Some inequalities for the triangle scenario

Tobias Fritz[1]

[1]*Perimeter Institute for Theoretical Physics*
Waterloo, Ontario, Canada, N2L 2Y5[*]

I sketch the derivation of some inequalities that hold for classical correlations in the triangle scenario.

I present several inequalities for binary outcomes together with a method of proof which has a combinatorial flavour. No quantum violations of any of these inequalities has been found to date.

In the following, an underscore stands for the corresponding marginal probability, like this: $P(\_0\_) := P(000) + P(001) + P(100) + P(101)$.

Theorem 1. *The following inequalities hold for all classical correlations in the triangle scenario:*

(a)

$$P(0\_\_)P(\_\_1) \leq P(00\_) + P(\_11)$$

(b)

$$P(001)P(010)P(100) \leq P(000) + P(11\_)P(001)P(0\_\_) + P(1\_1)P(010)P(\_\_0) + P(\_11)P(100)P(\_0\_)$$

(c)

$$P(001)P(010)P(100) \leq P(000)^2 + 2P(11\_)P(001) + 2P(1\_1)P(010) + 2P(\_11)P(100)$$

(d)

$$P(000)^2P(111) \leq P(001)P(010)P(100) + (2P(000) + P(111))(1 - P(000) - P(111))$$

(e)

$$P(000)^2P(111) \leq P(000)^3 + (2P(000) + P(111))(1 - P(000) - P(111))$$

(f)

$$P(\_\_1)P(\_1\_)P(1\_\_) \leq P(000) + P(11\_)P(\_\_1) + P(1\_1)P(\_1\_) + P(\_11)P(1\_\_)$$

(g)

$$P(\_\_1)P(\_1\_)P(1\_\_) \leq P(000)^2 + 2P(11\_)P(\_\_1) + 2P(1\_1)P(\_1\_) + 2P(\_11)P(1\_\_)$$

It is quite likely that some of these inequalities are dominated by the others, but I do not know for sure whether any of them are actually redundant.

Proof. We reduce the problem to counting cycles in a graph as follows.

The classical correlations are precisely those that can be written in the form

$$P(a,b,c) = \sum_{\lambda_1,\lambda_2,\lambda_3} P(a|\lambda_2,\lambda_3)P(b|\lambda_3,\lambda_1)P(c|\lambda_1,\lambda_2)P(\lambda_1)P(\lambda_2)P(\lambda_3).$$

By a suitable fine-graining, we can approximate such a classical model arbitrarily well by another one in which the hidden variables are uniformly distributed on a finite set Λ,

$$P(a,b,c) = \sum_{\lambda_1,\lambda_2,\lambda_3} P(a|\lambda_2,\lambda_3)P(b|\lambda_3,\lambda_1)P(c|\lambda_1,\lambda_2). \tag{1}$$

Hence it is sufficient to prove the inequalities for all distributions of this form. Moreover, we may assume without loss of generality that all response functions are deterministic, i.e. $P(a,b|\lambda_3) \in \{0,1\}$, etc. In this way,

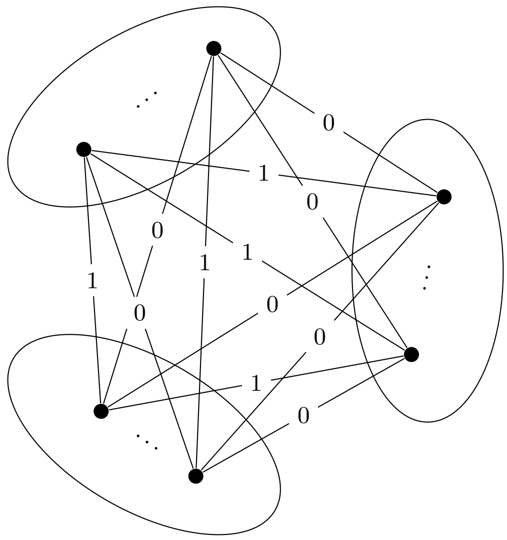

FIG. 1. Illustration of the labelled graph constructed in the proof.

the right-hand side of (1) simply counts the number of triples for which each conditional probability takes on the value 1, so that

$$P(a,b,c) = |\{\, (\lambda_1,\lambda_2,\lambda_3) \;|\; P(a|\lambda_2,\lambda_3) = P(b|\lambda_3,\lambda_1) = P(c|\lambda_1,\lambda_2) = 1 \,\}| \cdot |\Lambda|^{-3},$$

where the factor of $|\Lambda|^{-3}$ now is the probability of each triple of hidden variable values $(\lambda_1,\lambda_2,\lambda_3)$ to occur. Since all our inequalities can be made homogeneous (see below), we can simply omit this constant scalar factor. So it is sufficient to prove that the numbers

$$N(a,b,c) = |\{\, (\lambda_1,\lambda_2,\lambda_3) \;|\; P(a,b|\lambda_3) = P(b,c|\lambda_1) = P(c,a|\lambda_2) = 1 \,\}| \cdot |\Lambda|^{-3}, \tag{2}$$

in place of $P(a,b,c)$ satisfy our inequalities.

But now these numbers can be understood in terms of cycles on graphs: as illustrated in Figure 1, we consider the complete tripartite graph with one vertex for each possible value of each hidden variable λ_i. The edge between two hidden variable values λ_2 and λ_3 is labelled by the corresponding conditional probability $P(a|\lambda_2,\lambda_3)$, and similarly for $P(b|\lambda_3,\lambda_1)$ and $P(c|\lambda_1,\lambda_2)$. Then (2) counts nothing but the number of 3-cycles in this graph whose edge labels are precisely the outcomes a, b and c under consideration. With this in mind, the following arguments are all of the same form: for any cycle contributing to the left-hand side, there will be a corresponding cycle contributing to the right-hand side. The inequality then follows if the map from the left-hand side cycles to the right-hand side cycles is injective, since it follows that the cardinality of the former set is bounded by the cardinality of the latter. Now on to the various cases:

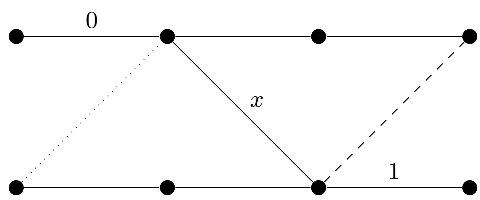

FIG. 2. Illustration of the proof of (3). In order for a sensible drawing to be possible, the cyclic structure of Figure 1 has now been cut open into a linear structure, so that each vertex on the right end is actually equal to the corresponding vertex on the left end.

(a) In homogeneous form, we need to show

$$N(0\_\_)N(\_\_1) \leq N(00\_)N(\_\_\_) + N(\_11)N(\_\_\_). \tag{3}$$

To see this, consider Figure 2: the left-hand side of the inequality counts all pairs of cycles for which the first edge in the first cycle is labelled 0 and the third edge in the second cycle by 1. For any such pair, we consider the label x in Figure 2: if it is 0, then we obtain a cycle with labels 00? by taking 0, x and then the dashed edge, and one with completely unknown labels ??? by taking the complementary path. But if $x = 1$, then we can obtain a cycle ?11 by taking the dashed edge first, together with a cycle ??? with completely unknown labels. Since these cases contribute to different terms on the right-hand side of (3), we obtain the desired injective map.

(b) In homogeneous form, this inequality says

$$
\begin{aligned}
N(001)N(010)N(100) \leq\, & N(000)N(\_\_\_)N(\_\_\_) \\
& + N(11\_)N(001)N(0\_\_) \\
& + N(1\_1)N(010)N(\_\_0) \\
& + N(\_11)N(100)N(\_0\_).
\end{aligned}
\tag{4}
$$

The proof is similar to the previous one; again we start with the left-hand side, which now counts triples of cycles with labels 001, 010 and 100 as illustrated as the horizontal lines in Figure 3. Now consider the labels of the dashed edges x, y and z. If $x = y = z = 0$, then we choose this cycle and rewire the rest in an arbitrary manner, resulting in a contribution to the first term on the right-hand side. Otherwise, at least one of x, y or z is equal to 1; suppose $y = 1$. Then we choose the cycle $1y?$ starting and ending at the bottom vertices, keep the top 001 one, which leaves a unique choice of 0?? for the third cycle; this gives a contribution to the second term. The other cases where $x = 1$ or $z = 1$ are analogous.

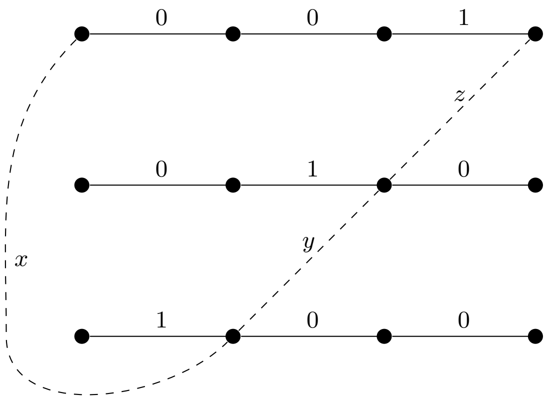

FIG. 3. Illustration of the proof of (4).

(c) . . . The proofs of all other inequalities should be similar. . .

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