A WEAK CHARACTERISATION OF THE DELAUNAY TRIANGULATION

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Abstract. We consider a new construction, the **weak Delaunay triangulation** of a finite point set in a metric space, which contains as a subcomplex the traditional (**strong**) Delaunay triangulation. The two simplicial complexes turn out to be equal for point sets in Euclidean space, as well as in the (hemi)sphere, hyperbolic space, and certain other geometries. There are weighted and approximate versions of the weak and strong complexes in all these geometries, and we prove equality theorems in those cases also. On the other hand, for discrete metric spaces the weak and strong complexes are decidedly different. We give a short empirical demonstration that weak Delaunay complexes can lead to dramatically clean results in the problem of estimating the homology groups of a manifold represented by a finite point sample.

1. Introduction

The Delaunay triangulation and its dual, the Voronoi diagram, are of central importance in the geometry of finite point sets, particularly in the realm of computational geometry [23]. The definitions are natural and clear, and the twin constructions have useful optimality properties.

The purpose of this paper is to distinguish two separate versions of the Delaunay triangulation: the strong form Del (which is the traditional version) and the weak form $\mathrm{Del}^{\mathrm{w}}$ (which is new). The terms 'strong' and 'weak' reflect the fact that $\mathrm{Del} \subseteq \mathrm{Del}^{\mathrm{w}}$, a priori. It is easier for a simplex to belong to $\mathrm{Del}^{\mathrm{w}}$ than to Del. In the case of finite point sets in Euclidean space, the two forms happen to agree: $\mathrm{Del} = \mathrm{Del}^{\mathrm{w}}$. In general, the two complexes can be quite different.

For reasons that will become clear, the equation $\mathrm{Del} = \mathrm{Del}^{\mathrm{w}}$ is called a 'weak witnesses theorem' in any context where it is true. The scope of this paper is limited to the following goals: (i) to define Del and $\mathrm{Del}^{\mathrm{w}}$ in a somewhat general setting; (ii) to prove the weak witnesses theorem for a range of geometrical spaces; (iii) to prove an ϵ-tolerant version of the weak witnesses theorem; (iv) conversely, to demonstrate by example that the difference between Del and $\mathrm{Del}^{\mathrm{w}}$ can matter.

In Chapter 10, we discuss the application which motivated these ideas: how to recover the topology of a manifold (or simplicial complex) from a finite sample of points. We present a strategy for topology recovery based on the weak Delaunay complex, and give a brief summary of results in this direction due to other authors.

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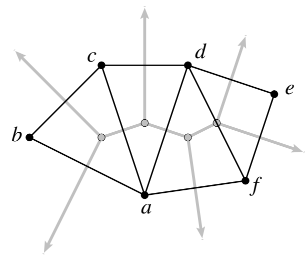

FIGURE 1. The Voronoi diagram and Delaunay triangulation for a set of six points.

2. Statement of the main result

We begin by recalling the traditional definitions. Consider a configuration of points in $\mathbb{R}^n$ labelled by the elements of a set A. We will use the subset notation $A \subset \mathbb{R}^n$, although the points in $\mathbb{R}^n$ need not be distinct. The set A determines a decomposition of $\mathbb{R}^n$ into **Voronoi cells**:

$$V_a = \{x \in \mathbb{R}^n \mid |x - a| \leq |x - b| \text{ for all } b \in A\}$$

The **Voronoi diagram** of A is the covering of $\mathbb{R}^n$ defined by these cells:

$$\mathrm{Vor}(A, \mathbb{R}^n) = \{V_a \mid a \in A\}$$

Finite intersections of Voronoi cells are labelled as follows. For each nonempty finite subset $\sigma \subset A$,

$$V_\sigma = \bigcap_{a \in \sigma} V_a.$$

The **(strong) Delaunay complex**, $\mathrm{Del}(A, \mathbb{R}^n)$, is the nerve of the Voronoi covering. This is the abstract simplicial complex on A defined by the condition

$$\sigma \in \mathrm{Del}(A, \mathbb{R}^n) \Leftrightarrow V_\sigma \neq \emptyset$$

for each nonempty finite subset $\sigma \subset A$. We will give a different, but equivalent, definition later on.

Notation. A nonempty finite subset $\sigma \subset A$ is called a *simplex with vertices in* A. A *p-simplex* is a simplex $\sigma = \{a_0, a_1, \ldots, a_p\}$ of cardinality $p + 1$. We will tend to write $\sigma = [a_0 a_1 \ldots a_p]$ instead of $\sigma = \{a_0, a_1, \ldots, a_p\}$ to emphasise that we are thinking of σ as a simplex. Similarly, when $\tau \subseteq \sigma$, we write $\tau \leq \sigma$ and say that τ is a *subsimplex* of σ.

If A is in general position, $\mathrm{Del}(A, \mathbb{R}^n)$ can be viewed as an n-dimensional geometric simplicial complex which triangulates the convex hull of A. This warrants the familiar name 'Delaunay triangulation'.

Example. In Figure 1, $A = \{a, b, c, d, e, f\}$ is a six-point set in $\mathbb{R}^2$. The edges of its (geometrically realised) Delaunay triangulation of $A = \{a, b, c, d, e, f\}$ are shown in black, and the edges of its Voronoi diagram are in grey.

Remark. For non-generic choices of $A \subset \mathbb{R}^n$, the complex $\mathrm{Del}(A, \mathbb{R}^n)$ may contain simplices of dimension greater than n. Its geometric realisation has a natural map to $\mathbb{R}^n$, but this map cannot be an embedding in such cases. This need not disturb us, provided that we regard $\mathrm{Del}(A, \mathbb{R}^n)$ purely as an abstract simplicial complex.

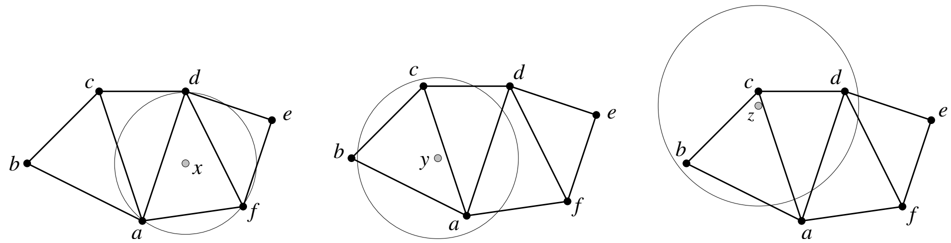

FIGURE 2. Strong and weak witnesses.

All of our theorems and proofs are insensitive to the question of genericity: we never need to presume that A is in general position.

Our theorems about Delaunay complexes are stated in terms of 'witnesses', which we now define.

Definition 2.1. Let σ be a simplex with vertices in A. We say that $x \in \mathbb{R}^n$ is a **strong witness** for σ with respect to A, if $|x - a| \leq |x - b|$ whenever $a \in \sigma$ and $b \in A$. (Note that this implies $|x - a| = |x - a'|$ whenever $a, a' \in \sigma$.)

Definition 2.2. Let σ be a simplex with vertices in A. We say that $x \in \mathbb{R}^n$ is a **weak witness** for σ with respect to A, if $|x - a| \leq |x - b|$ whenever $a \in \sigma$ and $b \in A \setminus \sigma$.

Remark. Let $\sigma = [a_0 a_1 \dots a_p]$ be a p-simplex with vertices in A. Then x is a strong witness for σ with respect to A iff it is a weak witness and additionally $|x - a_0| = |x - a_1| = \cdots = |x - a_p|$.

When the context is clear, we may omit the phrase 'with respect to A'.

Example. The witness conditions can be interpreted in an obvious way in terms of spheres centered on x. In the left panel of Figure 2, x is a strong witness for triangle $[adf]$. In the center panel, y is a weak witness for edge $[ac]$ and indeed for triangle $[abc]$. In the right panel, z is a weak witness for $[bcd]$. The circumcenter of $[bcd]$ lies close to a, so $[bcd]$ does not have a strong witness.

It follows from the definition of Voronoi cells that $\sigma \in \mathrm{Del}(A, \mathbb{R}^n)$ if and only if σ has a strong witness. Thus we can make the following alternative definition.

Definition 2.3. The **strong Delaunay complex**, $\mathrm{Del}(A, \mathbb{R}^n)$, is the abstract simplicial complex on A defined by the following condition:

$$\sigma \in \mathrm{Del}(A, \mathbb{R}^n) \Leftrightarrow \sigma \text{ has a strong witness in } \mathbb{R}^n$$

If σ has a strong witness x, then x is automatically a weak witness for every subsimplex $\tau \leq \sigma$. This suggests the following construction.

Definition 2.4. The **weak Delaunay complex**, $\mathrm{Del}^{\mathrm{w}}(A, \mathbb{R}^n)$, is the abstract simplicial complex on A defined by the following condition:

$$\sigma \in \mathrm{Del}^{\mathrm{w}}(A, \mathbb{R}^n) \Leftrightarrow \text{every subsimplex } \tau \leq \sigma \text{ has a weak witness in } \mathbb{R}^n$$

Thus we have $\mathrm{Del}(A, \mathbb{R}^n) \subseteq \mathrm{Del}^{\mathrm{w}}(A, \mathbb{R}^n)$, a priori. The main result of this paper is that the converse inclusion holds, so that $\mathrm{Del}(A, \mathbb{R}^n) = \mathrm{Del}^{\mathrm{w}}(A, \mathbb{R}^n)$:

Theorem 2.5 (weak witnesses theorem)**.** *Let $A \subset \mathbb{R}^n$ and let $\sigma \subset A$ be a simplex. Then σ has a strong witness in $\mathbb{R}^n$ if and only if every subsimplex $\tau \le \sigma$ has a weak witness in $\mathbb{R}^n$.*

Theorem 2.5 is proved in Section 3 using standard machinery from convex optimisation; and reproved in Section 4 more directly, by induction on σ.

Example. If $abcd$ is a convex quadrilateral in the plane, then each of the vertices $[a]$, $[b]$, $[c]$, $[d]$, each of the boundary edges $[ab]$, $[bc]$, $[cd]$, $[da]$ and each of the triangles $[abc]$, $[bcd]$, $[cda]$, $[dab]$ has a weak witness with respect to $\{a, b, c, d\}$. If $abcd$ is not cyclic, then exactly one of the diagonal edges $[ac]$, $[bd]$ has a weak witness. For instance, in Figure 2, edge $[ac]$ has a weak witness but edge $[bd]$ does not. The theorem predicts that the two triangles containing that edge have strong witnesses; and indeed the Delaunay triangulation consists of those two triangles joined along a common edge.

Theorem 2.5 was inspired by work of Martinetz and Schulten [18], who treated the special case where σ is an edge. In our language, they showed that $\mathrm{Del}(A, \mathbb{R}^n)$ and $\mathrm{Del}^{\mathrm{w}}(A, \mathbb{R}^n)$ have the same 1-skeleton. In their paper, they propose a dynamic algorithm for approximating the topology of a region of space, by a graph represented as a neural network. They define a 'competitive Hebbian' update rule: whenever a weak witness is found for an edge of the network, that edge is strengthened. The resulting graph is equivalent (in the ideal limit) to the Delaunay graph of the region, thanks to their lemma.

The present work was motivated by the need to build higher-dimensional simplicial complex approximations to point-cloud data sets. 'Witness complexes' of various types were introduced by Carlsson and the present author [11]. Theorem 2.5 provides a (heuristic) justification for using the weak witness complex to recover topological information. Such applications will be discussed briefly in Section 10. However, this is not a paper on manifold reconstruction; we focus on the weak witnesses theorem in its various forms.

We finish this section by expanding the context of definition of $\mathrm{Del}, \mathrm{Del}^{\mathrm{w}}$.

Definition 2.6. Let X be a metric space and $A \subset X$. Let $\sigma \subset A$ be a simplex. We can define weak and strong witnesses for σ exactly as before, using the metric in X in place of the Euclidean distance in $\mathbb{R}^n$:

$$\begin{aligned} x \text{ is a strong witness for } \sigma \quad &\Leftrightarrow \quad d(x,a) \le d(x,b) \text{ whenever } a \in \sigma \text{ and } b \in A \\ x \text{ is a weak witness for } \sigma \quad &\Leftrightarrow \quad d(x,a) \le d(x,b) \text{ whenever } a \in \sigma \text{ and } b \in A \setminus \sigma \end{aligned}$$

For any subset $B \subseteq X$, we can define strong and weak Delaunay complexes:

$$\begin{aligned} \sigma \in \mathrm{Del}(A, B) \quad &\Leftrightarrow \quad \sigma \text{ has a strong witness in } B \\ \sigma \in \mathrm{Del}^{\mathrm{w}}(A, B) \quad &\Leftrightarrow \quad \text{every subsimplex } \tau \le \sigma \text{ has a weak witness in } B \end{aligned}$$

Remark. The metric on X serves merely to provide real-valued functions f_x on A, namely $f_x(a) = d(x, a)$. More generally, if A is a set and X indexes a set of arbitrary real-valued functions $\{f_x\}$ on A, then strong and weak witnesses and Delaunay complexes can be defined in terms of the f_x. For instance, x is a strong witness for σ if $f_x(a) \le f_x(b)$ for all $a \in \sigma$ and $b \in A$. This point of view is adopted increasingly frequently from Section 5 onwards.

The rest of the paper is organised as follows. We will give two proofs of Theorem 2.5. The proof in section 3 reduces the problem to one of the standard convex duality theorems, using the well-known device of lifting the data to $\mathbb{R}^{n+1}$. The proof in section 4 constructs the strong witness to a simplex directly from the cluster of weak witnesses associated to the simplex, using an inductive argument. This second proof depends only on the convexity properties of Voronoi diagrams, and is extended to several non-Euclidean geometric settings in Section 5. Some necessary background in spherical and hyperbolic geometry is given in Section 6. Weights and approximations are introduced in Sections 7 and 8, respectively. Manifolds of constant curvature are discussed in Section 9. Finally, in Section 10, we briefly discuss how these ideas connect with the manifold reconstruction problem.

3. Proof of Theorem 2.5 by convex duality

There is a well-known 'lifting' procedure which expresses Delaunay geometry in $\mathbb{R}^n$ in terms of convex geometry in $\mathbb{R}^{n+1}$. Under this transformation, the weak witnesses theorem becomes a statement about the faces of a certain convex polytope. For our first proof, we will derive that result from Farkas' lemma in convex optimisation; specifically, from a general form of that lemma known as Motzkin's transposition theorem [19, 20].

We start with the convex polytope result. The following two definitions modify two familiar notions — faces of a polytope, separability by a hyperplane — with an extra condition of compatibility with a direction vector v.

Definition 3.1. Let $A \subset \mathbb{R}^n$ and let $v \in \mathbb{R}^n$ be any non-zero vector. A subset $\sigma \subset A$ is said to be a v**-face** of A if there exists $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}$ such that

$$\begin{array}{rcll} v \cdot \xi & > & 0 & \\ a \cdot \xi & = & \eta, & \text{for all } a \in \sigma \\ b \cdot \xi & \leq & \eta, & \text{for all } b \in A \setminus \sigma \end{array}$$

or, equivalently, if there exists $\xi \in \mathbb{R}^n$ with $v \cdot \xi > 0$, and $a \cdot \xi \geq b \cdot \xi$ whenever $a \in \sigma$ and $b \in A$.

Definition 3.2. A subset $\sigma \subset A$ is said to be v**-separable** in A if there exists $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}$ such that

$$\begin{array}{rcll} v \cdot \xi & > & 0 & \\ a \cdot \xi & \geq & \eta, & \text{for all } a \in \sigma \\ b \cdot \xi & \leq & \eta, & \text{for all } b \in A \setminus \sigma \end{array}$$

or, equivalently, if there exists $\xi \in \mathbb{R}^n$ with $v \cdot \xi > 0$, and $a \cdot \xi \geq b \cdot \xi$ whenever $a \in \sigma$ and $b \in A \setminus \sigma$.

If σ is a v-face then, in particular, its vertices are contained in a hyperplane (with the remaining points on one side of the hyperplane); whereas if it is merely v-separable then its vertices are separated by a hyperplane from the remaining points. The vector v indicates which side of the hyperplane contains the vertices of σ (the other side containing all the rest of the points). See Figure 3.

Clearly, if σ is a v-face then it is v-separable. More strongly, if σ is a v-face (evidenced by some choice of ξ, η) then any $\tau \subseteq \sigma$ is v-separable (by the same ξ, η). In fact, the converse is also true:

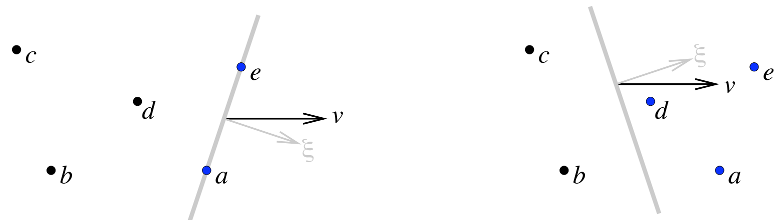

FIGURE 3. For this configuration of five points and vector v, we see that $[ae]$ is a v-face (left) and $[ade]$ is v-separable (right). The edge $[bc]$ is not a v-face, nor even v-separable, for this v.

Theorem 3.3. *Let $A \subset \mathbb{R}^n$ be finite, let $v \in \mathbb{R}^n$ be non-zero, and let $\sigma \subset A$. Then σ is a v-face of A if and only if every subset $\tau \subseteq \sigma$ is v-separable in A.*

Note that this theorem fails if we drop the condition involving the direction vector v. Consider, for example, the vertices of a non-degenerate triangle $[abc]$ in $\mathbb{R}^2$. Every subset of $\{a, b, c\}$ is separable (from its complement) by a line, but there is no line which contains all three points. This does not violate the theorem; for any particular choice of v, there is either a 'hidden' vertex $[a]$, $[b]$ or $[c]$, or else a single 'hidden' edge $[ab]$, $[bc]$ or $[ac]$, which is not v-separable in $\{a, b, c\}$.

Proof that (Theorem 3.3 ⇒ Theorem 2.5). (This assumes that A is finite.) Define the quadratic lift operation:

$$\begin{array}{rcll} a^+ & = & (a, \frac{1}{2}|a|^2) & \quad \text{for } a \in \mathbb{R}^n \\ A^+ & = & \{a^+ \mid a \in A\} & \quad \text{for } A \subset \mathbb{R}^n \end{array}$$

For $x \in \mathbb{R}^n$, it is easily verified that

$$|x - a| \leq |x - b| \;\Leftrightarrow\; a^+ \cdot \xi \geq b^+ \cdot \xi$$

where $\xi = (x, -1) \in \mathbb{R}^{n+1}$. Note that in Definitions 3.1 and 3.2 we can replace the condition $v \cdot \xi > 0$ with $v \cdot \xi = 1$ by rescaling. It follows that Theorem 2.5 for $A \in \mathbb{R}^n$ is equivalent to Theorem 3.3 for $A^+ \subset \mathbb{R}^{n+1}$ with $v = -e_{n+1} = (0, \ldots, 0, -1)$. □

Here is the duality result which we use to prove Theorem 3.3.

Theorem 3.4 (Motzkin's transposition theorem [19, 20])**.** *Let P, Q, R be matrices with the same number of columns. Then precisely one of the following alternatives holds:*

(1) *There is a solution w to the system $Pw > 0$, $Qw \geq 0$, $Rw = 0$.*
(2) *There exist p, q, r such that*

$$P^t p + Q^t q + R^t r = 0$$

where $p \geq 0$, $q \geq 0$ and p is not the zero vector.

Here w, p, q, r are column vectors of the appropriate lengths, and the equalities and inequalities are taken componentwise. □

Proof of Theorem 3.3. We define matrices P, Q, R with $n + 1$ columns as follows: P is the row-vector $[v^t, 0]$. The matrix Q has a row $[-b^t, 1]$ for every $b \in A \setminus \sigma$. The matrix R has a row $[-a^t, 1]$ for every $a \in \sigma$. Using the interpretation $w^t = [\xi^t, \eta]$, we see that alternative (1) of the Motzkin theorem is tautologically equivalent to the statement that σ is a v-face of A.

Suppose now that σ is not a v-face of A. Then alternative (2) must hold, for a positive scalar p, a non-negative vector q, and an arbitrary vector r. Note that q has an entry for each element $b \in A \setminus \sigma$, and r has an entry for each element $a \in \sigma$. Let τ denote the subset of σ corresponding to the negative entries of r. We now consider a new matrix $\tilde{Q}$ which has the following rows:

$$\begin{array}{ll} [-b^t, 1] & \text{for each } b \in A \setminus \sigma \\ [-a^t, 1] & \text{for each } a \in \sigma \setminus \tau \\ [a^t, -1] & \text{for each } a \in \tau \end{array}$$

We define a vector $\tilde{q}$ by concatenating q with $|r|$, the vector of absolute values of the entries of r. Then $\tilde{q} \geq 0$ and $P^t p + \tilde{Q}^t \tilde{q} = 0$ by construction. It follows that alternative (2) in the Motzkin theorem holds for the pair of matrices $P, \tilde{Q}$ (the third matrix is empty). Hence there is no solution w to the system $Pw > 0$, $\tilde{Q}w \geq 0$. Under the interpretation $w^t = [\xi^t, \eta]$ this precisely the assertion that τ is not v-separable.

We have shown that if σ is not a v-face of A then some $\tau \subseteq \sigma$ is not v-separable in A. The converse direction is trivial: if σ is a v-face then each $\tau \subseteq \sigma$ is v-separable using the same ξ, η. □

This completes the proof of Theorem 2.5 by convex duality.

4. Proof of Theorem 2.5 by induction

The strategy for our second proof[1] of Theorem 2.5 is to construct the strong witness for σ directly as a convex combination of weak witnesses for the subsimplices of σ. We use an induction argument based on the well-known case [18] where σ is an edge. In subsequent sections, we present several generalisations of the weak witnesses theorem which are based on this method of proof.

We strengthen the theorem slightly to give traction to the induction argument.

Theorem 4.1. *Let $A \subset \mathbb{R}^n$ and let $\sigma \subset A$ be a simplex. Suppose that for every subsimplex $\tau \leq \sigma$ there exists a weak witness $x_\tau \in \mathbb{R}^n$ for τ with respect to A. Then the convex hull of the points x_τ contains a strong witness for σ with respect to A.*

Proof. It is useful notation to write

$$R(a, b) = \{x \in \mathbb{R}^n \mid |x - a| \leq |x - b|\}$$

for $a, b \in A$. These sets are convex.

The proof proceeds by induction on $|\sigma|$. The base case $|\sigma| = 1$ is trivial. Suppose $|\sigma| \geq 2$, and let $a_0, a_1 \in \sigma$ be distinct. Consider the subspace

$$H = R(a_0, a_1) \cap R(a_1, a_0) = \{x \in \mathbb{R}^n \mid |x - a_0| = |x - a_1|\}$$

and let $\sigma' = \sigma \setminus \{a_0\}$ and $A' = A \setminus \{a_0\}$. We adopt the following plan. For each $\tau \leq \sigma'$ we find a weak witness $y_\tau \in H$ for τ with respect to A'. By the inductive hypothesis, we can find a strong witness x' for σ' with respect to A'. Since H is convex, the construction gives $x' \in H$. Then x' is a strong witness for σ with

[1] A third proof, given by Attali et al. [3], uses an attractive inductive argument of a more directly geometric flavour than the one given here.

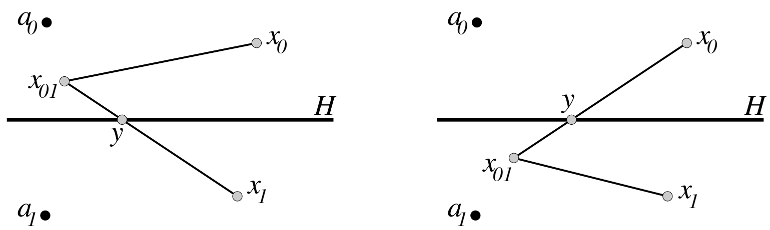

FIGURE 4. Locating y when $a_1 \in \tau$.

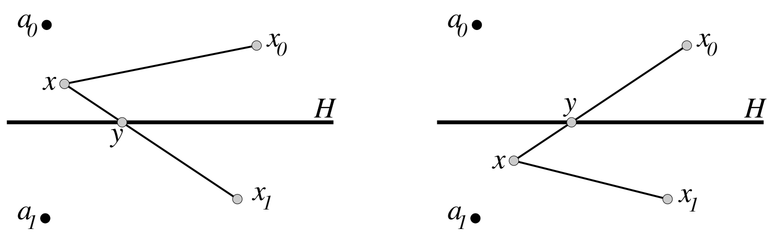

FIGURE 5. Locating y when $a_1 \notin \tau$.

respect to A. Moreover, the points y_τ lie in the convex hull of the original weak witnesses x_τ, so the same is true for x'.

It remains to locate a weak witness y_τ, for each $\tau \leq \sigma'$. The four cases are illustrated in Figures 4 and 5.

Suppose first that $a_1 \in \tau$, so we can write $\tau = \upsilon \cup \{a_1\}$ where $a_1 \notin \upsilon$. Let x_0, x_1, x_{01} be the weak witnesses for $\upsilon \cup \{a_0\}$, $\upsilon \cup \{a_1\}$, $\upsilon \cup \{a_0, a_1\}$ with respect to A. By definition $x_0 \in R(a_0, a_1)$ and $x_1 \in R(a_1, a_0)$. Now either $x_{01} \in R(a_0, a_1)$ or $x_{01} \in R(a_1, a_0)$. If $x_{01} \in R(a_0, a_1)$, it follows by continuity that the line segment $[x_{01}, x_1]$ meets H at some point y. Since $x_{01}, x_1 \in R(a, b)$ for all $a \in \tau$, $b \in A' \setminus \tau$, then the same is true for y (by convexity), so y is the required weak witness for τ. If, on the other hand, $x_{01} \in R(a_1, a_0)$, then the line segment $[x_{01}, x_0]$ must meet H at some point y. Since $x_{01}, x_0 \in R(a, b)$ for all $a \in \upsilon \cup \{a_0\}$, $b \in A' \setminus \tau$, then the same is true for y. Moreover, $|y - a_0| = |y - a_1|$, so $y \in R(a_1, b)$ for all $b \in A' \setminus \tau$, and again y is a weak witness for τ.

Now suppose that $a_1 \notin \tau$. Let x, x_0, x_1 be the weak witnesses for τ, $\tau \cup \{a_0\}$, $\tau \cup \{a_1\}$ with respect to A. By definition $x_0 \in R(a_0, a_1)$ and $x_1 \in R(a_1, a_0)$. Either $x \in R(a_0, a_1)$ or $x \in R(a_1, a_0)$. If $x \in R(a_0, a_1)$, then the line segment $[x, x_1]$ meets H at some point y. Since $x, x_1 \in R(a, b)$ for all $a \in \tau$, $b \in A \setminus (\tau \cup \{a_1\})$, then the same is true for y. Moreover, $|y - a_0| = |y - a_1|$, so $y \in R(a, a_1)$ for all $a \in \tau$, and therefore y is a weak witness for τ. If, on the other hand, $x \in R(a_1, a_0)$, then the line segment $[x, x_0]$ must meet H at some point y. Since $x, x_0 \in R(a, b)$ for all $a \in \tau$, $b \in A' \setminus \tau$, then the same is true for y, and once again y is a weak witness for τ.

This completes the proof of Theorem 4.1 and hence of Theorem 2.5. $\square$

Remark. Whereas Theorem 2.5 is equivalent to the assertion that $\mathrm{Del}(A, \mathbb{R}^n) = \mathrm{Del}^{\mathrm{w}}(A, \mathbb{R}^n)$, the slightly sharper Theorem 4.1 is equivalent to the assertion that $\mathrm{Del}(A, K) = \mathrm{Del}^{\mathrm{w}}(A, K)$ for all convex $K \subseteq \mathbb{R}^n$.

5. Voronoi convexity and non-euclidean geometries

Theorem 4.1 remains true when we move from $\mathbb{R}^n$ to certain standard non-Euclidean geometries. If we are careful, we can use the same proof. The following notion of 'Voronoi convexity' encapsulates a sufficient set of ingredients for the proof to go through.

Hypothesis 5.1 (Voronoi convexity). *Assume these four postulates:*

(1) *X is a topological space, A is a set.*
(2) *For each $a \in A$, there is a continuous function $d(a, -) : X \to \mathbb{R}$.*
(3) *Each unordered pair $x, y \in X$ is contained in a connected set $\gamma(x, y) \subset X$.*
(4) *For each $a, b \in A$ the set*
$$R(a, b) = \{x \in X \mid d(a, x) \leq d(b, x)\}$$
is convex, in the sense that $\gamma(x, y) \subset R(a, b)$ whenever $x, y \in R(a, b)$.

The first two postulates allow us to define strong and weak witnesses, and Delaunay complexes $\mathrm{Del}(A, X)$, $\mathrm{Del}^{\mathrm{w}}(A, X)$, by means of Definition 2.6.

The third postulate sets up a notion of convexity on X, and the fourth postulate amounts to the assertion that Voronoi cells
$$V_a = \{x \in X \mid d(a, x) \leq d(b, x) \text{ for all } b \in A\}$$
are convex. The weak witnesses theorem holds under these four assumptions.

Theorem 5.2 (weak witnesses theorem, general form). *Assume Hypothesis 5.1. A simplex $\sigma \subset A$ has a strong witness $x \in X$ with respect to A if and only if every subsimplex $\tau \leq \sigma$ has a weak witness $x_\tau \in X$ with respect to A. Moreover, x can be taken to lie in the convex hull of the points x_τ.*

(The convex hull is defined as the smallest convex set containing the given points.)

Proof. As before, any strong witness x for σ is necessarily a weak witness for each subsimplex $\tau \leq \sigma$. For the reverse implication, we follow the proof of Theorem 4.1 almost verbatim, replacing Euclidean distances $|x - a|$ with $d(a, x)$, and line segments $[x, y]$ with $\gamma(x, y)$, in each instance.

We note that the subspace $H = R(a_0, a_1) \cap R(a_1, a_0)$ is convex, being an intersection of two convex sets. We also note that if $x \in R(a_0, a_1)$ and $y \in R(a_1, a_0)$ then $\gamma(x, y)$ meets H. This follows because $R(a_0, a_1)$ and $R(a_1, a_0)$ are closed sets which cover X, whereas $\gamma(x, y)$ is connected and meets both of them. Subject to these comments, the proof goes through unchanged. $\square$

Remark. Theorem 5.2 is equivalent to the assertion that $\mathrm{Del}(A, K) = \mathrm{Del}^{\mathrm{w}}(A, K)$ for every convex $K \subseteq X$.

Example 5.3. Let X be a convex subset of $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$. Let $\gamma(x, y)$ denote the closed line segment from x to y, and let $d(a, x) = |x - a|$.

Example 5.4. Let X be a convex subset of hyperbolic space $\mathcal{H}^n$. Thus for each pair $x, y \in X$ the unique connecting geodesic $\gamma(x, y)$ is contained in X. Let $A \subset \mathcal{H}^n$ and let $d(a, x) = u(a, x)$ be the geodesic distance from a to x.

Example 5.5. Let X be a convex subset of an open hemisphere of the unit Euclidean sphere $\mathcal{S}^n \subset \mathbb{R}^{n+1}$. Let $A \subset \mathcal{S}^n$ and let $d(a, x) = \theta(a, x)$ be the geodesic distance from a to x. The interpretation of 'convex' is that X contains 'the' minimal geodesic $\gamma(x, y)$ whenever $x, y \in X$. Because of the restriction to an open hemisphere, X contains no antipodal pair and so $\gamma(x, y)$ is unique. This restriction is necessary for the weak witnesses theorem.

Example 5.6. Let $\mathcal{T}$ be a tree with edges of positive length, so any pair $x, y \in \mathcal{T}$ is connected by a unique path $\gamma(x, y)$, of length $\ell(x, y)$. Let X be any connected subset of $\mathcal{T}$, let $A \subset \mathcal{T}$, and let $d(a, x) = \ell(a, x)$.

Proposition 5.7. *The hypothesis of Voronoi convexity (Hypothesis 5.1) is satisfied in each of the Examples 5.3–5.6.*

Proof. In Examples 5.3 and 5.4, the region $R(a, b)$ is in each case a half-space intersected with X. Since half-spaces are convex, $R(a, b)$ is also convex. In Example 5.5, $R(a, b)$ is the intersection of a hemisphere with X. A closed hemisphere is not convex, but if it contains two non-antipodal points x, y then it necessarily contains the unique minimal geodesic $\gamma(x, y)$. Thus its intersection with X is convex.

For Example 5.6, fix $a, b \in A$ and for $x \in X$ let $\pi(x)$ denote the closest point on $\gamma(a, b)$ to x. Then

$$\ell(a, x) - \ell(b, x) = \ell(a, \pi(x)) - \ell(b, \pi(x)) \tag{1}$$

It follows that $R(a, b)$ is equal to the connected set $\pi^{-1}(\gamma(a, m))$, where m is the midpoint of (a, b). Since (connected $\Leftrightarrow$ convex) for subsets of a tree, it follows that $R(a, b)$ is convex. □

Corollary 5.8. *The weak witnesses theorem holds for the geodesic metric in Euclidean space $\mathbb{R}^n$, the hemisphere $\mathcal{S}^n_+$, hyperbolic space $\mathcal{H}^n$, and in any tree $\mathcal{T}$.* □ *(In the case $X = \mathcal{S}^n_+$, the set A is permitted to be any subset of the sphere $\mathcal{S}^n$.)*

Example 5.9. Let $A \subset \mathbb{R}^n$, let $X = \{x \in \mathbb{R}^n \mid x \cdot v > 0\}$ for some fixed vector $v \in \mathbb{R}^n$, and let $\gamma(x, y)$ be the line segment from x to y. If we define $d(a, x) = a \cdot x$ then Hypothesis 5.1 is clearly satisfied. The conclusion of the weak witnesses theorem in this case is precisely the conclusion of Theorem 3.3. This gives an independent proof of that result which does not require A to be finite.

Example 5.10 (Special relativity)**.** Let $\mathbb{R}^{1,n}$ denote the Minkowski space with signature $(+, -, \ldots, -)$. Let $A \subset \mathbb{R}^{1,n}$, let X be the positive light-cone, and let $\gamma(x, y)$ be the line segment from x to y. If we define $d(a, x) = a \star x$, where $\star$ is the Minkowski inner product, then Hypothesis 5.1 is satisfied. Each vector $x \in X$ specifies a direction for the space-time path of an inertial observer. If we view A as a set of events, then $a \star x$ can be interpreted as the time of occurrence of event $a \in A$ as measured in the frame specified by x (up to translation and rescaling). In these terms, the weak witnesses theorem amounts to the following fact:

Corollary 5.11. *Given a finite subset $\sigma \subset A$, the following statements are equivalent: (1) There exists an observer for whom the events in σ occur simultaneously in time, and are not preceded by any events in $A \setminus \sigma$. (2) For each $\tau \subseteq \sigma$ there exists an observer for whom none of the events in τ is preceded by any of the events in $A \setminus \tau$.* □

Example 5.12. Consider a collection of hyperplanes in $\mathbb{R}^n$ labelled by a set A. The hyperplanes are oriented in the sense of dividing $\mathbb{R}^n$ into designated positive and negative halfspaces. For a finite set $\sigma \subset A$, the following statements are equivalent. (1) There is a sphere in $\mathbb{R}^n$ which is tangent to each hyperplane in σ, and whose interior lies entirely on the positive side of every hyperplane in A. (2) For every $\tau \subseteq \sigma$ there is a sphere which meets every hyperplane in τ (not necessarily tangentially) and whose interior lies entirely on the positive side of every hyperplane in $A \setminus \tau$.

Proof. Let n_a denote the negative unit normal to the hyperplane $a \in A$; and set $d(a, x) = n_a \cdot x$, for $x \in \mathbb{R}^n$. The center of each sphere in (2) is a weak witness for τ, and the center of the sphere in (1) is a strong witness for σ, so the result follows from Theorem 5.2. □

6. Interlude: spherical and hyperbolic geometry

In subsequent sections, we consider weighted and approximate versions of the weak witnesses theorem. In this section, we state some basic information on spherical and hyperbolic geometries which will allow us to extend our results to these geometries quite transparently, with minimal difficulty. The two cases are treated in a closely parallel fashion.

The n-sphere $\mathcal{S}^n$ and hyperbolic n-space $\mathcal{H}^n$ have natural models inside $\mathbb{R}^{n+1}$. Vectors in $\mathbb{R}^{n+1}$ may be written in the form $x = (x_0, x_1, \ldots, x_n)$ where $x_i \in \mathbb{R}$; or in the form $(s; u)$ where $s \in \mathbb{R}$, $u \in \mathbb{R}^n$. For $x = (s; u)$ and $y = (t; v)$ in $\mathbb{R}^{n+1}$, we have the Euclidean inner product:

$$\begin{aligned} x \cdot y &= x_0 y_0 + x_1 y_1 + \cdots + x_n y_n \\ &= st + u \cdot v \end{aligned}$$

and the Minkowski inner product:

$$\begin{aligned} x \star y &= x_0 y_0 - (x_1 y_1 + \cdots + x_n y_n) \\ &= st - u \cdot v \end{aligned}$$

We make the following definitions:

$$\begin{aligned} \mathcal{S}^n &= \left\{ x \in \mathbb{R}^{n+1} \mid x \cdot x = 1 \right\} \\ \mathcal{H}^n &= \left\{ x \in \mathbb{R}^{n+1} \mid x \star x = 1 \text{ and } x_0 > 0 \right\} \end{aligned}$$

The spaces $\mathcal{S}^n$ and $\mathcal{H}^n$ have natural Riemannian metrics which are preserved under the respective groups of symmetries of the two inner products. These are unique up to a normalisation. In the standard normalisation, the metric on the tangent space at $(1; 0)$ agrees with the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$.

If V is a linear subspace of $\mathbb{R}^{n+1}$ then $V \cap \mathcal{S}^n$ is a great hypersphere of $\mathcal{S}^n$, and $V \cap \mathcal{H}^n$ is a hyperplane of $\mathcal{H}^n$ (or empty). For our purposes, the most useful formulation is this:

Proposition 6.1. *Halfspaces (i.e. hemispheres) of $\mathcal{S}^n$ are defined by inequalities in x of the form*

$$a \cdot x \geq 0$$

where $a \cdot a > 0$. Halfspaces of $\mathcal{H}^n$ are defined by inequalities in x of the form

$$a \star x \geq 0 \tag{2}$$

where $a \star a < 0$. In both cases, there is a one-to-one correspondence between halfspaces and normalised *vectors a (so $a \cdot a = 1$ or $a \star a = -1$).* □

Proposition 6.2. *If $a \star a \geq 0$ then the inequality $a \star x \geq 0$ in x is satisfied on all of $\mathcal{H}^n$ or none of it (according to the sign of a_0). Thus, for all values of a, the inequality $a \star x \geq 0$ is a convex condition on $x \in \mathcal{H}^n$.* □

Geodesic distances on $\mathcal{S}^n$ and $\mathcal{H}^n$ can be expressed in terms of the inner products.

Proposition 6.3. *If $\theta(x, y)$ denotes the geodesic distance between $x, y \in \mathcal{S}^n$ then:*

$$x \cdot y = \cos(\theta(x, y)) \tag{3}$$

If $u(x, y)$ denotes the geodesic distance between $x, y \in \mathcal{H}^n$ then:

$$x \star y = \cosh(u(x, y)) \tag{4}$$

□

The signed distance to an oriented great hypersphere or hyperplane can also be computed using the inner product. By 'oriented' we mean here that one side is designated the positive side; points on the other side will have negative signed distance from it. We can identify an oriented hypersphere with its positive hemisphere, and hence, by Proposition 6.1, with a unique vector $a \in \mathbb{R}^{n+1}$ satisfying $a \cdot a = 1$. Similarly we can identify an oriented halfspace of $\mathcal{H}^n$ with a unique vector $a \in \mathbb{R}^{n+1}$ satisfying $a \star a = -1$.

Proposition 6.4. *If $\theta(x, H_a)$ denotes the signed distance between x and the oriented hypersphere $H_a \subset \mathcal{S}^n$ corresponding to the vector $a \in \mathbb{R}^{n+1}$, then:*

$$a \cdot x = \sin(\theta(x, H_a)) \tag{5}$$

If $u(x, H_a)$ denotes the signed distance between x and the oriented hyperplane $H_a \subset \mathcal{H}^n$ corresponding to the vector $a \in \mathbb{R}^{n+1}$, then:

$$a \star x = \sinh(u(x, H_a)) \tag{6}$$

□

It may be helpful to think of Propositions 6.3 and 6.4 in terms of exponential maps. Given $s \in \mathbb{R}$, and a unit vector $u \in \mathbb{R}^n$, we define $\exp_{(1;0)}(su)$ to be the point reached after travelling from $(1; 0)$ a distance s in direction u. It can be shown by integration that

$$\begin{aligned} \exp_{(1;0)}(su) &= (\cos s;\, (\sin s)u) && \text{in } \mathcal{S}^n, \\ \exp_{(1;0)}(su) &= (\cosh s;\, (\sinh s)u) && \text{in } \mathcal{H}^n. \end{aligned}$$

The distance formulae can be read off from these equations in the case $y = (1; 0)$ and in the case $(1; 0) \in H_a$. By symmetry those formulae must be valid generally.

We finish with a brief discussion of spheres and hyperbolic spaces of non-unit radius r. These may be defined

$$\begin{aligned} r\mathcal{S}^n &= \left\{x \in \mathbb{R}^{n+1} \mid x \cdot x = r^2\right\} \\ r\mathcal{H}^n &= \left\{x \in \mathbb{R}^{n+1} \mid x \star x = r^2 \text{ and } x_0 > 0\right\} \end{aligned}$$

and each is equipped with the natural invariant metric that agrees at $(r; 0)$ with the induced Euclidean metric. Halfspaces of $r\mathcal{S}^n$ (respectively $r\mathcal{H}^n$) may be defined using vectors a with $a \cdot a = 1$ (respectively $a \star a = -1$) as before, and may also be matched with oriented hyperspheres (respectively hyperplanes).

Let $\Delta(x, y)$ denote the geodesic distance between x, y. We have formulae

$$\begin{aligned} x \cdot y &= r^2 \cos(\Delta(x, y)/r) \\ a \cdot x &= r \sin(\Delta(x, H_a)/r) \end{aligned}$$

for $r\mathcal{S}^n$, and

$$\begin{aligned} x \star y &= r^2 \cosh(\Delta(x, y)/r) \\ a \star x &= r \sinh(\Delta(x, H_a)/r) \end{aligned}$$

for $r\mathcal{H}^n$, generalising equations 3–6.

If $K(M)$ denotes the scalar curvatures of a Riemannian manifold M, normalised to equal the mean of the sectional curvatures, then

$$K(r\mathcal{S}^n) = 1/r^2, \quad K(r\mathcal{H}^n) = -1/r^2, \quad K(\mathbb{R}^n) = 0.$$

7. Weights

The Voronoi diagram for point sets in Euclidean space has a well-known generalisation to weighted point sets, called the Laguerre diagram, or weighted Voronoi diagram (see [12], for instance). In this section we extend this idea to spherical, hyperbolic, and tree geometries. We define Laguerre diagrams, strong and weak witnesses, and strong and weak Delaunay complexes. The weak witnesses theorem, $\mathrm{Del} = \mathrm{Del}^{\mathrm{w}}$, holds for weighted point sets in all four geometries. The active ingredients are Theorem 5.2, and the crucial fact that (in each geometry, with the correct definitions) weighted Voronoi cells are convex.

Remark. Spherical Laguerre diagrams were introduced by Sugihara [24]. Hyperbolic Laguerre diagrams appear to be new.

The general strategy for constructing these Laguerre diagrams is to replace the normal distance function $d(a, x)$ by a modified distance function $d_\mu(a, x)$, defined for a weighting μ. The modifed function d_μ is then used to define Voronoi diagrams, witnesses, and Delaunay complexes, according to the usual scheme.

Formally, consider a set A, a space X and a function $d(a, x)$, satisfying Hypothesis 5.1. A weighting μ on A associates a real number μ_a to every $a \in A$.

Definition 7.1. A function $d_\mu(a, x) = f(a, \mu_a, x)$ is called a *Laguerre function* extending $d(a, x)$ if the following conditions hold:

(1) The function $f(a, \mu_a, x)$ is continuous in x, for each a, μ_a.
(2) The region

$$R_\mu(a, b) = \{x \in X \mid d_\mu(a, x) \leq d_\mu(b, x)\}$$

is convex for all a, b and all choices of μ.
(3) If the weights μ_a are all equal, then $R_\mu(a, b)$ is equal to

$$R(a, b) = \{x \in X \mid d(a, x) \leq d(b, x)\}$$

for all a, b.

It is a monotone Laguerre function if, additionally:

(4) The function $f(a, \mu_a, x)$ is monotone decreasing in μ_a.

Conditions (1), (2) ensure that Theorem 5.2 applies immediately to the weighted distance functions $d_\mu(a,x)$. Thus we have a weak witnesses theorem valid for any choice of weights. Condition (3) ensures that the Laguerre diagram of a weighted point set specialises to the original Voronoi diagram, in the case where the points have equal weight. Condition (4) encapsulates the general desire that increasing the weight of a point should enlarge the corresponding Voronoi cell.

In the following examples, conditions (1), (3) are easily verified, so we focus on the more stringent condition (2).

Example 7.2 (Laguerre function for $\mathbb{R}^n$)**.** For $a, x \in \mathbb{R}^n$ define

$$d_\mu(a,x) = f(a,\mu_a,x) = \tfrac{1}{2}|x-a|^2 - \mu_a.$$

Then $f(a,\mu_a,x) \leq f(b,\mu_b,x)$ if and only if

$$(a-b)\cdot x \geq \tfrac{1}{2}(|a|^2 - |b|^2) - (\mu_a - \mu_b)$$

which is a convex condition on x, as required. The function is clearly monotone.

Example 7.3 (Laguerre function for $\mathcal{S}^n$)**.** For $a, x \in \mathcal{S}^n$ define

$$\begin{aligned} d_\mu(a,x) = f(a,\mu_a,x) &= -e^{\mu_a}\cos(\theta(a,x)) \\ &= -e^{\mu_a}(a\cdot x) \qquad \text{(by Eq. 3).} \end{aligned}$$

Then $f(a,\mu_a,x) \leq f(b,\mu_b,x)$ if and only if

$$(e^{\mu_a}a - e^{\mu_b}b)\cdot x \geq 0$$

which is satisfied on a hemisphere, or is vacuously true everywhere (if $a = b$ and $\mu_a = \mu_b$). For any open hemisphere $\mathcal{S}^n_+$, it follows that $R_\mu(a,b)\cap\mathcal{S}^n_+$ is convex.

The Laguerre function in this case is not monotone, since $\cos(\theta(a,x))$ may be negative. However, it becomes monotone if we restrict a, x to a subset of $\mathcal{S}^n$ of diameter at most $\pi/2$.

Example 7.4 (Laguerre function for $\mathcal{H}^n$)**.** For $a, x \in \mathcal{H}^n$ define

$$\begin{aligned} d_\mu(a,x) = f(a,\mu_a,x) &= e^{-\mu_a}\cosh(u(a,x)) \\ &= e^{-\mu_a}(a\star x) \qquad \text{(by Eq. 4).} \end{aligned}$$

Then $f(a,\mu_a,x) \leq f(b,\mu_b,x)$ if and only if

$$(e^{\mu_a}b - e^{\mu_b}a)\star x \geq 0$$

which is a convex condition on x, by Proposition 6.2. The function is monotone, since $\cosh(u(a,x)) > 0$.

Example 7.5 (Laguerre function for a tree)**.** Let $\mathcal{T}$ be a tree. For $a, x \in \mathcal{T}$ define

$$d_\mu(a,x) = f(a,\mu_a,x) = \ell(a,x) - \mu_a.$$

Then the set $R(a,b)$ defined by the condition $f(a,\mu_a,x) \leq f(b,\mu_b,x)$ is convex. Indeed, $R(a,b)\cap\gamma(a,b)$ is convex, so $R(a,b) = \pi^{-1}(R(a,b)\cap\gamma(a,b))$ must also be convex (see proof of Proposition 5.7). The function is clearly monotone.

Corollary 7.6. *The weak witnesses theorem holds for weighted point-sets (A,μ) in Euclidean space $\mathbb{R}^n$, the hemisphere $\mathcal{S}^n_+$, hyperbolic space $\mathcal{H}^n$ and any tree $\mathcal{T}$, with respect to the weighted distance functions d_μ defined in examples 7.2–7.5.*

Proof. In all cases, the regions $R(a,b)$ are convex. (This is not true for $\mathcal{S}^n$.) □

Remark. We can define Laguerre diagrams for spheres $r\mathcal{S}^n$ and hyperbolic spaces $r\mathcal{H}^n$ of non-unit radius r. The appropriate formulae are

$$d_\mu(a,x) = -e^{(\mu_a/r^2)} \cos\left(\Delta(a,x)/r\right)$$

for $r\mathcal{S}^n$ and

$$d_\mu(a,x) = e^{-(\mu_a/r^2)} \cosh\left(\Delta(a,x)/r\right)$$

for $r\mathcal{H}^n$. These definitions satisfy Voronoi convexity. The seemingly arbitrary normalisation factor $1/r^2$ in the exponent is explained in the next remark.

Remark. For each of the manifolds $\mathbb{R}^n$, $r\mathcal{S}^n$, $r\mathcal{H}^n$ the weighted distance d_μ is equivalent to a function

$$\tilde{d}_\mu = \tfrac{1}{2}\Delta^2 - \mu - \tfrac{1}{2}K\mu^2 + (\text{cubic and higher-order terms in } \Delta, \mu)$$

where Δ is geodesic distance and K is the scalar curvature of the manifold. By 'equivalent to' we mean that $d_\mu = c_1\tilde{d}_\mu + c_2$, where $c_1 > 0$ and c_2 are constants. Under those circumstances, d_μ and $\tilde{d}_\mu$ define the same sets $R(a,b)$.

We give one last example. (The analogous example for oriented hyperspheres $\mathcal{S}^n$ is not interesting, since each hypersphere may be identified with the pole in its positive hemisphere, and be treated exactly like it.)

Example 7.7 (Oriented hyperplanes in $\mathcal{H}^n$)**.** Let A label a set of oriented hyperplanes in $\mathcal{H}^n$, so $a \in A$ labels a vector in $\mathbb{R}^{n+1}$ with $a \star a = -1$, and H_a is the corresponding hyperplane. For $x \in \mathcal{H}^n$ define

$$\begin{aligned} d_\mu(H_a,x) = f(a,\mu_a,x) &= e^{-\mu_a}\sinh(u(H_a,x)) \\ &= e^{-\mu_a}(a \star x) \qquad \text{(by Eq. 6)}. \end{aligned}$$

Then $f(a,\mu_a,x) \le f(b,\mu_b,x)$ if and only if

$$(e^{\mu_a}b - e^{\mu_b}a) \star x \ge 0$$

which is a convex condition on x, as required. The function is not monotone, unless x is restricted to the intersection of the positive half-spaces of the H_a.

8. Approximate witnesses

In this section we relax the criteria for strong and weak witnesses, introducing an error tolerance ϵ. This has two benefits. First, it incorporates the idea that the distance measurements $d(a,x)$ used to define witnesses might not be known exactly. In applications this is always to be expected. Second, this naturally leads to families of complexes which grow monotonically as ϵ increases. These *filtered* complexes have proved to be useful targets of study in the theory of persistent homology [13, 25].

We now prepare to state a version of the weak witnesses theorem for approximate witnesses. This can be made to work in each of our four standard geometries.

Definition 8.1. We define relations $\le_\epsilon$, $\ge_\epsilon$, $=_\epsilon$ between real numbers α, β.

(1) $\alpha \le_\epsilon \beta$ means $\alpha \le \beta + \epsilon$.
(2) $\alpha \ge_\epsilon \beta$ means $\beta \le_\epsilon \alpha$, or equivalently $\alpha \ge \beta - \epsilon$.
(3) $\alpha =_\epsilon \beta$ means ($\alpha \le_\epsilon \beta$ and $\beta \le_\epsilon \alpha$), or equivalently $|\alpha - \beta| \le \epsilon$.

Approximate weak and strong witnesses can be defined using these ϵ-relations in place of the usual relations $\leq, \geq, =$. This works satisfactorily when the sets defined by ϵ-inequalities are convex. This is a strong condition. To achieve this in our usual examples, it is necessary to transform the distance metric. For this reason we use a new symbol λ in place of d in the following hypothesis.

Hypothesis 8.2 (strong Voronoi convexity)**.** *Assume these four postulates:*

(1) *X is a topological space and A is a set.*
(2) *For each $a \in A$, there is a continuous function $\lambda(a, -) : X \to \mathbb{R}$.*
(3) *Each unordered pair $x, y \in X$ is contained in a connected set $\gamma(x, y) \subset X$.*
(4) *For each $a, b \in A$ and $\epsilon \in \mathbb{R}$ the set*

$$R_\epsilon(a, b) = \{x \in X \mid \lambda(a, x) \leq_\epsilon \lambda(b, x)\}$$

is convex in the sense that $\gamma(x, y) \subset R_\epsilon(a, b)$ whenever $x, y \in R_\epsilon(a, b)$.

Definition 8.3. Assume items (1), (2) of Hypothesis 8.2. Let $\sigma \subset A$ be a simplex. We say that $x \in X$ is a **weak ϵ-witness** for σ with respect to A if $\lambda(a, x) \leq_\epsilon \lambda(b, x)$ whenever $a \in \sigma$ and $b \in A \setminus \sigma$. A weak ϵ-witness x is said to be a **strong ϵ-witness** if additionally $\lambda(a, x) =_\epsilon \lambda(b, x)$ for all $a, b \in \sigma$.

Definition 8.4. We can now define ϵ-relaxed versions of the strong and weak Delaunay complexes. For any subset $B \subset X$:

$$\sigma \in \mathrm{Del}(A, B; \epsilon) \quad \Leftrightarrow \quad \sigma \text{ has a strong } \epsilon\text{-witness in } B$$
$$\sigma \in \mathrm{Del}^{\mathrm{w}}(A, B; \epsilon) \quad \Leftrightarrow \quad \text{every subsimplex } \tau \leq \sigma \text{ has a weak } \epsilon\text{-witness in } B$$

Since $\mathrm{Del}(A, B; \epsilon) \subseteq \mathrm{Del}(A, B; \epsilon')$ whenever $\epsilon \leq \epsilon'$, and likewise for $\mathrm{Del}^{\mathrm{w}}$, we can view the families

$$\mathbf{Del}(A, B) = \{\mathrm{Del}(A, B; \epsilon)\}_{\epsilon \geq 0}, \quad \mathbf{Del^{w}}(A, B) = \{\mathrm{Del}^{\mathrm{w}}(A, B; \epsilon)\}_{\epsilon \geq 0},$$

as filtered complexes [25], with filtration parameter ϵ.

Theorem 8.5 (approximate weak witnesses theorem)**.** *Assume Hypothesis 8.2. A simplex $\sigma \subset A$ has a strong ϵ-witness $x \in X$ with respect to A if and only if every subsimplex $\tau \leq \sigma$ has a weak ϵ-witness $x_\tau \in X$ with respect to A. Moreover, x can be taken to lie in the convex hull of the points x_τ.*

Remark. Theorem 8.5 is equivalent to the assertion that the filtered complexes $\mathbf{Del}(A, K)$, $\mathbf{Del^{w}}(A, K)$ are equal for all convex $K \subseteq X$.

Remark. We can incorporate weights into Theorem 8.5 by replacing each $\lambda(a, x)$ with $\lambda(a, x) + \mu_a$. No further work is necessary.

The proof of Theorem 8.5 appears at the end of this section.

Remark. In the Euclidean setting, Attali et al. [3] have a slightly sharper result, the weak witnesses theorem for α-β complexes. Their parameter β corresponds to our ϵ, and α is the same parameter as in the theory of α-shapes [14]. The proof in [3] is geometric, whereas our proof of Theorem 8.5 is heavily algebraic.

In each of the following examples, it is easy to see that λ is related to the previously defined geodesic metric d via a monotone increasing transformation. Thus there is an equivalence:

$$\lambda(a, x) \leq \lambda(b, x) \;\Leftrightarrow\; d(a, x) \leq d(b, x)$$

Thus, when $\epsilon = 0$ these ϵ-witnesses are the same as the (exact) witnesses from before. Each function λ is derived in an obvious way from the corresponding Laguerre function d_μ.

Example 8.6 (approximate witnesses in $\mathbb{R}^n$)**.** For $a, x \in \mathbb{R}^n$ define

$$\lambda(a, x) = \tfrac{1}{2}|a - x|^2.$$

Then $\lambda(a, x) \le \lambda(b, x) + \epsilon$ is equivalent to

$$(a - b) \cdot x \ge \tfrac{1}{2}(|a|^2 - |b|^2) - \epsilon$$

which is a convex condition.

Example 8.7 (approximate witnesses in $\mathcal{S}^n$)**.** For $a, x \in \mathcal{S}^n$ such that $\theta(a, x) < \pi/2$, define

$$\lambda(a, x) = -\log(\cos(\theta(a, x)))$$

Then $\lambda(a, x) \le \lambda(b, x) + \epsilon$ is equivalent to

$$(e^\epsilon a - b) \cdot x \ge 0$$

which is a convex condition on any open hemisphere $\mathcal{S}^n_+$. In practice, to satisfy the condition on $\theta(a, x)$, we must work in a subset of $\mathcal{S}^n$ with diameter at most $\pi/2$.

Example 8.8 (approximate witnesses in $\mathcal{H}^n$)**.** For $a, x \in \mathcal{H}^n$, define

$$\lambda(a, x) = \log(\cosh(u(a, x)))$$

Then $\lambda(a, x) \le \lambda(b, x) + \epsilon$ is equivalent to

$$(e^{-\epsilon} a - b) \star x \le 0$$

which is a convex condition.

Example 8.9 (approximate witnesses in a tree)**.** For a tree, $\lambda(a, x) = \ell(a, x)$ does the job (see Example 7.5).

Corollary 8.10. *The approximate weak witnesses theorem holds in $\mathbb{R}^n$, in convex subsets of $\mathcal{S}^n$ with diameter at most $\pi/2$, in hyperbolic space $\mathcal{H}^n$, and in any tree $\mathcal{T}$, using the transformed geodesic distances $\lambda(a, x)$ defined in Examples 8.6–8.9.* $\square$

The proof of Theorem 8.5 is similar to the proof of Theorem 2.5. The main difference is that $=_\epsilon$ is not transitive, so we must keep track of all the points in σ even after we restrict the action to subspaces such as $H_\epsilon(a, b) = R_\epsilon(a, b) \cap R_\epsilon(b, a)$. We adopt the following abstract framework. The idea is to use an equivalence relation so that we can 'merge' points a, b without forgetting either.

Let $(A, \sim)$ be a set with an equivalence relation. We write $\sigma \subset (A, \sim)$ if $\sigma \subset A$ is a union of equivalence classes. Suppose we are in the setting of Hypothesis 8.2. For $\sigma \subset (A, \sim)$, a weak ϵ-witness for σ with respect to $(A, \sim)$ is defined to be a point $x \in X$ such that the following conditions hold:

$$\begin{array}{rcll} \lambda(a, x) & =_\epsilon & \lambda(b, x) & \quad \text{for all } a, b \in A \text{ such that } a \sim b \\ \lambda(a, x) & \le_\epsilon & \lambda(b, x) & \quad \text{for all } a \in \sigma,\ b \in A \setminus \sigma \end{array}$$

Let $\approx$ be obtained from $\sim$ by merging distinct equivalence classes $[a_0]$, $[a_1]$, say, to a single class $[\![a_0]\!] = [\![a_1]\!]$. Note that if $\tau \subset (A, \approx)$ then $\tau \subset (A, \sim)$.

Lemma 8.11. *In the preceding situation, let $\tau \subset (A, \approx)$ be given.*

(1) *If τ does not include the class $[\![a_0]\!] = [\![a_1]\!]$, define $\tau_0, \tau_1 \subset (A, \sim)$ by $\tau_0 = \tau \cup [a_0]$, and $\tau_1 = \tau \cup [a_1]$.*

(2) *If* τ *includes the class* $[\![a_0]\!] = [\![a_1]\!]$*, define* $\tau_0, \tau_1 \subset (A, \sim)$ *by* $\tau_0 = \tau \setminus [a_1]$*, and* $\tau_1 = \tau \setminus [a_0]$.

Suppose τ, τ_0, τ_1 *have weak* ϵ*-witnesses* x, x_0, x_1 *with respect to* $(A, \sim)$*. Then* τ *has a weak* ϵ*-witness with respect to* $(A, \approx)$ *in the convex hull of* x, x_0, x_1*.*

Proof. (We can treat both cases simultaneously, except at the very end.) One of the following must be true at x:

$$\begin{array}{rcll} \lambda(b_0, x) & =_\epsilon & \lambda(b_1, x) & \text{for all } b_0 \in [a_0],\ b_1 \in [a_1] \\ \lambda(b_0, x) & > & \lambda(b_1, x) + \epsilon & \text{for some } b_0 \in [a_0],\ b_1 \in [a_1] \\ \lambda(b_1, x) & > & \lambda(b_0, x) + \epsilon & \text{for some } b_0 \in [a_0],\ b_1 \in [a_1] \end{array}$$

The first possibility immediately implies that x is a weak ϵ-witness for τ with respect to $(A, \approx)$. It remains to consider the second possibility. (The third possibility is treated symmetrically.)

Consider the closed set

$$R = \{z \in X \mid \lambda(b_0, z) \leq_\epsilon \lambda(b_1, z) \text{ for all } b_0 \in [a_0],\ b_1 \in [a_1]\}.$$

Then $x_0 \in R$ and $x \notin R$. Since $\gamma(x, x_0)$ is connected, there exists a point $y \in \gamma(x, x_0)$ which is simultaneously in R and the closure of $\gamma(x, x_0) \setminus R$. We must have

$$\lambda(M_0, y) = \lambda(m_1, y) + \epsilon \qquad \text{for some } M_0 \in [a_0],\ m_1 \in [a_1]$$

where, moreover, $\lambda(M_0, y)$ is the largest of the terms $\lambda(b_0, y)$, for $b_0 \in [a_0]$, and $\lambda(m_1, y)$ is the smallest of the terms $\lambda(b_1, y)$, for $b_1 \in [a_1]$. This implies that:

$$\lambda(b_0, y) =_\epsilon \lambda(b_1, y) \qquad \text{for all } b_0 \in [a_0],\ b_1 \in [a_1]$$

Convexity allows us to deduce statements about y from corresponding statements about x and x_0. Specifically:

$$\begin{array}{rrcll} \text{both cases:} & \lambda(a, y) & =_\epsilon & \lambda(b, y) & \text{for all } a, b \in A \text{ such that } a \sim b \\ \text{case (1):} & \lambda(a, y) & \leq_\epsilon & \lambda(b, y) & \text{for all } a \in \tau,\ b \in A \setminus \tau,\ b \notin [a_0] \\ \text{case (2):} & \lambda(a, y) & \leq_\epsilon & \lambda(b, y) & \text{for all } a \in \tau,\ b \in A \setminus \tau,\ a \notin [a_1] \end{array}$$

In case (1), if $a \in \tau$ and $b_0 \in [a_0]$, then

$$\lambda(a, y) \leq \lambda(m_1, y) + \epsilon = \lambda(M_0, y) \leq \lambda(b_0, y) + \epsilon$$

so $\lambda(a, y) \leq_\epsilon \lambda(b_0, y)$. In case (2), if $c_1 \in [a_1]$ and $b \in A \setminus \tau$, then

$$\lambda(c_1, y) \leq \lambda(m_1, y) + \epsilon = \lambda(M_0, y) \leq \lambda(b, y) + \epsilon$$

so $\lambda(c_1, y) \leq_\epsilon \lambda(b, y)$.

Collating these statements, we find that y is a weak ϵ-witness for τ with respect to $(A, \approx)$, in both cases (1) and (2). □

Proof of Theorem 8.5. Consider equivalence relations on A which are σ-supported, meaning that a nontrivial relation $a \sim b$ is permitted only if $a, b \in \sigma$. We say that $\sim$ is σ-complete if every nonempty $\tau \subset \sigma$ such that $\tau \subset (A, \sim)$ has a weak ϵ-witness with respect to $(A, \sim)$.

Let $\sim$ be σ-supported and σ-complete, and let $\approx$ be the σ-supported relation obtained from $\sim$ by merging some two of the classes in σ. Then Lemma 8.11 implies that $\approx$ is σ-complete. The hypothesis of Theorem 8.5 asserts that the trivial relation (where $a \sim b$ only if $a = b$) is σ-complete. After a finite sequence of mergers, we deduce that the maximal σ-supported relation $\sim_\sigma$ (where $a \sim_\sigma b$ for all $a, b \in \sigma$) is itself σ-complete. In particular, there is a weak ϵ-witness x for σ

with respect to $(A, \sim_\sigma)$. It follows from the definitions that x is a strong ϵ-witness for σ with respect to A. By construction, x lies in the convex hull of the original weak ϵ-witnesses. □

9. Manifolds of constant curvature

If X is a complete Riemannian manifold of constant sectional curvature, and $A \subset X$ is densely sampled, then $\mathrm{Del}(A, X) = \mathrm{Del}^{\mathrm{w}}(A, X)$. This is not surprising, because such an X is locally isometric to $\mathbb{R}^n$, $\mathcal{S}^n$ or $\mathcal{H}^n$; and if A is sufficiently dense then the weak and strong Delaunay complexes are completely determined by local metric information.

In this section we give a precise theorem of this form. To state the result, we need to quantify 'sufficiently dense' and 'local'.

Definition 9.1. Let X be a metric space. We say that a subset $A \subset X$ is ϵ-dense if for every $x \in X$ there exists $a \in A$ such that $d(a, x) < \epsilon$. Equivalently, for all $x \in X$,

$$B_\epsilon(x) \cap A \neq \emptyset$$

where $B_\epsilon(x)$ denotes the open ball with center x and radius ϵ.

Definition 9.2. Let X be a complete Riemannian manifold. At each $x \in X$, consider the exponential map $\exp_x : T_x X \to X$. The injectivity radius of X is the following quantity:

$$\rho_X = \inf_{x \in X} \max \{\rho \in \mathbb{R} \mid \exp_x \text{ is a smooth embedding on the open ball } B_\rho(0)\}$$

Theorem 9.3 (localised weak witnesses theorem)**.** *Let X be a complete Riemannian manifold with constant sectional curvature, not equal to a sphere. Suppose $A \subset X$ is ϵ-dense, where $\epsilon \leq \rho_X/3$. Then* $\mathrm{Del}(A, X) = \mathrm{Del}^{\mathrm{w}}(A, X)$.

Remark. The present argument can be made to work for $X = \mathcal{S}^n$ under the stronger condition $\epsilon \leq \rho_X/6$. It is plausible that $\epsilon \leq \rho_X/3$ is sufficient for spheres. The most conspicuous counterexample to $\mathrm{Del} = \mathrm{Del}^{\mathrm{w}}$ is the set of $n + 2$ vertices of a regular simplex, in $\mathcal{S}^n$. This is (just barely) ruled out by the weaker condition $\epsilon \leq (\rho_X/\pi) \cos^{-1}(1/n)$, and one would suspect this to be the correct bound. I have not been able to prove it.

Our strategy is to transfer the problem to the universal cover $\tilde{X}$ of X. If X is a complete Riemannian manifold with constant sectional curvature, then $\tilde{X}$ is isometrically equal to $\mathbb{R}^n$, or (up to a constant scalar multiple) to $\mathcal{S}^n$ or $\mathcal{H}^n$. Thus we can apply the weak witnesses theorem in $\tilde{X}$.

We begin with some notation and preliminaries. The universal cover comes with a covering map $\pi : \tilde{X} \to X$ which is invariant under the group Γ of deck transformations of $\tilde{X}$. For any subset $A \subset X$, let $\tilde{A} = \pi^{-1}(A)$. If $x \in X$, then $\hat{x}$ is used to denote any particular lift of x to $\tilde{X}$; so $\pi(\hat{x}) = x$. The shortest-path metric in X is related to the metric in $\tilde{X}$ by the formula

$$d(x, y) = \min_{g \in \Gamma} d(\hat{x}, g(\hat{y})), = \min_{g \in \Gamma} d(g(\hat{x}), \hat{y}),$$

where $\hat{x}, \hat{y}$ are lifts of x, y.

Note that if $\pi(\hat{x}) = x$ then $\pi(B_r(\hat{x})) = B_r(x)$ for any value of r, since paths can be projected to X or lifted to $\tilde{X}$. One consequence is that if A is ϵ-dense in X then $\tilde{A}$ is ϵ-dense in $\tilde{X}$.

If $r \leq \rho_X$ then the map $\pi : B_r(\hat{x}) \to B_r(x)$ is 1–1. In that case, $\pi^{-1}(B_r(x))$ is a disjoint union of copies of $B_r(\hat{x})$:

$$\pi^{-1}(B_r(x)) = \coprod_{g \in \Gamma} B_r(g(\hat{x})) = \coprod_{g \in \Gamma} g(B_r(\hat{x}))$$

Lemma 9.4. *Let X be a complete Riemannian manifold. If $A \subset X$ is ϵ-dense and $\sigma \in \mathrm{Del}^{\mathrm{w}}(A, X)$, then* $\mathrm{diam}(\sigma) < 2\epsilon$.

Proof. For any $a, b \in \sigma$, let $x \in X$ be a weak witness for $[ab]$. Assume without loss that $d(b, x) \leq d(a, x) =: R$, and let γ be a minimal geodesic from x to a. By continuity there exists $y \in \gamma$ with $d(a, y) = d(b, y) =: r$. Since $d(x, y) = R - r$ we have an inclusion $B_r(y) \subset B_R(x)$. Since $B_r(y)$ does not meet $\{a, b\}$ and $B_R(x)$ does not meet $A \setminus \{b\}$ it follows that $B_r(y)$ does not meet A. This would violate the sampling condition unless $r < \epsilon$. Therefore $d(a, b) \leq d(a, y) + d(y, b) = 2r < 2\epsilon$. $\square$

Lemma 9.5. *Let X be a complete Riemannian manifold, with universal cover $\tilde{X}$, covering group Γ, and injectivity radius $\rho = \rho_X$.*

(1) *For any $\hat{x} \in \tilde{X}$, $g \in \Gamma$, $g \neq 1$, we have $d(\hat{x}, g(\hat{x})) \geq 2\rho$.*
(2) *Let $\hat{x}, \hat{y} \in \tilde{X}$ be lifts of $x, y \in X$. If $d(\hat{x}, \hat{y}) \leq \rho$ then $d(x, y) = d(\hat{x}, \hat{y})$.*

Proof. The balls $B_\rho(\hat{x})$ and $B_\rho(g(\hat{x}))$ are disjoint. It follows that any path from $\hat{x}$ to $g(\hat{x})$ must have length at least 2ρ, implying (1). If $g \neq 1$ then

$$d(\hat{x}, g(\hat{y})) \geq d(\hat{y}, g(\hat{y})) - d(\hat{x}, \hat{y}) \geq 2\rho - \rho = \rho \geq d(\hat{x}, \hat{y})$$

Thus $d(x, y) = \min_{g \in \Gamma} d(\hat{x}, g(\hat{y})) = d(\hat{x}, \hat{y})$. $\square$

Proof of Theorem 9.3, assuming $X \neq \mathcal{S}^n$. Suppose $\sigma \in \mathrm{Del}^{\mathrm{w}}(A, X)$. We must prove that $\sigma \in \mathrm{Del}(A, X)$. Let $a \in \sigma$; then Lemma 9.4 implies that $\sigma \subset B_{2\epsilon}(a)$. For $\tau \subseteq \sigma$, let $x_\tau \in X$ be a weak witness for τ. Then $x_\tau \in B_{3\epsilon}(a)$, since A is ϵ-dense. Now consider a fixed lift $\hat{a} \in \tilde{X}$ of a. Let $\hat{\tau}$ be the unique lift of τ to $B_{2\epsilon}(\hat{a})$, and let $\hat{x}_\tau$ be the unique lift of x_τ to $B_{3\epsilon}(\hat{a})$. Uniqueness is guaranteed by $3\epsilon \leq \rho_X$.

The rest of the proof is divided into three steps:

(1) For each $\tau \subseteq \sigma$, the lift $\hat{x}_\tau$ is a weak witness for $\hat{\tau}$ with respect to $\tilde{A}$;
(2) therefore, there exists a strong witness $\hat{x} \in B_{3\epsilon}(\hat{a})$ for $\hat{\sigma}$ with respect to $\tilde{A}$;
(3) therefore, $x = \pi(\hat{x})$ is a strong witness for σ with respect to A.

(1): Suppose $R \geq 0$ is such that that the closed ball $\bar{B}_R(x_\tau)$ contains τ, and the open ball $B_R(x_\tau)$ does not meet $A \setminus \tau$. It immediately follows that $B_R(\hat{x}_\tau)$ does not meet $\pi^{-1}(A \setminus \tau) = \tilde{A} \setminus \pi^{-1}(\tau)$. This accounts for most of the points of $\tilde{A}$. Step (1) will be complete if we can show that $\bar{B}_R(\hat{x}_\tau) \cap \pi^{-1}(\tau) = \hat{\tau}$. It is enough to show that $\bar{B}_R(\hat{x}_\tau)$ does not meet any of the balls $g(B_{2\epsilon}(\hat{a})) = B_{2\epsilon}(g(\hat{a}))$, for $g \neq 1$, which are the alternative lifts of $B_{2\epsilon}(a)$.

In order for $\bar{B}_R(\hat{x}_\tau)$ to meet $B_{2\epsilon}(g(\hat{a}))$ it is necessary that

$$d(\hat{a}, \hat{x}_\tau) + R + 2\epsilon > d(\hat{a}, g(\hat{a}))$$

and so $d(\hat{a}, \hat{x}_\tau) + R > 4\epsilon$ (by Lemma 9.5). We will see that this is impossible.

Construct a point $\hat{y}$ by extending the minimal geodesic from $\hat{a}$ to $\hat{x}$ to a total length 3ϵ. The ball $B_\epsilon(\hat{y})$ does not meet $B_{2\epsilon}(\hat{a})$; nor does it meet $B_{2\epsilon}(g(\hat{a}))$ for any $g \neq 1$, because $d(\hat{a}, \hat{y}) + \epsilon = 4\epsilon$. On the other hand, it must meet the ϵ-dense set $\tilde{A}$. So, in fact, $B_\epsilon(\hat{y})$ meets $\tilde{A} \setminus \pi^{-1}(\tau)$. A consequence is that $B_\epsilon(\hat{y})$ is not contained in $B_R(\hat{x}_\tau)$, since that ball does not meet $\tilde{A} \setminus \pi^{-1}(\tau)$. However, if $d(\hat{a}, \hat{x}_\tau) + R > 4\epsilon$, then

$$d(\hat{x}_\tau, \hat{y}) = d(\hat{a}, \hat{y}) - d(\hat{a}, \hat{x}_\tau) < 3\epsilon + R - 4\epsilon = R - \epsilon$$

which implies $B_\epsilon(\hat{y}) \subset B_R(\hat{x}_\tau)$, a contradiction.

(2): Except when $\tilde{X} = \mathcal{S}^n$, it is clear that $B_{3\epsilon}(\hat{a})$ is convex, so Theorem 5.2 applies. If $X \neq \tilde{X} = \mathcal{S}^n$, then the group Γ has cardinality at least two. The different lifts $B_{3\epsilon}(g(\hat{a}))$ are disjoint and have equal volume, which must therefore be at most half the volume of $\mathcal{S}^n$. This implies that $B_{3\epsilon}(\hat{a})$ is contained in a hemisphere, so again Theorem 5.2 applies. The only untreatable case is $X = \tilde{X} = \mathcal{S}^n$, where $|\Gamma| = 1$.

(3): Let $R < 3\epsilon$ be the common distance from $\hat{x}$ to the points $\hat{b} \in \hat{\sigma}$. By Lemma 9.5, $d(x, b) = d(\hat{x}, \hat{b}) = R$ for any $b \in \sigma$. Finally, $B_R(\hat{x}) \cap \tilde{A} = \emptyset$ and hence $B_R(x) \cap A = \emptyset$. Thus x is the required strong witness. $\square$

Theorem 9.6. *Let X be a complete Riemannian manifold with constant sectional curvature. If $A \subset X$ is ϵ-dense, where $\epsilon \leq \rho_X/2$, then* $\mathrm{Del}(A, X) \simeq X$.

Remark. A much more impressive theorem was announced in [17], which asserts that $\mathrm{Del}(A, X)$ is homeomorphic to X for an arbitrary Riemannian manifold X, provided that A is a generic ϵ-dense finite subset of X, for a suitable explicitly-defined ϵ. Unfortunately, the full proof was never published.

Proof. It is enough to show that the Voronoi cells V_a are convex, in the sense that if $x, y \in V_a$ then x, y are connected by a unique minimal geodesic γ in X, and $\gamma \subset V_a$. This implies that finite intersections of Voronoi cells are empty or contractible; and so $\mathrm{Del}(A, X) \simeq X$ follows from the nerve theorem.

Let $a \in A$, and let $\hat{a} \in \tilde{X}$ be a lift of a. The ball $B_\epsilon(\hat{a})$ is certainly convex; the only doubtful case is $\tilde{X} = \mathcal{S}^n$, but $B_{2\epsilon}(\hat{a})$ is known not to be the whole sphere, so $B_\epsilon(\hat{a})$ is contained in a hemisphere. Next, if $K \subset B_\epsilon(\hat{a})$ is convex then so is $\pi(K) \subset B_\epsilon(a)$. Indeed, if $x, y \in \pi(K)$, then their lifts $\hat{x}, \hat{y} \in B_\epsilon(\hat{a})$ are connected by a minimal geodesic $\gamma \subset K$ of length less than ρ_X. The geodesic $\pi \circ \gamma \subset \pi(K)$ connects x to y and is minimal: any other path from x to y lifts to a path from $\hat{x}$ to some $g(\hat{y})$ and therefore must be longer than γ.

Finally, we know that Voronoi cells in $\mathbb{R}^n, \mathcal{S}^n_+, \mathcal{H}^n$ are convex. Let $V_{\hat{a}}$ denote the Voronoi cell of $\hat{a}$ with respect to $\tilde{A}$. For $x \in B_\epsilon(a)$, with lift $\hat{x} \in B_\epsilon(\hat{a})$, let $R = d(x, a) = d(\hat{x}, \hat{a})$. Then $B_R(x) \cap A = \emptyset$ if and only if $B_R(\hat{x}) \cap \tilde{A} = \emptyset$; in other words $V_a = \pi(V_{\hat{a}})$. From the previous paragraph, V_a is convex. $\square$

Corollary 9.7 (reconstruction theorem)**.** *Let X be a complete Riemannian manifold with constant sectional curvature. Suppose $A \subset X$ is ϵ-dense, where $\epsilon \leq \rho_X/3$, or $\epsilon \leq \rho_X/6$ if X is a sphere of dimension at least 2. Then* $\mathrm{Del}(A, X) = \mathrm{Del}^{\mathrm{w}}(A, X) \simeq X$. $\square$

Remark. The results in this section have direct analogues in the case where X is a graph and ρ_X is half the girth of the graph (that is, half the length of the shortest cycle). In this case the universal cover of X is a tree.

10. Topological data analysis

What happens to the weak witnesses theorem when the ambient space X is a submanifold of Euclidean space? The proofs in this paper make heavy use of special geometric properties of the constant curvature manifolds $\mathbb{R}^n$, $\mathcal{S}^n$ and $\mathcal{H}^n$. However, the original motivation for this work was a desire to construct Delaunay triangulations on Euclidean submanifolds, as part of a strategy for *manifold reconstruction* [1, 2, 9] and *homotopy reconstruction* [11, 13, 18].

The set-up is the same for both problems. Let $X \subset \mathbb{R}^n$ be an unknown submanifold of Euclidean space, and let $B \subset X$ be a known finite sample. The problem is to recover from B alone the topological type of X (manifold reconstruction) or the homotopy type of X (homotopy reconstruction). More concretely, construct a simplicial complex $S = S(B)$ which is provably homeomorphic or homotopy equivalent to X under reasonable sampling conditions on B.

Solutions now exist to both problems. For manifold reconstruction, Cheng et al. [9] use 'cocone' methods to identify a suitable subcomplex of the Delaunay triangulation $\mathrm{Del}(B, \mathbb{R}^n)$. To exclude near-degenerate simplices called 'slivers', the Delaunay triangulation is computed with respect to a carefully chosen system of weights. When this is done, the resulting subcomplex is homeomorphic to X. For homotopy reconstruction, Niyogi et al. [21] prove that if each point in B is thickened to an ϵ-ball, then the resulting union of balls is homotopy equivalent to X. It follows that the nerve of that union of balls (also known as the Čech complex) is homotopy equivalent to X. This complex tends to be rather large, but the corresponding α-complex [14], obtained by replacing each ball with its intersection with the Voronoi cell for its center, is much smaller and has the same homotopy type.

In more recent work, Chazal et al. [7, 8] obtain powerful generalisations of the reconstruction results of [21] to the case where X is a compact subset of $\mathbb{R}^n$. One major technical challenge is to quantify the irregularity of the boundary of X so as to give effective sampling conditions for successful reconstruction. This is comparatively easily achieved in the case of a smooth manifold using the concept of 'reach' or 'feature size', which is the injectivity radius of the normal exponential map. Chazal et al. develop a parametrised weak feature size which is applicable to arbitrary compact subsets and leads to the desired sampling conditions.

The troublesome common feature of all of these constructions is that they require Delaunay triangulation computations in the ambient space. As such their complexity is subject to a curse of dimensionality in the ambient dimension n. Ideally, the curse should apply to the (smaller) intrinsic dimension of X, but the necessary techniques for such a reduction have not yet been developed for these algorithms.

Witness complexes were introduced in [11] as a strategy for building homotopy reconstructions relatively cheaply: using a small set of vertices, and depending only on very crude geometric information. The method is very simple. Select a well-distributed set of *landmark* points $A \subset B$, and construct $\mathrm{Del}(A, B)$ or $\mathrm{Del}^{\mathrm{w}}(A, B)$ using the ambient Euclidean metric. More robustly, construct the filtered complexes $\mathbf{Del}(A, B)$, $\mathbf{Del}^{\mathbf{w}}(A, B)$ discussed in Section 8. Homological features of X are detected (at least in principle) by computing the homology or persistent homology of these complexes.

For a heuristic justification of this procedure, consider the approximation argument represented by the diagram in Figure 6. The complexes $\mathrm{Del}(A, B)$, $\mathrm{Del}^{\mathrm{w}}(A, B)$ on the right are constructible from the available data B. The maps in the diagram relate them to the unknown space X via intermediate spaces $\mathrm{Del}(A, X)$, $\mathrm{Del}^{\mathrm{w}}(A, X)$. We now consider each of these maps in turn.

$$\begin{tikzcd}
X \arrow[r, "\text{(a)}"] & \mathrm{Del}(A,X) \arrow[d, "\text{(d)}"'] & \mathrm{Del}(A,B) \arrow[l, dotted, "\text{(b)}"'] \arrow[d, dotted, "\text{(e)}"] \\
& \mathrm{Del}^{\mathrm{w}}(A,X) & \mathrm{Del}^{\mathrm{w}}(A,B) \arrow[l, "\text{(c)}"']
\end{tikzcd}$$

FIGURE 6. A diagram of approximations to X.

(a) The restricted Delaunay complex $\mathrm{Del}(A, X)$ is the nerve of the covering of X by restricted Voronoi cells $V_a \cap X$. By the Čech nerve theorem [16], X is homotopy equivalent to $\mathrm{Del}(A, X)$ provided that each nonempty finite intersection $V_\sigma \cap X = \bigcap_{a\in\sigma} V_a \cap X$ is contractible. Moreover, Edelsbrunner and Shah [15] show that X is homeomorphic to $\mathrm{Del}(A, X)$ provided that each $V_\sigma \cap X$ is a closed ball of the expected dimension.

(b) Since X itself is unavailable, the sample set B is used as a approximation. This works if $V_\sigma \cap B$ is nonempty whenever $V_\sigma \cap X$ is nonempty. *This almost always fails*, because B is discrete and V_σ has positive codimension (except when σ is just a single vertex). Thus $\mathrm{Del}(A, B)$ is almost surely 0-dimensional, and looks nothing like $\mathrm{Del}(A, X)$.

(c) On the other hand, the weak witness regions

$$W_\sigma = \{x \in \mathbb{R}^n \mid x \text{ is a weak witness for } \sigma\}$$

generically have codimension 0. There is a nonzero probability that a given nonempty $W_\sigma \cap X$ has a representative element in $W_\sigma \cap B$. It is therefore reasonable to hope that $\mathrm{Del}^{\mathrm{w}}(A, X) = \mathrm{Del}^{\mathrm{w}}(A, B)$. This is the principal reason for introducing weak witness complexes in the first place.

(d) The desired equivalence $\mathrm{Del}(A, X) = \mathrm{Del}^{\mathrm{w}}(A, X)$, if true, would be the weak witnesses theorem for the nonlinear submanifold $X \subset \mathbb{R}^n$. One argument runs as follows (following [4]). If every subsimplex $\tau \leq \sigma$ has a weak witness $x_\tau \in X$, then σ has a strong witness x in the convex hull of the x_τ, by Theorem 5.2. If this convex hull lies close to X, then one can seek a nearby strong witness $x' \in X$. For this to work, V_σ must be sufficiently transverse to X.

(e) Since $\mathrm{Del}(A, B)$ is almost surely 0-dimensional, this map is as useless as (b).

Thus, the sequence (a)–(d)–(c) gives us a plausible chain of equivalences:

$$X = \mathrm{Del}(A, X) = \mathrm{Del}^{\mathrm{w}}(A, X) = \mathrm{Del}^{\mathrm{w}}(A, B)$$

How well is this supported by theoretical guarantees? When X is a smoothly-embedded curve or surface, Attali et al. [4] establish that $X = \mathrm{Del}^{\mathrm{w}}(A, X)$ for any sufficiently dense landmark set A. Difficulties arise in higher dimensions: Oudot has constructed a 3-dimensional submanifold $X \subset \mathbb{R}^n$ and arbitrarily dense $A \subset X$ such that the restricted Voronoi intersection $V_\sigma \cap X$ (for a particular 3-simplex $\sigma \subset A$) is not connected [22]. It can be confirmed in this case that $\mathrm{Del}(A, X)$ is not even homotopy equivalent to X. The good news is that all such counterexamples can be disarmed using the weight-pumping strategy of Cheng et al. [9]. Boissonat et al. [5] establish equality at (a), (d) for suitable choices of weights. To complete their manifold reconstruction theorem, they establish equality at (c) with $\mathrm{Del}^{\mathrm{w}}(A, B)$

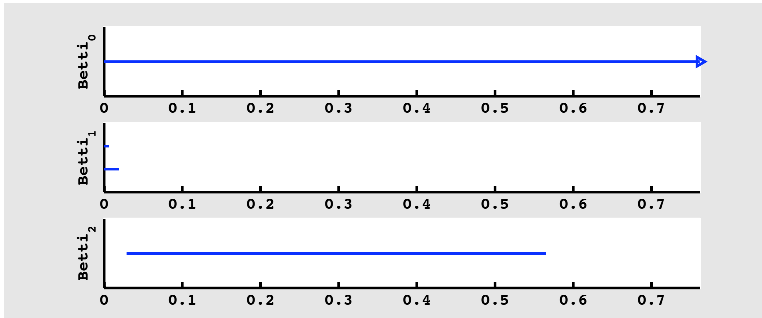

FIGURE 7. Persistence barcode for $\mathbf{Del^w}(A, B)$ with $A \subset B$ sampled from $X = S^2$; $|A| = 20$, $|B| = 400$.

replaced by $\mathrm{Del}^{\mathrm{w}}(A, \hat{B})$, where $\hat{B}$ is a union of balls centered at the points of B. Since $\hat{B}$ is an open subset of $\mathbb{R}^n$, this last step necessitates a (localised) Delaunay computation in $\mathbb{R}^n$. The problem remains open: to find a manifold reconstruction strategy that avoids any sort of Delaunay computation in dimension n.

Some of the technical difficulties dissolve when the goal is merely to extract homological invariants of X. The rather fragile witness complexes $\mathrm{Del}(A, B)$ and $\mathrm{Del}^{\mathrm{w}}(A, B)$ can be replaced by their corresponding filtered complexes $\mathbf{Del}(A, B)$ and $\mathbf{Del^w}(A, B)$. These must be shown to encode the homology of X in their persistent homology (under suitable sampling conditions). This is an easier task for various reasons. One immediate advantage is that ϵ-Voronoi intersections $V_\sigma(\epsilon)$ have positive volume when $\epsilon > 0$, so map (b) can actually be well behaved. More importantly, persistent homology is fundamentally more robust than homology [10]. It is not necessary to show that any individual complex $\mathrm{Del}^{\mathrm{w}}(A, B; \epsilon)$ is homotopy equivalent to X. Instead, one can show that $\mathbf{Del^w}(A, B)$ interleaves with some other filtered complex known to correctly recover the homology of X. One strategy is to compare $\mathbf{Del}(A, B)$ with the filtered Čech complex on A. Combining this idea with the persistent homology stability theorem of Cohen-Steiner et al. [10] and the results of Niyogi et al. [21] for the Čech complex, one gets reconstruction theorems for $\mathbf{Del^w}(A, B)$. Of course, the weak witness complex is supposed to *improve* on the Čech complex, so such a proof necessarily leads to somewhat unsatisfactory reconstruction sampling conditions. For the current state of the art regarding the filtered weak witness complex (and much more besides), we refer the reader to work by Chazal and Oudot [6].

Example. We finish with an empirical demonstration: it seems that the filtered strong and weak complexes perform well at recovering homology in simple examples, but the weak witness complex gives cleaner results. Figures 7 and 8 compare persistent homology barcodes [13, 25] for weak and strong witness complexes of a point sample taken from the 2-sphere in $\mathbb{R}^3$. The horizontal axis corresponds to the filtration parameter ϵ. The correct answer for the 2-sphere is one long bar in Betti 0 and one long bar in Betti 2. Both complexes exhibit this, but the strong witness complex has to overcome some initial noise for small values of ϵ — inevitably, since the complex has no edges at $\epsilon = 0$. When ϵ is the only parameter, this homological noise is not a serious thing, but if there are other parameters involved then it becomes much more important to have clean readings which are reliable across a

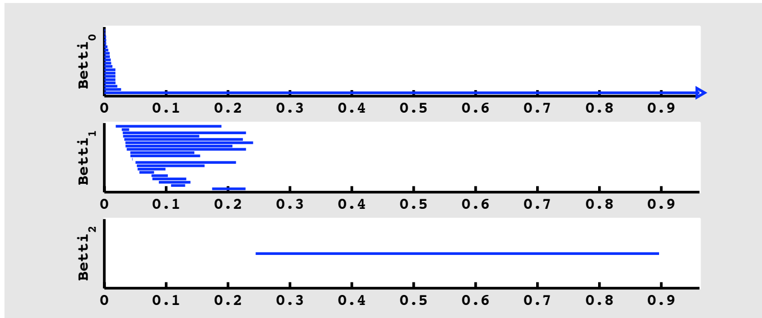

FIGURE 8. Persistence barcode for $\mathbf{Del}(A, B)$ with $A \subset B$ sampled from $X = S^2$; $|A| = 20$, $|B| = 400$.

large range in ϵ. It is desirable to quantify this homological 'cleanness'; Chazal and Oudot [6] have the first results in this direction.

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