UTA Method

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June 20, 2017

Abstract

This report contains the summary of the chapter **UTA Methods** from the Book : Multiple Criteria Decision Analysis. This document was made during my internship at LAMSADE in the summer of 2017.

Contents

Chapter 1

Introduction

In 1982, E. Jacquet-Lagrze and J. Siskos proposed a decision mehtod called UTA. This method is proposed by the Multi-Attribute Utility Theory (MAUT) that build a utility function based on the DM[1] preferences.
Assuming that the decision is given, the UTA method will find the rational basis for the decision being made. Or how can we use the DM's preference leading to the exact same or "similar" decision.

The UTA method is used to solve a multi-criteria problem. It build a utility function based on the preferences of the DM and it consist in solving a linear program (LP).

UTA methods adpot the aggregation-disaggregation principles. The aggregation is where the model is known and the preferneces are unknown, while the disaggregation is where the model is based on a given preferences

[1]Decision Maker

Chapter 2

Method

2.1 Principles and Notation

2.1.1 Weighted form

A weighted form of the UTA Method has the following form:

$$u(g) = \sum_{i=1}^{n} p_i u_i(g_i) \tag{2.1}$$

subject to normalization constraints:

$$\begin{cases} \sum_{i=1}^{n} p_i = 1 \\ u_i(g_{i*}) = 0, u_i(g_{i*}) = 1, \forall i = 1, 2, ..., n \end{cases} \tag{2.2}$$

where $u_i, i = 1, 2, ..., n$ are non decreasing real valued functions, and p_i is the weight of u_i.

2.1.2 Unweighted form

An unweighted form of the UTA Method has the following form:

$$u(g) = \sum_{i=1}^{n} u_i(g_i) \tag{2.3}$$

subject to normalization constraints:

$$\begin{cases} \sum_{i=1}^{n} u_i(g_{i*}) = 1 \\ u_i(g_{i*}) = 0, \forall i = 1, 2, ..., n \end{cases} \tag{2.4}$$

2.2 Development

The value of each alternative $a \in A_R$ may be written as:

$$u^{'}[g(a)] = \sum_{i=1}^{n} u_i[g_i(a)] + \sigma(a) \forall a \in A_R \tag{2.5}$$

where $\sigma(a)$ is a potential error relative to $u^{'}[g(a)]$.

We use the linear interpolation to estimate the marginal value function. For each criteria, the interval $[g_{i*}, g_i^*]$ is cut into $(\alpha_i - 1)$ equals interval, and the end points g_i^j are given by the formula:

$$g_i^j = g_{i*} + \frac{j-1}{\alpha_i - 1}(g_i^* - g_{i*}) \forall j = 1, 2, ..., \alpha_i \tag{2.6}$$

The marginal value of an action a is calculated by a linear interpolation

$$u_i[g_i(a)] = u_i(g_i^j) + \frac{g_i(a) - g_i^j}{g_i^{j+1} - g_i^j}[u_i(g_i^{j+1}) - u_i(g_i^j)] \tag{2.7}$$

The set of reference action $A_R = a_1, a_2, ..., a_m$ is arranged where a_1 is the best action and a_m is the worst action. Which means :

$$\begin{cases} \Delta(a_k, a_{k+1}) \geq \delta \\ \Delta(a_k, a_{k+1}) = 0 \end{cases} \tag{2.8}$$

where $\Delta(a_k, a_{k+1}) = u^{'}[g(a_k)] - u^{'}[g(a_{k+1})]$ and δ is a small positive number.

The marginal value $u_i(g_i)$ must satisfy these constraints:

$$u_i(g_i^{j+1}) - u_(g_i^j) \geq s_i \forall j = 1, 2, ..., \alpha_i - 1, i = 1, 2, ..., n \tag{2.9}$$

where $s_i \geq 0$ and being indifference for each criteria.

The marginal value functions are estimated by means of the Linear Programm with (2.3), (2.4), (2.8), (2.9) as constraints, and an objective function depending on the $\sigma(a)$:

$$[min]F = \sum_{a \in A_R} \sigma(a)$$

subject to :

$$\begin{cases} \Delta(a_k, a_{k+1}) \geq \delta \quad or \quad \Delta(a_k, a_{k+1}) = 0 \\ u_i(g_i^{j+1}) - u_(g_i^j) \geq 0 \quad \forall \quad i \quad and \quad j \\ \sum_{i=1}^n u_i(g_i^*) = 1 \\ u_i(g_i^*) = 0 \quad u_i(g_i^j) \geq 0 \quad \sigma(a) \geq 0 \quad \forall a \in A_R \quad \forall iandj \end{cases} \tag{2.10}$$

The analysis found by this LP is a post-optimality analysis.

2.3 UTASTAR

The UTASTAR is an improved version of the UTA.

2.3.1 Principles and Notations

In UTA we used a single error $\sigma(a)$ in UTASTAR we use a double positive error function :

$$u^{'}[g(a)] = \sum_{i=1}^{n} u_i[g_i(a)] - \sigma^{+}(a) + \sigma^{-}(a) \quad \forall a \in A_R \tag{2.11}$$

where $\sigma^{+}(a)$ and $\sigma^{-}(a)$ are the underestimation and overstimation of the error function.

So, the UTASTAR algorithm has the following steps :

1. Find the value of reference actions $u[g(a_k)]$ in terms of marginal values $u_i(g_i)$ and in terms of variables w_{ij}

$$\begin{cases} u_i(g_i^1) = 0 \quad \forall i = 1,2,...,n \\ u_i(g_i^j) = \sum_{t=1}^{j-1} w_{it} \quad \forall i = 1,2,...,n \quad and \quad j = 2,3,...,\alpha_i - 1 \end{cases} \tag{2.12}$$

2. Write for each pair of consecutive action by including the errors function $\sigma^{+}(a)$ and $\sigma^{-}(a)$

$$\Delta(a_k, a_{k+1}) = u[g(a_k)] - \sigma^{+}(a_k) + \sigma^{-}(a_k) - u[g(a_{k+1})] + \sigma^{+}(a_{k+1}) - \sigma^{-}(a_{k+1}) \tag{2.13}$$

3. Solve the Linear Program

$$[min]z = \sum_{k=1}^{m} [\sigma^{+}(a_k) + \sigma^{-}(a_k)]$$

subject to :

$$\begin{cases} \Delta(a_k, a_{k+1}) \geq \delta \quad or \quad \Delta(a_k, a_{k+1}) = 0 \\ \sum_{i=1}^{n} \sum_{j=1}^{\alpha_i - 1} w_{ij} = 1 \\ w_{ij} \geq 0, \quad \sigma^{+}(a_k) \geq 0, \quad \sigma^{-}(a_k) \geq 0, \quad \forall i, jandk \end{cases} \tag{2.14}$$

4. Test the exitence of the optimal solution of the LP. You can find the mean additive value fuction of the optimal solutions which maximise the objective functions (in case of multiple optimal solutions) :

$$u_i(g_i^*) = \sum_{j=1}^{\alpha_i - 1} w_{ij} \quad \forall i = 1,2,...,n \tag{2.15}$$

With the additional constraint of (2.14), we add the following constaint :

$$\sum_{k=1}^{m} [\sigma^{+}(\alpha_k) + \sigma^{-}(\alpha_k)] \geq z^* + \varepsilon \tag{2.16}$$

where z^* is the optimal value of the LP found in step 3 and ε is a very small positive value

2.3.2 Comparison

The UTASTAR has been considered a better algorithm than UTA. Better result were found using the UTASTAR algorithm.

Chapter 3

Examples

3.1 Analyzing the choice of transportation

3.1.1 Problem

A DM wants to analyse the choice of transportation. The DM is interstered in the following criteria

1. price
2. time (min)
3. comfort (possibility to have a seat)

The evaluation of the previous criteria :

Means of transportation	Price	Time	Comfort	Ranking of the DM
RER	3	10	+	1
METRO (1)	4	20	++	2
METRO (2)	2	20	0	2
BUS	6	40	0	3
TAXI	30	30	+++	4

DM's preferences : $RER \succ Metro1 \approx Metro2 \succ Bus \succ Taxi$

3.1.2 Solution

Scale for each criteria

For each criteria, the interval $[g_i^*, g_{i*}]$ is cut into $(\alpha_i - 1)$ equal intervals. So in this case we have :

- Price $\rightarrow$ $[30, 16, 2]$
- Time $\rightarrow$ $[40, 30, 20, 10]$
- Comfort $\rightarrow$ $[0, +, ++, +++]$

Marginal value by linear interpolation

The marginal value is calculated by a linear interpolation. In this case we have:

- $u[g(RER)] = 0.07u_1(16) + 0.93u_1(2) + u_2(10) + u_3(+)$
- $u[g(METRO1)] = 0.14u_1(16) + 0.86u_1(2) + u_2(20) + u3(++)$
- $u[g(METRO2)] = u_1(2) + u_2(20) + u3(0) = u_1(2) + u_2(20)$
- $u[g(BUS)] = 0.29u_1(16) + 0.71u_1(2) + u_2(40) + u3(0) = 0.29u_1(16) + 0.71u_1(2)$
- $[u[g(TAXI)] = u_1(30) + u_2(30) + u3(+++) = u_2(30) + u3(+++)$

Replace u_i with w_{ij}

- $u[g(RER)] = w_{11} + 0.93w_{12} + w_{21} + w_{22} + w23 + w_{31}$
- $u[g(METRO1)] = w_{11} + 0.86w_{12} + w_{21} + w_{22} + w_{31} + w_{32}$
- $u[g(METRO2)] = w_{11} + w_{12} + w_{21} + w_{22}$
- $u[g(BUS)] = w_{11} + 0.71w_{12}$
- $[u[g(TAXI)] = w_{21} + w_{31} + w_{32} + w_{33}$

Difference between each pair of consecutive actions

- $\Delta(RER, METRO1) = 0.07w_{12} + w_{23} - w_{32} + \sigma_{RER} - \sigma_{METRO1} \geq \delta$
- $\Delta(METRO1, METRO2) = -0.14w_{12} + w_{31} + w_{32} + \sigma_{METRO1} - \sigma_{METRO2} = 0$
- $\Delta(METRO2, BUS) = 0.29w_{12} + w_{21} + w_{22} + \sigma_{METRO2} - \sigma_{BUS} \geq \delta$
- $\Delta(BUS, TAXI) = w_{11} + 0.71w_{12} - w_{21} - w_{31} - w_{32} - w_{33} + \sigma_{BUS} - \sigma_{TAXI} \geq \delta$

Linear Program

With $[min]F = \sum_{a \in A_R} \sigma(a)$ as the main objectif, we have the following linear program to solve:

Desc	w_{11}	w_{12}	w_{21}	w_{22}	w_{23}	w_{31}	w_{32}	w_{33}	Result
$\Delta(RER, METRO1) \geq \delta$	0	0.07	0	0	1	0	-1	0	$\geq \delta$
$\Delta(METRO1, METRO2) = 0$	0	-0.14	0	0	0	1	1	0	$= 0$
$\Delta(METRO2, BUS) \geq \delta$	0	0.29	1	1	0	0	0	0	$\geq \delta$
$\Delta(BUS, TAXI) \geq \delta$	1	0.71	-1	0	0	-1	-1	-1	$\geq \delta$
$\sum_{i=1}^{n} u_i(g_i^*) = 1$	1	1	1	1	1	1	1	1	$= 1$

3.2 Analyzing the choice of transportation

3.2.1 Problem

Lebanon is in need of a new location for the placement of a recycling center. The locations has been studied on the following criteria:

- Technical
- Economical activity
- Environmental
- Socio-Demographic
- Financial

A specialized team has selected 6 possible locations. The following table contains the evaluations of those locations on the previous criterias :

Location	Technical	Economical activity	Environmental	Socio-Demographic	Financial	Ranking
L1	3	3	3	1	3	6
L2	3	3	3	2	4	4
L3	3	4	3	3	3	2
L4	2	2	3	2	4	5
L5	5	3	4	4	3	1
L6	4	1	3	4	4	3

DM's preferences : $L5 \succ L3 \succ L6 \succ L2 \succ L4 \succ L1$

3.2.2 Solution

Scale for each criteria

For each criteria, the interval $[g_i^*, g_{i*}]$ is cut into $(\alpha_i - 1)$ equal intervals. So in this case we have :

- Technical $\rightarrow$ $[1, 2, 3, 4, 5]$
- Economical activity $\rightarrow$ $[1, 2, 3, 4, 5]$
- Environmental $\rightarrow$ $[1, 2, 3, 4, 5]$
- Socio-Demographic $\rightarrow$ $[1, 2, 3, 4, 5]$
- Financial $\rightarrow$ $[1, 2, 3, 4, 5]$

Marginal value by linear interpolation

The marginal value is calculated by a linear interpolation. In this case we have:

- $u[g(L1)] =$
- $u[g(L2)] =$
- $u[g(L3)] =$
- $u[g(L4)] =$
- $u[g(L5)] =$
- $u[g(L6)] =$

Replace u_i with w_{ij}

- $u[g(L1)] =$
- $u[g(L2)] =$
- $u[g(L3)] =$
- $u[g(L4)] =$
- $u[g(L5)] =$
- $u[g(L6)] =$

Difference between each pair of consecutive actions

- $\Delta(L1, L2) = \geq \delta$
- $\Delta(L2, L3) = \geq \delta$
- $\Delta(L3, L4) = \geq \delta$
- $\Delta(L4, L5) = \geq \delta$
- $\Delta(L5, L6) = \geq \delta$

Linear Program

Chapter 4

Conclusion

The UTA method build a utility function based on the preferences of the DM and it consist in solving a linear program (LP) to solve a multi-criteria problem.

This method will elaborate a model of preferences which is as similiar as possible to the DM's preferences.

An improved version of the UTA is the UTASTAR. In UTA we used a single error $\sigma(a)$ in UTASTAR we use a double positive error function. The updated version has performed better than the regular method.

Bibliography

[1] Salvatore Greco, Matthias Ehrgott, Jose Rui Figueira *Multiple Criteria Decision Analysis.* State of the Art Surveys, 2004.

[2] Siskos, Yannacopoulos *UTASTAR: an ordinal regression method for building additive value functions*

[3] Siskos *Cahier du LAMSADE : Les programmes UTA*, 1979

[4] Abdelkader Hammami, These du grade Philosophiae Doctor PHD de la Faculte des Sciences et de Genie *Modelisation Technico-Economique d'une chaine logistique dans une entreprise reseau*, 2003