurrence of $\Gamma_1, \psi \wedge \chi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_1$). From this, it immediately follows that, for any such R, $\mathrm{cd}_R(\mathcal{E}^*) < \mathrm{cd}_R(\mathcal{D})$.

Subcase 3c. A dual argument holds if the cut formula is of the form $\psi \vee \chi$.

Subcase 3d. The cut formula is of the form $\psi \rightarrow \chi$. Then the $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction $\mathcal{D}$ has the form:

$$
\cfrac{
\cfrac{
\begin{array}{c}\mathcal{D}'_0\\ \Gamma_0, \psi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_0, \chi\end{array}
}{\Gamma_0 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_0, \psi \rightarrow \chi} \ \rightarrow\text{-R}
\qquad
\cfrac{
\begin{array}{c}\mathcal{D}'_1\\ \Gamma'_1 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta'_1, \psi\end{array}
\quad
\begin{array}{c}\mathcal{D}''_1\\ \Gamma''_1, \chi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta''_1\end{array}
}{\Gamma_1, \psi \rightarrow \chi \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_1} \ \rightarrow\text{-L}
}{\Gamma_0, \Gamma_1 \;\Rightarrow_{\mathbf{I\kappa T}^\omega}\; \Delta_0, \Delta_1} \ \text{S}
$$

which we can transform into the $\Rightarrow_{\mathbf{I\kappa T}^\omega}$-deduction $\mathcal{E}$:

$$
\cfrac{
\cfrac{\mathcal{D}'_1}{\Gamma'_1 \;\Rightarrow_{\mathbf{IKT}^\omega}\; \Delta'_1, \psi}
\qquad
\cfrac{
\cfrac{\mathcal{D}'_0}{\Gamma_0, \psi \;\Rightarrow_{\mathbf{IKT}^\omega}\; \Delta_0, \chi}
\qquad
\cfrac{\mathcal{D}''_1}{\Gamma''_1, \chi \;\Rightarrow_{\mathbf{IKT}^\omega}\; \Delta''_1}
}{\Gamma_0, \psi, \Gamma''_1 \;\Rightarrow_{\mathbf{IKT}^\omega}\; \Delta_0, \Delta''_1}\ \text{S}
}{\Gamma_0, \Gamma_1 \;\Rightarrow_{\mathbf{IKT}^\omega}\; \Delta_0, \Delta_1}\ \text{S}
$$

where, for every R monotonic over $\langle \mathcal{D}, \mathcal{E}\rangle$, $\mathrm{cd}_R(\mathcal{E}') < \mathrm{cd}_R(\mathcal{D})$ (with $\mathcal{E}'$ being the maximal subtree of $\mathcal{E}$ whose root is the displayed occurrence of $\Gamma_0, \psi, \Gamma''_1 \Rightarrow_{\mathbf{IKT}^\omega} \Delta_0, \Delta''_1$). Thus, by the induction hypothesis, the upper instance of S in $\mathcal{E}$ can be eliminated producing a S-free $\Rightarrow_{\mathbf{IKT}^\omega}$-deduction $\mathcal{E}'^*$ of $\Gamma_0, \psi, \Gamma''_1 \Rightarrow_{\mathbf{IKT}^\omega} \Delta_0, \Delta''_1$ and a corresponding $\Rightarrow_{\mathbf{IKT}^\omega}$-deduction $\mathcal{E}^*$ of $\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{IKT}^\omega} \Delta_0, \Delta_1$ having only one instance of S at its last step. Now, let $\mathcal{E}'^{*-1}$ be the $\Rightarrow_{\mathbf{IKT}^\omega}$-deduction from which $\mathcal{E}'^*$ has immediately been obtained (clearly, $\mathcal{E}'^*$ must have been obtained from $\mathcal{E}'^{*-1}$ by one of the transformations defined in main case 1 or subsubsubcase 3eba). Take any R suitable for the elimination of the upper instance of S and such that $R(\mathcal{E}'^*)$ coincides with $R(\mathcal{E}'^{*-1})$ wherever possible (identifying a node in $\mathcal{E}'^{*-1}$ with its \$-variant in $\mathcal{E}'^*$). It is then easy to verify that the transformations defined in this proof have the property that, for any such R, $|\mathrm{fld}(\mathcal{E}^{*'}) \cap \mathrm{fld}(R(\mathcal{E}^*))| \leq |\mathrm{fld}(\mathcal{D}') \cap \mathrm{fld}(R(\mathcal{D}))| + |\mathrm{fld}(\mathcal{D}''_1) \cap \mathrm{fld}(R(\mathcal{D}))|$ (with $\mathcal{E}^{*'}$ being the maximal subtree of $R(\mathcal{E}^*)$ whose root is the right premise* of the instance of S in $\mathcal{E}^*$ and with $\mathcal{D}'$ being the maximal subtree of $\mathcal{D}$ whose root is the displayed occurrence of $\Gamma_0 \Rightarrow_{\mathbf{IKT}^\omega} \Delta_0, \psi \rightarrow \chi$). From this, it immediately follows that, for any such R, $\mathrm{cd}_R(\mathcal{E}^*) < \mathrm{cd}_R(\mathcal{D})$.

Subcase 3e. The cut formula is of the form $\forall\xi\psi$. Then the $\Rightarrow_{\mathbf{IKT}^\omega}$-deduction $\mathcal{D}$ has the form:

$$
\cfrac{
\cfrac{
\cfrac{\mathcal{D}'_0}{\Gamma'_0 \Rightarrow_{\mathbf{IKT}^\omega} \Delta'_0, \psi_{v_0/\xi}}
\quad
\cfrac{\mathcal{D}''_0}{\Gamma''_0 \Rightarrow_{\mathbf{IKT}^\omega} \Delta''_0, \psi_{v_1/\xi}}
\quad
\cfrac{\mathcal{D}'''_0}{\Gamma'''_0 \Rightarrow_{\mathbf{IKT}^\omega} \Delta'''_0, \psi_{v_2/\xi}\dots}
}{\Gamma_0 \Rightarrow_{\mathbf{IKT}^\omega} \Delta_0, \forall\xi\psi}\ \forall\text{-R}
\qquad
\cfrac{\mathcal{D}'_1}{\cfrac{\Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\dots \Rightarrow_{\mathbf{IKT}^\omega} \Delta_1}{\Gamma_1, \forall\xi\psi \Rightarrow_{\mathbf{IKT}^\omega} \Delta_1}\ \forall\text{-L}}
}{\Gamma_0, \Gamma_1 \Rightarrow_{\mathbf{IKT}^\omega} \Delta_0, \Delta_1}\ \text{S}
$$

Subsubcase 3ea. The denumerable multiset of premises is introduced by an instance of $\forall$-R. Then $\mathcal{D}'_1$ has the form:

$$
\cfrac{
\cfrac{\mathcal{D}^{\#}_1}{\Gamma^{\#}_1, \psi_{v_0/\xi} \Rightarrow_{\mathbf{IKT}^\omega} \Delta^{\#}_1}
\quad
\cfrac{\mathcal{D}^{\#\#}_1}{\Gamma^{\#\#}_1, \psi_{v_1/\xi} \Rightarrow_{\mathbf{IKT}^\omega} \Delta^{\#\#}_1}
\quad
\cfrac{\mathcal{D}^{\#\#\#}_1}{\Gamma^{\#\#\#}_1, \psi_{v_2/\xi} \Rightarrow_{\mathbf{IKT}^\omega} \Delta^{\#\#\#}_1}\dots
}{
\cfrac{
\cfrac{\Gamma^*_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\dots \Rightarrow_{\mathbf{IKT}^\omega} \Delta^*_1}{\mathcal{D}^*_1}
}{\Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi}\dots \Rightarrow_{\mathbf{IKT}^\omega} \Delta_1}
}\ \forall\text{-R}
$$

so that, for instance, we can transform $\mathcal{D}$ into the $\Rightarrow_{\mathbf{IKT}^\omega}$-deduction $\mathcal{E}$:

$$\cfrac{\mathcal{D}_0'^{\$}}{\cfrac{\Gamma_0 \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_0, \psi_{v_0/\xi}}{} \quad \cfrac{\cfrac{\mathcal{D}_1^{\#\$}}{\Gamma_1^*, \psi_{v_0/\xi} \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_1^*}}{\cfrac{\mathcal{D}_1^{*\pounds}}{\Gamma_1, \psi_{v_0/\xi} \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_1}}}{\Gamma_0, \Gamma_1 \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_0, \Delta_1}} \ \text{s}$$

where $\mathcal{D}_0'^{\$}$ and $\mathcal{D}_1^{\#\$}$ are suitable variants of $\mathcal{D}_0'$ and $\mathcal{D}_1^{\#}$ respectively (both obtained by fiddling with $\mathrm{I^K}$), where $\mathcal{D}_1^{*\pounds}$ is a suitable variant of $\mathcal{D}_1^{*}$ (obtained by substituting $\psi_{v_0/\xi}$ for $\psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi} \ldots$ in $\mathcal{D}_1^{*}$) and where, for every R monotonic over $\langle \mathcal{D}, \mathcal{E} \rangle$ (identifying a node in $\mathcal{D}$ with its $\$$- or $\pounds$-variant in $\mathcal{E}$) and such that $R(\mathcal{D})$ includes a subtree of $\mathcal{D}_0'$ and a subtree of $\mathcal{D}_1^{\#}$, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$. (I said 'for instance' because, of course, the choice of taking $\psi_{v_0/\xi}$ is arbitrary—for every $i \neq 0$, taking $\psi_{v_i/\xi}$ instead would produce an equally acceptable transformation.)

Subsubcase 3eb. The denumerable multiset of premises is introduced by an instance of $\mathrm{I^K}$.

Subsubsubcase 3eba. A wff which is not a member of $[\psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi} \ldots]$ occurs both as a premise and as a conclusion of the relevant instance of $\mathrm{I^K}$. Then $\mathcal{D}_1'$ has the form:

$$\cfrac{\cfrac{}{\Gamma_1^*, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi} \ldots \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_1^*} \ \mathrm{I^K}}{\begin{array}{c} \mathcal{D}_1^* \\ \Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi} \ldots \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_1 \end{array}}$$

so that we can transform $\mathcal{D}$ into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$\cfrac{\cfrac{}{\Gamma_1^*, \Gamma_0 \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_1^*, \Delta_0} \ \mathrm{I^K}}{\begin{array}{c} \mathcal{D}_1^{*\$\pounds} \\ \Gamma_1, \Gamma_0 \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_1, \Delta_0 \end{array}}$$

where $\mathcal{D}_1^{*\$\pounds}$ is a suitable variant of $\mathcal{D}_1^{*}$ (obtained by fiddling with $\mathrm{I^K}$ and by substituting $\varnothing$ for $\psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi} \ldots$ in $\mathcal{D}_1^{*}$) and where, for every R, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

Subsubsubcase 3ebb. No wff which is not a member of $[\psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi} \ldots]$ occurs both as a premise and as a conclusion of the relevant instance of $\mathrm{I^K}$. Then, for some i, $\mathcal{D}_1'$ has the form:

$$\cfrac{\cfrac{}{\Gamma_1^*, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi} \ldots \psi_{v_i/\xi} \ldots \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_1^*, \psi_{v_i/\xi}} \ \mathrm{I^K}}{\begin{array}{c} \mathcal{D}_1^* \\ \Gamma_1, \psi_{v_0/\xi}, \psi_{v_1/\xi}, \psi_{v_2/\xi} \ldots \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \Delta_1 \end{array}}$$

so that we can transform $\mathcal{D}$ into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$\cfrac{\mathcal{D}_0^{'i\$}\qquad\qquad\cfrac{\overline{\psi_{v_i/\xi},\Gamma_1^* \ \Rightarrow_{\mathbf{I^K T}^\omega} \ \psi_{v_i/\xi},\Delta_1^*}\ \mathrm{I^K}}{\mathcal{D}_1^{*\pounds}}}{\cfrac{\Gamma_0 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0,\psi_{v_i/\xi}\qquad \psi_{v_i/\xi},\Gamma_1 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_1}{\Gamma_0,\Gamma_1 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0,\Delta_1}}\ \mathrm{S}$$

where '$\mathcal{D}_0^{'i}$' is the result of superscripting '$\mathcal{D}_0$' with $i+1$ occurrences of '$'$', where $\mathcal{D}_0^{'i\$}$ is a suitable variant of $\mathcal{D}_0^{'i}$ (obtained by fiddling with $\mathrm{I^K}$), where $\mathcal{D}_1^{*\pounds}$ is a suitable variant of $\mathcal{D}_1^*$ (obtained by substituting $\psi_{v_i/\xi}$ for $\psi_{v_0/\xi},\psi_{v_1/\xi},\psi_{v_2/\xi}\ldots$ in $\mathcal{D}_1^*$) and where, for every R monotonic over $\langle\mathcal{D},\mathcal{E}\rangle$ (identifying a node in $\mathcal{D}$ with its \$- or £-variant in $\mathcal{E}$) and such that $R(\mathcal{D})$ includes a subtree of $\mathcal{D}_0^{'i}$ and a subtree of $\mathcal{D}_1^*$, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

Subcase 3f. A dual argument holds if the cut formula is of the form $\exists\xi\psi$.

Subcase 3g. The cut formula is of the form $T^{\ulcorner}\psi^{\urcorner}$. Then the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ has the form:

$$\cfrac{\cfrac{\mathcal{D}_0'}{\cfrac{\Gamma_0 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0,\psi}{\Gamma_0 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0,T^{\ulcorner}\psi^{\urcorner}}\ T\text{-R}}\qquad\cfrac{\mathcal{D}_1'}{\cfrac{\Gamma_1,\psi \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_1}{\Gamma_1,T^{\ulcorner}\psi^{\urcorner} \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_1}\ T\text{-L}}}{\Gamma_0,\Gamma_1 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0,\Delta_1}\ \mathrm{S}$$

which we can transform into the $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$:

$$\cfrac{\mathcal{D}_0'\qquad\qquad\mathcal{D}_1'}{\cfrac{\Gamma_0 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0,\psi\qquad \Gamma_1,\psi \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_1}{\Gamma_0,\Gamma_1 \ \Rightarrow_{\mathbf{I^K T}^\omega}\ \Delta_0,\Delta_1}}\ \mathrm{S}$$

where, for every R monotonic over $\langle\mathcal{D},\mathcal{E}\rangle$, $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$.

This proves that, for every $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{D}$ containing an instance of S only at its last step, it can be transformed into a $\Rightarrow_{\mathbf{I^K T}^\omega}$-deduction $\mathcal{E}$ with the same root as $\mathcal{D}$ for which there is a route R such that $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$. This is so because clearly, for each of the defined transformations, there is at least one route R that satisfies the conditions which are imposed in the transformation and whose satisfaction suffices for its being the case that $\mathrm{cd}_R(\mathcal{E}) < \mathrm{cd}_R(\mathcal{D})$. To complete the proof, it suffices to observe that the defined transformations have the property that, for any sequence S obtained by repeatedly applying them, there is at least one route R through S that can be used as a constant witness for the claim that, for each element $\mathcal{D}_i$ of S for which $\mathrm{cd}_R(\mathcal{D}_i) > 0$, $\mathrm{cd}_R(\mathcal{D}_{i+1}) < \mathrm{cd}_R(\mathcal{D}_i)$. Keeping the monotonicity condition fixed, this is so because, for any sequence S of $\Rightarrow_{\mathbf{I^K T}^\omega}$-deductions $\{\mathcal{D}_i : i < j\}$ obtained by repeatedly applying the defined transformations, if, on the one hand, none of subcases 3b, 3c and 3d occurs, for every $i < j - 3$, the conditions imposed on the transformation from $\mathcal{D}_{i+1}$ to $\mathcal{D}_{i+2}$ are in effect at worst (i.e. in subsubcase 3ea, subsubsubcase 3ebb and sometimes in main case 2) simply a further satisfiable specification of the conditions imposed on the transformation from $\mathcal{D}_i$ to $\mathcal{D}_{i+1}$ (somewhat pictorially, a further satisfiable specification of how the

pertinent subtrees in R should behave in some parts that, being "further up", were *not* concerned by the conditions imposed on the transformation from $\mathcal{D}_i$ to $\mathcal{D}_{i+1}$). If, on the other hand, either of subcases 3b, 3c or 3d occurs, and in the subinduction for one of these cases either main case 2 or subsubcase 3ea occurs, it might be worried that the conditions imposed on the transformation defined for the latter cases can enter into conflict with the conditions later imposed on one of the transformations in the superinduction, in particular if in the superinduction either main case 2 or subsubcase 3ea or subsubsubcase 3ebb occurs (somewhat pictorially, it is possible that the conditions imposed on the transformations defined for these cases in the superinduction are a further specification of how the pertinent subtrees in R should behave *in the same parts* that were concerned by the conditions imposed on the transformations defined for main case 2 or subsubcase 3ea in the subinduction, so that it might be worried that there would no longer be a guarantee that this further specification is satisfiable). However, the clause of definition 13 regarding $\forall$-R or $\exists$-L (which are in effect the rules of inference creating this worry) clearly does guarantee that, in any instance of $\forall$-R or $\exists$-L, there are enough subtrees included in a pertinent subtree as to avoid any conflict between the conditions imposed on one of the transformations in a subinduction and the conditions later imposed on one of the transformations in the corresponding superinduction.

$\square$

4.2   Applications of the *Hauptsatz*

From theorem 20 we easily obtain a series of interesting consistency properties for $\mathbf{IKT}^\omega$.

Corollary 7. $\mathbf{IKT}^\omega$ *is* minimally consistent*, in the sense that $\oslash \vdash_{\mathbf{IKT}^\omega} \oslash$ does not hold.*

Proof. Clear from theorem 20 and the properties of S-free $\mathbf{IKT}^\omega$-deductions.

$\square$

Corollary 8. $\mathbf{IKT}^\omega$ *is* positively conservative*, in the sense that, for no wff φ with no occurrence of $\neg$, $\rightarrow$ and T, $\oslash \vdash_{\mathbf{IKT}^\omega} \varphi$ holds.*

Proof. Clear from theorem 20 and the properties of S-free $\mathbf{IKT}^\omega$-deductions.

$\square$

Corollary 9. $\mathbf{IKT}^\omega$ *is* non-trivial*, in the sense that, for some φ and ψ, $\varphi \vdash_{\mathbf{IKT}^\omega} \psi$ does not hold.*

Proof. If, for every φ and ψ, $\varphi \vdash_{\mathbf{IKT}^\omega} \psi$ held, $\varphi \lor \neg\varphi \vdash_{\mathbf{IKT}^\omega} \varphi \land \neg\varphi$ would hold. But, by (LEM), $\oslash \vdash_{\mathbf{IKT}^\omega} \varphi \lor \neg\varphi$ holds and, by (LNC), $\varphi \land \neg\varphi \vdash_{\mathbf{IKT}^\omega} \oslash$ holds, and hence, by S, $\oslash \vdash_{\mathbf{IKT}^\omega} \oslash$ would hold, which contradicts corollary 7 to theorem 20.

$\square$

Corollary 10. $\mathbf{IKT}^\omega$ *is* non-contradictory, *in the sense that either* $\oslash \vdash_{\mathbf{IKT}^\omega} \varphi$ *or* $\oslash \vdash_{\mathbf{IKT}^\omega} \neg\varphi$ *does not hold.*

Proof. If both $\oslash \vdash_{\mathbf{IKT}^\omega} \varphi$ and $\oslash \vdash_{\mathbf{IKT}^\omega} \neg\varphi$ held, by $\wedge$-R $\oslash \vdash_{\mathbf{IKT}^\omega} \varphi \wedge \neg\varphi$ would hold. But, by (LNC), $\varphi \wedge \neg\varphi \vdash_{\mathbf{IKT}^\omega} \oslash$ holds, and hence, by S, $\oslash \vdash_{\mathbf{IKT}^\omega} \oslash$ would hold, which contradicts corollary 7 to theorem 20.

$\square$

Corollary 11. W-L *and* W-R *are not admissible for either* $\mathbf{IK}^\omega$ *or* $\mathbf{IKT}^\omega$.

Proof. Clearly, it suffices to show that W-L and W-R are not admissible for $\mathbf{IKT}^\omega$. That in turn follows from the fact that, as the semantic paradoxes presented in sections 2.1 and 2.2 show, the addition of either W-L and W-R would generate a consequence relation which is not even minimally consistent, which contradicts corollary 7 to theorem 20.

$\square$

5 Conclusion and Glimpses Beyond

This paper has presented a novel naive theory of truth. It has done so by, after briefly motivating philosophically the failure of contraction, developing a formal non-contractive naive theory of truth and proving its consistency. The focus has been on demonstrating the logical and truth-theoretic strength and coherence of the theory, especially in the surprisingly many respects of philosophically interesting strength in which it outperforms its naive rivals.

The success, however, as the reader will certainly have noticed, has only been partial in many important respects. Let me conclude by listing and briefly commenting on what I regard as the most pressing open problems (clearly, there are important connections among many of them—the order is not fortuitous):

- The metaphysical picture sketched in section 2.3, although appealing, is certainly in need of further elaboration and defence;

- The theory I've offered has for simplicity presupposed a countable domain, and should be extended to domains of larger cardinalities;

- The theory also does not include a theory of identity or of higher-order quantification, and should be so extended;

- The theory, which I've developed in a broadly proof-theoretical fashion, should be provided with a suitable semantics;

- Even if I've proven in section 4 that the theory is in itself consistent, I haven't proven that it is consistent with other theories we might also wish to endorse. Stronger conservativeness results are highly desirable;

- Even if I've indicated in sections 2.1 and 2.2 where some of the most venerable semantic paradoxes are blocked by the theory, I haven't done so for each and every semantic paradox that has been discussed in the literature, in spite of the fact that, for some of these, to do so would shed significant additional light into the inner workings of the theory (the consistency proof I've offered in section 4 shows of course that all such paradoxes expressible in the system developed in section 3 are effectively blocked one way or another);

- I haven't tried to discuss the important question of whether and to what extent the theory is subject to "revenge" worries;

- I haven't discussed at all semantic paradoxes generated using intuitively correct principles governing other semantic notions (such as the notions of being true of, satisfying, denoting etc). I think that it is a requirement on the kind of theory I've presented that it be smoothly extendible to other semantic notions;

- Nor have I discussed broadly "self-referential" paradoxes generated using intuitively correct principles governing other non-semantic notions (such as the notions of set, property, knowledge etc). I'm open to the idea that at least some of these paradoxes require differential treatments (I've briefly touched on this in fn 11).

Many important problems remain thus open. I hope that this paper has done enough to justify the idea that the theory giving rise to such problems is a new worthy candidate in the debate among naive theories of truth.

References

JC Beall. *Spandrels of Truth*. Oxford University Press, Oxford, 2009.

Wayne Blizard. Multiset theory. *Notre Dame Journal of Formal Logic*, 30:36–66, 1989.

Ross Brady. *Universal Logic*. CSLI Publications, Stanford, 2006.

Haskell Curry. The inconsistency of certain formal logics. *The Journal of Symbolic Logic*, 7:115–117, 1942.

Hartry Field. Truth and the unprovability of consistency. *Mind*, 115:567–605, 2006.

Hartry Field. *Saving Truth from Paradox*. Oxford University Press, Oxford, 2008.

Dov Gabbay. Theoretical foundations for non-monotonic reasoning in expert systems. In Krzysztof Apt, editor, *Logics and Models of Concurrent Systems*, pages 439–459. Springer, Berlin, 1985.

Gerhard Gentzen. Untersuchungen über das logische Schließen I. *Mathematische Zeitschrift*, 39:176–210, 1934.

Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

Jean-Yves Girard. Light linear logic. *Information and Computation*, 143:175–204, 1998.

Vjačeslav Grišin. A nonstandard logic and its applications to set theory. In *Studies in Formalized Languages and Nonclassical Logics*, pages 135–171. Nauka, Moscow, 1974.

Vjačeslav Grišin. Predicate and set-theoretic calculi based on logics without contractions. *Izvestija Akademij Nauk SSSR Serija Matematičeskaja*, 45:47–68, 1981.

Anil Gupta and Nuel Belnap. *The Revision Theory of Truth*. MIT Press, Cambridge MA, 1993.

Hans Herzberger. Naive semantics and the liar paradox. *The Journal of Philosophy*, 79: 479–497, 1982.

David Kaplan. Demonstratives. In Joseph Almog, John Perry, and Howard Wettstein, editors, *Themes from Kaplan*, pages 481–563. Oxford University Press, Oxford, 1989.

Diogenes Laertius. *De vitis, dogmatibus et apophtegmatibus clarorum philosophorum*.

Dan López de Sa and Elia Zardini. Truthmakers, knowledge and paradox. *Analysis*, 67: 242–250, 2007.

Dan López de Sa and Elia Zardini. No-no. Paradox and consistency. *Analysis*, 2010. Forthcoming.

Tim Maudlin. *Truth and Paradox*. Oxford University Press, Oxford, 2004.

Vann McGee. *Truth, Vagueness, and Paradox*. Hackett, Indianapolis, 1991.

Robert Meyer, Richard Routley, and Michael Dunn. Curry's paradox. *Analysis*, 39: 124–128, 1979.

Charles Parsons. The liar paradox. *Journal of Philosophical Logic*, 3:381–412, 1974.

Uwe Petersen. Logic without contraction as based on inclusion and unrestricted abstraction. *Studia Logica*, 64:365–403, 2000.

Uwe Petersen. *Diagonal Method and Dialectical Logic*. Der Andere, Osnabrück, 2002.

Graham Priest. *In Contradiction*. Oxford University Press, Oxford, 2nd edition, 2006a.

Graham Priest. *Doubt Truth to Be a Liar*. Oxford University Press, Oxford, 2006b.

Anne Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, Cambridge, 2nd edition, 2000.

Richard White. A demonstrably consistent type-free extension of the logic BCK. *Mathematica Japonica*, 32:149–169, 1987.

Richard White. A consistent theory of attributes in a logic without contraction. *Studia Logica*, 52:113–142, 1993.

Elia Zardini. Truth and what is said. *Philosophical Perspectives*, 22:545–574, 2008.

Elia Zardini. The role of utterances in Bradwardine's theory of truth. In Christoph Kann, Benedikt Löwe, Christian Rode, and Sara Uckelman, editors, *Modern Views of Medieval Logic*, Recherches de théologie et philosophie médiévales. Peeters, Leuven, 2010a. Forthcoming.

Elia Zardini. Logical consequence and the preservation of truth. ms, 2010b.

Elia Zardini. Consequence and context. Bearers, truth preservation and structural properties. ms, 2010c.