A WEAK DEFINITION OF DELAUNAY TRIANGULATION

VIN DE SILVA

Abstract. We show that the traditional criterion for a simplex to belong to the Delaunay triangulation of a point set is equivalent to a criterion which is *a priori* weaker. The argument is quite general; as well as the classical Euclidean case, it applies to hyperbolic and hemispherical geometries and to Edelsbrunner's weighted Delaunay triangulation. In spherical geometry, we establish a similar theorem under a genericity condition. The weak definition finds natural application in the problem of approximating a point-cloud data set with a simplical complex.

1. Strong and weak witnesses

Let $A \subset \mathbb{R}^n$ be a set of points, not necessarily finite. The *Voronoi diagram* of A, denoted $\mathrm{Vor}(A)$, is the decomposition of $\mathbb{R}^n$ into Voronoi cells $\{V_a : a \in A\}$ defined as follows:

$$V_a = \{x \in \mathbb{R}^n : |x - a| \leq |x - b| \text{ for all } b \in A\}$$

The dual of the Voronoi diagram is the *Delaunay triangulation* $\mathrm{Del}(A)$, an abstract simplicial complex which contains the p-simplex $[a_0 a_1 \dots a_p]$ with vertices $a_0, a_1, \dots, a_p \in A$ if and only if $V_{a_0} \cap V_{a_1} \cap \cdots \cap V_{a_p} \neq \emptyset$, for all $p \geq 0$. When A is a finite set of points in general position, there are no cells with $p > n$, and $\mathrm{Del}(A)$ is geometrically realised as a triangulation of the convex hull of A. In this paper our interest is in the abstract simplicial complex, which is defined under all circumstances. An example of the geometrically-realisable case is given in Figure 1: the edges of the Delaunay triangulation of the set $A = \{a, b, c, d, e, f\}$ are shown in black, and the Voronoi diagram is in grey.

In order to state our theorem, we reformulate the definition of $\mathrm{Del}(A)$ in terms of *witnesses*.

Definition. Let $\sigma = [a_0 a_1 \dots a_p]$ be a p-simplex with vertices in A. We say that $x \in \mathbb{R}^n$ is a *weak witness* for σ with respect to A if $|x - a| \leq |x - b|$ whenever $a \in \{a_0, a_1, \dots, a_p\}$ and $b \in A \setminus \{a_0, a_1, \dots, a_p\}$.

Definition. Let $\sigma = [a_0 a_1 \dots a_p]$ be a p-simplex with vertices in A. We say that $x \in \mathbb{R}^n$ is a *strong witness* for σ with respect to A if x is a weak witness for σ with respect to A and additionally there is equality $|x - a_0| = |x - a_1| = \cdots = |x - a_p|$.

Examples are given in Figure 2. In the left panel, x is a strong witness for triangle $[adf]$. In the center panel, y is a weak witness for edge $[ac]$ and indeed for triangle $[abc]$. On the right, z is a weak witness for $[bcd]$. The circumcenter of $[bcd]$ lies close to a, so $[bcd]$ does not have a strong witness.

When the context is unambiguous, we may occasionally drop the phrase "with respect to A". It follows from the definition of Voronoi cells that $\sigma \in \mathrm{Del}(A)$ if and only if σ has a strong witness.

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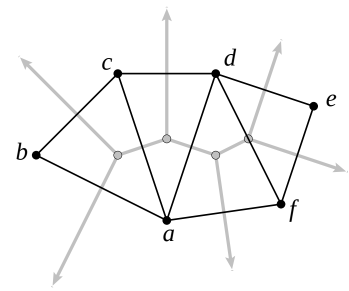


FIGURE 1. The Voronoi diagram and Delaunay triangulation for a 6 point set.

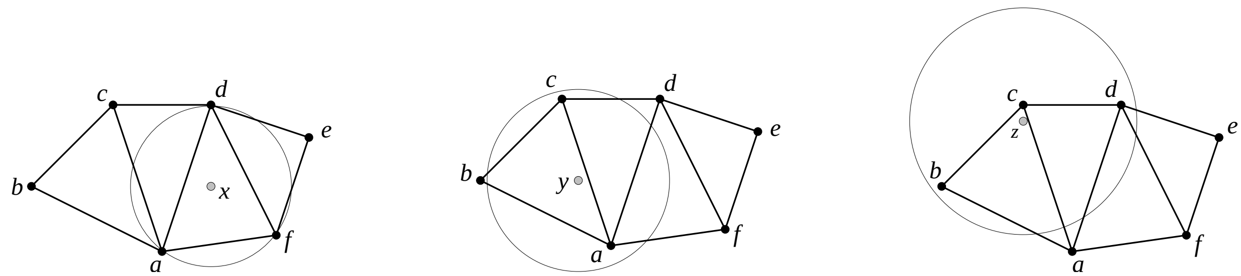


FIGURE 2. Strong and weak witnesses.

Theorem 1. *Let $A \subset \mathbb{R}^n$ and let $\sigma = [a_0 a_1 \dots a_p]$ be a simplex with vertices in A. Then σ has a strong witness if and only if every subsimplex $\tau \leq \sigma$ has a weak witness.*

A consequence of this theorem is that the Delaunay triangulation can be defined using weak witnesses only. Note that one direction is trivial: if x is a strong witness for σ then it is a strong (and hence weak) witness for every subsimplex $\tau \leq \sigma$.

Example. If $abcd$ is a convex quadrilateral in the plane, then each of the vertices $[a]$, $[b]$, $[c]$, $[d]$, each of the boundary edges $[ab]$, $[bc]$, $[cd]$, $[da]$ and each of the triangles $[abc]$, $[bcd]$, $[cda]$, $[dab]$ has a weak witness with respect to $\{a, b, c, d\}$. If $abcd$ is not cyclic, then exactly one of the diagonal edges $[ac]$, $[bd]$ has a weak witness. For instance, in Figure 2, edge $[ac]$ has a weak witness but edge $[bd]$ does not. The theorem predicts that the two triangles containing that edge have strong witnesses; and indeed the Delaunay triangulation consists of those two triangles joined along a common edge.

The special case of an edge $\sigma = [a_0 a_1]$ is well-known, and appeared in [MS94] to justify a construction of a topology-approximating graph. When their construction is generalised to topology-approximating simplicial complexes [CdS03], the corresponding justification is provided by the full theorem as it appears here. The terminology of weak and strong witnesses was introduced in the latter paper.

2. Proof of Theorem 1

The proof of Theorem 1 is an inductive generalisation of the proof in [MS94] for the case of an edge. We strengthen the statement of the theorem, in order to facilitate the induction.

Theorem 2. *Let $A \subset \mathbb{R}^n$ and let $\sigma = [a_0 a_1 \dots a_p]$ be a simplex with vertices in A. For every nonempty $I \subseteq \{a_0, a_1, \dots, a_p\}$, suppose there exists a weak witness x_I for the subsimplex*

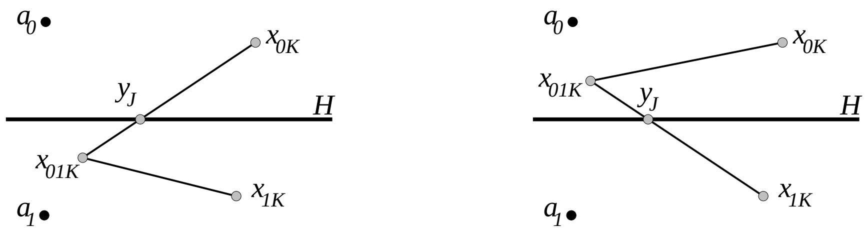


FIGURE 3. Finding y_J: the two possibilities in the case $a_1 \in J$.

$\sigma_I \le \sigma$ with vertex set I. Then the convex hull of the points x_I contains a strong witness for σ.

Proof. The proof proceeds by induction on p. The base case $p = 0$ is trivial. Now suppose $p \ge 1$. Consider the subspace $H = \{x \in \mathbb{R}^n : |x - a_0| = |x - a_1|\}$ and the $(p-1)$-simplex $\tau = [a_1 a_2 \dots a_p]$. The following statements are easily seen to be equivalent:

$$\begin{aligned} & x \in H \text{ is a strong witness for } \sigma \text{ with respect to } A \\ \Leftrightarrow\ & x \in H \text{ is a strong witness for } \tau \text{ with respect to } A \\ \Leftrightarrow\ & x \in H \text{ is a strong witness for } \tau \text{ with respect to } A' = A \setminus \{a_0\} \end{aligned}$$

For every nonempty $J \subseteq \{a_1, \dots, a_p\}$, we will find a weak witness $y_J \in H$ with respect to A' for the subsimplex $\tau_J \le \tau$ having vertex set J. By the inductive hypothesis, this implies the existence of a strong witness x for τ with respect to A', in the convex hull of the points y_I, and hence in H. The weak witnesses y_J are constructed as convex combinations of the points x_I, so x itself belongs to the convex hull of the points x_I. This will complete the inductive step. It remains to locate the weak witnesses y_J.

The first case is $a_1 \in J$. Write $J = \{a_1\} \cup K$ where $K \subseteq \{a_2, \dots, a_p\}$, and consider weak witnesses x_{0K}, x_{1K} and x_{01K} for the simplices with vertex sets $\{a_0\} \cup K$, $\{a_1\} \cup K$ and $\{a_0, a_1\} \cup K$ respectively. Suppose $|x_{01K} - a_1| \ge |x_{01K} - a_0|$. Since $|x_{1K} - a_1| \le |x_{1K} - a_0|$ it follows by continuity that there is some point y in the closed line segment $\ell = [x_{01K}, x_{1K}]$ at which $|y - a_1| = |y - a_0|$. Moreover, for every $a \in J$ and $b \in A' \setminus J$, the convex inequality $|x - a| \le |x - b|$ is valid at the endpoints of ℓ and hence throughout ℓ. It follows that $y_J = y$ is the required weak witness for τ_J. If instead $|x_{01K} - a_1| \le |x_{01K} - a_0|$, there is a similar argument with the roles of a_0 and a_1 interchanged, giving y_J in $[x_{01K}, x_{0K}] \cap H$. The two possibilities are illustrated in Figure 3.

The second case is $a_1 \notin J$, so $\emptyset \ne J \subseteq \{a_2, \dots, a_p\}$. Consider weak witnesses x_J, x_{0J} and x_{1J} for the simplices with vertex sets J, $\{a_0\} \cup J$ and $\{a_1\} \cup J$ respectively. Suppose $|x_J - a_0| \ge |x_J - a_1|$. Since $|x_{0J} - a_0| \le |x_{0J} - a_1|$, there is some point y in $\ell = [x_J, x_{0J}]$ at which $|y - a_0| = |y - a_1|$. Moreover, for every $a \in J$ and $b \in A' \setminus J$, the inequality $|x - a| \le |x - b|$ is valid throughout ℓ. Thus the required weak witness is $y_J = y$. If instead $|x_J - a_0| \le |x_J - a_1|$, we can interchange the roles of a_0 and a_1 to find y_J in $[x_J, x_{1J}] \cap H$.

This completes the proof of Theorem 2 and hence of Theorem 1. □

3. VARIATIONS

The proof of Theorem 2 is quite general. Here is a set of sufficient requirements, in the form of axioms.

(1) Let R be a set with a notion of convexity; so every unordered pair $x, y \in R$ determines a subset $[x, y] \subset R$ of intermediate elements.
(2) Let A be a set. For every ordered pair (a, b) in A, there is a *convex* subset $R_{ab} \subset R$, interpreted as "the set of points whose distance from a is no greater than their distance from b". Convex here means closed under taking intermediate elements.
(3) Define $H_{ab} = R_{ab} \cap R_{ba}$. If $x \in R_{ab}$ and $y \in R_{ba}$ then we require that $[x, y] \cap H_{ab} \neq \emptyset$. This rule stands in for the intermediate value theorem.
(4) For all $a, b, c \in A$, we require that $H_{ab} \cap R_{ac} = H_{ab} \cap R_{bc}$ and $H_{ab} \cap R_{ca} = H_{ab} \cap R_{cb}$. In other words, a and b can be treated alike on the subspace of points equidistant from them.

One can define weak and strong witnesses in terms of the sets R_{ab}, and formulate analogous versions of Theorems 1 and 2. The proof goes through unchanged. Here are three important special cases.

- Let $R = \mathcal{H}^n$, hyperbolic n-space, and define $R_{ab} = \{x \in \mathcal{H}^n : d(x, a) \leq d(x, b)\}$ using the hyperbolic metric.
- Let $R = S^n_+$, the open hemisphere in Euclidean $n+1$-space, defined by
$$S^n_+ = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 = 1, x_1 > 0\}$$
and set $R_{ab} = \{x \in S^n_+ : d(x, a) \leq d(x, b)\}$ using the geodesic metric.
- Let $R = \mathbb{R}^n$ and let $a \mapsto w_a$ be a real-valued function on A. Define $R_{ab} = \{x \in \mathbb{R}^n : |x - a|^2 + w_a \leq |x - b|^2 + w_b\}$.

In each case, $[x, y]$ is the set of points on the unique geodesic arc from x to y. The remaining axioms are easily verified; in particular the sets R_{ab} are convex. It follows that weak witnesses can be used to define the Delaunay triangulation in hyperbolic space and the Euclidean hemisphere; as well as Edelsbrunner's weighted Delaunay triangulations [Ede95] in Euclidean space. In the latter case, it is not necessarily true that there is a weak witness for every 0-dimensional simplex $[a]$, so one must not fail to check this. In the other cases, a itself is always a witness for $[a]$.

Remark. A stronger version of Axiom 4 is *transitivity*: $R_{ab} \cap R_{bc} \subset R_{ac}$. This holds in all our examples.

4. The Euclidean sphere

In this section and the next, we study the case of the Euclidean sphere S^n, with weak and strong witnesses defined in terms of the intrinsic geodesic metric. The direct analogue of Theorem 1 is false, so we must weaken the claim. We begin by reformulating Theorem 1 itself.

Theorem 3. *Let $A \subset \mathbb{R}^n$ and let $\sigma = [a_0 a_1 \ldots a_p]$ be a simplex with vertices in A. Suppose that for every subsimplex $\tau \leq \sigma$ there exists a closed Euclidean ball B_τ, such that B_τ contains the vertices of τ and the interior of B_τ contains no other points of A. Then there exists a closed Euclidean ball B, such that the boundary of B contains the vertices of σ and the interior of B is disjoint from A.*

Proof. The existence of B_τ is equivalent to the existence of a weak witness x_τ for τ: take x_τ to be the center of B_τ, or conversely take B_τ to be the smallest ball centered on x_τ which contains all the vertices of τ. Similarly, the existence of B is equivalent to the existence of a strong witness for σ. In this way Theorem 3 is equivalent to Theorem 1. $\square$

For the sphere S^n we have the following result.

Theorem 4. *Let $A \subset S^n$ and let $\sigma = [a_0 a_1 \ldots a_p]$ be a simplex with vertices in A. Suppose that for every subsimplex $\tau \leq \sigma$ there exists a closed metric ball B_τ, such that B_τ contains the vertices of τ and the interior of B_τ contains no other points of A. Suppose further that the union of the balls $\{B_\tau : \tau \leq \sigma\}$ is not equal to the whole sphere S^n. Then there exists a closed metric ball B, such that the boundary of B contains the vertices of σ and the interior of B is disjoint from A.*

Proof. Let $s \in S^n \setminus \bigcup\{B_\tau : \tau \leq \sigma\}$. Regard s as a vector in $\mathbb{R}^{n+1}$ and let $V \subset \mathbb{R}^{n+1}$ be a hyperplane perpendicular to s. It is well known that the stereographic projection $S^n \setminus \{s\} \to V$ maps closed metric balls in S^n which are disjoint from s onto closed Euclidean balls in V; and that this is a bijective correspondence. Under this correspondence, Theorem 4 is equivalent to Theorem 3. $\square$

Example. Theorem 4 is false without the extra condition. Consider the poles $n, s \in S^2$. Let $a_1, a_2, \ldots, a_p$ belong to a circle of latitude C, and let a_0 be any point lying north of C. Every simplex $\tau \leq \sigma = [a_0 a_1 \ldots a_p]$ then has a weak witness: if $a_0 \in \tau$ then n is a weak witness for τ, and if $a_0 \notin \tau$ then s is a weak (and indeed a strong) witness for τ. The corresponding disks are the northern and southern polar caps bounded by C; these cover S^2 between them. If $p \geq 3$ then σ has no strong witness.

The proof of Theorem 2 breaks down because of the failure of convexity on S^n. There is no unique shortest geodesic connecting a pair of antipodal points on S^n. More seriously the 0-sphere S^0 is disconnected, so there is not even one geodesic on which to find the intermediate point required by Axiom 3. The inductive argument works by reducing the problem to subspheres of successively lower dimension, so the S^0 case is unavoidable when $p > n$. On the other hand, for subspheres S^k with $k \geq 1$, the failure of convexity is non-generic, since it is only a problem when there is an antipodal pair of witnesses at some stage in the proof. We will develop this idea in Section 5.

Corollary 5. *Suppose $A = \{a_0, a_1, \ldots, a_{n+1}\} \subset S^n$ is not contained in an $(n-1)$-sphere of any radius. For every $J \subset A$ suppose there is a closed metric ball B_J containing J and whose interior is disjoint from $A \setminus J$. Then the balls B_J cover the sphere S^n.* $\square$

Example. For $0 < k < n$, identify $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ with the tangent space of $S^n \subset \mathbb{R}^{n+1}$ at the north pole. For $1 \gg \epsilon > \epsilon' > 0$, let $\{a_0, a_1, \ldots, a_k\}$ be the vertices of a regular k-simplex on the sphere of radius ϵ in $\mathbb{R}^k$, and let $\{b_0, b_1, \ldots, b_{n-k}\}$ be the vertices of a regular $(n-k)$-simplex on the sphere of radius ϵ' in $\mathbb{R}^{n-k}$, projected onto a neighbourhood of the north pole in S^n. If ϵ' is sufficiently close to ϵ, it can be checked that every simplex $\tau \leq \sigma = [a_0 \ldots a_k b_0 \ldots b_{n-k}]$ has a small witness disk B_τ contained in a neighbourhood of the north pole, *except* for $\tau_a = [a_0 a_1 \ldots a_k]$. Since σ does not have a strong witness, it follows that every witness disk for the k-simplex τ_a is necessarily large, and in particular contains the entire southern hemisphere.

5. The Euclidean sphere, continued

We will prove the following theorem.

Theorem 6. *For a generic finite subset $A \subset S^n$, and $p \le n$, a simplex $\sigma = [a_0 a_1 \dots a_p]$ with vertices in A has a strong witness if and only if every subsimplex $\tau \le \sigma$ has a weak witness. Here "generic" means that, for all $k < n$, no set of $k+3$ points in A is contained in a k-sphere (of any radius).*

Thus if $A \subset S^n$ is generic we can use weak witnesses to define its Delaunay triangulation in S^n correctly as far as the n-skeleton. On the other hand, the true Delaunay triangulation is at most n-dimensional if A is generic: the existence of an $(n+1)$-cell would imply that its $n+2$ vertices lay on a common $(n-1)$-sphere. The conclusion is that the Delaunay triangulation in S^n of a generic set A is equal to the n-skeleton of the complex defined using weak witnesses. Since a randomly-chosen $A \subset S^n$ is generic with probability 1, this is a satisfactory result.

Definition. Let $A \subset S^n$ (not necessarily finite), and let $\sigma = [a_0 a_1 \dots a_p]$ be a p-simplex with vertices in A. We say that $x \in S^n$ is a *robust witness* for σ with respect to A if x has a neighbourhood in which every point is a weak witness for σ with respect to A.

The next proposition may clarify the situation.

Proposition 7. *For $n \ge 1$, let $A \subset S^n$ be a finite set and let $\sigma = [a_0 a_1 \dots a_p]$ be a p-simplex with vertices in A. Then x is a robust witness for σ if and only if there is strict inequality $d(x,a) < d(x,b)$ whenever $a \in \{a_0, a_1, \dots, a_p\}$ and $b \in A \setminus \{a_0, a_1, \dots, a_p\}$.*

Proof. Assume first that x is a robust witness, and $a \in \{a_0, a_1, \dots, a_p\}$ and $b \in A \setminus \{a_0, a_1, \dots, a_p\}$. Suppose $d(x,a) = d(x,b) < \pi$. (The possibility $d(x,a) = d(x,b) = \pi$ is ruled out since x has only one antipode.) Let $y \ne x, b$ be any point on the minimal geodesic $[x,b]$. Then

$$d(y,b) = d(x,b) - d(x,y)$$

but

$$d(y,a) < d(x,a) - d(x,y)$$

since y is not on the minimal geodesic $[x,a]$. Thus $d(y,b) < d(y,a)$ for points y which can be arbitrarily close to x. Since this would contradict the robustness of x it must be that $d(x,a) < d(x,b)$ in all cases.

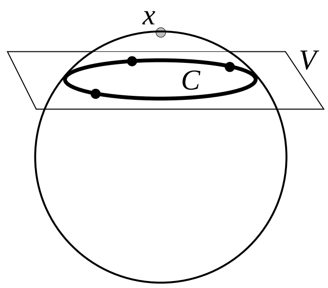
In the converse direction, note that the inequalities $d(x,a) < d(x,b)$ are open conditions and there are finitely many of them. If all of the appropriate inequalities are satisfied at x (making x a weak witness), then they are also satisfied on an open neighbourhood of x. Thus x is a robust witness. $\square$

Theorem 8. *Let $A \subset S^n$ (not necessarily finite), let $p \le n$, and let $\sigma = [a_0 a_1 \dots a_p]$ be a p-simplex with vertices in A. If every subsimplex $\tau \le \sigma$ has a robust witness, then σ has a strong witness.*

Proof. For $k = 0, 1, \dots, p$, let $\sigma_k = [a_k a_{k+1} \dots a_p]$, let $A_k = A \setminus \{a_i : i < k\}$ and let H_k be the great sphere defined by the equations:

$$d(x, a_0) = d(x, a_1) = \dots = d(x, a_k)$$

Note that $\dim(H_k) \ge n - k$. We will show inductively that every subsimplex $\tau \le \sigma_k$ has a robust witness in H_k with respect to A_k. The case $k = 0$ is our hypothesis, and the case



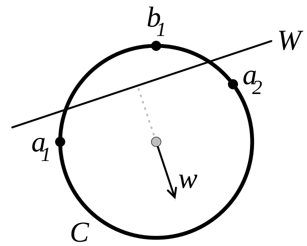


FIGURE 4. [left] The n-plane V determined by the sphere C; [right] the separating hyperplane W determines a suitable direction for perturbation.

$k = p$ implies the conclusion, since a witness $x \in H_p$ (robust or otherwise) for $[a_p]$ with respect to A_p is automatically a strong witness for σ.

Suppose the assertion is proved for some $k < p$. For each $\tau \leq \sigma_k$ we have a robust witness $x_\tau \in H_k$ with respect to A_k. We may assume that no two of the x_τ are antipodal; since $\dim(H_k) \geq 1$ we can perturb the points x_τ to make this true. We now apply the argument in the proof of Theorem 2 to find weak witnesses $y_\tau \in H_{k+1}$ for $\tau \leq \sigma_{k+1}$ with respect to A_{k+1}. If an inequality $d(x,a) \leq d(x,b)$ is satisfied on neighbourhoods of points x_0, x_1 which are not antipodal, then it is satisfied on a neighbourhood of the minimal geodesic interval $[x_0, x_1]$. Thus the new witnesses y_τ are themselves robust. This completes the inductive step. $\square$

If A is generic we can manufacture robust witnesses from weak witnesses.

Lemma 9. *Suppose $A \subset S^n$ is a finite set satisfying the condition that, for all $k < n$, no set of $k+3$ points in A is contained in a k-sphere. Let σ be a simplex with vertices in A. If σ has a weak witness, then σ has a robust witness.*

Proof. Let x be a weak witness for σ. If x is not robust, then there are points $a_1, a_2, \ldots, a_k$ which are vertices of σ, and $b_1, b_2, \ldots, b_l \in A$ which are not, such that

$$d(x, a_1) = \cdots = d(x, a_k) = d(x, b_1) = \cdots = d(x, b_l),$$

so all these points lie on a $(n-1)$-sphere C centered on x. The other vertices of σ may be assumed to lie inside C, and the remaining points of A outside C. By the condition, $k + l < n + 2$.

Regarding S^n as a subspace of $\mathbb{R}^{n+1}$, consider the n-plane V defined by $V \cap S^n = C$. This can naturally be identified with the tangent space of S^n at x. We will construct a vector $w \in V$, and show that, for a small perturbation $x' = x + \epsilon w + O(\epsilon^2)$, there are strict inequalities $d(x', a_i) < d(x', b_j)$ for all i, j. When ϵ is small enough, the relevant inequalities for the remaining points of A remain unchanged, so x' will be the required robust witness.

Inside V, we claim that the convex hull of $\{a_1, \ldots, a_k, b_1, \ldots, b_l\}$ is an embedded $(k+l-1)$-simplex. If this were false, then the $k+l$ points would lie on a $(k+l-2)$-plane in $\mathbb{R}^n$ and hence on a $(k+l-3)$-sphere in S^n, contradicting the hypothesis. It follows that the convex hulls of $\{a_1, a_2, \cdots, a_k\}$ and $\{b_1, b_2, \ldots, b_l\}$ are disjoint, so they can be strictly separated by an $(n-1)$-plane W. Let w be the unit normal vector of $W < V$, pointing towards the half-space containing the points a_i. See Figure 4.

We now determine the effect of the perturbation $x' = x + \epsilon w + O(\epsilon^2)$ along S^n, on the distances. It is simplest to compute the derivative of the squared Euclidean distance function

$f_a : \epsilon \mapsto |x+\epsilon w-a|^2$, namely $f_a'(0) = -2\langle w, a\rangle$ (since $\langle w, x\rangle = 0$). Since w was chosen as the normal to a separating hyperplane, we deduce the inequality $f_{a_i}'(0)-f_{b_j}'(0) = 2\langle w, b_j-a_i\rangle < 0$ for all i, j. Thus for small ϵ we have $|x'-a_i|^2 < |x'-b_j|^2$ and hence $d(x', a_i) < d(x', b_j)$ since there is a monotonic relationship between the two distance functions. This completes the proof. ☐

Proof of Theorem 6. We have the following sequence of implications:

$$\begin{aligned} \sigma \text{ has a strong witness} \quad &\Rightarrow \quad \text{every } \tau \leq \sigma \text{ has a weak witness} \\ [\text{by Lemma 9, generically}] \quad &\Rightarrow \quad \text{every } \tau \leq \sigma \text{ has a robust witness} \\ [\text{by Theorem 8, since } p \leq n] \quad &\Rightarrow \quad \sigma \text{ has a strong witness} \end{aligned}$$

This proves the theorem. ☐

6. Application

It is a standard problem in computational geometry to recover the topology of a manifold $M \subset \mathbb{R}^n$ from a discrete sample of points $A \subset M$. Recovery is often taken to mean the construction of a simplical complex with vertices in A, which is homeomorphic to the unknown space M. This problem has a rich literature.

When M is regarded as known, it is natural to consider is the restricted Delaunay triangulation $\mathrm{Del}(A, M)$, which is dual to the partitioning of M by Voronoi cells. Unlike the full Delaunay complex $\mathrm{Del}(A)$, which is always contractible, $\mathrm{Del}(A, M)$ carries information about the topology of M and may even be homeomorphic to it, if A is sampled sufficiently finely. According to the traditional definition, a simplex σ with vertices in A belongs to $\mathrm{Del}(A, M)$ if there is a strong witness for σ on M.

When M is not known, one may attempt to approximate $\mathrm{Del}(A, M)$. This approach, introduced in [MS94] for approximation by graphs, is extended to higher dimensional complexes in [CdS03]. The idea is to select a subsample $L \subset A$ to serve as the vertex set, and to let the remaining points stand in for the unknown manifold M. Unfortunately, it is useless to define $\mathrm{Del}(L, A)$ by mimicking the traditional definition of $\mathrm{Del}(L, M)$, since strong witnesses in A exist with probablility 0. The solution is to replace definitions involving strong witnesses with definitions involving weak witnesses, which do exist with nonzero probability.

The goal of this paper has been to establish the credentials of weak witness definitions, proving equivalence in the archetypal cases of Delaunay triangulation in Euclidean, hyperbolic and spherical geometries, and paying attention to some of the pitfalls. All this may be regarded as groundwork for the following question: under what conditions is the weak-witness complex $\mathrm{Del}_w(L, A)$ homeomorphic (or homotopy equivalent) to M?

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Department of Mathematics, Building 380, Stanford University, CA 94305-2125, USA.
E-mail: silva@math.stanford.edu