Some inequalities for the triangle scenario

Tobias Fritz[1]
[1] *Perimeter Institute for Theoretical Physics*
Waterloo, Ontario, Canada, N2L 2Y5[*]

I sketch the derivation of some inequalities that hold for classical correlations in the triangle scenario.

I present several inequalities for binary outcomes together with a method of proof which has a combinatorial flavour. No quantum violations of any of these inequalities has been found to date.

In the following the complement of a value is marked by an empty circle accent, so $\mathring{b}$ means "anything but b", and accordingly $P(\mathring{a}\mathring{b}) = P(A{\neq}a, B{\neq}b)$. Additionally, an underscore stands for the corresponding marginal probability, like this: $P(\_b\_) := P(abc) + P(ab\mathring{c}) + P(\mathring{a}bc) + P(\mathring{a}b\mathring{c})$.

Theorem 1. *The following inequalities hold for all classical correlations in the triangle scenario:*

(a) $P(a)P(c) \leq P(ab) + P(\mathring{b}c)$

(b) $P(ab\mathring{c})P(a\mathring{b}c)P(\mathring{a}bc) \leq P(abc) + P(\mathring{a}\mathring{b})P(ab\mathring{c})P(a) + P(\mathring{a}\mathring{c})P(a\mathring{b}c)P(c) + P(\mathring{b}\mathring{c})P(\mathring{a}bc)P(b)$

(c) $P(001)P(010)P(100) \leq P(000)^2 + 2P(11\_)P(001) + 2P(1\_1)P(010) + 2P(\_11)P(100)$

(d) $P(000)^2P(111) \leq P(001)P(010)P(100) + (2P(000) + P(111))(1 - P(000) - P(111))$

(e) $P(000)^2P(111) \leq P(000)^3 + (2P(000) + P(111))(1 - P(000) - P(111))$

(f) $P(a)P(b)P(c) \leq P(\mathring{a}\mathring{b}\mathring{c}) + P(ab)P(c) + P(ac)P(b) + P(bc)P(a)$

(g) $P(a)P(b)P(c) \leq P(\mathring{a}\mathring{b}\mathring{c})^2 + 2\,(P(ab)P(c) + P(ac)P(b) + P(bc)P(a))$

It is quite likely that some of these inequalities are dominated by the others, but I do not know for sure whether any of them are actually redundant.

Proof. We reduce the problem to counting cycles in a graph as follows.

The classical correlations are precisely those that can be written in the form

$$P(abc) = \sum_{\lambda_{AB},\lambda_{BC},\lambda_{AC}} P(a|\lambda_{AB}\lambda_{AC})P(b|\lambda_{AB}\lambda_{BC})P(c|\lambda_{BC}\lambda_{AC})P(\lambda_{AB})P(\lambda_{BC})P(\lambda_{AC}).$$

By a suitable fine-graining, we can approximate such a classical model arbitrarily well by another one in which the hidden variables are uniformly distributed on a finite set Λ,

$$P(abc) = \sum_{\lambda_{AB},\lambda_{BC}\lambda_{AC}} P(a|\lambda_{AB}\lambda_{AC})P(b|\lambda_{AB}\lambda_{BC})P(c|\lambda_{BC},\lambda_{AC}). \tag{1}$$

Hence it is sufficient to prove the inequalities for all distributions of this form. Moreover, we may assume without loss of generality that all response functions are deterministic, i.e. $P(a|\lambda_{AB}\lambda_{AC}) \in \{0,1\}$, etc. In this way, the right-hand side of (1) simply counts the number of triples for which each conditional probability takes on the value 1, so that

$$P(abc) = |\{\, (\lambda_{AB},\lambda_{BC}\lambda_{AC}) \;|\; P(a|\lambda_{AB}\lambda_{AC}) = P(b|\lambda_{AB}\lambda_{BC}) = P(c|\lambda_{BC},\lambda_{AC}) = 1 \,\}| \cdot |\Lambda|^{-3},$$

where the factor of $|\Lambda|^{-3}$ now is the probability of each triple of hidden variable values $(\lambda_{AB},\lambda_{BC}\lambda_{AC})$ to occur. Since all our inequalities can be made homogeneous (see below), we can simply omit this constant scalar factor. So it is sufficient to prove that the numbers

$$N(abc) = |\{\, (\lambda_{AB},\lambda_{BC}\lambda_{AC}) \;|\; P(a|\lambda_{AB}\lambda_{AC}) = P(b|\lambda_{AB}\lambda_{BC}) = P(c|\lambda_{BC},\lambda_{AC}) = 1 \,\}|\,, \tag{2}$$

in place of $P(a,b,c)$ satisfy our inequalities.

But now these numbers can be understood in terms of cycles on graphs: as illustrated in Figure 1, we consider the complete tripartite graph with one vertex for each possible value of each hidden variable λ_i. The edge between two hidden variable values λ_2 and λ_3 is labelled by the corresponding conditional probability $P(a|\lambda_2,\lambda_3)$, and similarly for $P(b|\lambda_3,\lambda_1)$ and $P(c|\lambda_1,\lambda_2)$. Then (2) counts nothing but the number of 3-cycles in this graph

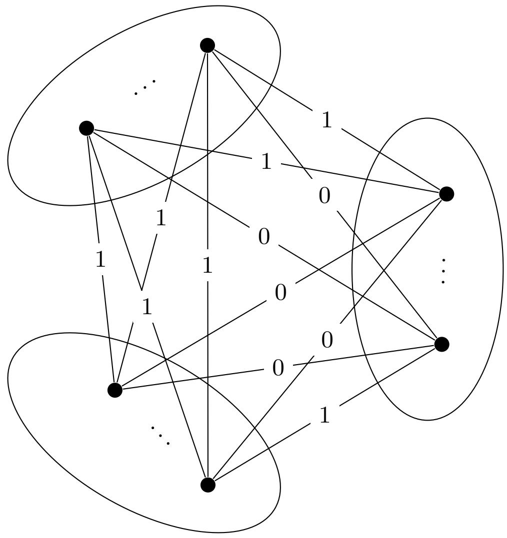

FIG. 1. Illustration of the labelled graph constructed in the proof.

whose edge labels are precisely the outcomes a, b and c under consideration. With this in mind, the following arguments are all of the same form: for any cycle contributing to the left-hand side, there will be a corresponding cycle contributing to the right-hand side. The inequality then follows if the map from the left-hand side cycles to the right-hand side cycles is injective, since it follows that the cardinality of the former set is bounded by the cardinality of the latter. Now on to the various cases:

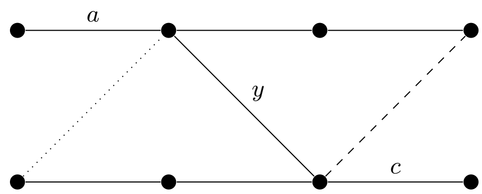

FIG. 2. Illustration of the proof of (3). In order for a sensible drawing to be possible, the cyclic structure of Figure 1 has now been cut open into a linear structure, so that each vertex on the right end is actually equal to the corresponding vertex on the left end.

(a) In homogeneous form, we need to show

$$N(a\_\_)N(\_\_c) \leq N(ab\_)N(\_\_\_) + N(\_\mathring{b}c)N(\_\_\_). \tag{3}$$

To see this, consider Figure 2: the left-hand side of the inequality counts all pairs of cycles for which the first edge in the first cycle is labelled a and the third edge in the second cycle by c. For any such pair, we consider the label y in Figure 2: if it is b, then we obtain a cycle with labels $ab?$ by taking a, y and then the dashed edge, and one with completely unknown labels ??? by taking the complementary path. But if $y = \mathring{b}$, then we can obtain a cycle $?\mathring{b}c$ by taking the dashed edge first, together with a cycle ??? with completely unknown labels. Since these cases contribute to different terms on the right-hand side of (3), we obtain the desired injective map.

(b) In homogeneous form, this inequality says

$$\begin{aligned} N(ab\mathring{c})N(a\mathring{b}c)N(\mathring{a}bc) \leq & N(abc)N(\_\_\_)N(\_\_\_) \\ & + N(\mathring{a}\mathring{b}\_)N(ab\mathring{c})N(a\_\_) \\ & + N(\mathring{a}\_\mathring{c})N(a\mathring{b}c)N(\_\_c) \\ & + N(\_\mathring{b}\mathring{c})N(\mathring{a}bc)N(\_b\_). \end{aligned} \tag{4}$$

The proof is similar to the previous one; again we start with the left-hand side, which now counts triples of cycles with labels $ab\mathring{c}$, $a\mathring{b}c$ and $\mathring{a}bc$ as illustrated as the horizontal lines in Figure 3. Now consider the labels of the dashed edges x, y and z. If $x = a, y = b, z = c$, then we choose this cycle and rewire the rest in an arbitrary manner, resulting in a contribution to the first term on the right-hand side. Otherwise, at least one of x, y or z corresponds to a complimentary value; suppose $y = \mathring{b}$. Then we choose the cycle $\mathring{a}y?$ starting and ending at the bottom vertices, keep the top $ab\mathring{c}$ one, which leaves a unique choice of $a??$ for the third cycle; this gives a contribution to the second term. The other cases where $x = \mathring{a}$ or $z = \mathring{c}$ are analogous.

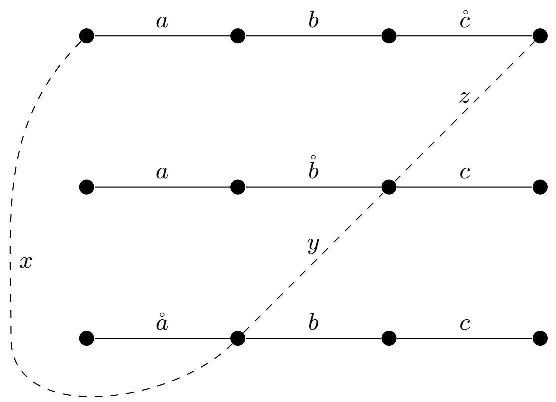

FIG. 3. Illustration of the proof of (4).

(c) . . . The proofs of all other inequalities should be similar. . . □

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