Algorithmic Robotics
COMP/ELEC/MECH 450/550
Homework 2

DUE: September 14th at the beginning of class (1pm). Submit your answers as a PDF to Canvas.

Please read the honor code and the additions described in the course syllabus. Present your work and your work only. You must *explain* all of your answers. Answers without explanation will be given no credit.

1. **(20 points)** A set S and an operator $\cdot$ form a *group* if the following properties are satisfied:

Closure	For all s_1 and s_2 in S, $s_1 \cdot s_2$ is also an element of S.
Associativity	For all s_1, s_2 and s_3 in S, $(s_1 \cdot s_2) \cdot s_3 = s_1 \cdot (s_2 \cdot s_3)$.
Identity	There exists an element of S denoted by I such that $I \cdot s_1 = s_1 \cdot I = s_1$ for all s_1 in S.
Inverse	For each s_1 there exists a s_2 in S such that $s_1 \cdot s_2 = s_2 \cdot s_1 = I$.

   Let $\mathcal{T}$ be the set of all rigid body transformations in 2D in homogeneous coordinates. Prove that $\mathcal{T}$ and regular matrix multiplication form a *group*. That is, prove that $\mathcal{T}$ and regular matrix multiplication satisfy the properties listed above. In your answers, use the facts that:

   (a) Rotation matrices with matrix multiplication form a group.

   (b) Translation vectors and vector addition form a group.

   In your answers, you can write a rigid body transformation:

$$T_i = \begin{pmatrix} R_i & p_i \\ 0 & 1 \end{pmatrix}$$

2. **(10 points)**

   Figure 1 shows a top-down perspective of a UR5 robot hand at two different orientations. The orientation of the hand shown in 1a) is the result of applying a rotation, represented with the unit quaternion $q_1 = \frac{\sqrt{2}}{2} + 0i + \frac{\sqrt{2}}{2}j + 0k$, to the world's reference frame (not shown here). Your goal is to calculate the unit quaternion q_2 required to rotate the world's reference frame to achieve the orientation of the hand as shown in Figure 1b). To this end, answer the following questions:

   (a) **(5.0 points)** Calculate the Euler Angles ZYX that correspond to q_1[1] and use them to recover the world's reference frame (i.e., the frame before q_1 was applied). Show a figure of the world's reference frame.

   (b) **(5.0 points)** Find the Euler Angles ZYX that had to be applied to the world's reference frame to achieve the robot orientation shown in Figure 1b). Describe how you found them and use them to calculate the corresponding unit quaternion q_2[1].

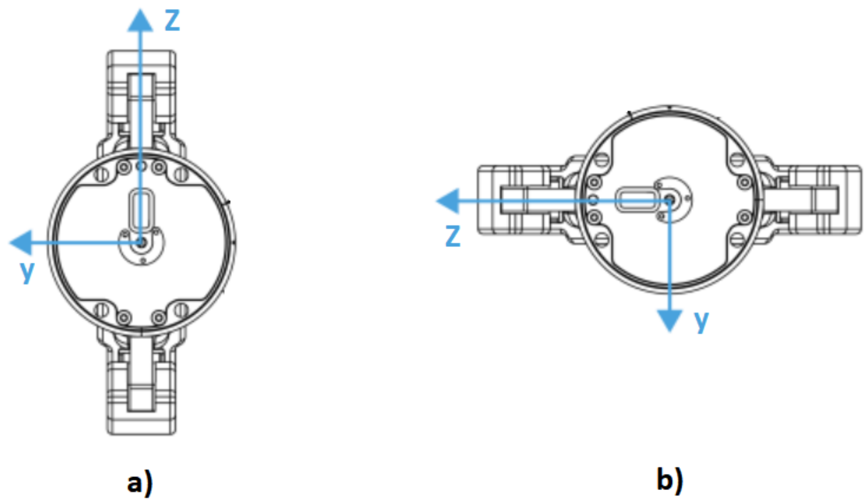

Figure 1: Figure for Problem 2

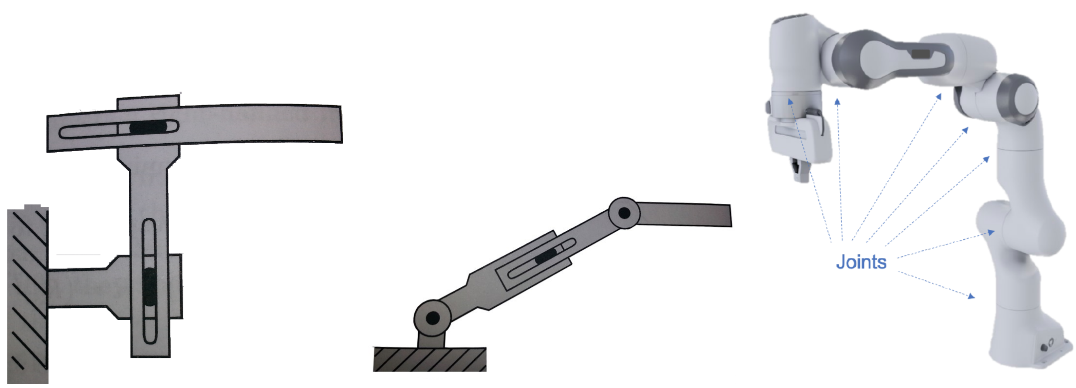

Figure 2: From left to right: a manipulator with two prismatic joints, a manipulator with two revolute joints and a prismatic joint, and a manipulator with seven revolute joints.

3. **(15 points)**

   For each of the three manipulators shown in Figure 2, determine the topology and dimension of the manipulator's configuration space.

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[1]You can use this site https://quaternions.online/. Make sure you choose ZYX order. This means that the rotations are make in thar order: first around the Z axis, then around the new Y axis and finally around the new X axis.

4. **(30 points)** Figure 3 shows a three-link kinematic chain in 2D. The lengths of link A_1, A_2 and A_3 are l_1, l_2 and l_3, respectively. The joint angles of the chain are θ_2 and θ_3, as shown in Figure 3. For each link, we attach a local frame to the base end of that link (e.g., for link A_1, the axes of frame 1, x_1 and y_1 are labeled).

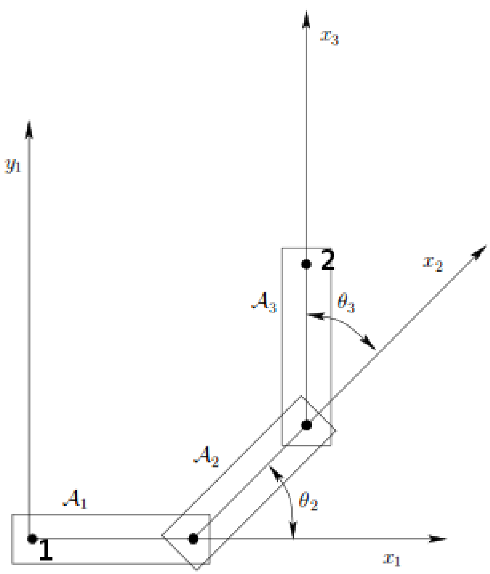

Figure 3: The Three-Link Chain

(a) **(5 points)** Determine the topology and dimension of the configuration space for this manipulator.

(b) **(5 points)** Determine the Homogeneous coordinates v_3 of the point **2** in the local frame of link A_3, in terms of l_1, l_2, l_3, θ_2 and θ_3.

(c) **(5 points)** Determine the forward kinematics of this three-link chain. That is, calculate the homogeneous coordinates v_1 of the point 2 in the local frame of link A_1, in terms of l_1, l_2, l_3, θ_2 and θ_3.

(d) **(5 points)** Determine the homogeneous transformations from the local frame of A_3 to the local frame of A_1. That is, determine the transformation matrices T_2 and T_3 such that, T_2 moves A_2 from its local frame to the local frame of A_1, and T_3 moves A_3 from its local frame to the local frame of A_2. Then the transformation matrix $T_2 \cdot T_3$ moves A_3 from its local frame to the local frame of A_1.

(e) **(10 points)** Show that $v_1 = T_2 \cdot T_3 \cdot v_3$.

5. **(15 points)** Consider a workspace $\mathcal{W}$ with a convex obstacle and a convex robot. Show that the C-space obstacle is convex.

Hint: A set $\mathcal{S}$ is a *convex* set if the line segment between any two points in $\mathcal{S}$ lies in $\mathcal{S}$, i.e., $\forall x_1, x_2 \in \mathcal{S}, \forall \lambda \in [0,1]$,

$$\lambda x_1 + (1-\lambda)x_2 \in \mathcal{S}$$

6. **(10 points)** Suppose five polyhedral bodies float freely in a 3D world. They are each capable of rotating and translating. If these are treated as "one" composite robot, what is the topology of the resulting configuration space (assume that the bodies are not attached to each other)? What is the dimension of the composite configuration space?