Expressible sets and expressible assignments

RWS

I. EXPRESSIBLE SETS

Example 1 Incompatibility of Pienaar distribution with DAG #16

Consider the DAG of Fig. 1. Henson, Lal and Pusey showed that this DAG is a candidate for being 'interesting', that is, the compatible distributions satisfy constraints over and above the conditional independence relations that follow from d-separation relations in the DAG.

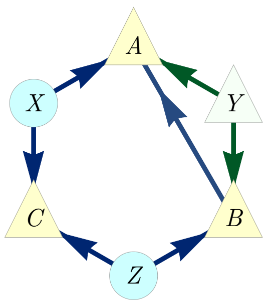

FIG. 1. DAG #16 in Ref. [?].

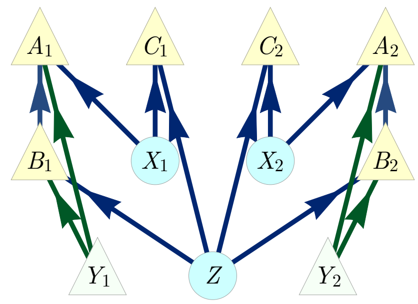

FIG. 2. The Rocket inflation of ??.

?] identified a distribution which satisfies the CI relations among the observed variables in DAG #16, namely, $Y \perp\!\!\!\perp C$ and $A \perp\!\!\!\perp B|Y$ [?], but is nonetheless incompatible with it:

$$P^{\text{Pien}}_{ABCY} := \frac{[0000]+[0110]+[0001]+[1011]}{4}, \quad \text{i.e.,} \quad P^{\text{Pien}}_{YABC}(yabc) := \begin{cases} \frac{1}{4} & \text{if } y\cdot c = a \text{ and } (y\oplus 1)\cdot c = b, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that we can rewrite Eq. (1) as

$$P^{\text{Pien}}_{ABCY} = \frac{1}{2}([00]_{BC}+[11]_{BC})[0]_A[0]_Y + \frac{1}{2}([00]_{AC}+[11]_{AC})[0]_B[1]_Y, \tag{2}$$

which makes it evident that the distribution can be described as follows: if $Y=0$, then B and C are in a maximally correlated state and $A=0$, while if $Y=1$, then A and C are maximally correlated and $B=0$.

Here, we will establish this incompatibility using the inflation technique. To do so, we use the inflation of DAG #16 depicted in Fig. 2. To do so, we will make use of the fact that $\{B_2C_2Y_2\}, \{A_1C_1Y_1\}$ and $\{B_2C_1Y_2\}$ are injectable sets, together with the fact that $\{A_1C_2Y_1\}$ is an expressible set.

We begin by demonstrating how the d-separation relations in the inflation imply that $\{A_1C_2Y_1\}$ is expressible. The expressibility of $\{A_1C_2Y_1\}$ follows from the expressibility of $\{A_1B_1C_2Y_1\}$ and the fact that the distribution on the former can be obtained from the distribution on the latter by marginalization. $\{A_1B_1C_2Y_1\}$ is expressible because the d-separation relation $A_1 \perp C_2|B_1Y_1$ implies that

$$P_{A_1B_1C_2Y_1} = \frac{P_{A_1B_1Y_1}P_{C_2B_1Y_1}}{P_{B_1Y_1}}, \tag{3}$$

and each of the sets $\{A_1B_1Y_1\}, \{C_2B_1Y_1\}$, and $\{B_1Y_1\}$ are injectable. We therefore have

$$P_{A_1C_2Y_1}(acy) = \sum_b \frac{P^{\text{Pien}}_{ABY}(aby)P^{\text{Pien}}_{CBY}(cby)}{P^{\text{Pien}}_{BY}(by)}, \tag{4}$$

From the injectability of $\{B_2C_2Y_2\}, \{A_1C_1Y_1\}$ and $\{B_2C_1Y_2\}$, we can infer that

$$\begin{aligned} P_{B_2C_2|Y_2}(bc|y) &= P^{\text{Pien}}_{BC|Y}(bc|y) \\ P_{A_1C_1|Y_1}(ac|y) &= P^{\text{Pien}}_{AC|Y}(ac|y) \\ P_{B_2C_1|Y_2}(bc|y) &= P^{\text{Pien}}_{BC|Y}(bc|y) \end{aligned} \tag{5}$$

which implies that

$$P_{B_2C_2|Y_2}(\cdot\cdot|0) = \frac{1}{2}([00]_{B_2C_2} + [11]_{B_2C_2}) \tag{6}$$

$$P_{A_1C_1|Y_1}(\cdot\cdot|1) = \frac{1}{2}([00]_{A_1C_1} + [11]_{A_1C_1}) \tag{7}$$

$$P_{B_2C_1|Y_2}(\cdot\cdot|0) = \frac{1}{2}([00]_{B_2C_1} + [11]_{B_2C_1}) \tag{8}$$

For the expressible set $\{A_1C_2Y_1\}$, Eq. (4) implies that

$$\begin{aligned} P_{A_1C_2|Y_1}(ac|y) &= \sum_b \frac{P^{\text{Pien}}_{ABY}(aby)P^{\text{Pien}}_{CBY}(cby)}{P^{\text{Pien}}_{BY}(by)P^{\text{Pien}}_{Y}(y)} \\ &= \sum_b \frac{P^{\text{Pien}}_{AB|Y}(ab|y)P^{\text{Pien}}_{CB|Y}(cb|y)}{P^{\text{Pien}}_{B|Y}(b|y)}, \end{aligned} \tag{9}$$

where we have simply used the definition of conditioning.

Now suppose that $Y_2 = 0$ and $Y_1 = 1$. From Eq. (8), we infer that

$$\text{With probability } 1/2,\ B_2 = 0 \text{ and } C_1 = 0. \tag{10}$$

From Eq. (6), we infer that

$$\text{if } B_2 = 0 \text{ then } C_2 = 0. \tag{11}$$

From Eq. (7), we infer that

$$\text{if } C_1 = 0 \text{ then } A_1 = 0. \tag{12}$$

These three results imply that

$$\text{The probability } p \text{ that } C_2 = 0 \text{ and } A_1 = 0 \text{ must be } \geq 1/2. \tag{13}$$

However, from Eq. (14), we infer that the probability of $C_2 = 0$ and $A_1 = 0$ is only $p = 1/4$. Explicitly,

$$\begin{aligned} P_{A_1C_2|Y_1}(00|1) &= \sum_b \frac{P^{\text{Pien}}_{AB|Y}(0b|1)P^{\text{Pien}}_{CB|Y}(0b|1)}{P^{\text{Pien}}_{B|Y}(b|1)} \\ &= \frac{P^{\text{Pien}}_{AB|Y}(00|1)P^{\text{Pien}}_{CB|Y}(00|1)}{P^{\text{Pien}}_{B|Y}(0|1)} + \frac{P^{\text{Pien}}_{AB|Y}(01|1)P^{\text{Pien}}_{CB|Y}(01|1)}{P^{\text{Pien}}_{B|Y}(1|1)} \\ &= \frac{1}{4} \end{aligned} \tag{14}$$

We have therefore arrived at a contradiction. This establishes the incompatibility of the Pienaar distribution with DAG #16.

II. INSTRUMENTAL INEQUALITY VIA EXPRESSIBLE ASSIGNMENTS IN THE BELL SCENARIO

Consider the instrumental scenario of Fig. 7.

Pearl has found causal compatibility inequalities, termed the instrumental inequalities. If the observed variables are binary, then they have the following form:

$$\begin{aligned} &P_{XY|Z}(00|0) + P_{XY|Z}(00|0) \leq 1, \\ &P_{XY|Z}(10|0) + P_{XY|Z}(11|1) \leq 1, \\ &P_{XY|Z}(01|0) + P_{XY|Z}(00|1) \leq 1, \\ &P_{XY|Z}(11|0) + P_{XY|Z}(10|1) \leq 1. \end{aligned} \tag{15}$$

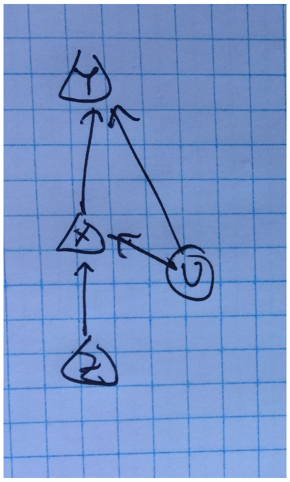

FIG. 3. The instrumental scenario.

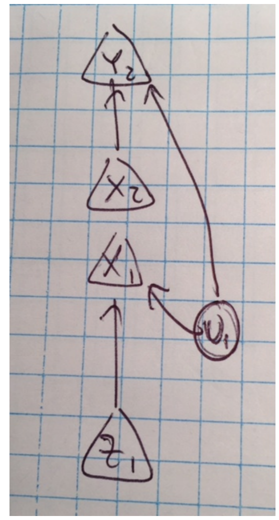

FIG. 4. The Bell scenario.

This can be summarized as

$$\forall x : \sum_y P_{XY|Z}(xy|y) \leq 1, \tag{16}$$

$$\forall x : \sum_y P_{XY|Z}(x(y \oplus 1)|y) \leq 1, \tag{17}$$

If we label the two pairs of cause-effect-related observed variables in the Bell scenario by X_2, Y_2 and Z_1, X_1 respectively, then we can think of the Bell scenario as supporting a quasi-inflation of the instrumental scenario: for every causal model in the instrumental scenario, we can define a causal model in the Bell scenario where every variable except X_2 depends on its parents in exactly the manner that the corresponding variable (i.e., the one where the index is dropped) did in the Instrumental scenario. In this mapping, X_2 is presumed to be a root variable that is distributed in the same manner as X is in the Instrumental scenario.

We then note that although the set $\{Y_2 Z_1 X_1 X_2\}$ is not an expressible set, assignments of the form $P_{Y_2 Z_1|X_1 X_2}(yz|xx)$, where X_1 and X_2 take the same value, *are* expressible, in the sense that

$$P_{Y_2 Z_1|X_1 X_2}(yz|xx) = P_{YZ|X}(yz|x). \tag{18}$$

This equality follows from considering the consequences of conditioning on X in the Instrumental Scenario.

Eq. (18) in turn implies that

$$P_{XY|Z}(xy|z) = P_{Y_2 X_1|X_2 Z_2}(yx|xz). \tag{19}$$

The proof is as follows. One notes that

$$P_{YZ|X} = \frac{P_{XY|Z} P_Z}{P_X} \tag{20}$$

and that

$$\begin{aligned} P_{Y_2 Z_1|X_1 X_2} &= \frac{P_{Y_2 X_1|X_2 Z_2} P_{Z_2}}{P_{X_1|Z_2}} \\ &= \frac{P_{Y_2 X_1|X_2 Z_2} P_{Z_2}}{P_{X_1}} \end{aligned} \tag{21}$$

where the second equality follows from the fact that $X_1 \perp X_2$ in the Bell scenario. It is then sufficent to note that $\{X_1\}$ and $\{Z_1\}$ are injectable in order to complete the proof.

We will shortly demonstrate how the Bell scenario implies the following causal compatibility inequalities:

$$\sum_y P_{Y_2X_1|X_2Z_2}(yx|xy) \leq 1 \tag{22}$$

$$\sum_y P_{Y_2X_1|X_2Z_2}((y\oplus 1)x|xy) \leq 1 \tag{23}$$

$$\tag{24}$$

Combining these with Eq. (19), we obtain the Instrumental inequalities of (16) and (16).

We now show how these causal compatibility inequalities in the Bell scenario are instances of bounds on the performance of a distributed guessing game.

A. Distributed guessing games

We recall the definition of a distributed guessing game from Ref. [1]

> [A distributed guessing game is] a non-local game in which a referee has access to a set of vectors of n symbols with values in $\{0,...,d-1\}$. Denote this set by S and by $|S|$ its size, which can be less than d^n in general. Now, the referee chooses a vector $(\tilde{a}_1,\ldots,\tilde{a}_n)$ uniformly at random from S, and encodes it into a new vector of, again, n symbols using a function f. However, the new symbols can now take m values and, thus, $f : S \to \{0,...,m-1\}^n$. The resulting vector is $(x_1,...,x_n) = f(\tilde{a}_1,...,\tilde{a}_n)$. These n symbols are distributed among n distant players who cannot communicate and must produce individual guesses $a_1,...,a_n$. Their goal is to guess the initial input to the function, that is, they win whenever $a_j = \tilde{a}_j$ for all j. Note that the encoding function f and the set S are known in advance to all the players.

The game known as "Guess your neighbour's input", abbreviated GYNI, is an instance [2]. We here make use of a simplified version of GYNI wherein one of the inputs is fixed, so that only one of the player's guessing tasks is nontrivial.

The probability of succes in such a game corresponds to a probability of achieving particular outcomes in the Bell scenario, where the hidden common cause can be considered the strategy of the two players. The game is defined as follows; Alice's binary setting, Z_2, is chosen uniformly at random. Bob's binary setting, X_2 is fixed with value x. Thus $S = \{(0,x),(1,x)\}$, and the input pair is chosen uniformly from this set. The probability of success in the game is clearly

$$\begin{aligned} P_{\text{succ}} &= \sum_{y=0}^{1} P_{Y_2X_1|X_2Z_2}(yx|xy)P(y) \\ &= \frac{1}{2}\sum_{y=0}^{1} P_{Y_2X_1|X_2Z_2}(yx|xy), \end{aligned} \tag{25}$$

It follows that to derive Eq. (22), it suffices to demonstrate that

$$P_{\text{succ}} \leq \frac{1}{2}. \tag{26}$$

This inequality has, in fact, a highly intuitive proof. Given that Bob's setting is fixed to be x, the optimal strategy involves Alice outputing x with probability 1. However, because there is no causal influence from Alice's setting variable to Bob's outcome variable, any classical strategy can do no better than the one wherein Bob simpy guesses Alice's input, in which case the probability of guessing correctly is $\frac{1}{2}$.

Note that if one modifies the game to one wherein Bob must guess the *negation* of Alice's input (BGNAI), the probability of success is still bounded above by $\frac{1}{2}$, and we derive Eq. (23).

III. FAILED ATTEMPT TO UNDERSTAND INSTRUMENTAL INEQUALITY VIA INFLATION

A. Alternate form of the inequality

Consider the instrumental inequalities for binary variables:

$$
\begin{aligned}
&P_{XY|Z}(00|0) + P_{XY|Z}(00|0) \leq 1,\\
&P_{XY|Z}(10|0) + P_{XY|Z}(11|1) \leq 1,\\
&P_{XY|Z}(01|0) + P_{XY|Z}(00|1) \leq 1,\\
&P_{XY|Z}(11|0) + P_{XY|Z}(10|1) \leq 1.
\end{aligned} \tag{27}
$$

It turns out that these can be obtained as special cases of the following set of causal compatibility inequalities for the instrumental scenario:

$$P_{XY|Z} \leq P_Y. \tag{28}$$

THIS IS WRONG. As Elie has noted, the latter inequality cannot be a valid causal compatibility inequality because of the following example: if $X = Z$, and $Y = X$ and Z is uniformly distribution, then we have $P_Y(0) = P_Y(1) = 1/2$, but $P_{XY|Z}(00|0) = 1$.

To see that Eq. (28) implies (27), note that it implies

$$\forall x \forall z \forall y : P_{XY|Z}(xy|z) \leq P_Y(y), \tag{29}$$

which in turn implies that

$$\forall x \forall y : P_{XY|Z}(xy|y) \leq P_Y(y), \tag{30}$$

and

$$\forall x \forall y : P_{XY|Z}(xy|y \oplus 1) \leq P_Y(y), \tag{31}$$

Summing over y in each case, we obtain:

$$\forall x : \sum_y P_{XY|Z}(xy|y) \leq 1, \tag{32}$$

and

$$\forall x : \sum_y P_{XY|Z}(xy|y \oplus 1) \leq 1, \tag{33}$$

respectively. These are the instrumental inequalities for binary variables.

The usual way of representing the instrumental inequality is as follows:

$$\max_x \sum_y \max_z P(xy|z) \leq 1. \tag{34}$$

It too can be obtained as a special case of Eq. (28).

To see that Eq. (28) implies (34), note that it implies

$$\forall x \forall y \forall z : P_{YX|Z}(yx|z) \leq P_Y(y), \tag{35}$$

which in turn implies

$$\forall x \forall y : \max_z P_{YX|Z}(yx|z) \leq P_Y(y), \tag{36}$$

and therefore

$$\forall x : \sum_y \max_z P_{YX|Z}(yx|z) \leq 1, \tag{37}$$

which entails finally

$$\max_x \sum_y \max_z P_{YX|Z}(yx|z) \leq 1, \tag{38}$$

which is the standard form of the instrumental inequality.

B. Deriving the instrumental inequality using inflation technique and *expressible assignments*

Consider the instrumental scenario, depicted in Fig. 7, and the inflation thereof depicted in Fig. 8.

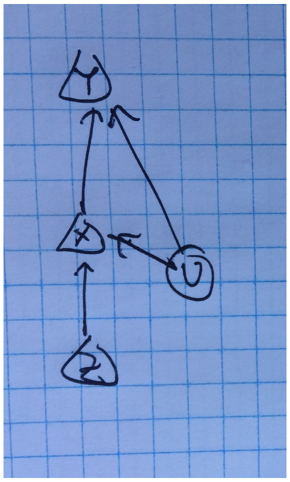

FIG. 5. The instrumental scenario.

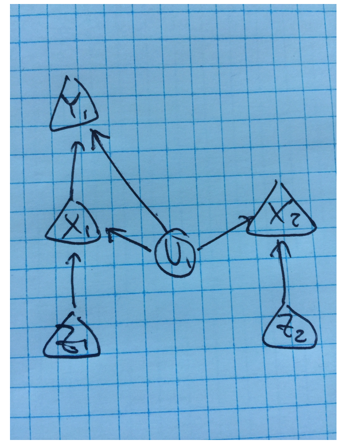

FIG. 6. The Greyhound inflation of the instrumental scenario.

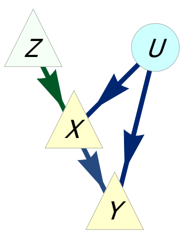

FIG. 7. The instrumental scenario.

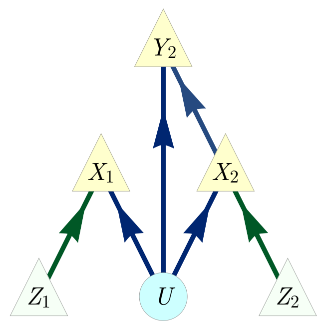

FIG. 8. The Greyhound inflation of the instrumental scenario, with the opposite labelling convention to the one I use below.

It is straightforward to verify that

$$P_{Y_1 Z_2|X_2 X_1}(\cdot\cdot|xx) = P_{YZ|X}(\cdot\cdot|x). \tag{39}$$

In this case, we say that $P_{Y_1 Z_2|X_2 X_1}$ is an *expressible assignment.*
Note that Eq. (39) implies that

$$\frac{P_{Y_1 Z_2 X_2 X_1}(\cdot\cdot xx)}{P_{X_1}(x)P_{X_2}(x)} = \frac{P_{YZX}(\cdot\cdot x)}{P_X(x)}, \tag{40}$$

and because X_1 and X_2 are injectable, we infer that

$$P_{Y_1 Z_2 X_2 X_1}(\cdot\cdot xx) = P_{YZX}(\cdot\cdot x), \tag{41}$$

so $P_{Y_1 Z_2 X_2 X_1}(\cdot\cdot xx)$ is also an expressible assignment.
THE ABOVE IS INCORRECT. Eq. (39) implies that

$$\frac{P_{Y_1 Z_2 X_2 X_1}(\cdot\cdot xx)}{P_{X_1 X_2}(xx)} = \frac{P_{YZX}(\cdot\cdot x)}{P_X(x)}. \tag{42}$$

Because $\{X_1X_2\}$ is not an injectable set, neither is $P_{Y_1Z_2X_2X_1}(\cdot\cdot xx)$ an expressible assignment.
IT FOLLOWS THAT THE REST OF THE PROOF DOES NOT GO THROUGH

The d-separation relation $Y_1 \perp Z_2$ in the inflation implies that

$$\sum_{X_1,X_2} P_{Y_1Z_2X_2X_1} = P_{Y_1}P_{Z_2} \tag{43}$$

This can be rewritten as

$$\forall y_1 \forall z_2 : \sum_x P_{Y_1Z_2X_2X_1}(y_1z_2xx) + \sum_{x_1\neq x_2} P_{Y_1Z_2X_2X_1}(\cdot\cdot x_1x_2) \leq P_{Y_1}(y_1)P_{Z_2}(z_2) \tag{44}$$

which implies that

$$\forall y_1 \forall z_2 : \sum_x P_{Y_1Z_2X_2X_1}(y_1z_2xx) \leq P_{Y_1}(y_1)P_{Z_2}(z_2). \tag{45}$$

Given that every term in the sum on the LHS is positive, the inequality holds for each such term,

$$\forall y_1 \forall z_2 \forall x : P_{Y_1Z_2X_2X_1}(y_1z_2xx) \leq P_{Y_1}(y_1)P_{Z_2}(z_2). \tag{46}$$

Conditioning on Z_2, we obtain

$$\forall y_1 \forall z_2 \forall x : P_{Y_1X_2X_1|Z_2}(y_1xx|z_2) \leq P_{Y_1}(y_1). \tag{47}$$

Finally, given that the singleton set $\{Y_1\}$ is an injectable set and given that $P_{Y_1X_2X_1|Z_2}(y_1xx|z_2)$ is an expressible assignment (described in Eq. (41)), we conclude that

$$\forall y \forall z \forall x : P_{YX|Z}(yx|z) \leq P_Y(y), \tag{48}$$

which we showed previously to imply the instrumental inequality.

[1] Fritz, T., Sainz, A. B., Augusiak, R., Brask, J. B., Chaves, R., Leverrier, A., and Acn, A., Local orthogonality as a multipartite principle for quantum correlations. Nature communications **4**, 2263 (2013).

[2] Acn, A., Almeida, M. L., Augusiak, R., and Brunner, N., Guess your neighbours input: no quantum advantage but an advantage for quantum theory. In Quantum Theory: Informational Foundations and Foils (pp. 465-496). Springer Netherlands (2-16).