Towards First-Order Tolerant Logic

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Abstract: The paper develops an extension to the theory of 1st-order quantification and identity of tolerant logics. These logics offer a new solution to the sorites paradox, by placing principled restrictions on the transitivity of the consequence relation. In so doing, they are hospitable to the naive theory of vagueness, according to which the vagueness of an expression consists, roughly, in the existence of both positive and negative cases of application of the expression and in the non-existence of a sharp boundary between them. It is shown that tolerant logics allow not only for the consistency of the 1st-order fragment of the naive theory, but also for its extension to a naive theory of vague identity which is based on naive abstraction principles and under which identity is transitive (and indeed obeys the law of indiscernibility of identicals). These results are also proved to hold for a suitable tolerant counterpart to classical logic ($\mathbf{KII}^1$). However, it is shown that under $\mathbf{KII}^1$ the 1st-order fragment of the naive theory, albeit non-trivial and consistent, entails some quite problematic claims. Hence, the paper finally argues for a retreat to a weaker, non-distributive tolerant logic $\mathbf{LKII}^1$, which preserves all the virtues of $\mathbf{KII}^1$ while being free of the latter's undesired strength.

Keywords: abstraction, distributivity, identity, lattice theory, sorites paradox, tolerant logics, vagueness.

According to the *naive theory of vagueness*, the vagueness of an expression consists, roughly and at least in the most basic and fundamental cases, in having both positive and negative cases of application, whilst not drawing any sharp boundary between them. Hence, for example, the vagueness of the predicate 'A man with ξ hairs is bald' consists in having both positive and negative cases of application (say, 0 and 1,000,000 respectively), whilst not drawing any sharp boundary between them. The last clause is crucial, and is meant very seriously: for no i, a man with i hairs is bald and a man with $i + 1$ hairs is not bald, which we can assume to entail that, for every i, if a man with i hairs is bald, so is a man with $i + 1$ hairs. Lack of a sharp boundary in this sense is what it is for an expression to be "*tolerant*" (see Wright 1975: 333–334).

There is no denying that the naive theory of vagueness enjoys a great *prima facie* plausibility. It is also clear that it scores high on many usual parametres of theory choice, ranging from simplicity to explanatory power. It is even arguable that it is only concepts governed by tolerance principles that make certain thoughts about, experiences of and interactions with the world possible (see Zardini 2009a). All this however has long seemed to be shattered by the simple observation that the naive theory is inconsistent. The most straightforward route to the inconsistency is afforded by so-called '*sorites paradoxes*'. Consider the premises:

(1) A man with 0 hairs is bald;

(2) A man with 1,000,000 hairs is not bald;

(3) If a man with i hairs is bald, so is a man with $i + 1$ hairs.

The naive theory is committed to all these three premises. However, from (3) we have that, if a man with 0 hairs is bald, so is a man with 1 hair, which, together with (1), yields that a man with 1 hair is bald. Yet, from (3) we also have that, if a man with 1 hair is bald, so is a man with 2 hairs, which, together with the previous lemma that a man with 1 hair is bald, yields that a man with 2 hairs is bald. With another 999,997 structurally identical arguments, we reach the conclusion that a man with 999,999 hairs is bald. From (3) we also have that, if a man with 999,999 hairs is bald, so is a man with 1,000,000 hairs, which, together with the previous lemma that a man with 999,999 hairs is bald, yields that a man with 1,000,000 hairs is bald. It would then seem that the contradictory of (2) follows simply from (1) and (3).

Since, as far as I can tell, the problem represented by the sorites paradox is the only major problem confronting the naive theory of vagueness, which, for the reasons adumbrated at the beginning the previous paragraph, I find otherwise extremely appealing, in Zardini 2008 I motivated, proposed and developed a family of 0th-order logics—*tolerant logics*—in which (a suitable 0th-order regimentation of) the naive theory is consistent (and indeed conservative over a background mathematical theory). Dealing with the sorites paradox and other conundrums of vagueness by weakening classical logic was by no means a novelty. However, traditional non-classical approaches have been characterized by the weakening of one or the other of the *operational* properties of the consequence relation—that is, roughly, properties attaching to a specific set of logical constants (such as e.g. the law of excluded middle). Contrary to such tradition, the key idea of tolerant logics is a principled weakening of one of the *structural* properties of the consequence relation—that is, roughly, properties attaching to the relation itself in abstraction from any specific logical constant (or variable) that may occur in the formulae of the language over which the consequence relation is defined (such as e.g. reflexivity). More specifically, the key idea of tolerant logics is a principled weakening of the structural property of *transitivity*,[1] which is implicitly appealed to in the idea of a *lemma* so crucial in the informal reasoning presented at the end of the previous paragraph.

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[1] Tarski 1930 very famously and influentially argued that a *consequence relation* should correspond to a *closure operation* clos from sets of formulae to sets of formulae satisfying:

(i) $X \subseteq \text{clos}(X)$ (increment);

(ii) If $X \subseteq Y$, then $\text{clos}(X) \subseteq \text{clos}(Y)$ (monotonicity);

(iii) $\text{clos}(\text{clos}(X) \cup Y) \subseteq \text{clos}(X \cup Y)$ (union-adjoint subidempotency).

Letting $\Rightarrow$ stand for the relevant consequence relation, the operation-theoretic properties (i)–(iii) correspond to the relation-theoretic properties:

Although it made sense in a first presentation of such logics to restrict attention to the 0th-order fragment, where the workings of that key idea can be appreciated in particularly pure conditions without the distraction of additional complexities, on reflection it should be clear that a complete logical solution to the sorites paradox should at least extend to the theories of *1st-order quantification* and *identity*. Indeed, some problematic claims expressible in a vague language, such as the claim that *there is* a sharp boundary between the positive and the negative cases, can only be adequately regimented in a 1st-order language, and the notion of *identity* itself is apparently subject to sorites paradoxes which only employ plausible principles pertaining to the identity predicate, so that a specific treatment of that predicate would appear to be called for. Such an extension of tolerant logics to 1st-order quantification and identity is the project of this paper. As we will see, carrying out the project is by no means trivial and will indeed help to bring out some very interesting features and insights of tolerant logics that were not so clearly recognizable by restricting attention to their 0th-order fragment.

While tolerant logics are all marked by the failure of the unrestricted transitivity property of the consequence relation, their algebraic framework is flexible enough as to accommodate very different logics. That is welcome, since, in addition to the sorites paradox, vagueness presents other conundrums that have seemed to many theorists to call for yet other revisions of classical logic (for example, vagueness-generated borderline cases have seemed to many theorists to call into the question the validity of the law of excluded middle, see e.g. Field 2003). Analogously to what I did in Zardini 2008, I will proceed by first providing an extremely minimal framework, which, while already encapsulating the key idea of the failure of the transitivity property, does not prejudge either way any of the other logical issues arising from vagueness. I will then build up stepwise to very strong systems, recovering as many of the standard operational properties as is compatible with maintaining the failure of transitivity characteristic of tolerant logics. I will not argue for the philosophical underpinnings of the general algebraic framework (I did so at some length in Zardini 2008; suffice it to say that, contrary to a widespread trend, I do not regard the elements of the algebra as modelling "degrees of truth"). I will argue, however, for particular choices within that framework, ending up settling for a very specific 1st-order tolerant logic which—I will contend—once used as the background logic for the naive theory of vagueness exhibits an optimal balance between desired *strength* (getting all the classical consequences that a naive theorist would still like to have) and desired *weakness* (avoiding all the classical consequences that a naive theorist would rather not have).

Along the way, I will show that tolerant logics allow for a strong version of *consistency* (and indeed conservativity over a background mathematical theory) of the 1st-order fragment of the

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(i′) If $\varphi \in \Gamma$, then $\Gamma \Rightarrow \varphi$ (reflexivity);

(ii′) If $\Gamma \subseteq \Delta$ and $\Gamma \Rightarrow \varphi$, then $\Delta \Rightarrow \varphi$ (weakening);

(iii′) If, for every $\varphi \in \Delta$, $\Gamma \Rightarrow \varphi$ and $\Lambda, \Delta \Rightarrow \psi$, then $\Lambda, \Gamma \Rightarrow \psi$ (transitivity).

Tarski and his many followers' view has an undeniable intuitive force, which a tolerant logician must somehow defuse. This calls for an extended philosophical discussion of a non-transitive logic's viability and import I cannot hope to undertake here. I try to make some progress on these issues in Zardini 2009b.

naive theory of vagueness. Moreover, I will show that tolerant logics also allow for the naive theory's extension to a naive theory of *vague identity* based on naive *abstraction* principles such as:

(ABS) The baldness status of a man with *i* hairs is the same as the baldness status of a man with *j* hairs iff [a man with *i* hairs has roughly the same number of hairs as a man with *j* hairs]

and under which identity is *transitive* (although not *chain transitive*) and indeed obeys the law of *indiscernibility of identicals*. These results will also be proved to hold for a suitable tolerant counterpart to classical logic ($\mathbf{KII^1}$), one of the strongest logics definable in this framework. However, I will also show that under $\mathbf{KII^1}$ the 1st-order fragment of the naive theory, even though non-trivial, consistent and conservative, entails some quite problematic claims. Consequently, and as mentioned in the previous paragraph, I will finally argue for a retreat to a weaker, *non-distributive* tolerant logic $\mathbf{LKII^1}$, which preserves all the virtues of $\mathbf{KII^1}$ while being free of the latter's undesired strength.

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