Deriving Pearl's instrumental inequalities using the inflation DAG technique

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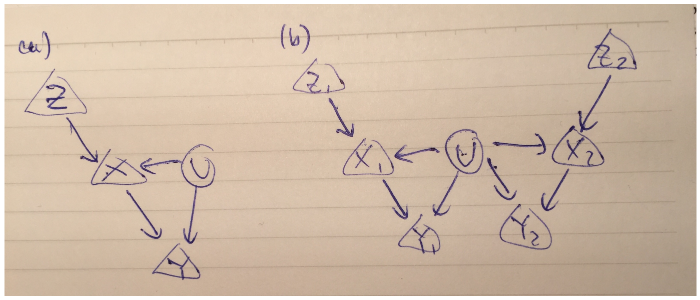
The instrumental inequality is a constraint for the DAG depicted in Fig. 8.1 of Pearl's book, reproduced here as Fig. (a). Consider the inflation DAG depicted in Fig. (b).



Noting that the ancestral subgraph of $\{Z_1 Z_2 X_1 Y_1\}$ is isomorphic to that of $\{Z_1 Z_2 X_2 Y_2\}$, we can conclude that the associated distributions are equal,

$$P_{Z_1 Z_2 X_1 Y_1} = P_{Z_1 Z_2 X_2 Y_2}. \tag{1}$$

Now note that for any pair of values a, b,

$$P_{Z_1 Z_2}(a, b) = \sum_{c,d} P_{Z_1 Z_2 X_1 Y_1}(a, b, c, d), \tag{2}$$

which implies that for any value of c,

$$\sum_{d} P_{Z_1 Z_2 X_1 Y_1}(a, b, c, d) \leq P_{Z_1 Z_2}(a, b). \tag{3}$$

As one concrete example, we can take $(a, b, c) = (0, 1, 0)$, in which case we obtain

$$P_{Z_1 Z_2 X_1 Y_1}(0, 1, 0, 0) + P_{Z_1 Z_2 X_1 Y_1}(0, 1, 0, 1) \leq P_{Z_1 Z_2}(0, 1). \tag{4}$$

Next comes the critical step. We use Eq. (1) to rewrite the second term on the LHS, thereby obtaining

$$P_{Z_1 Z_2 X_1 Y_1}(0, 1, 0, 0) + P_{Z_1 Z_2 X_2 Y_2}(0, 1, 0, 1) \leq P_{Z_1 Z_2}(0, 1). \tag{5}$$

Finally, we use the marginal independence relations implied by the inflation DAG to infer that

$$\begin{aligned}
&P_{Z_1 Z_2}(0, 1) = P_{Z_1}(0) P_{Z_2}(1), \\
&P_{Z_1 Z_2 X_1 Y_1}(0, 1, 0, 0) = P_{Z_1 X_1 Y_1}(0, 0, 0) P_{Z_2}(1), \\
&P_{Z_1 Z_2 X_2 Y_2}(0, 1, 0, 1) = P_{Z_2 X_2 Y_2}(1, 0, 1) P_{Z_1}(0).
\end{aligned} \tag{6}$$

This yields

$$P_{Z_1 X_1 Y_1}(0, 0, 0) P_{Z_2}(1) + P_{Z_2 X_2 Y_2}(1, 0, 1) P_{Z_1}(0) \leq P_{Z_1}(0) P_{Z_2}(1). \tag{7}$$

Finally, noting that the terms of this inequality refer only to injectable sets, we can infer that on the original DAG we have

$$P_{ZXY}(0, 0, 0) P_Z(1) + P_{ZXY}(1, 0, 1) P_Z(0) \leq P_Z(0) P_Z(1), \tag{8}$$

which is equivalent to

$$P_{XY|Z}(0, 0|0) + P_{XY|Z}(0, 1|1) \leq 1, \tag{9}$$

which is an example of one of Pearl's instrumental inequalities (Eq. 8.21 in Pearl's book).