INF562 – Géométrie Algorithmique et Applications

Curve and surface reconstruction

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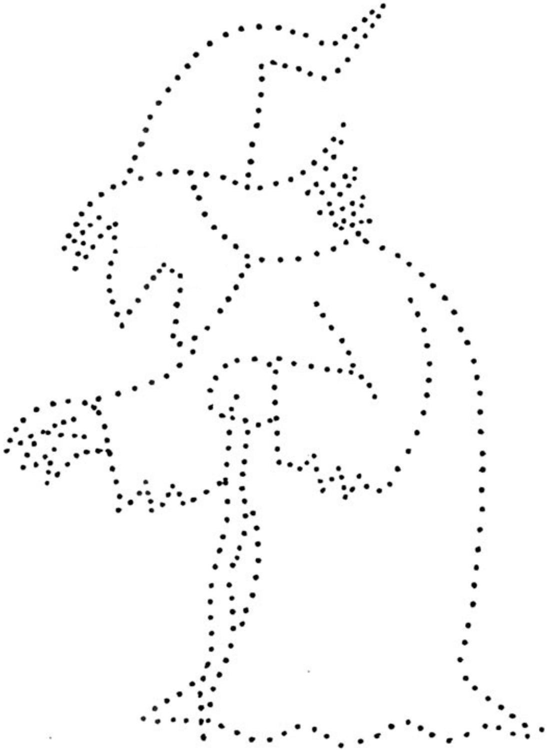
Reconstruction Paradigm

Q What do you see?

Why?



http://www.pointapoint.com

Reconstruction Paradigm

Q What do you see?

Why?

http://jolicoloriage.free.fr

Reconstruction Paradigm

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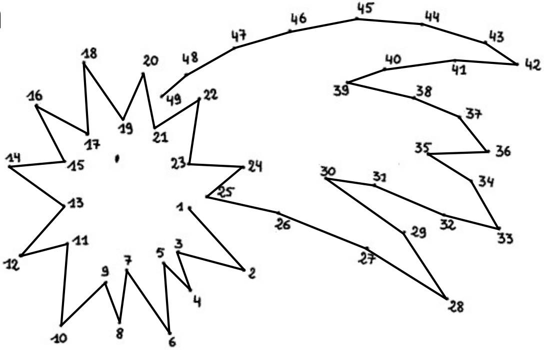


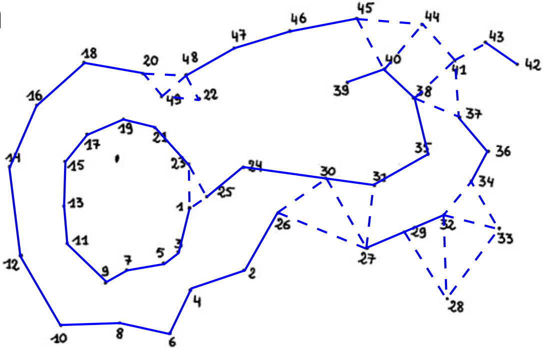
http://jolicoloriage.free.fr

Reconstruction Paradigm

Q What do you see?

Why?

without the numbers...



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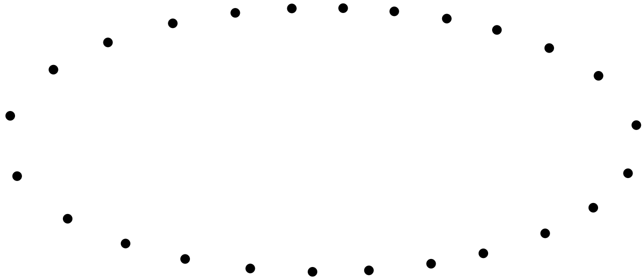
Reconstruction Paradigm (Cont'd)



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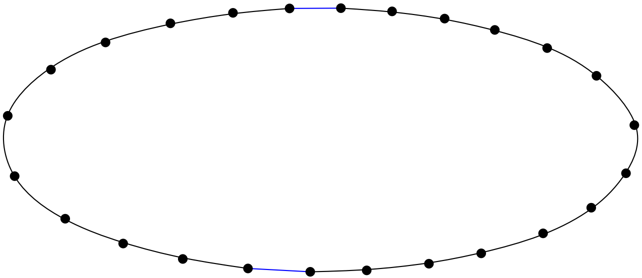
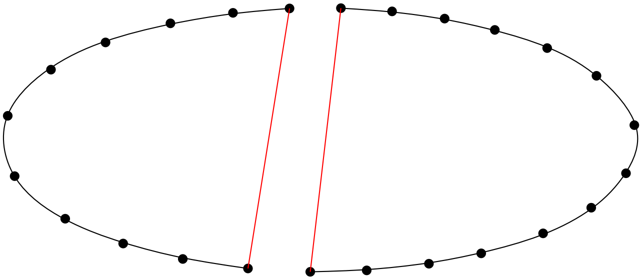


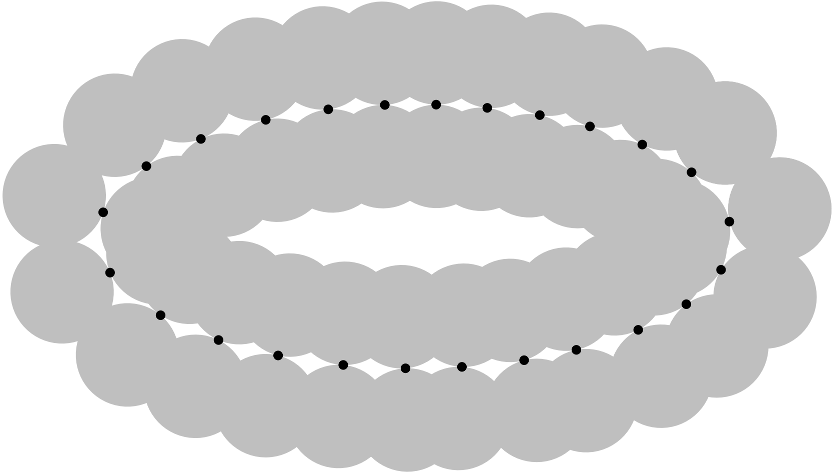
Q Given a point cloud, build a *faithful* (implicit, PL, ...) approximation of the shape underlying the data.

Reconstruction Paradigm (Cont'd)



Reconstruction problem is ill-posed by nature.

Reconstruction Paradigm (Cont'd)



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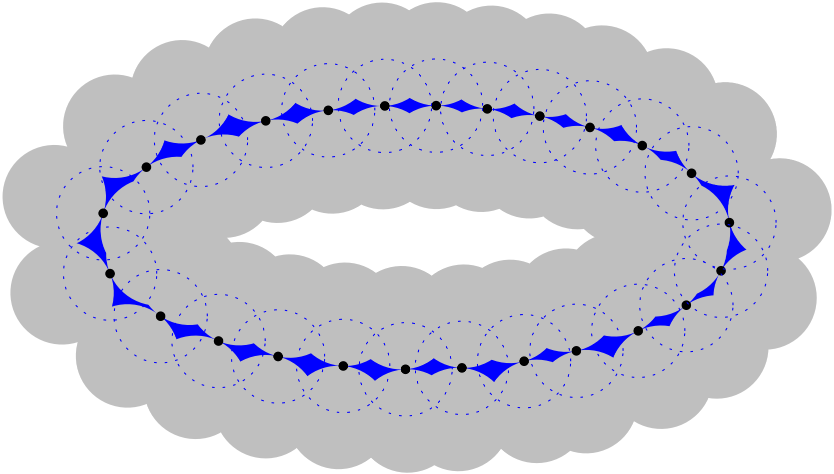
→ make assumptions on the underlying shape, *e.g.*: fix dimension, topological type, regularity (differentiability), Hausdorff distance to input...

Reconstruction Paradigm (Cont'd)



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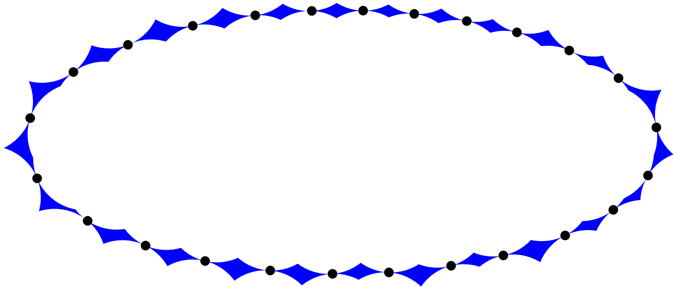
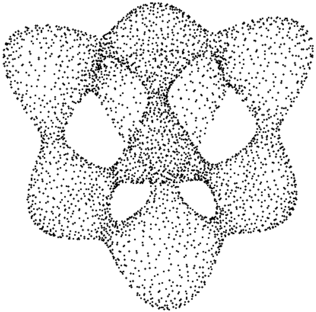


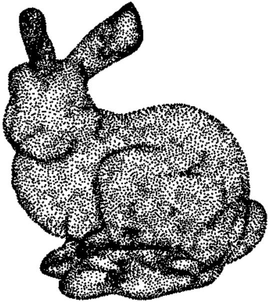
Reconstruction problem is ill-posed by nature.

→ make assumptions on the underlying shape, *e.g.*: fix dimension, topological type, regularity (differentiability), Hausdorff distance to input...

→ for a suitable choice of hypotheses, the solution becomes unique **up to a set of local regular deformations** (solution never unique!)

Other (weaker) forms of reconstruction





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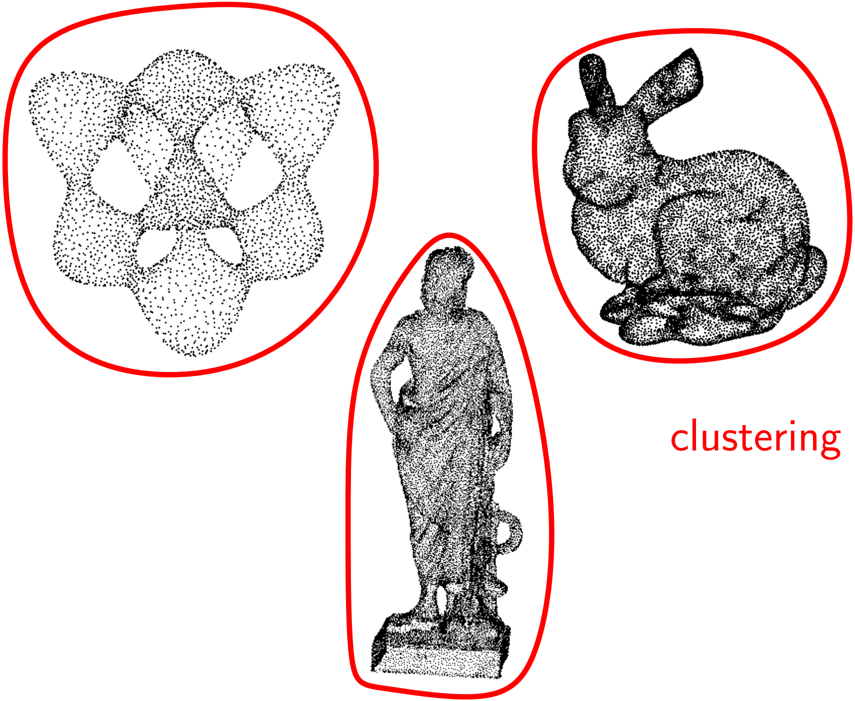
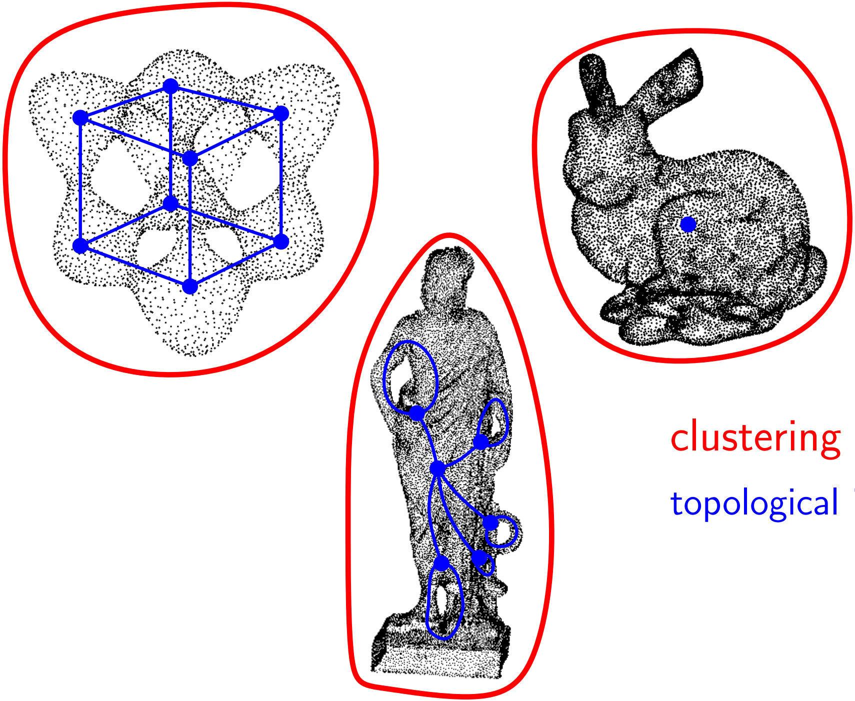


clustering

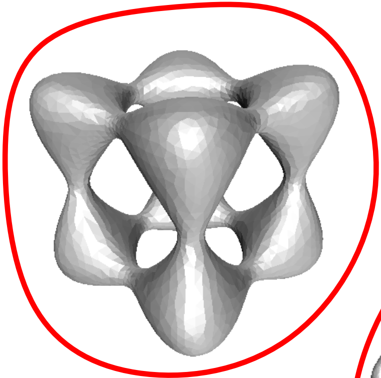
Other (weaker) forms of reconstruction

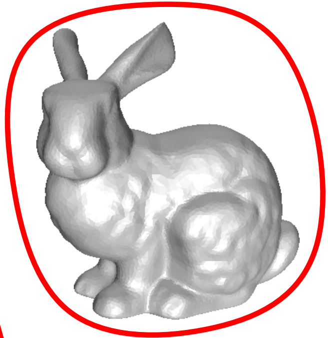


clustering

topological inference

Other (weaker) forms of reconstruction





clustering

topological inference

reconstruction

Where do the data come from?

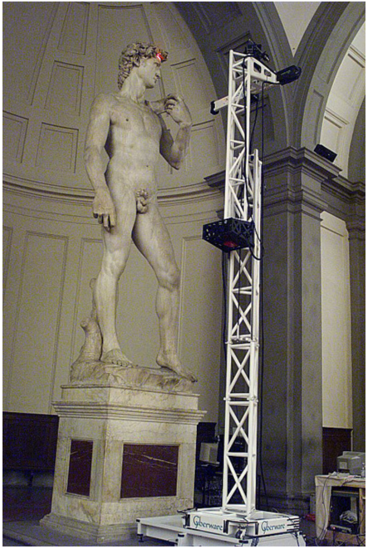
3D scans

Sources

- LASER
- stereo vision
- mechanical sensor

Applications

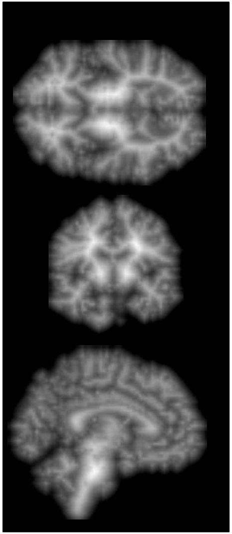
- Reverse engineering
- Prototyping
- Quality control



Stanford Michelangelo Project

Where does the data come from?

Medical Imaging



Sources

- MRI scan
- echograph
- ...

Applications

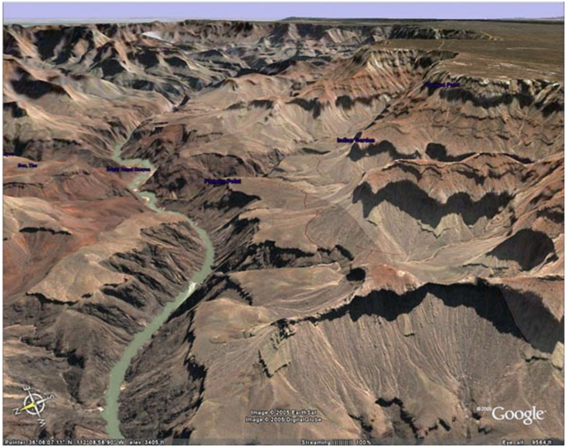
- Diagnostic
- Endoscopy simulation
- Chirurgical intervention planning

Where does the data come from?

Geography, Geology

Sources

- satellite/aerial images
- ground probing
- seismograph



Applications

- Maps making / Terrain modeling
- Prospection (tunnels, oil)

Where does the data come from?

Higher-Dimensions

Sources

- Data bases
- Simulations

Applications

- Machine Learning
- Path planning
- Pattern recognition
- Image processing
- ...

$\mathcal{S}^7$

Various reconstruction techniques

Delaunay-based

- Crust / Power Crust
- Cocone
- Gabriel / α-shape / β-skeleton
- flow complex

Implicitization

- Local polynomial fitting
- Natural Neighbors (Voronoi-based)
- Radial Basis Functions

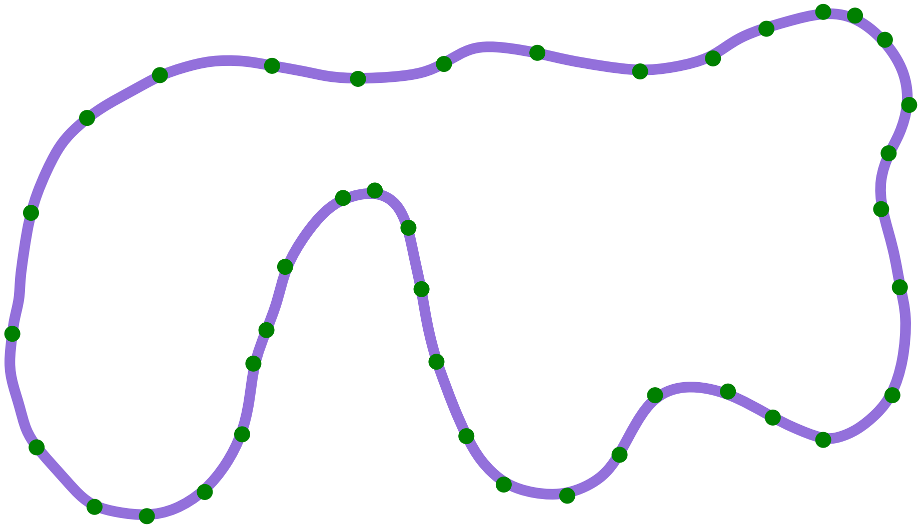
Projection operators

- Moving Least Squares
- Extremal surfaces

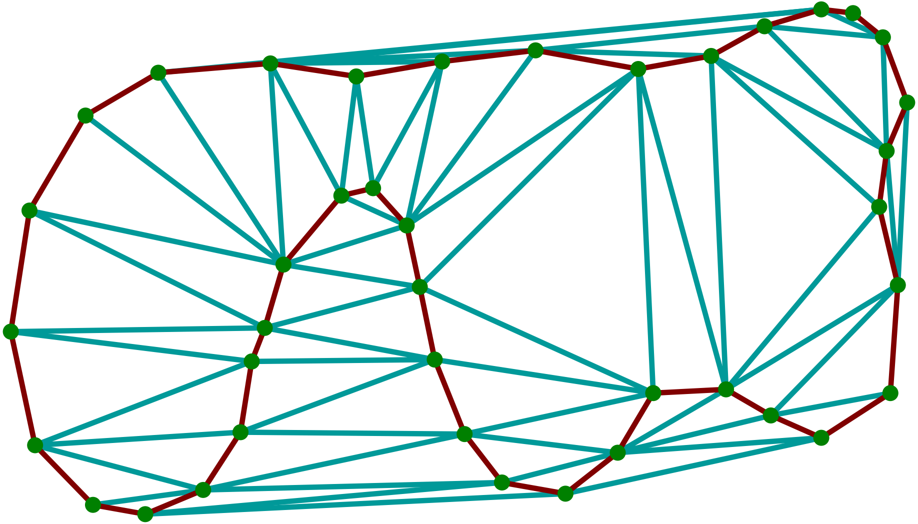
For arbitrary dimensions and co-dimensions

- Unions of balls / nerves
- Witness Complex

What Delaunay has to do with reconstruction



What Delaunay has to do with reconstruction



→ a faithful approximation of the curve appears as a subcomplex of the Delaunay

→ this should hold whenever the point cloud is sufficiently densely sampled along the curve

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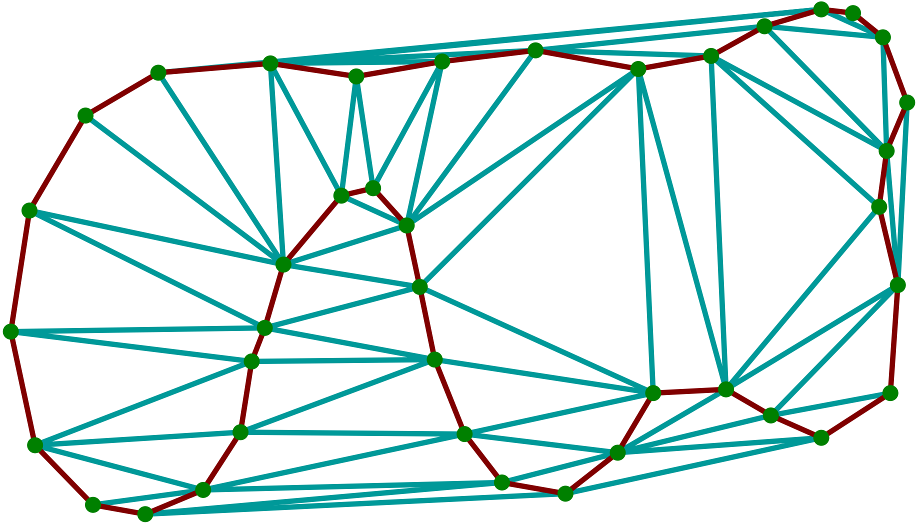
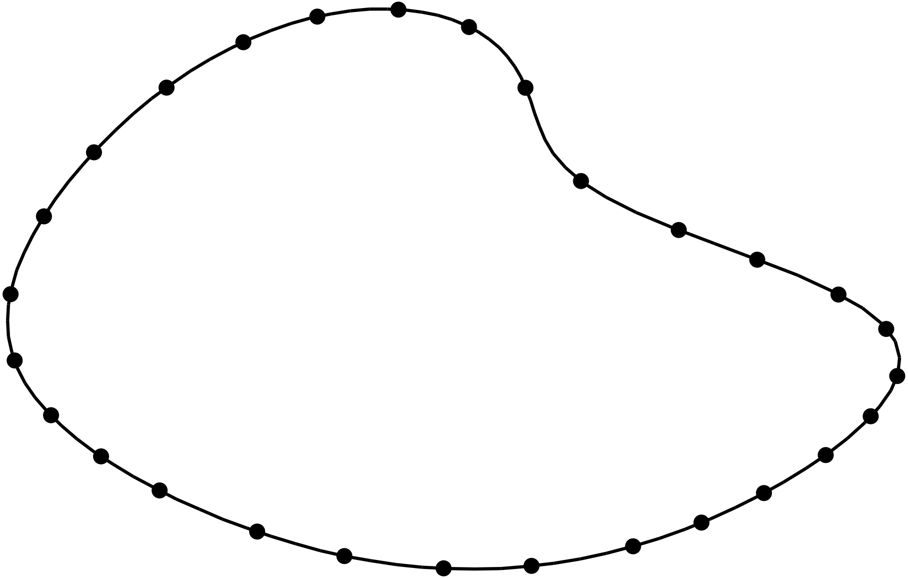


→ a faithful approximation of the curve appears as a subcomplex of the Delaunay

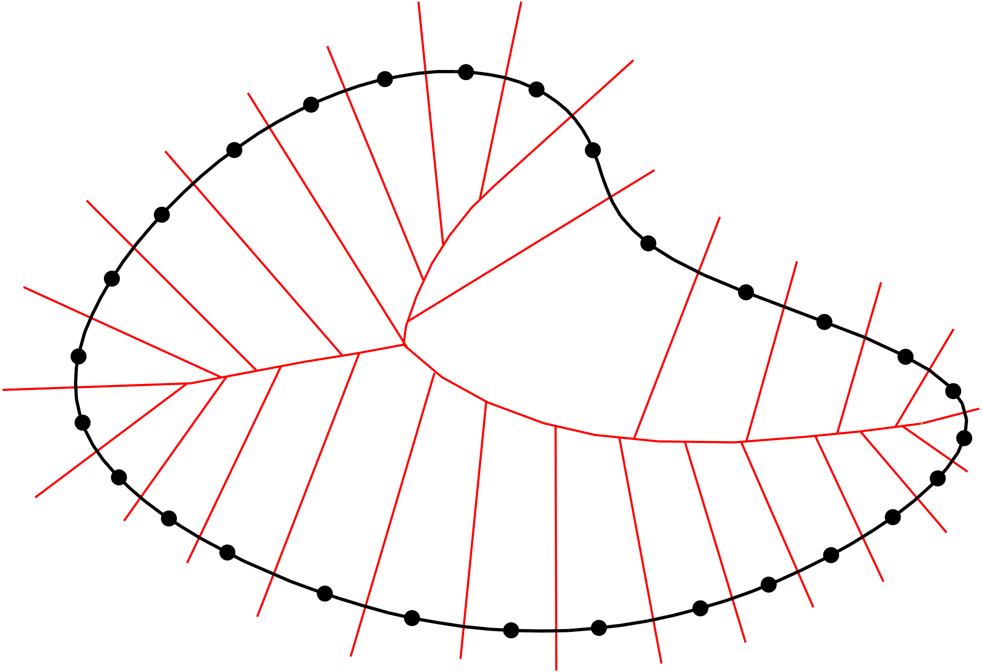
→ this should hold whenever the point cloud is sufficiently densely sampled along the curve

Q What is this *good* subcomplex? Can it be defined in some canonical way?

Restricted Delaunay triangulation

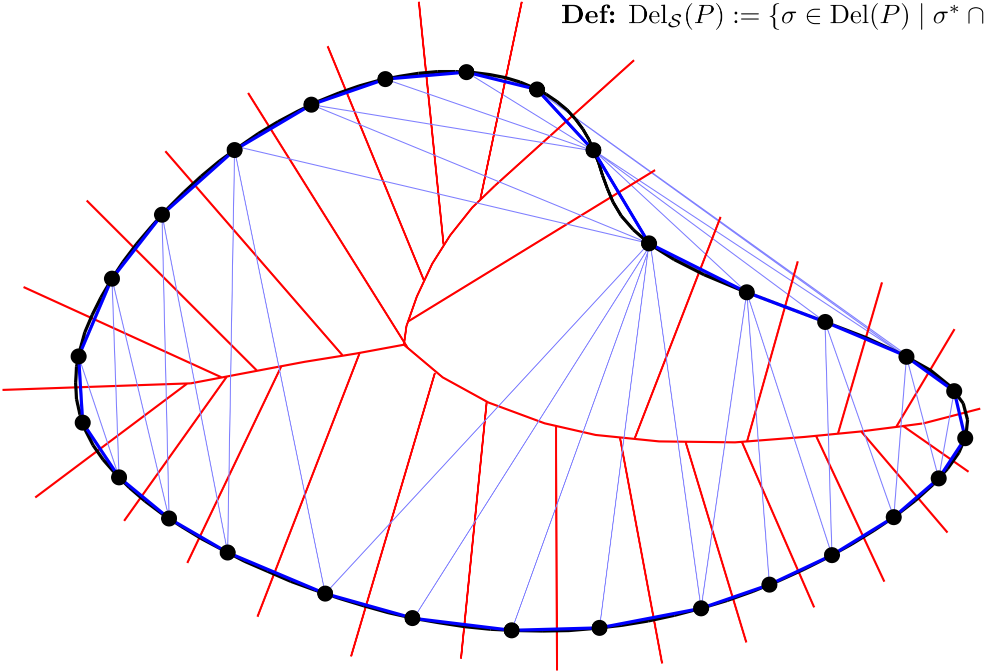


Restricted Delaunay triangulation



Restricted Delaunay triangulation

Def: $\mathrm{Del}_{\mathcal{S}}(P) := \{\sigma \in \mathrm{Del}(P) \mid \sigma^* \cap \mathcal{S} \neq \emptyset\}$



Approximation power of the restricted Delaunay

$\rightarrow$ Our assumptions:

1. the underlying shape $\mathcal{S}$ is a closed curve or surface with positive *reach* $\varrho_{\mathcal{S}}$

2. the point cloud P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon \in O(\varrho_{\mathcal{S}})$.

Approximation power of the restricted Delaunay

$\rightarrow$ Our assumptions:

$\rightarrow$ analogy with 1-d signal theory (Shannon's reconstruction theorem):

1. the underlying shape $\mathcal{S}$ is a closed curve or surface with positive *reach* $\varrho_{\mathcal{S}}$

1'. the underlying signal is a weighted sum of sinusoids

2. the point cloud P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon \in O(\varrho_{\mathcal{S}})$.

2'. the sampling has ≥ 2 samples per period (signal has bounded bandwidth)

Approximation power of the restricted Delaunay

Theorem: [Amenta et al. 1998-99]
If $\mathcal{S}$ is a curve or surface with positive *reach*, and if P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon < \varrho_{\mathcal{S}}$ (curve) or $\varepsilon < 0.1\varrho_{\mathcal{S}}$ (surface), then:

- $\mathrm{Del}_{\mathcal{S}}(P)$ is homeomorphic to $\mathcal{S}$,
- $\mathrm{d_H}(\mathrm{Del}_{\mathcal{S}}(P), \mathcal{S}) \in O(\varepsilon^2)$,
- $\forall f \in \mathrm{Del}_{\mathcal{S}}(P)$, $\forall v \in f$, $\angle n_f n_v \mathcal{S} \in O(\varepsilon)$,
- $\cdots$ (similar areas, curvature estimation, etc.)

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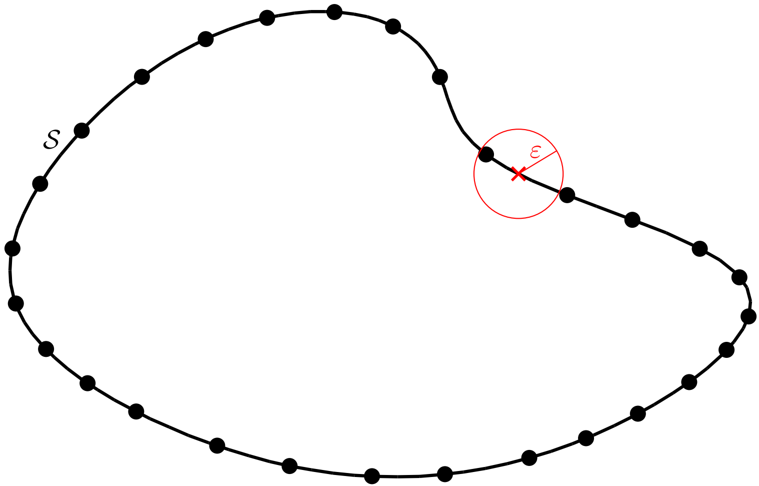
- $\cdots$ (similar areas, curvature estimation, etc.)

$\rightarrow$ to be explicited: ε-sampling, reach

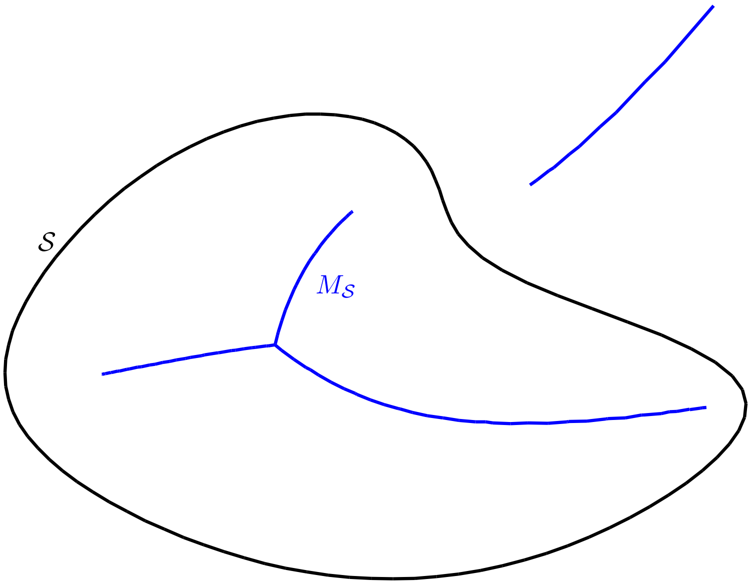
ε-samples

Def: P is an ε-sample of $\mathcal{S}$ if $\forall x \in \mathcal{S}$, $\min\{\|x - p\| \mid p \in P\} \leq \varepsilon$.



Shapes with positive reach [Federer 1958]

Def: $M_{\mathcal{S}}$ is the closure of the set of points of $\mathbb{R}^d$ that have ≥ 2 nearest neighbors on $\mathcal{S}$.



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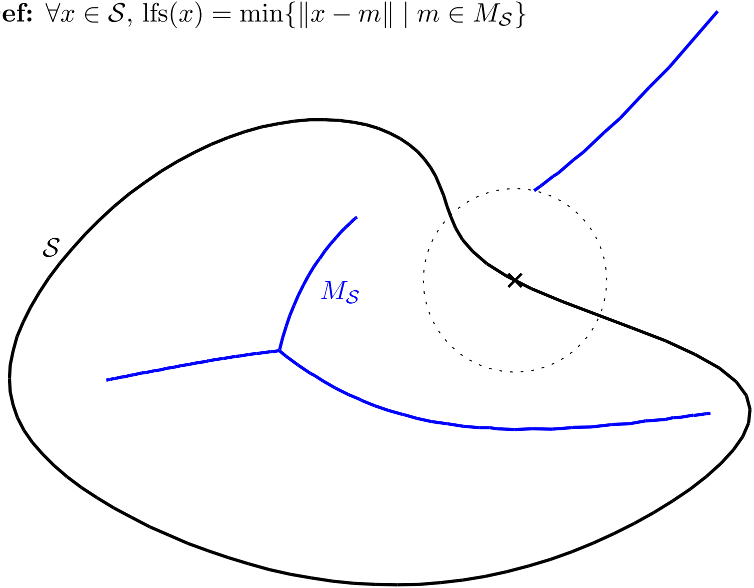

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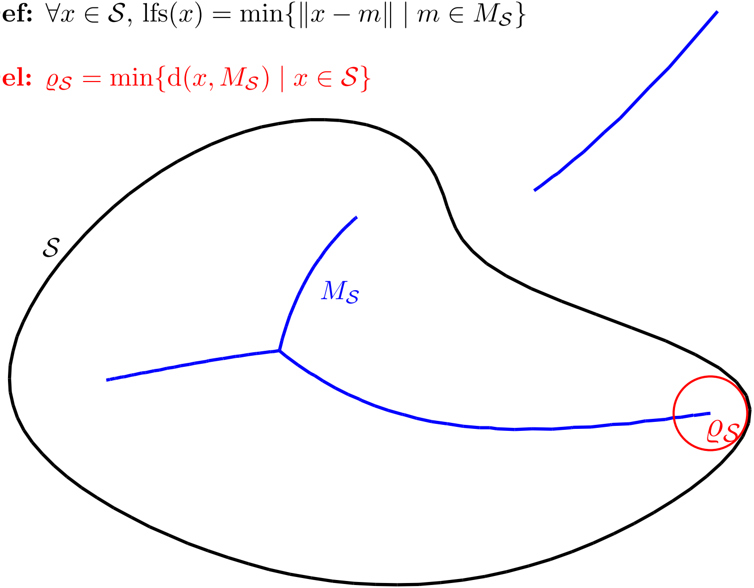
Del: $\varrho_{\mathcal{S}} = \min\{\mathrm{d}(x, M_{\mathcal{S}}) \mid x \in \mathcal{S}\}$



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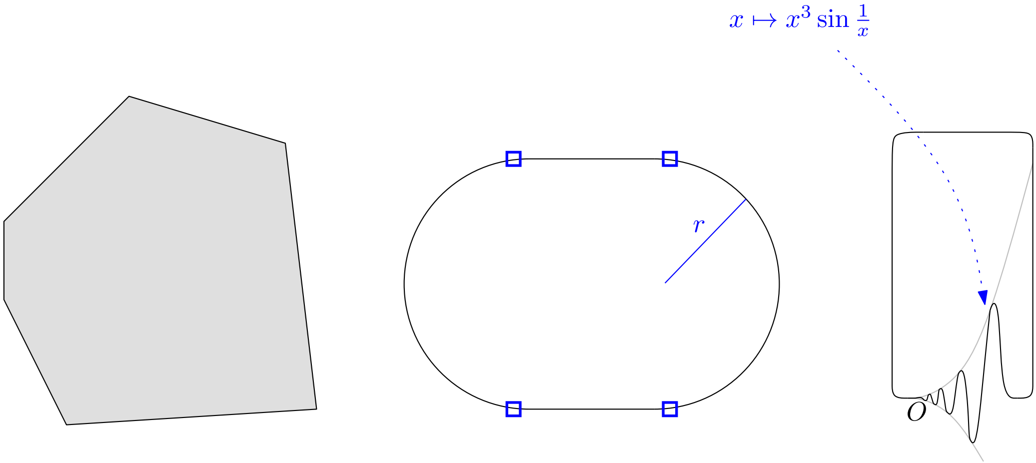

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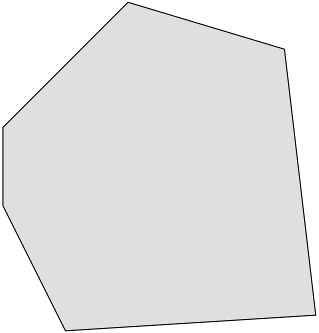
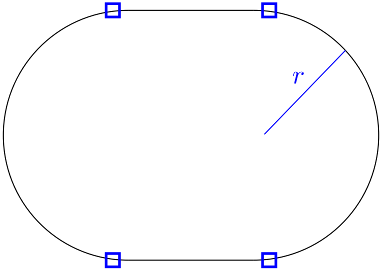


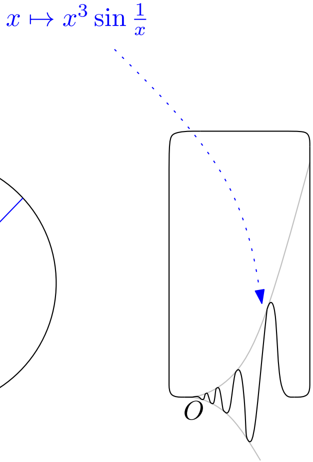
$\varrho_{\mathcal{S}} = +\infty$

(convex)



$\varrho_{\mathcal{S}} = r$

$C^{1,1}$ but not C^2)



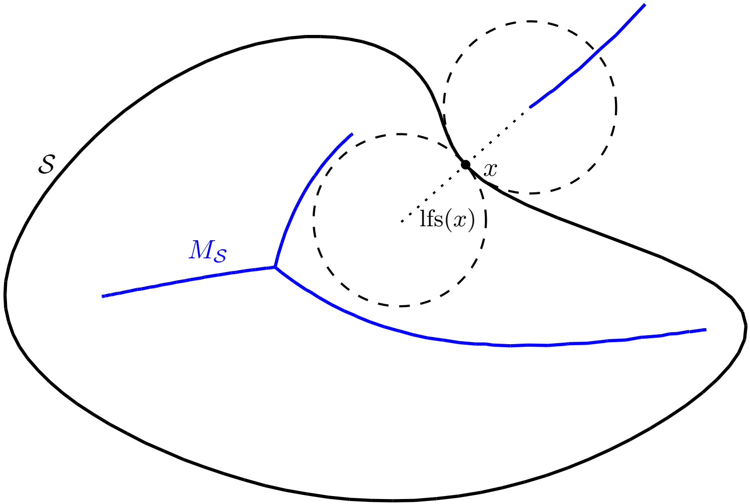
$\varrho_{\mathcal{S}} = 0$

(C^1 but not $C^{1,1}$)

Shapes with positive reach (Cont'd)

$\rightarrow$ Fundamental properties: (see [Federer 1958])

Tangent Ball Lemma: $\forall x \in \mathcal{S}$, $\forall c \in n_x\mathcal{S}$, $\|x - c\| < \mathrm{lfs}(x) \Rightarrow B(c, \|x - c\|) \cap \mathcal{S} = \emptyset$.



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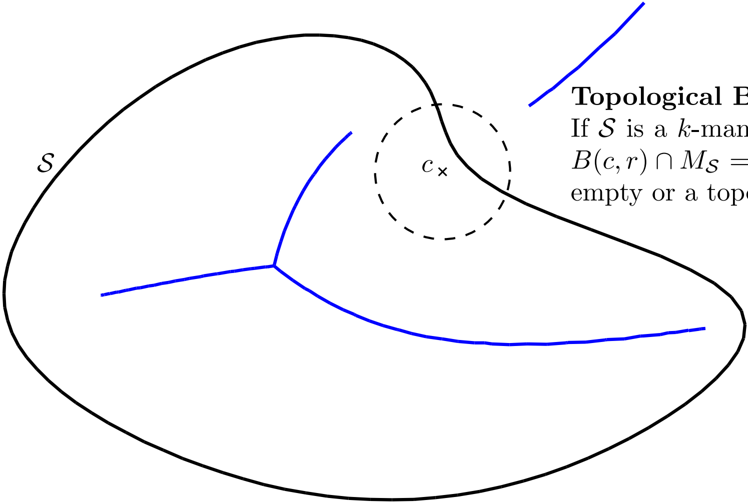

Topological Ball Lemma:
If $\mathcal{S}$ is a k-manifold, then $\forall B(c, r)$ s.t. $B(c, r) \cap M_{\mathcal{S}} = \emptyset$, $B(c, r) \cap \mathcal{S}$ is either empty or a topological k-ball.

Approximation power of the restricted Delaunay

Theorem: [Amenta et al. 1998-99]
If $\mathcal{S}$ is a curve or surface with positive *reach*, and if P is an ε-*sample* of $\mathcal{S}$ with $\varepsilon < \varrho_{\mathcal{S}}$ (curve) or $\varepsilon < 0.1\varrho_{\mathcal{S}}$ (surface), then:

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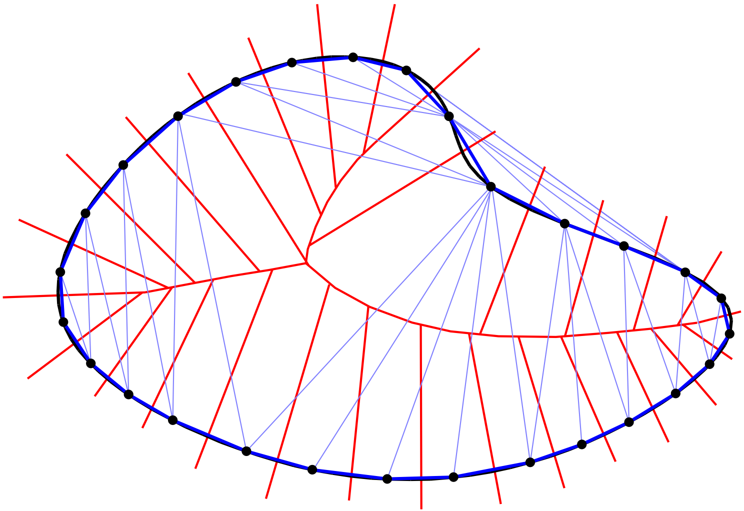
Approximation power of the restricted Delaunay

Proof for curves:

$\rightarrow$ show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa



Approximation power of the restricted Delaunay

Proof for curves:

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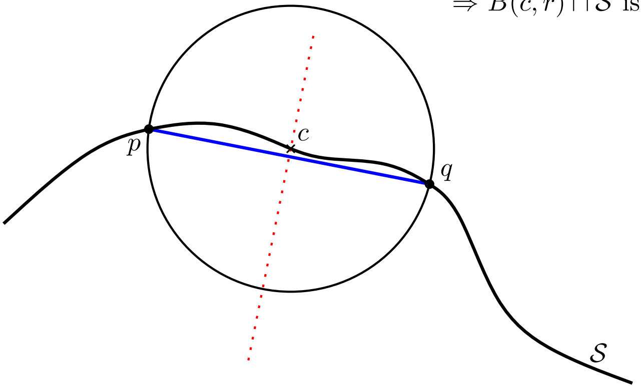
Let $c \in pq^* \cap \mathcal{S}$.

$r = \|c - p\| = \|c - q\| = \mathrm{d}(c, P) \leq \varepsilon < \varrho_{\mathcal{S}} \leq \mathrm{lfs}(c)$

$\Rightarrow B(c, r) \cap \mathcal{S}$ is a topological arc



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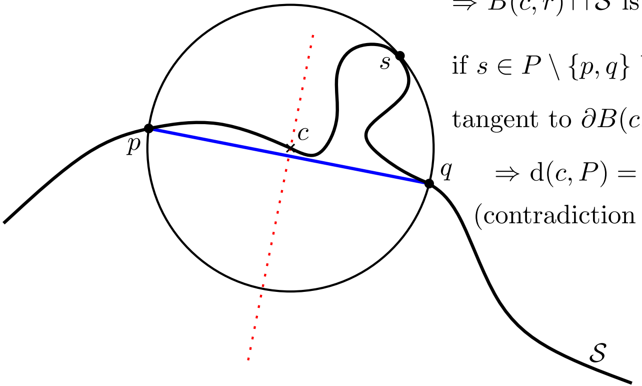

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$\Rightarrow B(c, r) \cap \mathcal{S}$ is a topological arc

if $s \in P \setminus \{p, q\}$ belongs to this arc, then the arc is tangent to $\partial B(c, r)$ in p, q or s (say s)

$\Rightarrow \mathrm{d}(c, P) = r = \|c - s\| \geq \mathrm{lfs}(s) > \varepsilon$.

(contradiction with the hypothesis of the theorem)

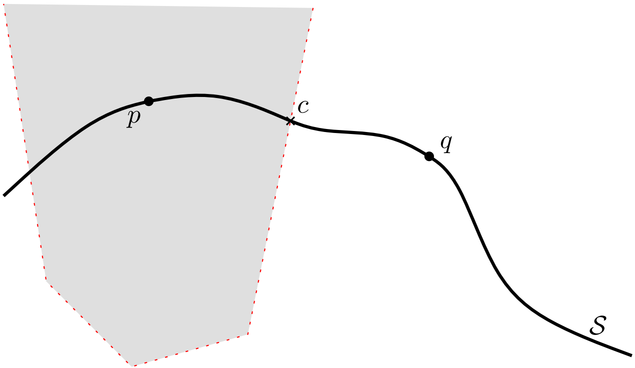
Approximation power of the restricted Delaunay

Proof for curves:

$\rightarrow$ show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa

Let $c \in \mathrm{arc}_{\mathcal{S}}(pq) \cap \partial p^*$.

$c \in ps^*$ for some $s \in P \setminus \{p\}$



Approximation power of the restricted Delaunay

Proof for curves:

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Let $c \in \mathrm{arc}_{\mathcal{S}}(pq) \cap \partial p^*$.

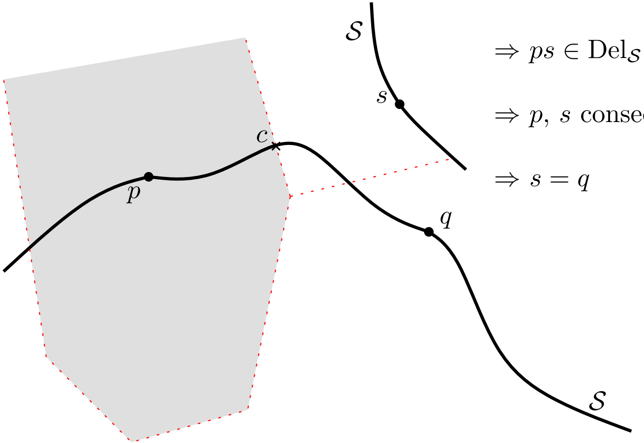
$c \in ps^*$ for some $s \in P \setminus \{p\}$

$\Rightarrow ps \in \mathrm{Del}_{\mathcal{S}}(P)$

$\Rightarrow p$, s consecutive along $\mathcal{S}$, with $c \in \mathrm{arc}_{\mathcal{S}}(ps)$

(by previous part of the proof)

$\Rightarrow s = q$

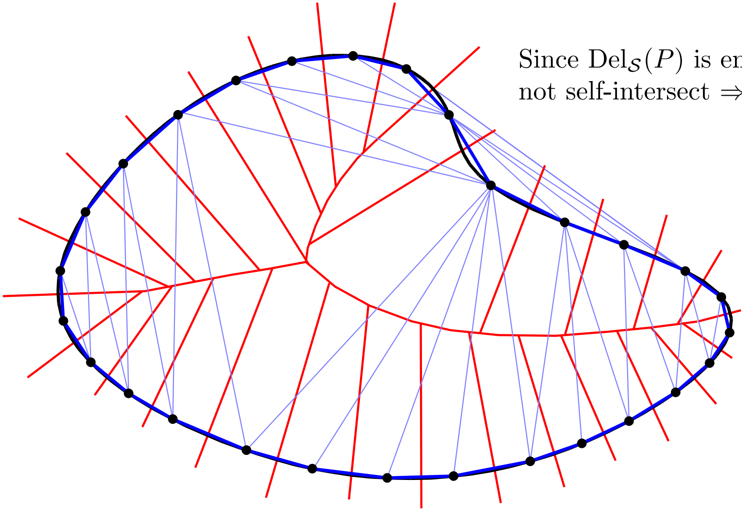


Approximation power of the restricted Delaunay

Proof for curves:

$\rightarrow$ show that every edge of $\mathrm{Del}_{\mathcal{S}}(P)$ connects consecutive points of P along $\mathcal{S}$, and vice-versa

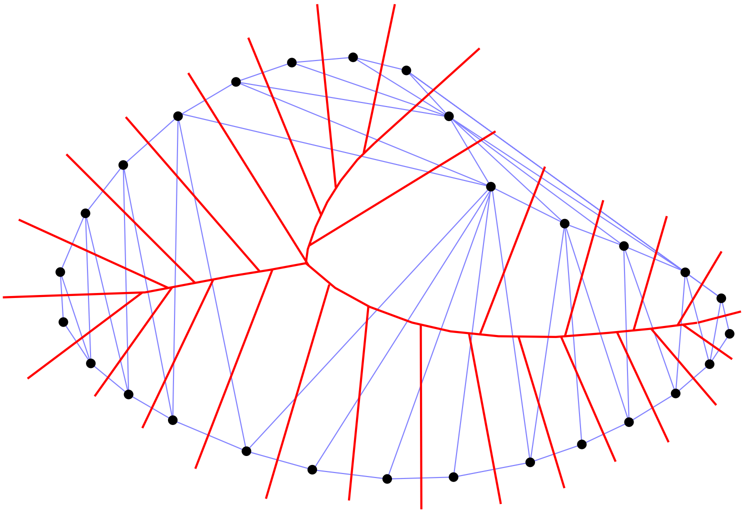
$\Rightarrow$ $\mathrm{Del}_{\mathcal{S}}(P)$ is homeomorphic to $\mathcal{S}$ between each pair of consecutive points of P



Since $\mathrm{Del}_{\mathcal{S}}(P)$ is embedded in $\mathrm{Del}(P)$, it does not self-intersect $\Rightarrow$ global homeomorphism

Computing the restricted Delaunay

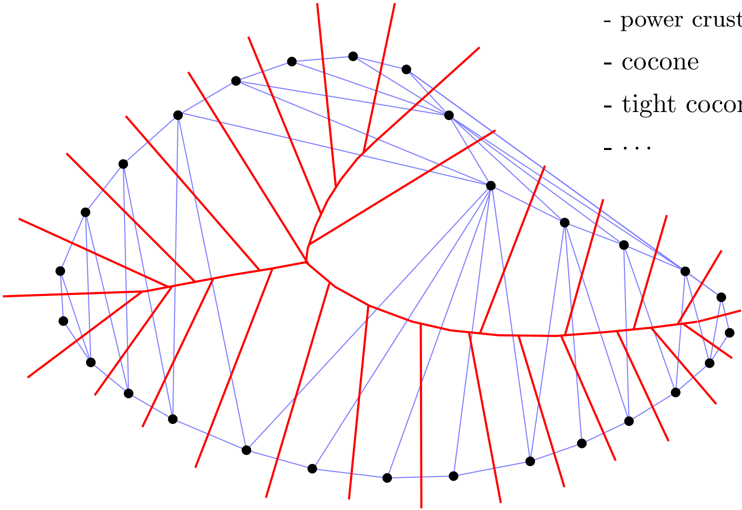
Q How to compute $\mathrm{Del}_{\mathcal{S}}(P)$ when $\mathcal{S}$ is unknown?



Computing the restricted Delaunay

Q How to compute $\mathrm{Del}_{\mathcal{S}}(P)$ when $\mathcal{S}$ is unknown?

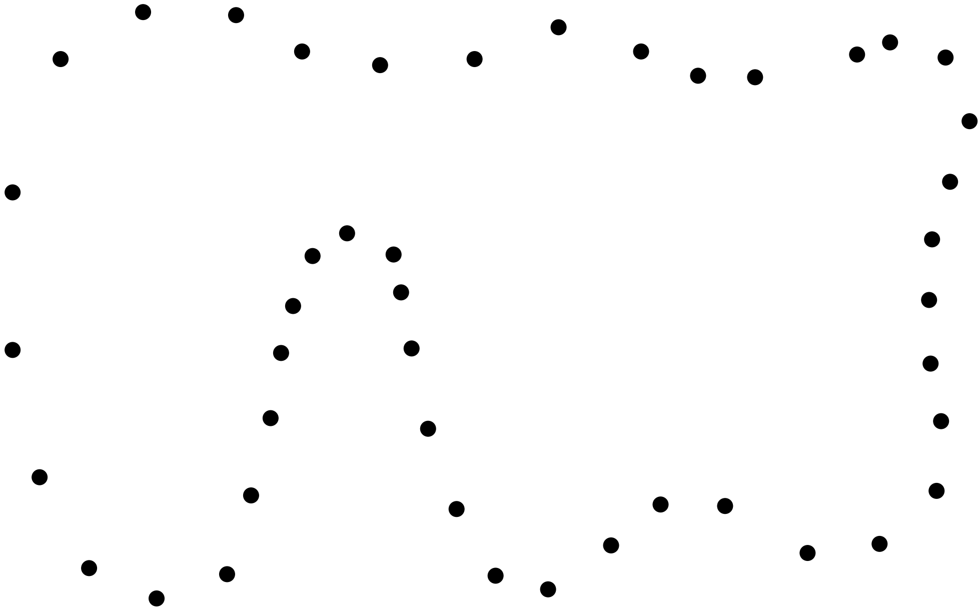
$\rightarrow$ a whole family of algorithms use various Delaunay extraction criteria:



- crust
- power crust
- cocone
- tight cocone
- $\cdots$

Crust algorithm

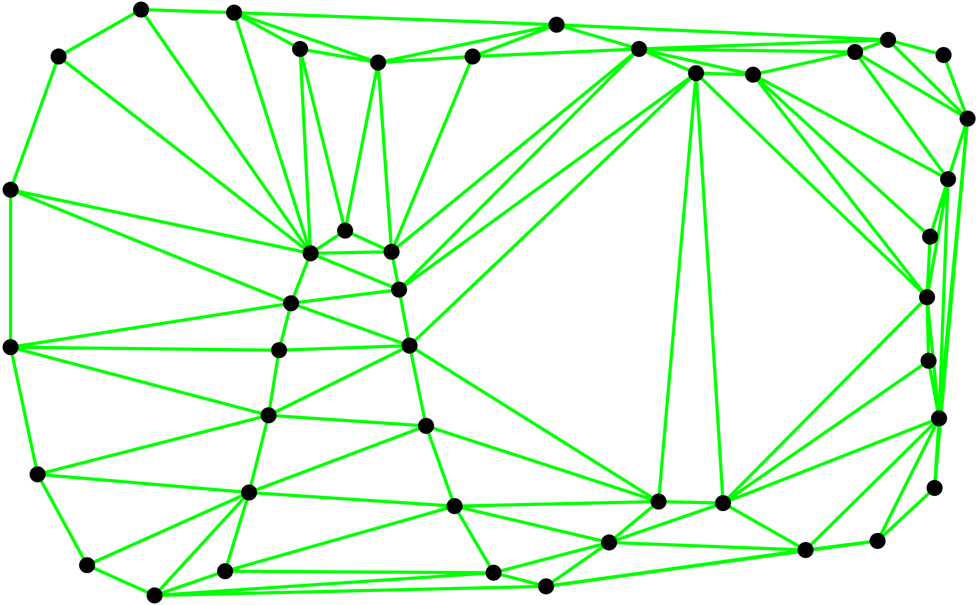
[Amenta et al. 1997-98]



Crust algorithm

[Amenta et al. 1997-98]

1. Compute Delaunay triangulation of P



Crust algorithm

[Amenta et al. 1997-98]

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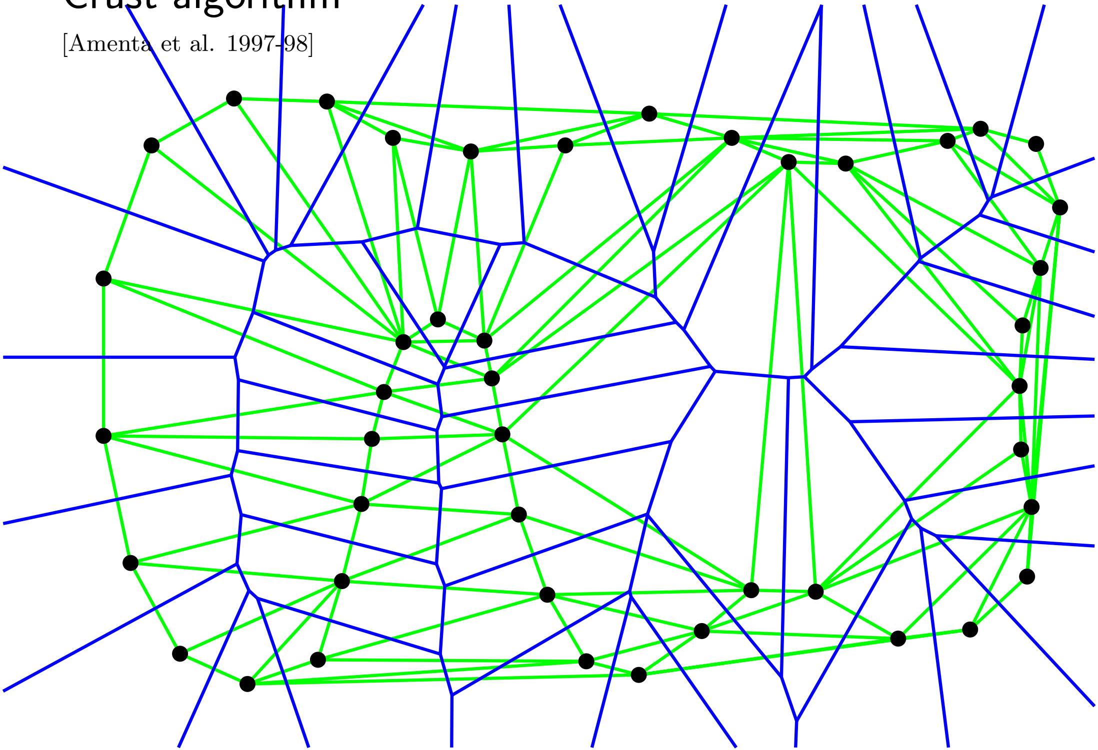
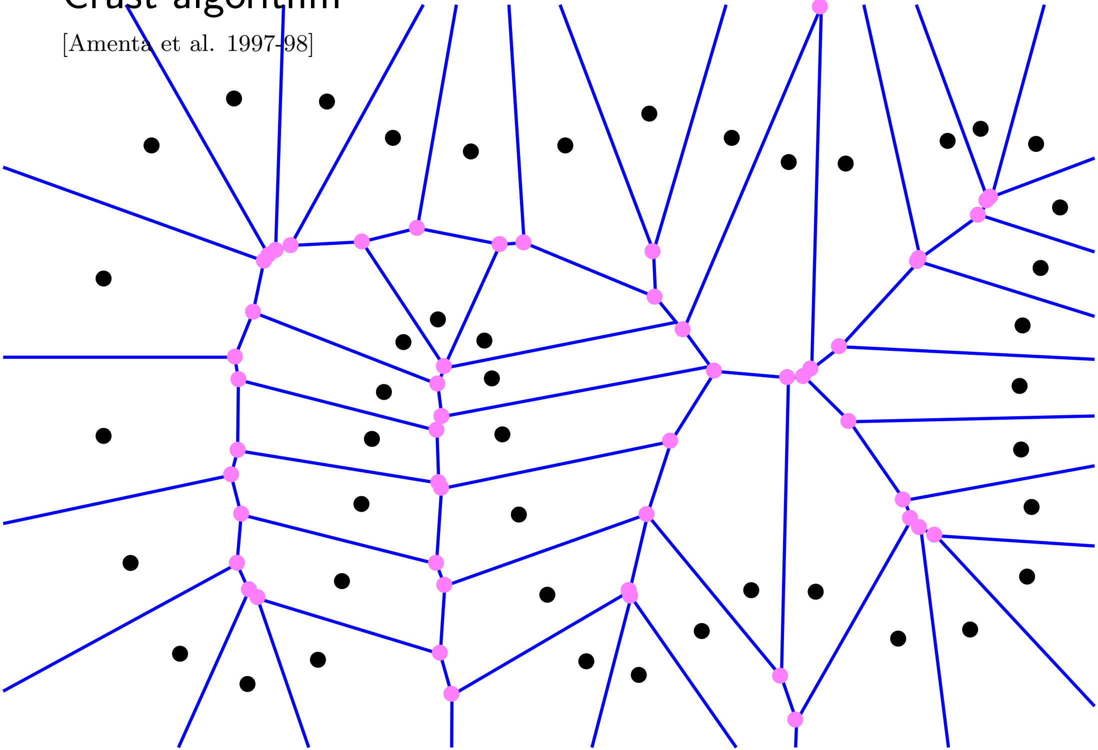


Crust algorithm

[Amenta et al. 1997-98]

2. Compute *poles* (furthest Voronoi vertices)



Crust algorithm

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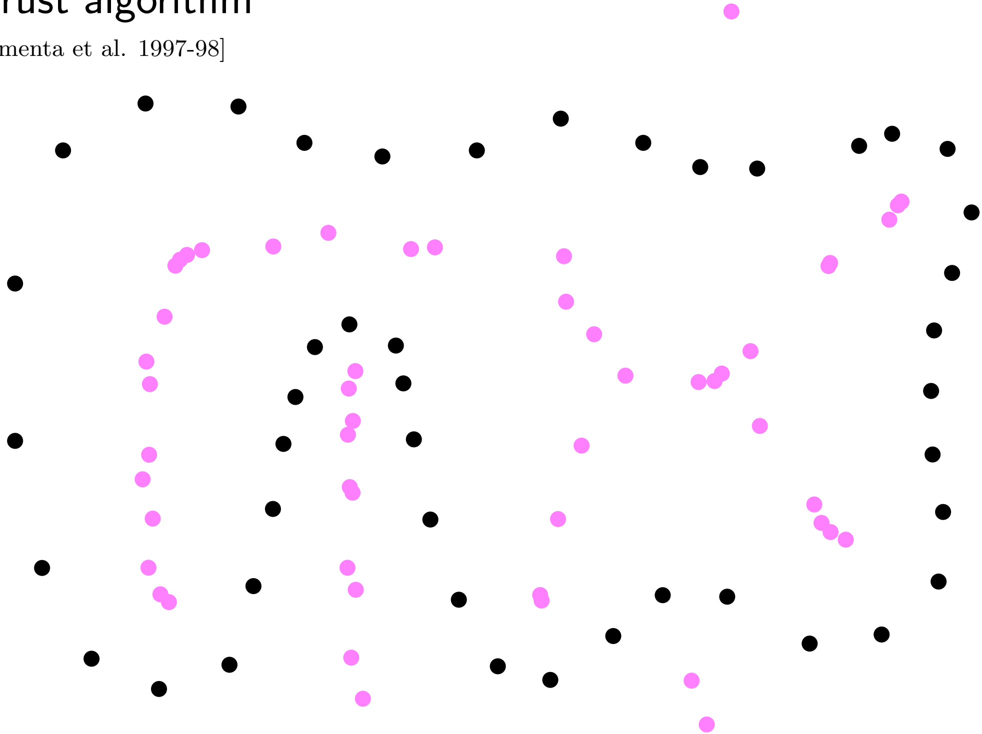
3. Add poles to the set of vertices



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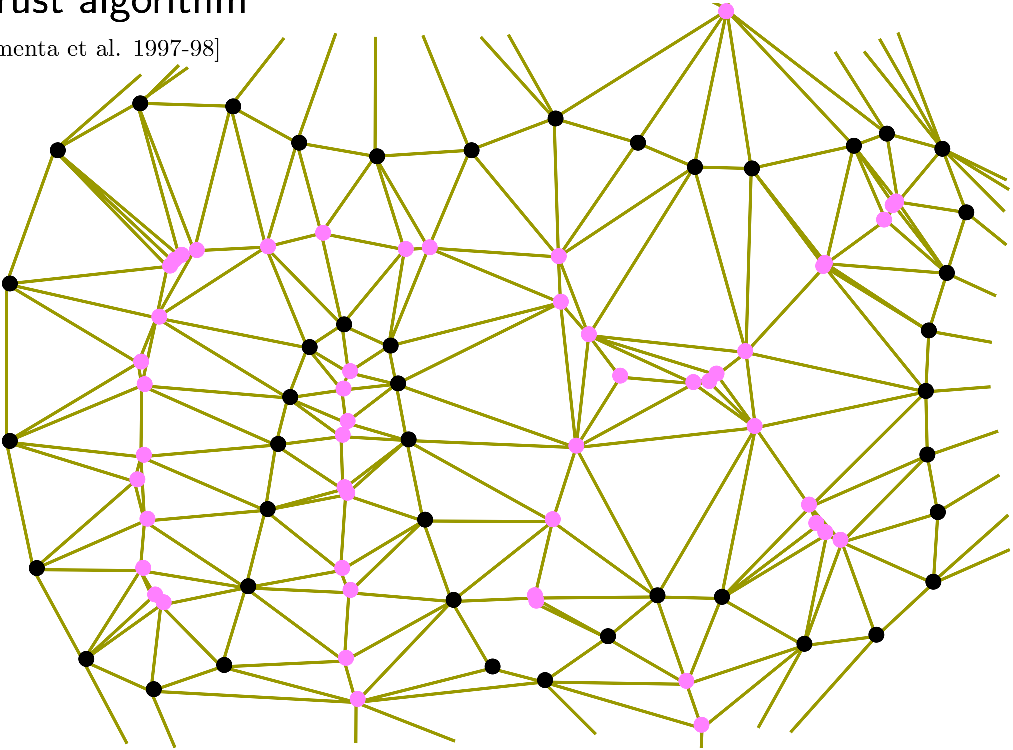
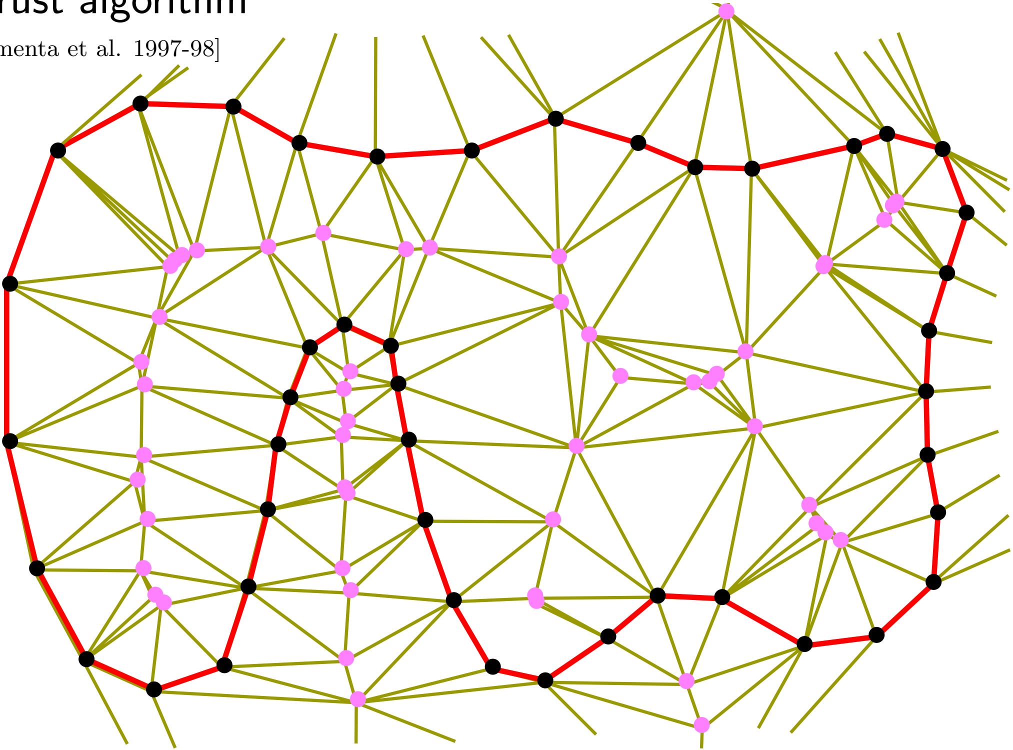


Crust algorithm

[Amenta et al. 1997-98]

4. Keep Delaunay simplices whose vertices are in P



Crust algorithm

$\rightarrow$ in 2-d, crust $= \mathrm{Del}_{\mathcal{S}}(P) \approx \mathcal{S}$

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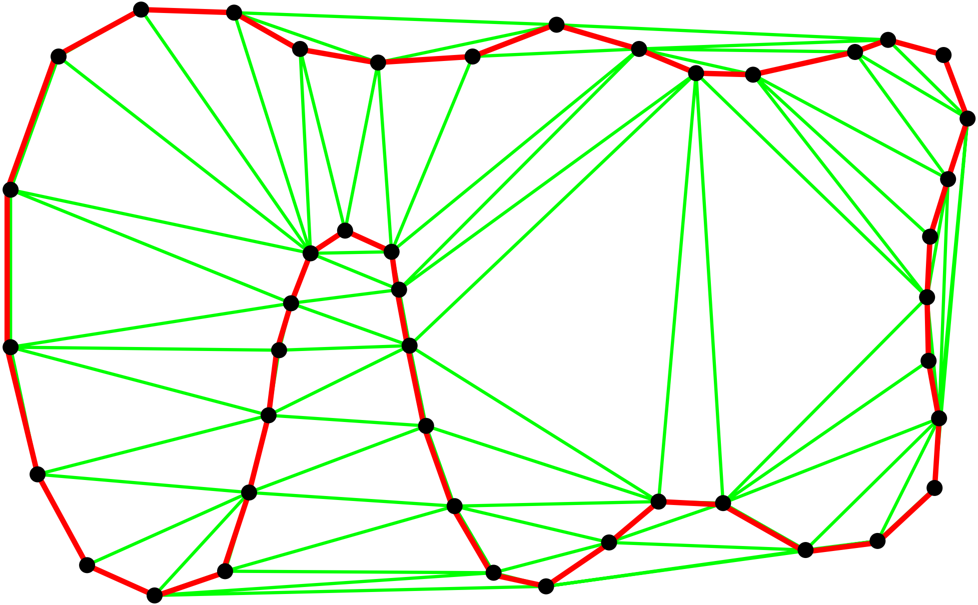

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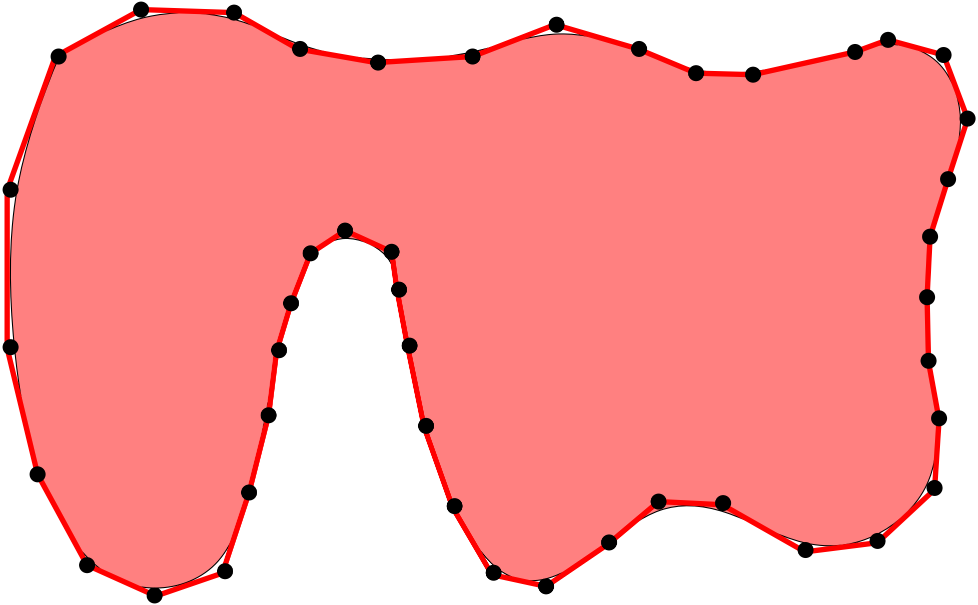

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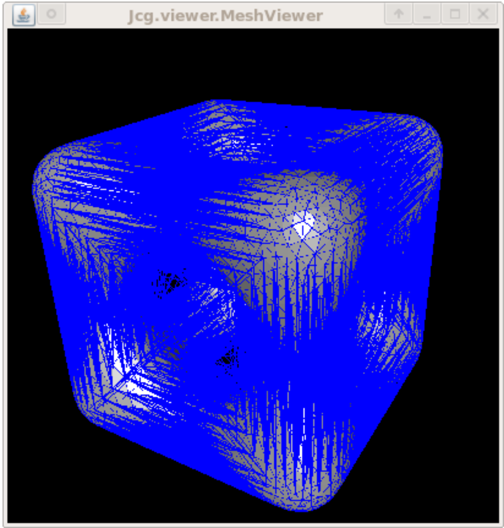

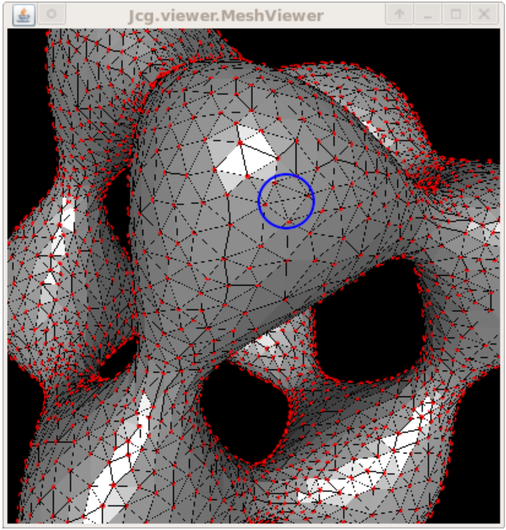

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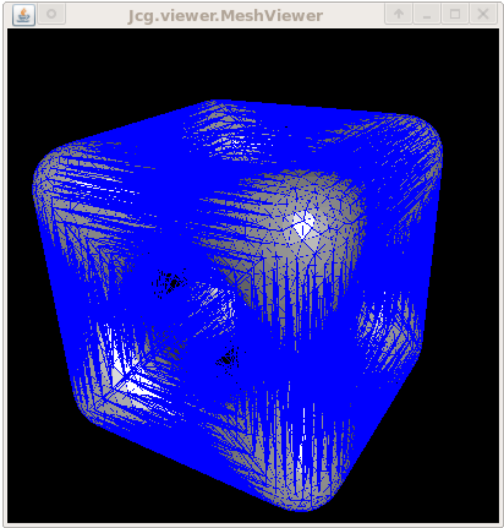
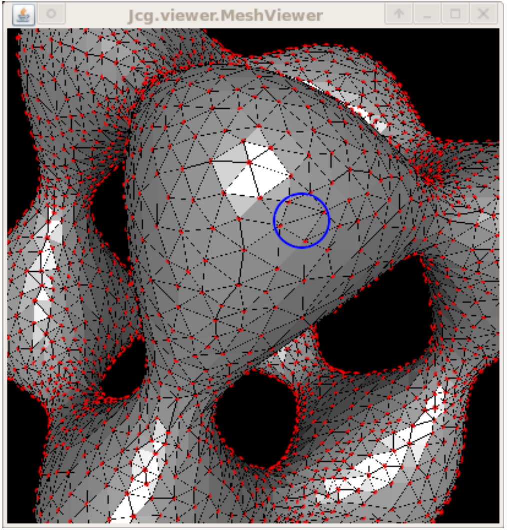
$\Rightarrow$ manifold extraction step in post-processing

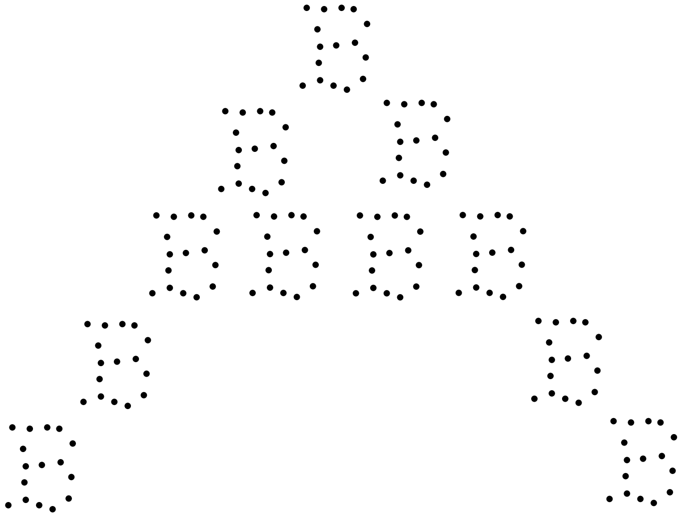




Back to the reconstruction paradigm

Q What do you see?

Why?



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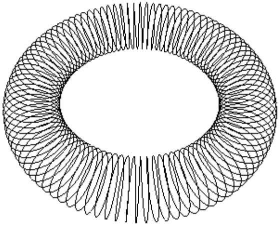
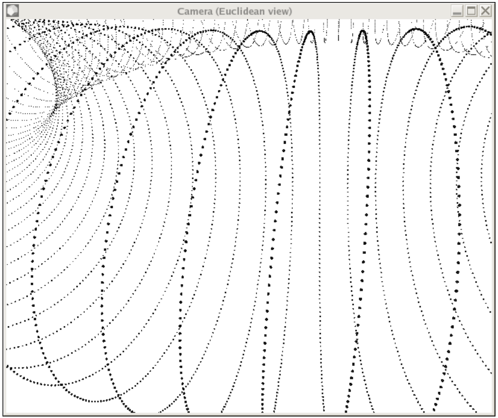
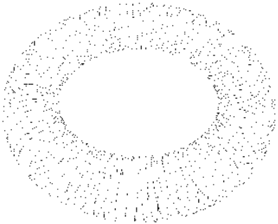






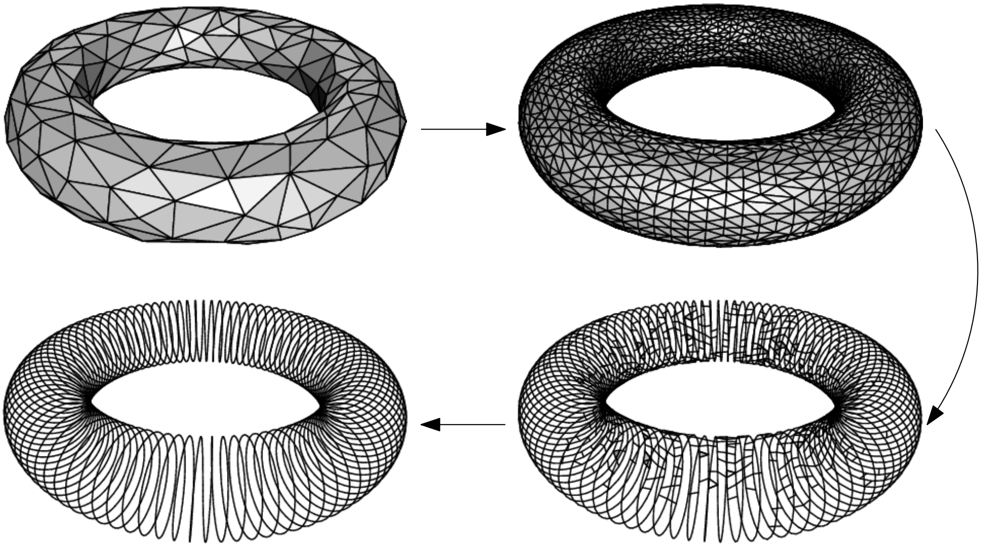
Back to the reconstruction paradigm

$\rightarrow$ **When the dimensionality of the data is unknown or there is noise, the reconstruction result depends on the scale at which the data is looked at.**

$\rightarrow$ need for multi-scale reconstruction techniques

Multi-scale approach in a nutshell

$\rightarrow$ build a one-parameter family of complexes approximating the input at various scales



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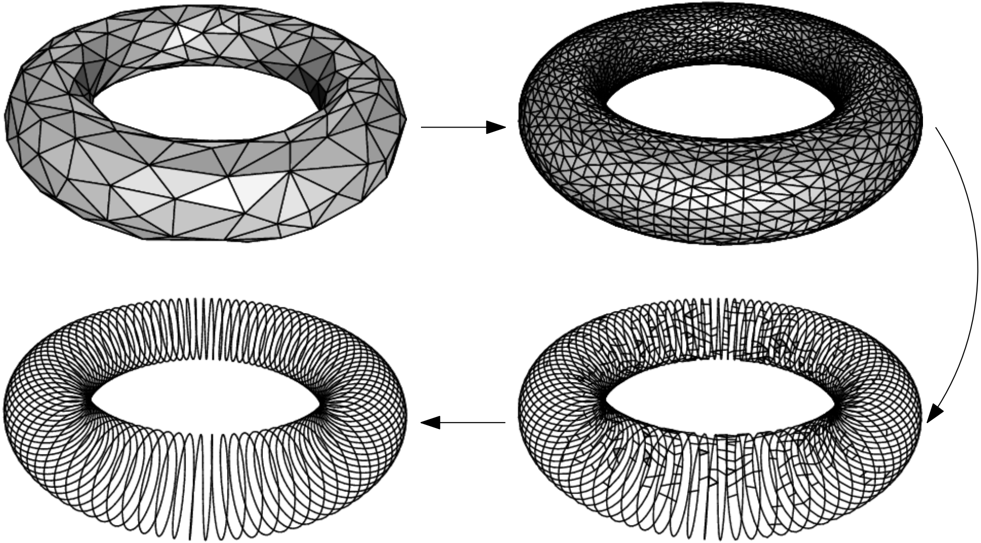
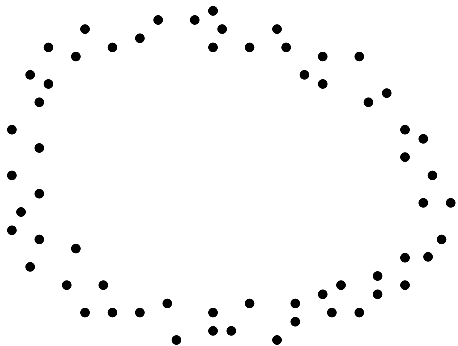


→ connections with manifold learning and topological persistence

Multi-scale algorithm [Guibas, Oudot 2007]

Input: a finite point set $W \subset \mathbb{R}^n$

$\rightarrow$ resample W iteratively, and maintain a simplicial complex:



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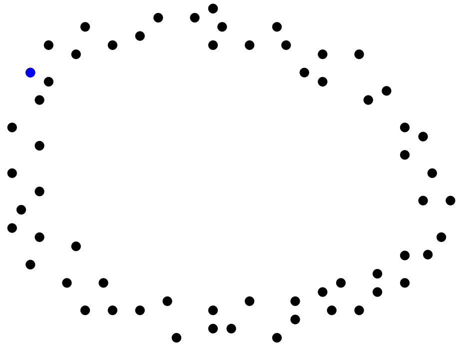

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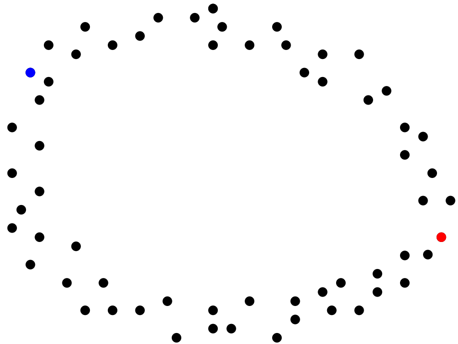

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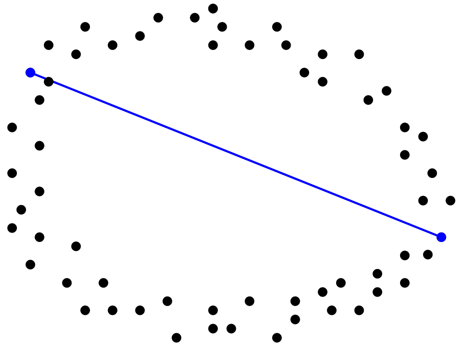

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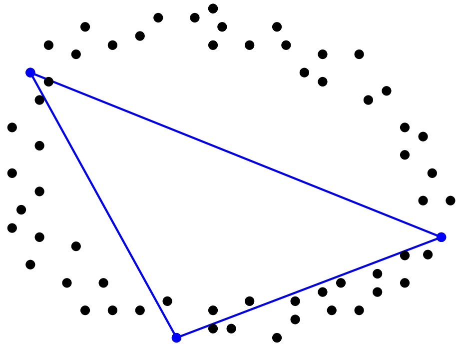

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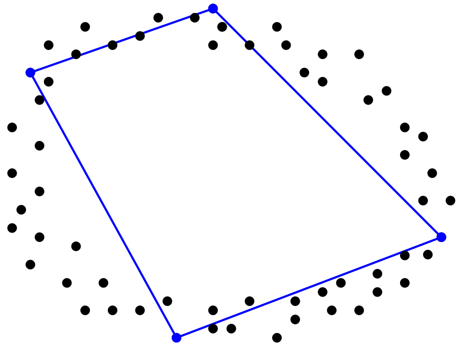
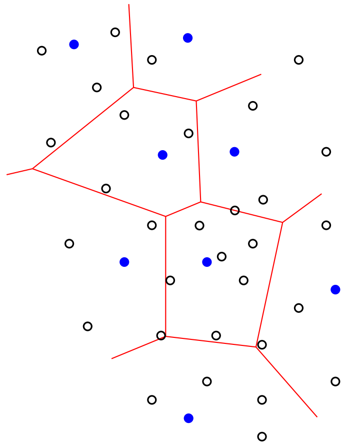


Output: the sequence of simplicial complexes

The simplicial complex to maintain

$\rightarrow$ maintain the **witness complex** $\mathrm{C}^W(L)$ [de Silva 2003]:

Let $L \subseteq \mathbb{R}^d$ (landmarks) s.t. $|L| < +\infty$ and $W \subseteq \mathbb{R}^d$ (witnesses)



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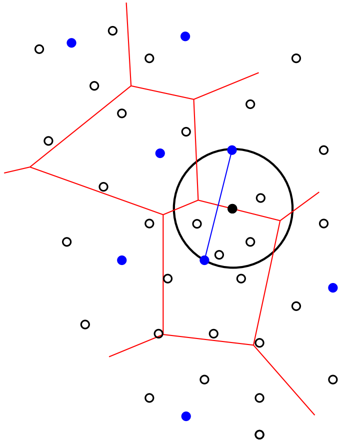

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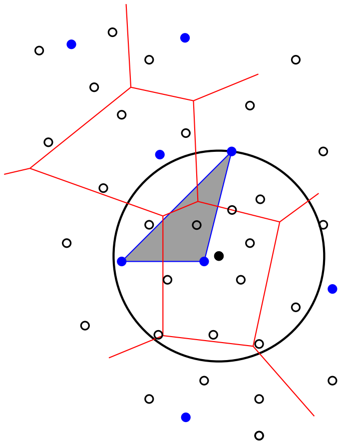

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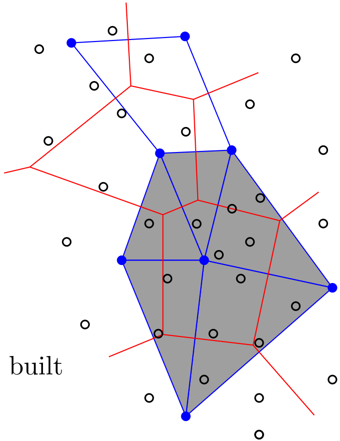

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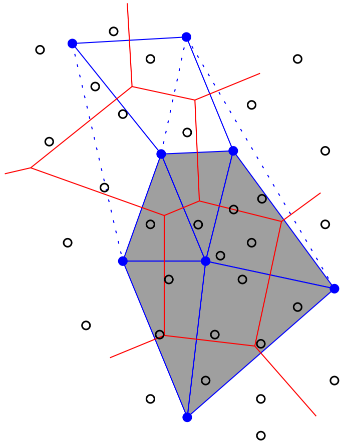
Def. $C^W(L)$ is the largest abstract simplicial complex built over L, whose faces are weakly witnessed by points of W.

The witness complex (properties)

Thm. 1 [de Silva 2003] $\forall W, L$, $\forall \sigma \in \mathrm{C}^W(L)$, $\exists c \in \mathbb{R}^d$ that strongly witnesses σ.

$\Rightarrow \mathrm{C}^W(L)$ is a subcomplex of $\mathrm{Del}(L)$

$\Rightarrow \mathrm{C}^W(L)$ is embedded in $\mathbb{R}^d$



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- The size of $\mathrm{C}^W(L)$ is $O(d|W|)$
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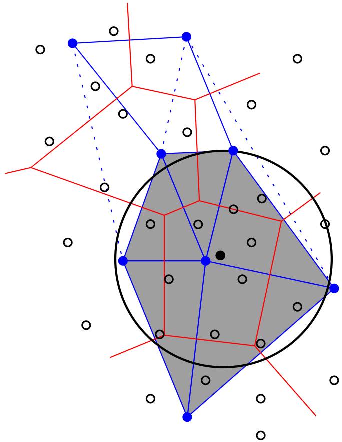

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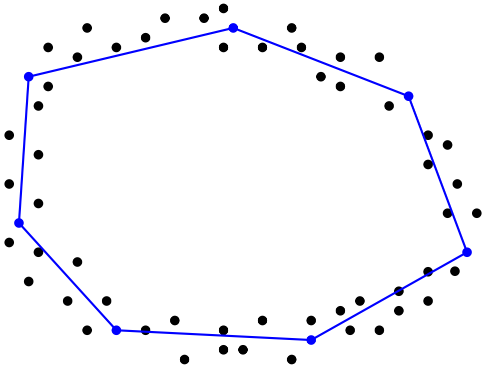

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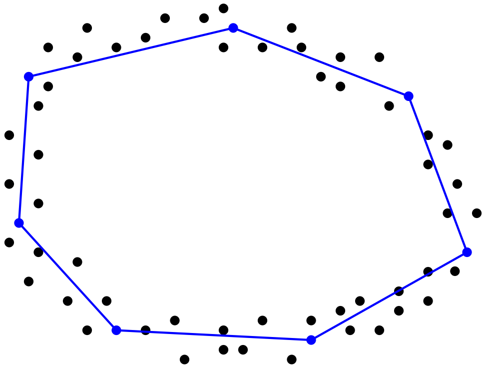
Thm. 3 [Guibas, Oudot 2007]
[Attali, Edelsbrunner, Mileyko 2007]
Under some conditions, $\mathrm{C}^W(L) = \mathrm{Del}_{\mathcal{S}}(L) \approx \mathcal{S}$

The witness complex (properties)

$\rightarrow$ connection with reconstruction:

- $W \subset \mathbb{R}^d$ is given as input
- $L \subseteq W$ is generated
- underlying manifold $\mathcal{S}$ unknown
- only distance comparisons

$\Rightarrow$ algorithm is applicable in any metric space



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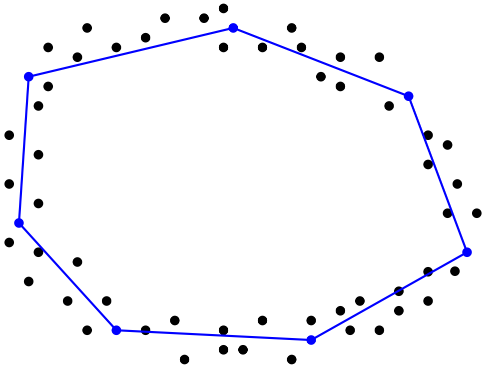
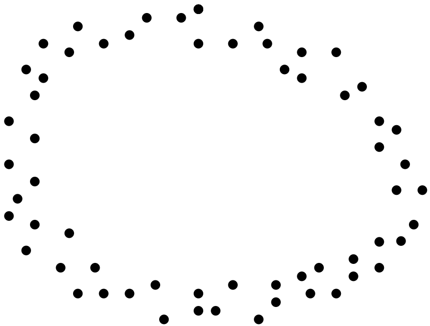


- In $\mathbb{R}^d$, $C^W(L)$ can be maintained by updating, for each witness w, the list of $d+1$ nearest landmarks of w.

$$\Rightarrow \begin{array}{lcl} \text{space} & \leq & O\left(d|W|\right) \\ \text{time} & \leq & O\left(d|W|^2\right) \end{array}$$

The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.



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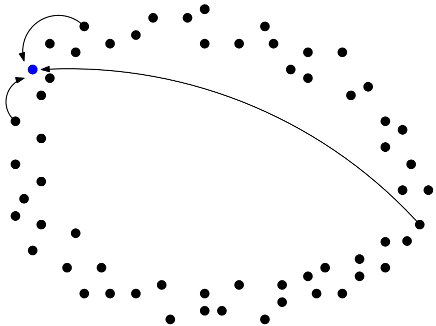

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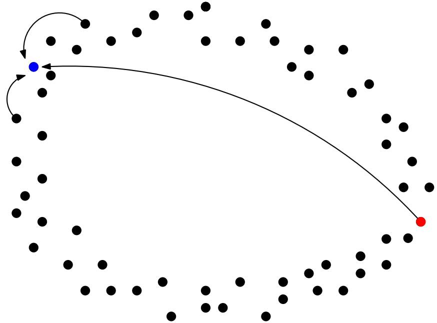

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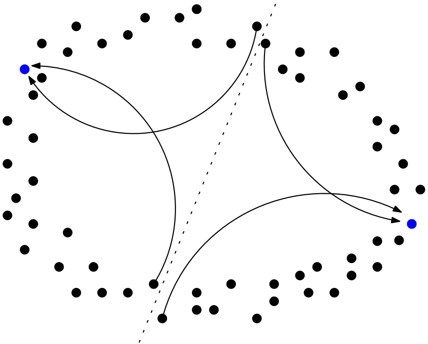

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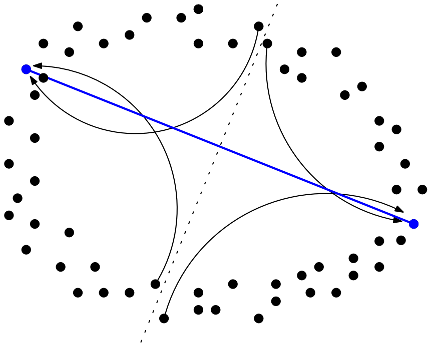

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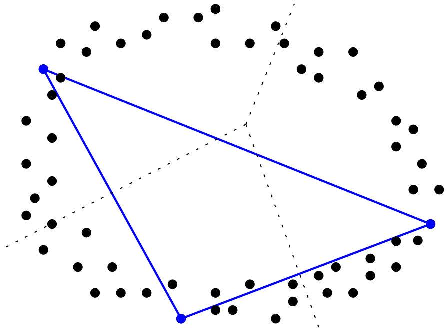

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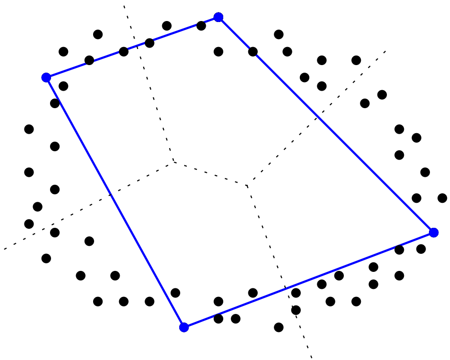
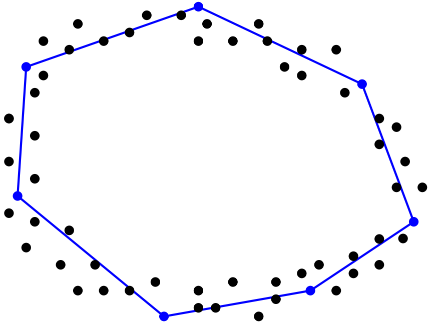
Output: the sequence of complexes $\mathrm{C}^W(L)$



Theoretical guarantees

$\rightarrow$ case of curves:

Conjecture [Carlsson, de Silva 2004]:
$\mathrm{C}^W(L)$ coincides with $\mathrm{Del}_{\mathcal{S}}(L)$...



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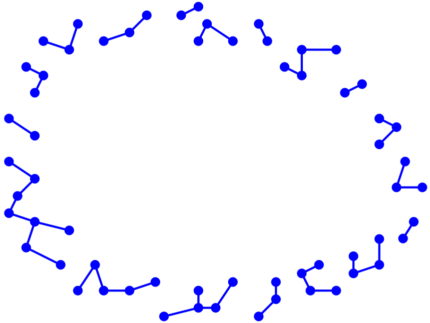
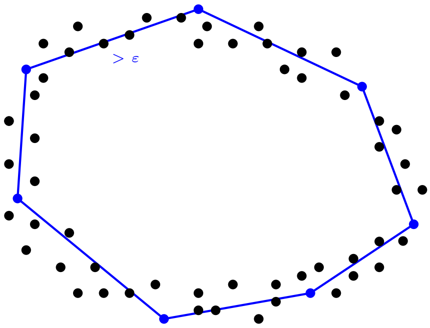


... under some conditions on W and L

Theoretical guarantees

$\rightarrow$ case of curves:

Thm. 3 If $\mathcal{S}$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $\mathrm{d_H}(W, \mathcal{S}) \leq \delta$, $L \subseteq W$ ε-sparse ε-sample of W with $\delta \ll \varepsilon \ll \varrho_{\mathcal{S}}$, then $\mathrm{C}^W(L) = \mathrm{Del}_{\mathcal{S}}(L) \approx \mathcal{S}$.



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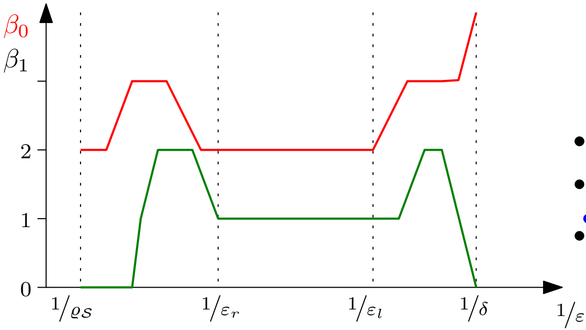

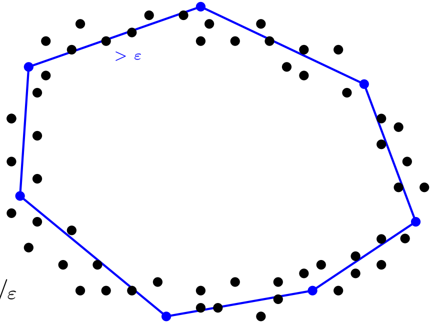

$\rightarrow$ There is a plateau in the diagram of Betti numbers of $\mathrm{C}^W(L)$.

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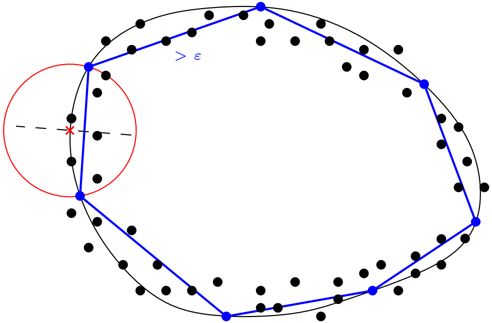

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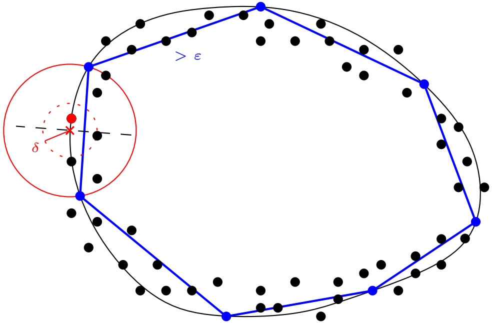

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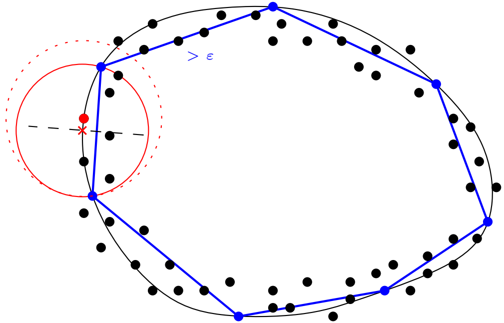

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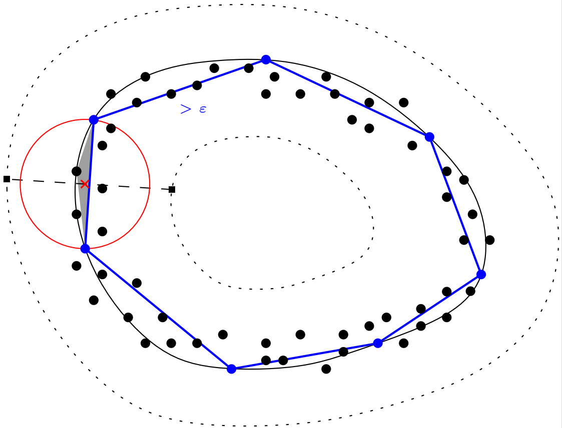

Theoretical guarantees

$\to$ case of surfaces:

Thm [Attali, Edelsbrunner, Mileyko]
If $\varepsilon \ll \varrho_{\mathcal{S}}$, then $\forall W \subseteq \mathcal{S}$, $\mathrm{C}^W(L) \subseteq \mathrm{Del}_{\mathcal{S}}(L)$.

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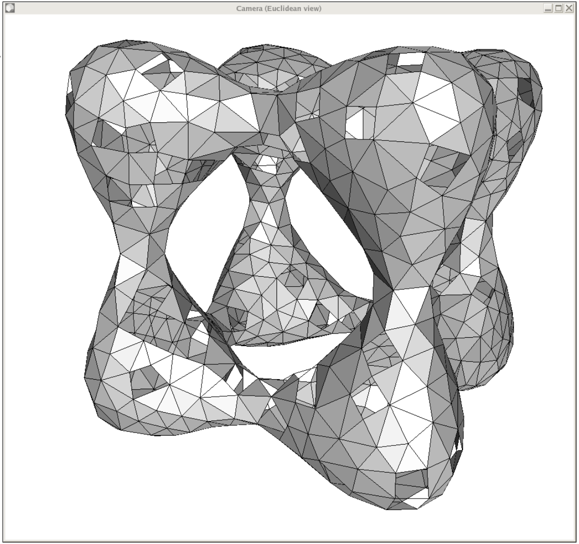

$\varepsilon = 0.2,\ \varrho_{\mathcal{S}} \approx 0.25$

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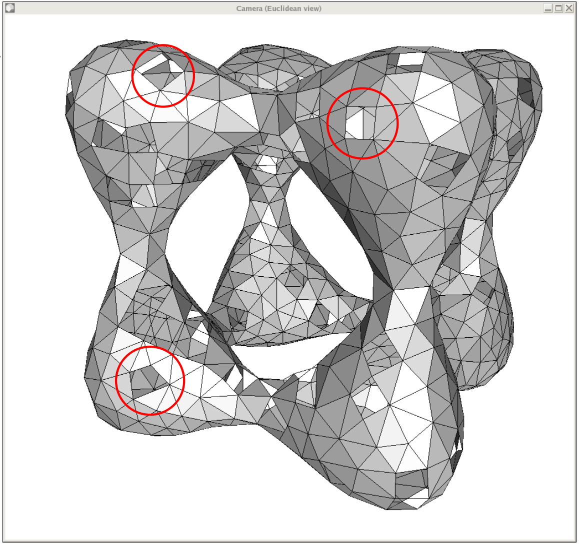

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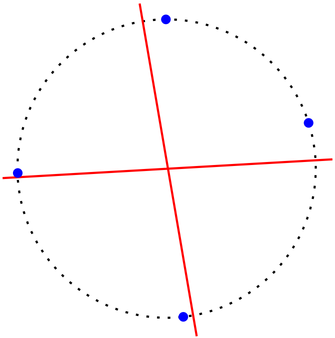
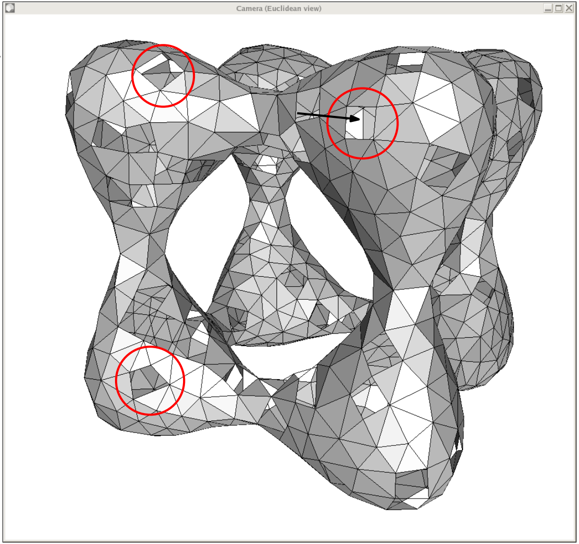


order-2 Voronoi diagram



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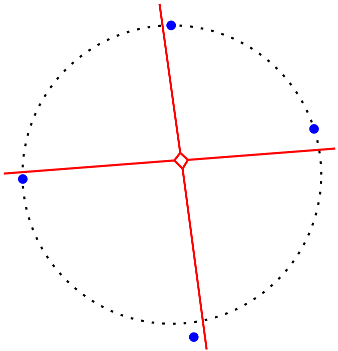

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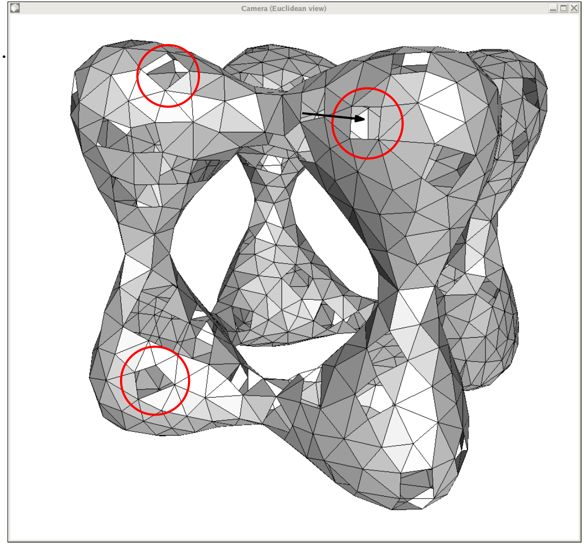

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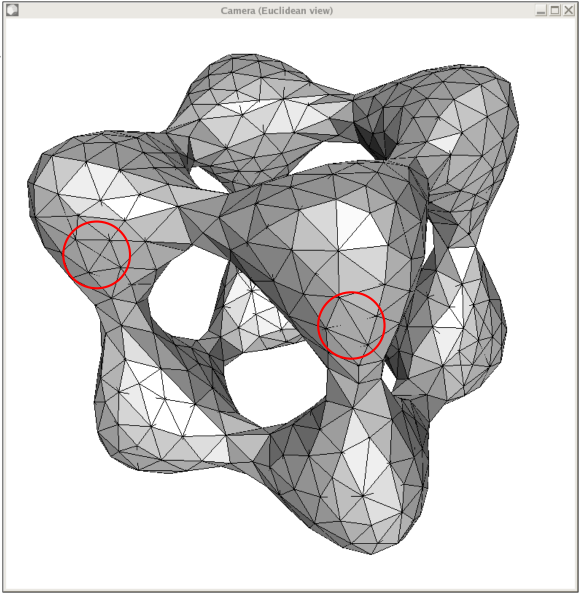
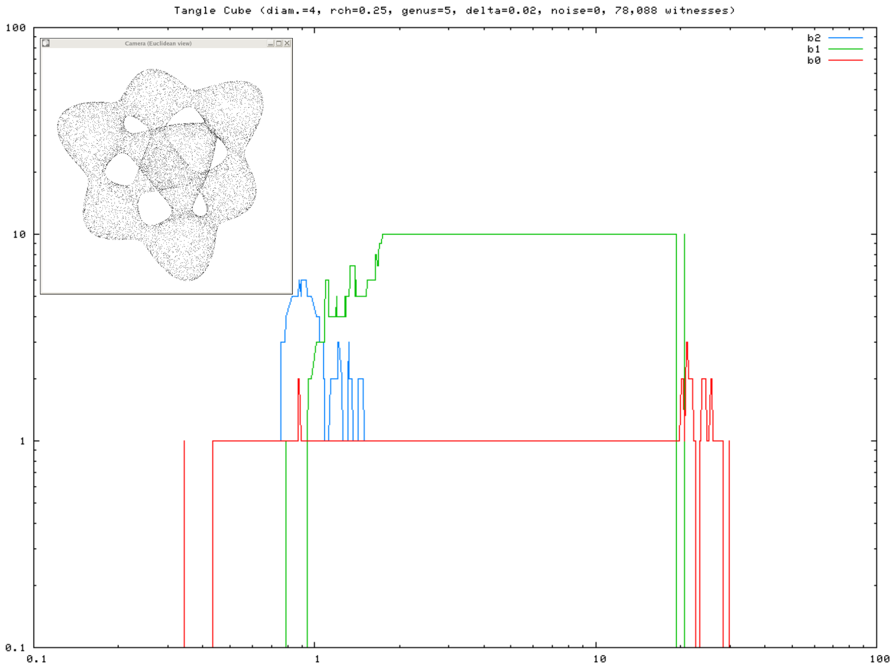
Solution relax witness test
[Guibas, Oudot]

$\Rightarrow \mathrm{C}^W_\nu(L) = \mathrm{Del}_{\mathcal{S}}(L) +$ slivers
$\Rightarrow \mathrm{C}^W_\nu(L) \not\subseteq \mathrm{Del}(L)$
$\Rightarrow \mathrm{C}^W_\nu(L)$ not embedded.

Post-process extract manifold M from $\mathrm{C}^W_\nu(L) \cap \mathrm{Del}(L)$
[Amenta, Choi, Dey, Leekha]



Some results



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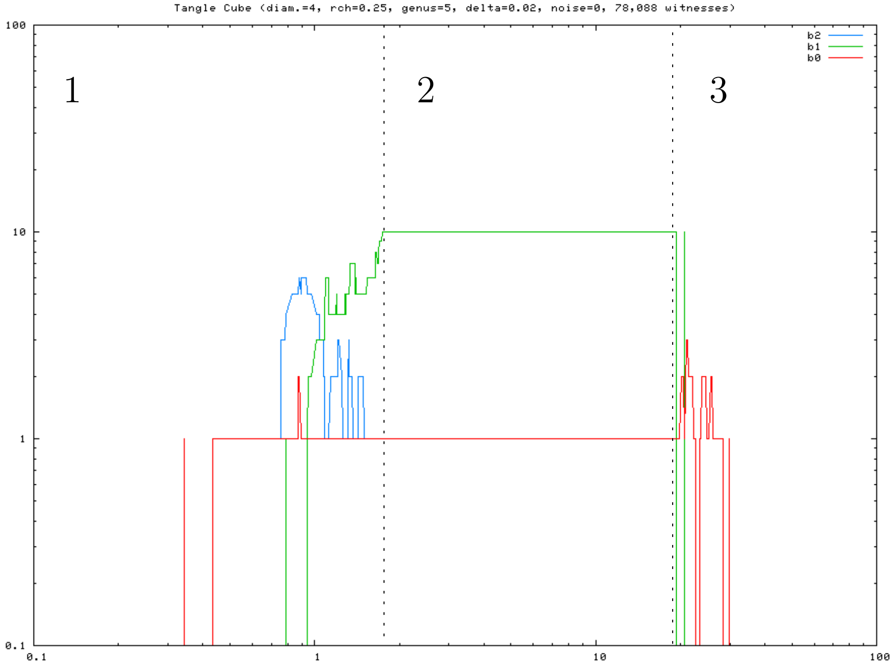

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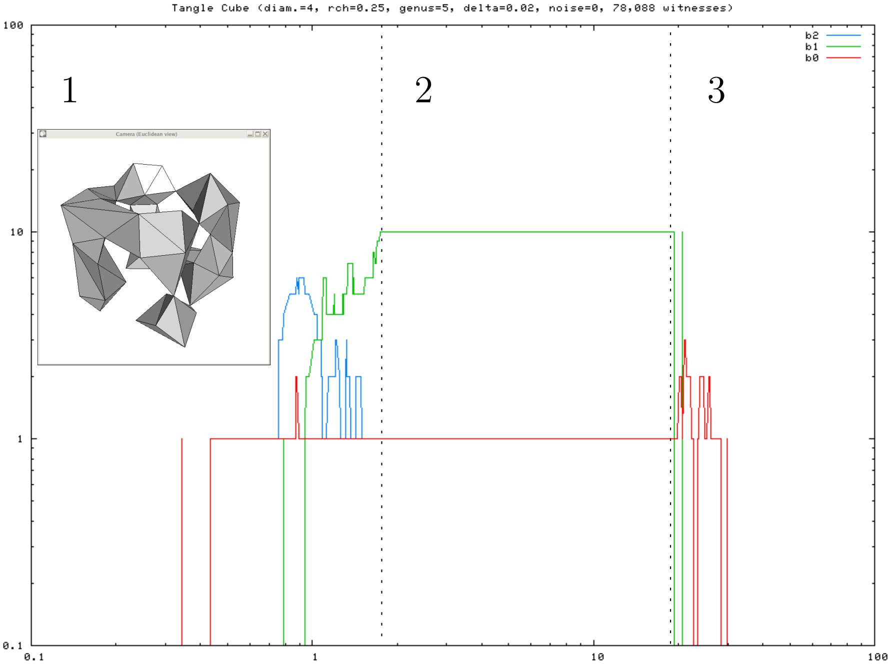

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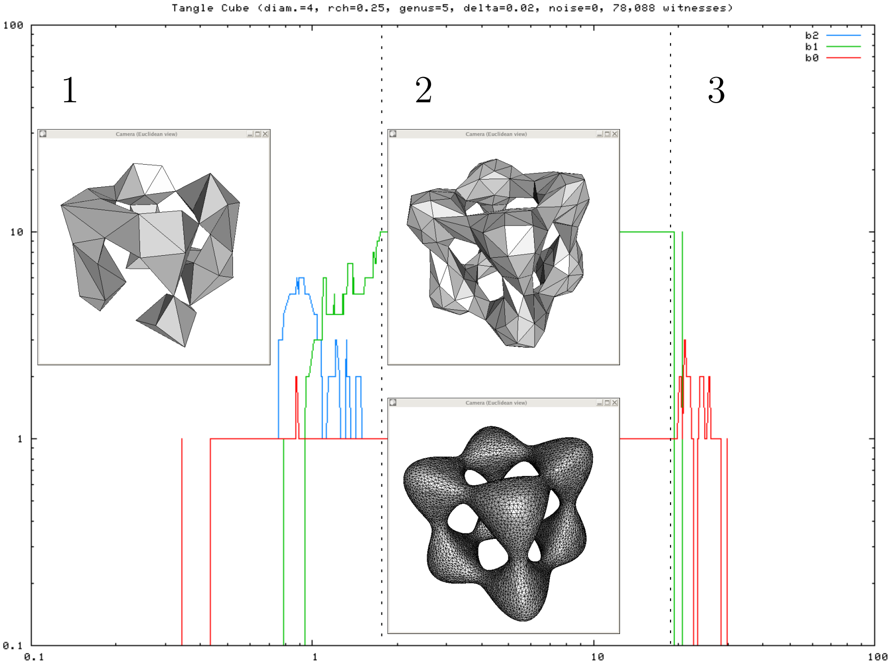

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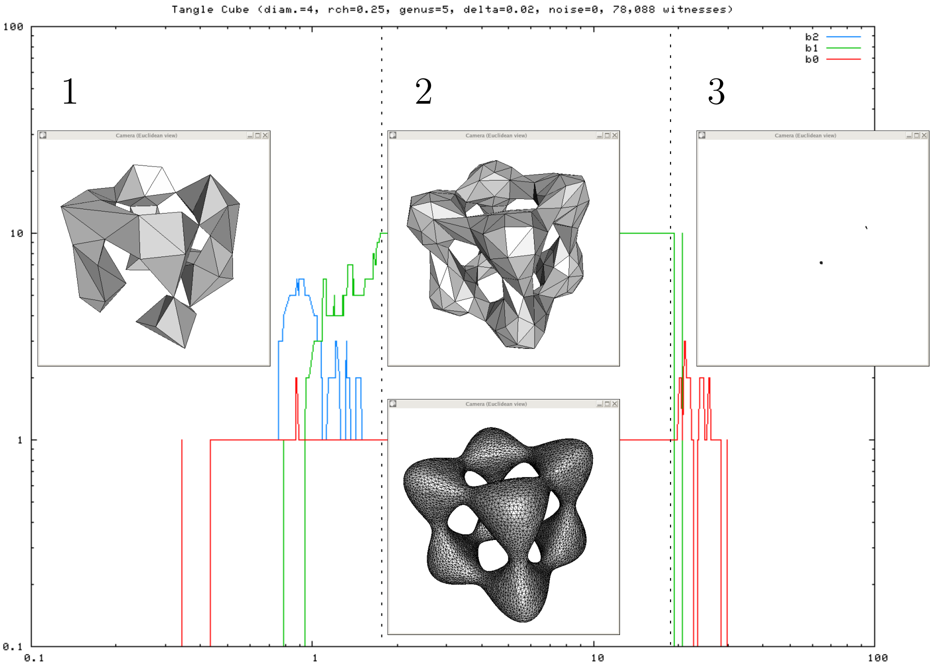
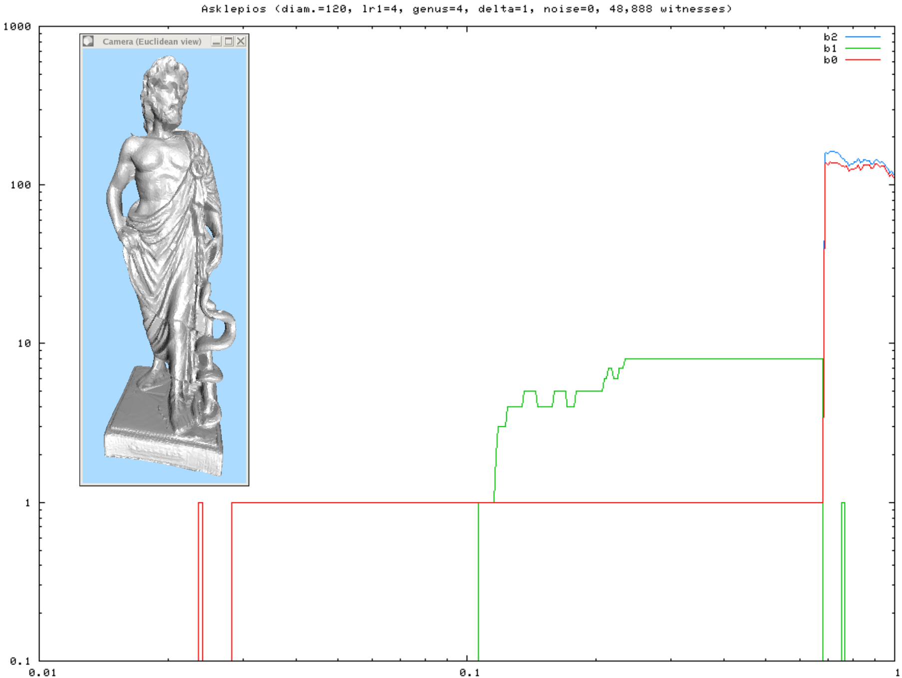
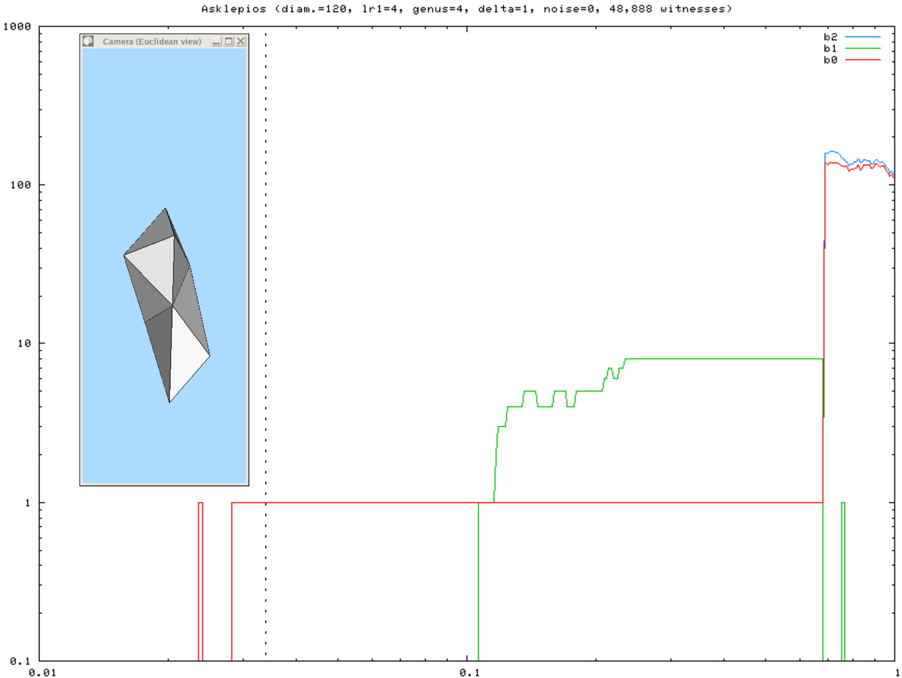


Some results (cont'd)



input model provided courtesy of IMATI by the Aim@Shape repository

Some results (cont'd)



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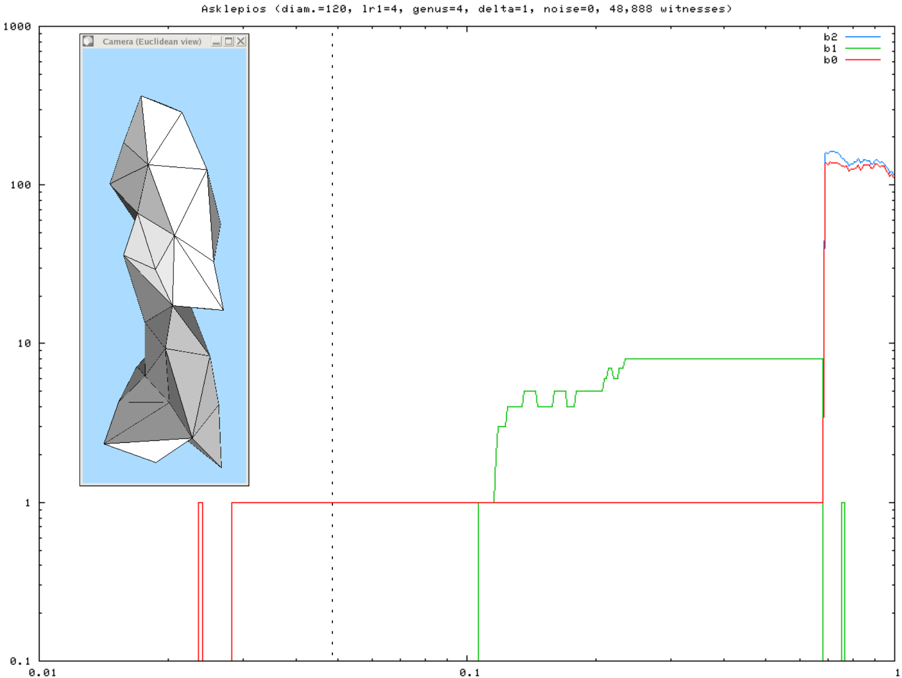

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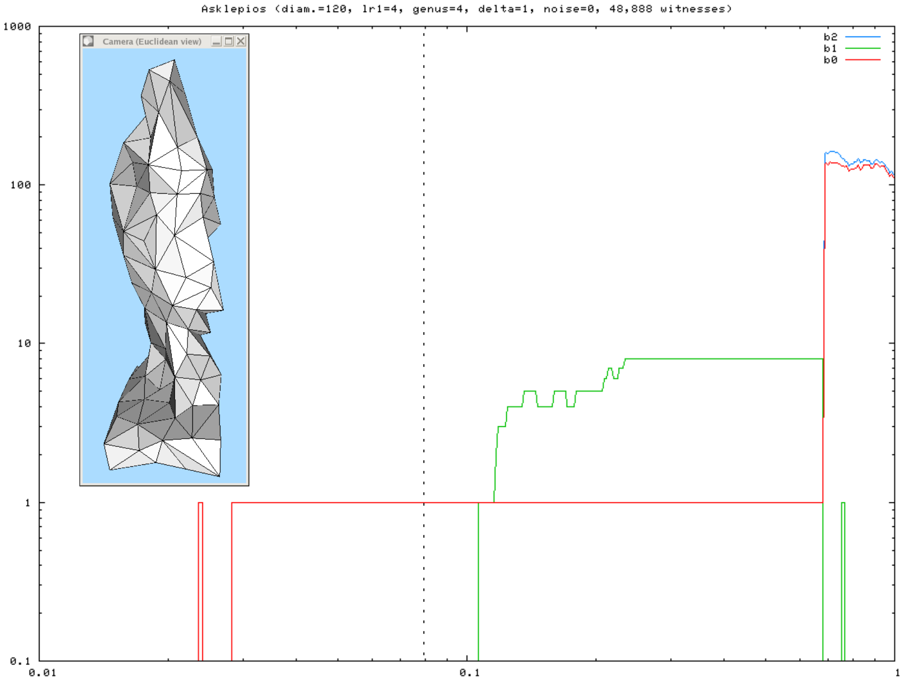

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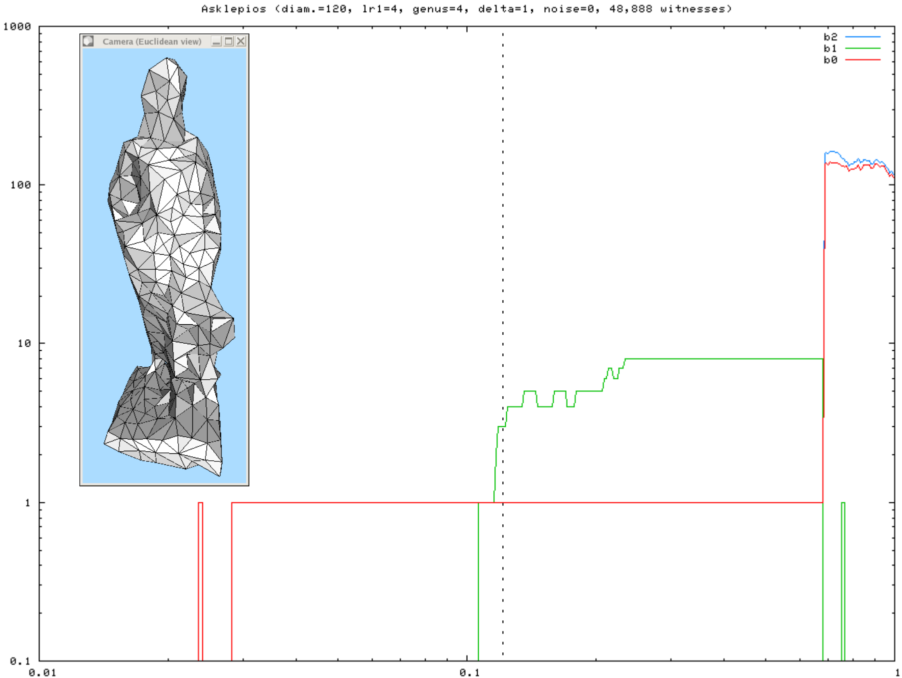

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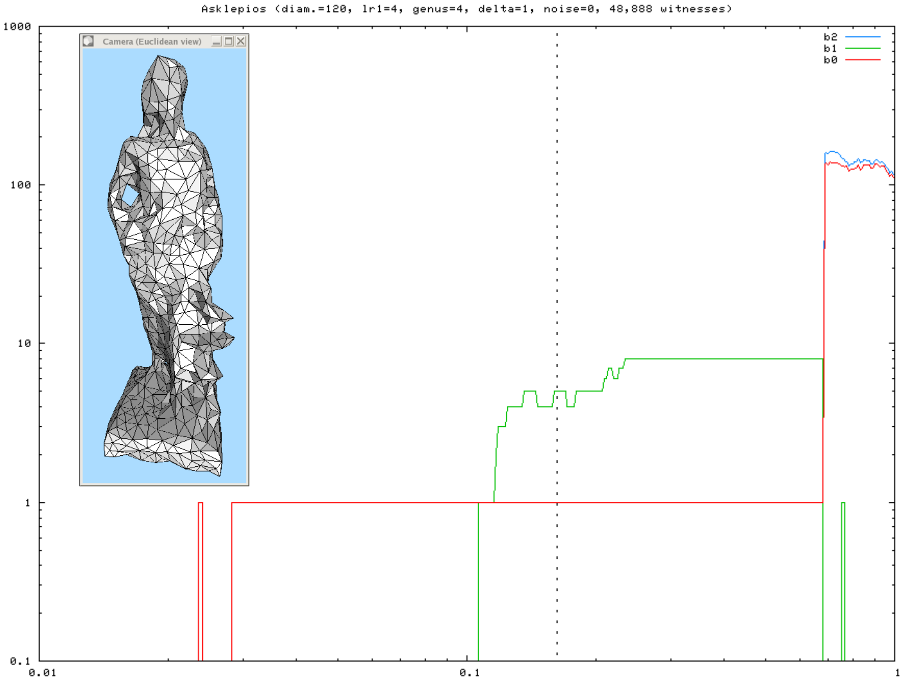

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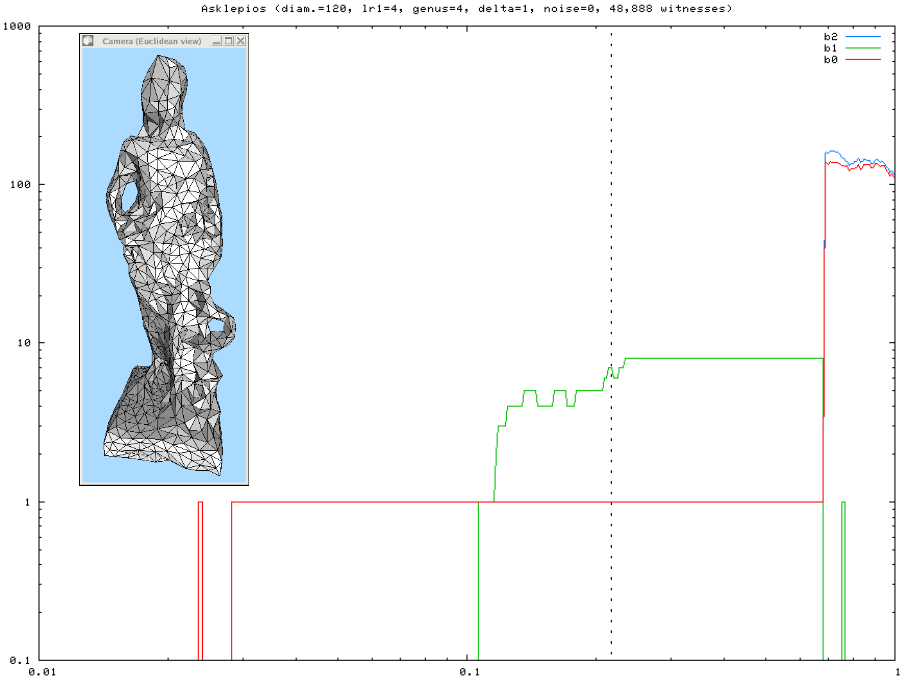

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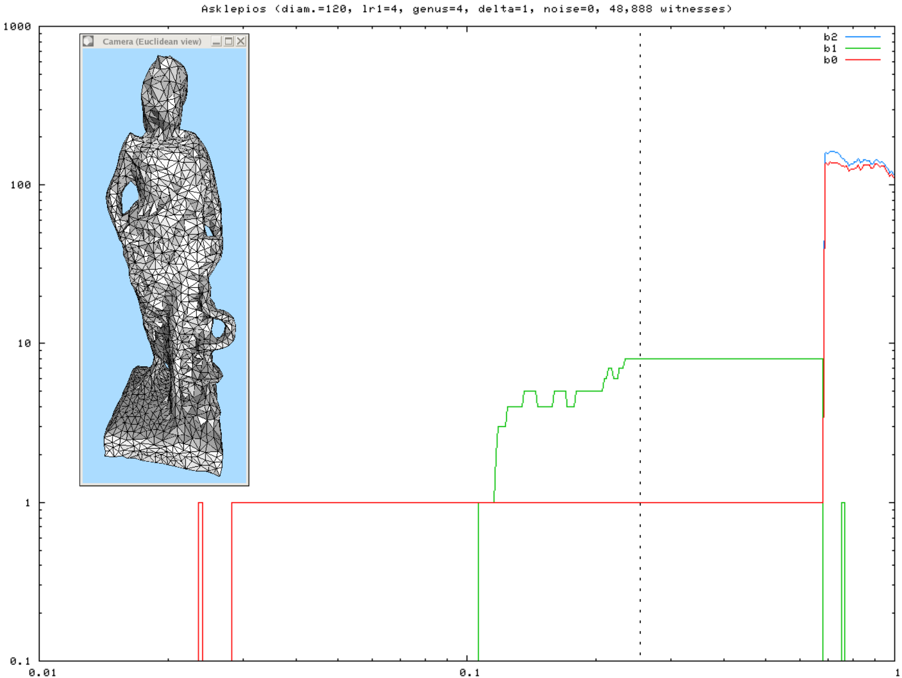

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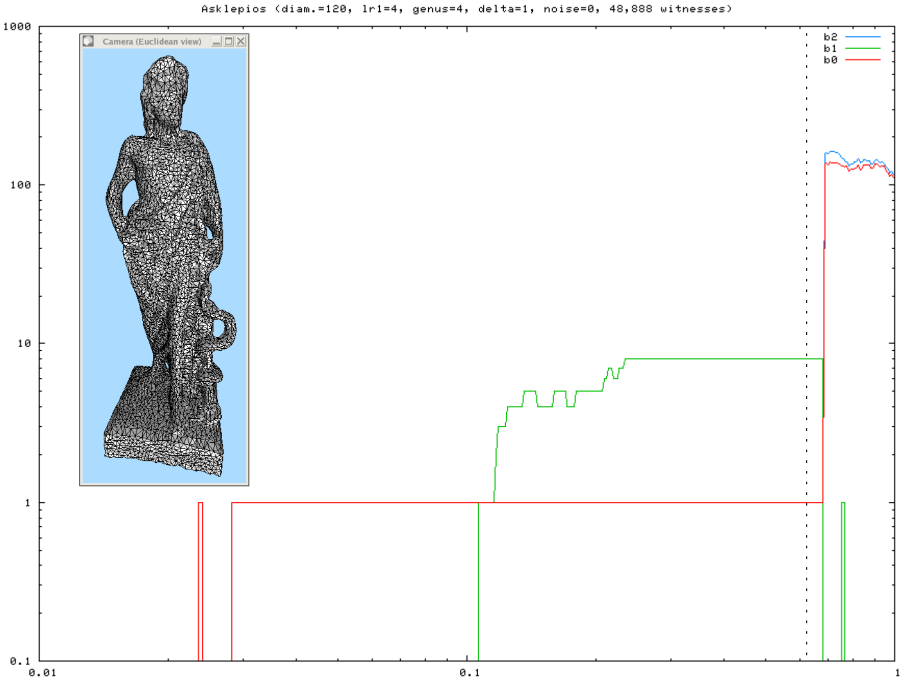

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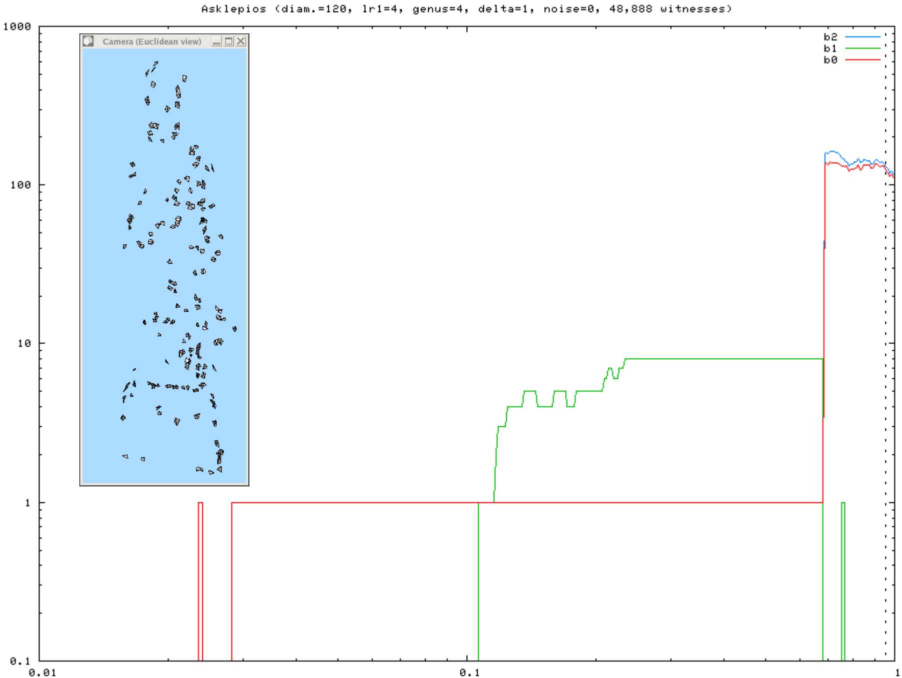

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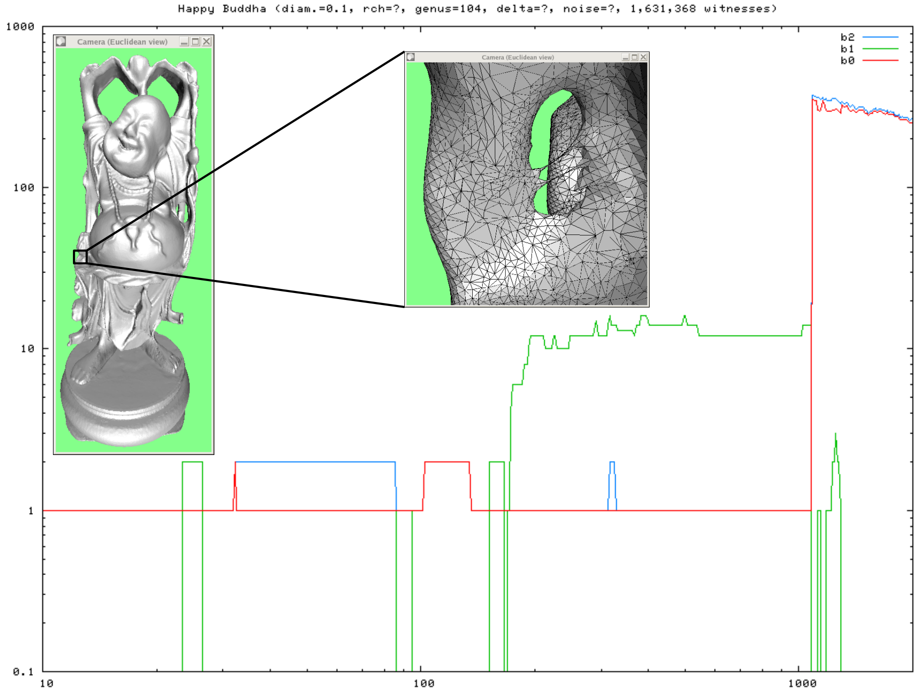

input data set courtesy of the Graphics Lab@Stanford

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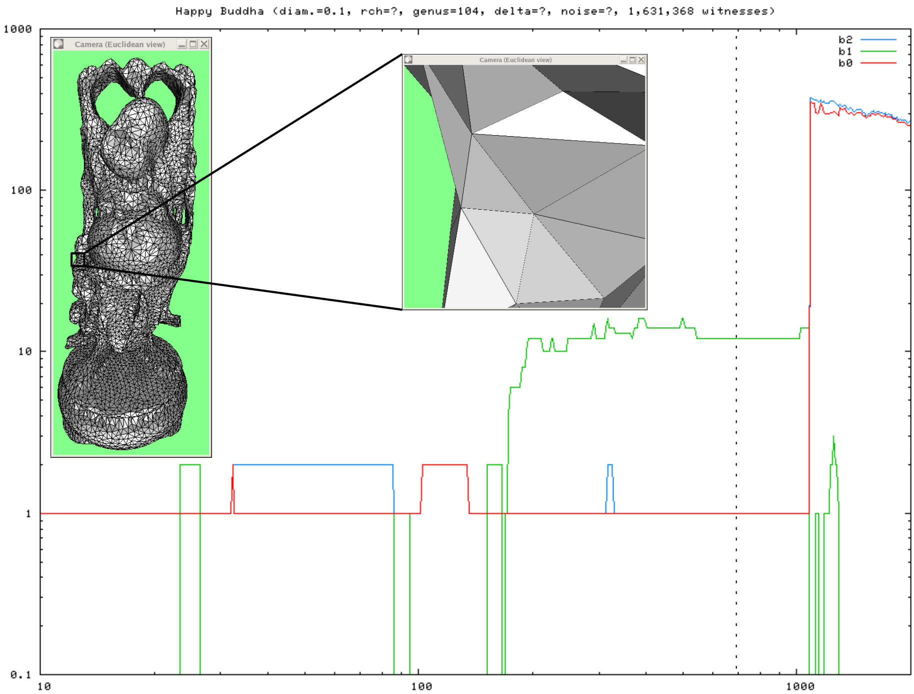

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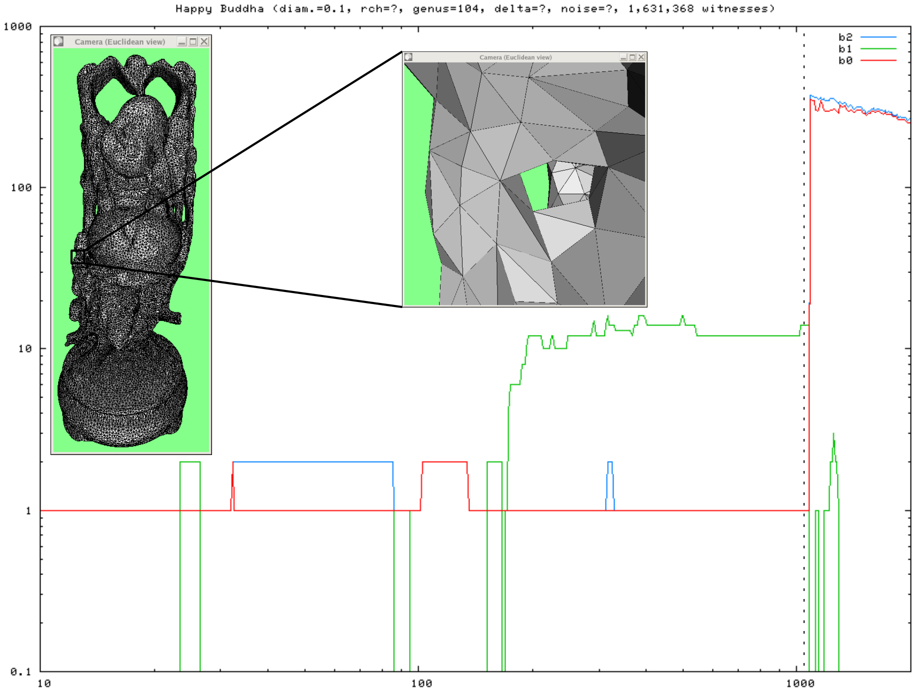

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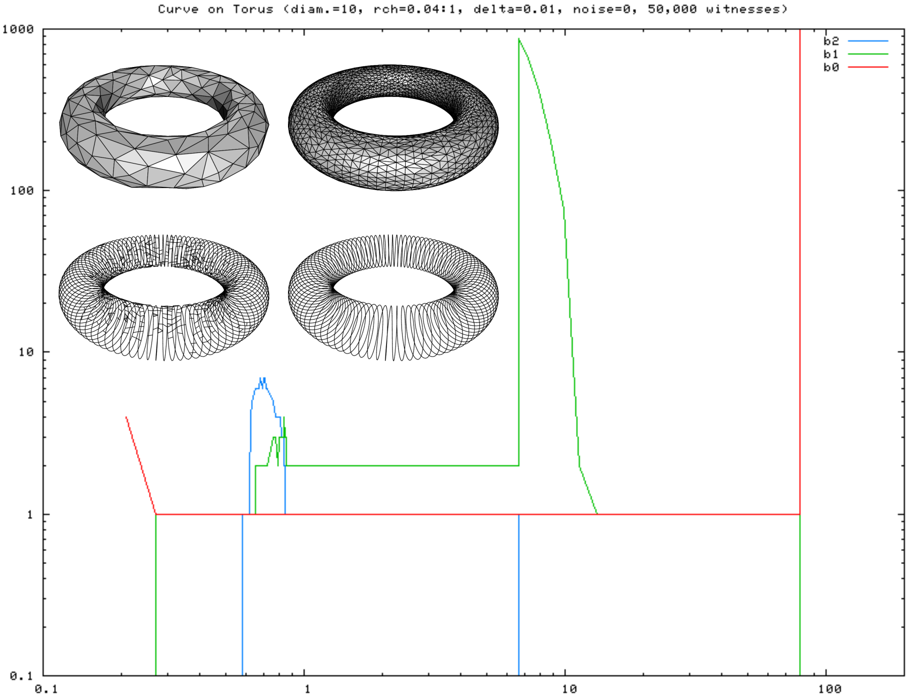

Higher dimensions

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Under some sampling conditions, $\mathrm{C}^W(L) = \mathrm{Del}_{\mathcal{S}}(L) \approx \mathcal{S}$

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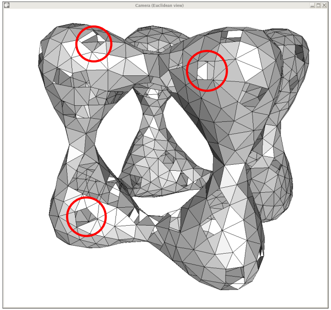

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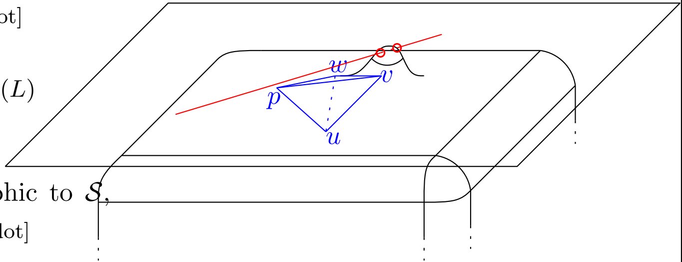

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Higher-dimensional reconstruction is still widely open