Alternatives to Entropic Inequalities and the Triangle Scenario

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Given some hypothesis of causal structure it is desirable to determine the set of probability distributions compatible with the hypothesis. For certain causal structures, such as Bell scenarios, the compatible set corresponds to the convex hull of various deterministic distributions, and admits a necessary and sufficient description in terms of conditional probability linear inequalities. For more general causal structures, however, it is far easier to derive entropic inequalities instead of probabilistic inequalities. Unfortunately there typically exist distributions which are genuinely incompatible with the causal structure but which satisfy all the structure's entropic inequalities. A tight characterization of all distributions which can be explained in terms of classical latent variables is critical in quantum information theory in order to recognize and exploit uniquely quantum distributions. The insufficiency of entropic inequalities, therefore, motivates us to explore alternative means of deriving compatibility tests for general causal structures, ideally directly at the level of probabilities. Uniquely quantum distributions are known to exist in the Triangle scenario; the methods presented herein may assist in isolating the criteria which distinguish quantum from classical distributions in that scenario.

INTRODUCTION

needs introduction

INFERENTIAL COMPATIBILITY CRITERIA

Inspired by Hardy-type proofs on nonlocality [1–3] we have developed a method to derive compatibility criteria from causal structures. The method is illustrated by examples. We follow the convention that upper-case indicates random variables while lower-case indicates some value associated with the random variable. Thus $p(ab|xy\lambda)$ should be understood as $p(A{=}a, B{=}b|X{=}x, Y{=}y, \Lambda{=}\lambda)$. We'll often also use lower-case subscripts to indicate conditioning upon a particular value, such that for instance $p(a_x b_y|\lambda) = p(ab|xy\lambda)$.

BELL SCENARIO

Consider the causal structure associated to the Bell experiment, depicted in Fig. 1. The assumption of causal structure dictates that

$$p(ab|xy\lambda) = p(a_x|\lambda)p(b_y|\lambda) \tag{1}$$

and accordingly that

$$p(a_x b_y) = \sum_{\text{all } \lambda} p(a_x|\lambda)p(b_y|\lambda)p(\lambda). \tag{2}$$

It is furthermore trivially true that

$$\begin{aligned} \forall_{y'}: \quad p(a_x|\lambda) &= \sum_{b'} p(a_x|b'_{y'})p(b'_{y'}|\lambda), \\ \forall_{x'}: \quad p(b_y|\lambda) &= \sum_{a'} p(b_y|a'_{x'})p(a'_{x'}|\lambda). \end{aligned} \tag{3}$$

As such, for any x', y' we have $p(a_x b_y) =$

$$\sum_{\text{all } \lambda, a', b'} p(a_x|b'_{y'})p(b'_{y'}|\lambda)p(b_y|a'_{x'})p(a'_{x'}|\lambda)p(\lambda). \tag{4}$$

where we can consider specific terms in the sum for convenience, such as one particular a' or one particular b'. In this manner we obtain multiple upper bounds for $p(a_x b_y)$. Formally, $p(a_x b_y) \geq$

$$\begin{aligned} \forall_{a'}: \; p(b_y|a'_{x'}) \sum_{\lambda, b'} p(a_x|b'_{y'})p(b'_{y'}|\lambda)p(a'_{x'}|\lambda)p(\lambda), \\ \forall_{b'}: \; p(a_x|b'_{y'}) \sum_{\lambda, a'} p(b_y|a'_{x'})p(b'_{y'}|\lambda)p(a'_{x'}|\lambda)p(\lambda). \end{aligned} \tag{5}$$

It is possible to evaluate the sums over λ in Eq. (5) to obtain

Proposition 1 *The Bell causal structure (Fig. 1) implies*

$$\begin{aligned} \forall_{x',y',a'}: \; p(a_x b_y) &\geq p(b_y|a_{x'}) \sum_{b'} p(a_x|b'_{y'})p(a'_{x'}b'_{y'}) \\ \forall_{x',y',b'}: \; p(a_x b_y) &\geq p(a_x|b_{y'}) \sum_{a'} p(b_y|a'_{x'})p(a'_{x'}b'_{y'}) \end{aligned} \tag{6}$$

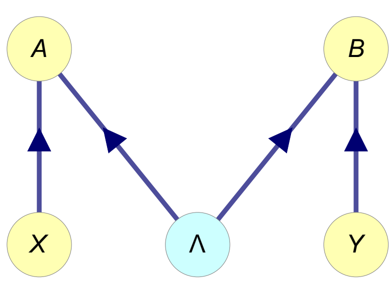

FIG. 1. The causal structure of the Bell scenario, on which Bell's theorem is based.

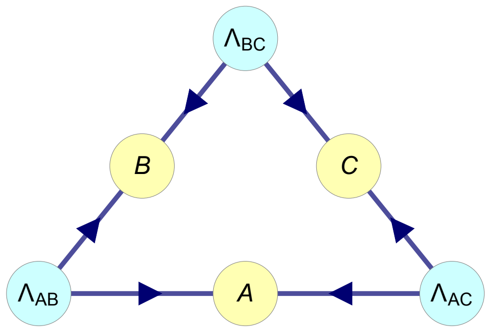

FIG. 2. The causal structure of the Triangle scenario, for which the three observed variables lack a common ancestor.

Of course, Prop. 1 can also be weakened by considering individual terms from the sums. Thus

Corollary 1.1 *The Bell causal structure (Fig. 2) implies*

$$\forall_{x',y',a',b'}:\; p(a_x b_y) \geq p(a_x|b_{y'})p(b_y|a'_{x'})p(a'_{x'}b'_{y'}) \tag{7}$$

Consider the following special case, where $p(a_1|b_0) = p(b_1|a_0) = 1$. For this case, taking $a'{=}a$, $b'{=}b$, $x'{=}y'{=}0$, and $x{=}y{=}1$, then Corr. 1.1 implies $p(a_1,b_1) \geq p(a_0,b_0)$. This special case is identically Cabello's variant of Hardy's nonlocality proof [1–3]. An example of a conditional probability distribution which is rejected by this special case of Prop. 1 is the PR-box [4, 5] distribution,

$$p(a_x b_y) = \begin{cases} 1/2 & \text{if } \; a \oplus b = xy \\ 0 & \text{otherwise} \end{cases}, \tag{8}$$

where $a,b,x,y \in \{0,1\}$.

TRIANGLE SCENARIO

This technique for deriving compatibility criteria can also be applied to more general scenarios. We illustrate this by considering the causal structure of the Triangle scenario, depicted in Fig. 2. Here the subscript notation is used merely to label the pair of variables A, B to which Λ_{AB} is a common cause.

The Triangle scenario's causal structure dictates that

$$\begin{aligned} &p(abc|\lambda_{AB}\lambda_{BC}\lambda_{AC}) \\ &\quad = p(a\lambda_{AB}\lambda_{AC})p(b\lambda_{AB}\lambda_{BC})p(c|\lambda_{BC}\lambda_{AC}) \end{aligned} \tag{9}$$

and accordingly that

$$\begin{aligned} &p(abc) = \\ &\sum_{\{\lambda\}} \begin{pmatrix} p(a|\lambda_{AB}\lambda_{AC})p(b|\lambda_{AB}\lambda_{BC})p(c|\lambda_{BC}\lambda_{AC}) \\ \times p(\lambda_{AB})p(\lambda_{BC})p(\lambda_{AC}) \end{pmatrix} \end{aligned} \tag{10}$$

where the sum is over all three latent variables, i.e. $\{\lambda\} = \lambda_{AB}, \lambda_{BC}, \lambda_{AC}$. Once again the first step is to replace each conditional probability with a product sum analogous to Eq. (3). An example would be

$$p(a|\lambda_{AB}\lambda_{AC}) = \sum_{b'} p(a|b')p(b'|\lambda_{AB}\lambda_{AC}) \tag{11}$$

but we can employ conditional independence relations implied by the causal structure to simplify such terms, namely $p(b'|\lambda_{AB}\lambda_{AC}){=}p(b'|\lambda_{AB})$ etc. A complete list of allowable substitutions is

$$p(a|\lambda_{AB}\lambda_{AC}) = \sum_{b'} p(a|b')p(b'|\lambda_{AB}), \tag{12}$$

$$p(a|\lambda_{AB}\lambda_{AC}) = \sum_{c'} p(a|c')p(c'|\lambda_{AC}), \tag{13}$$

$$p(b|\lambda_{AB}\lambda_{BC}) = \sum_{a'} p(b|a')p(b'|\lambda_{AB}), \tag{14}$$

$$p(b|\lambda_{AB}\lambda_{BC}) = \sum_{c'} p(b|c')p(c'|\lambda_{BC}), \tag{15}$$

$$p(c|\lambda_{BC}\lambda_{AC}) = \sum_{a'} p(c|a')p(a'|\lambda_{AC}), \tag{16}$$

$$p(c|\lambda_{BC}\lambda_{AC}) = \sum_{b'} p(c|b')p(b'|\lambda_{BC}). \tag{17}$$

There are five ways to select equalities from Eqs. (12-17) for substitution into Eq. (10) to yield novel compatibility criteria. For the sake of example, let's select Eqs. (12,15,16). This yields

$$\begin{aligned} p(abc) &= \sum_{\lambda_{AB},\lambda_{BC},\lambda_{AC},a',b',c'} p(a|b')p(b'|\lambda_{AB})p(b|c')p(c'|\lambda_{BC})p(c|a')p(a'|\lambda_{AC})p(\lambda_{AB})p(\lambda_{BC})p(\lambda_{AC}) \\ &= \left(\sum_{b'}\sum_{\lambda_{AB}} p(a|b')p(b'|\lambda_{AB})p(\lambda_{AB})\right)\left(\sum_{c'}\sum_{\lambda_{BC}} p(b|c')p(c'|\lambda_{BC})p(\lambda_{BC})\right)\left(\sum_{a'}\sum_{\lambda_{AC}} p(c|a')p(a'|\lambda_{AC})p(\lambda_{AC})\right) \\ &= \left(\sum_{b'} p(a|b')p(b')\right)\left(\sum_{c'} p(b|c')p(c')\right)\left(\sum_{a'} p(c|a')p(a')\right) \end{aligned} \tag{18}$$

which, being an equality on the observable probabilities, is an exceptionally strict necessary compatibility criterion.

Proposition 2 *The Triangle causal structure (Fig. 2) implies that* $p(abc)$

$$\begin{aligned}
&= \left(\sum_{b'} p(a|b')p(b')\right)\left(\sum_{c'} p(b|c')p(c')\right)\left(\sum_{a'} p(c|a')p(a')\right)\\
&= \left(\sum_{c'} p(a|c')p(c')\right)\left(\sum_{a'} p(b|a')p(a')\right)\left(\sum_{b'} p(c|b')p(b')\right)\\
&= p(c)\sum_{a',b'} p(a|b')p(b|a')p(a'b')\\
&= p(a)\sum_{b',c'} p(b|c')p(c|b')p(b'c')\\
&= p(b)(\sum_{a',c'} p(a|c')p(c|a')p(a'c')
\end{aligned}$$

where the additionally equalities come from choosing different identities from Eqs. (12-17) for substitution into Eq. (10), such as the triple of Eqs. (13,14,17) or the pair of Eqs. (12,14), etc. Of course, Prop. 2 can also be weakened by considering individual terms from the sums. Thus

Corollary 2.1 *The Triangle causal structure (Fig. 2) implies that for any* a', b', c' *we have*

$$\begin{aligned}
p(abc) &\geq p(a|b')p(b')p(b|c')p(c')p(c|a')p(a')\\
p(abc) &\geq p(a|c')p(c')p(b|a')p(a')p(c|b')p(b')\\
p(abc) &\geq p(c)p(a|b')p(b|a')p(a'b')\\
p(abc) &\geq p(a)p(b|c')p(c|b')p(b'c')\\
p(abc) &\geq p(b)p(a|c')p(c|a')p(a'c')
\end{aligned}$$

Consider the following special case, where $p(A{=}0|B{=}1) = p(C{=}0|B{=}1) = p(A{=}0|C{=}1) = 1$. For this case, taking $a{=}b{=}c{=}1$ and $a'{=}b'{=}c'{=}0$, then the first inequality in Corr. 2.1 implies $p(A{=}B{=}C{=}1) \geq p(A{=}0)p(C{=}0)p(C{=}0)$. An example of a probability distribution which is rejected by this special case of Corr. 2.1 is the W-distribution,

$$p(abc) = \begin{cases} 1/3 & \text{if } \; a+b+c=1 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

where $a, b, c \in \{0, 1\}$. The W-distribution states that the in any event in which A, B, C are observed, precisely one of them will be found to equal 1 while the other two will equal 0. The identity of the variable which takes the value 1 is uniformly random.

FAILURE OF ENTROPIC INEQUALITIES

It is interesting to note that entropic inequalities [6, 7] fail to recognize the PR-box per Eq. (8) as incompatible with the Bell scenario and also the W-type distribution per Eq. (19) as incompatible with the Triangle scenario, whereas our alternative reasoning is capable of doing so. To reiterate from the abstract, enumeration of entropic inequalities is considered state-of-the-art derivation of necessary albeit insufficient causal structure compatibility criteria [8]. The insufficiency is a pressing concern in quantum information theory as there are uniquely-quantum distributions which cannot be certified as non-classical by means of entropic inequalities [9].

The entropic inequalities associated with the Bell scenario are given by

$$\begin{aligned} &H(A_1, B_1) + H(A_0) + H(B_0) \\ &\quad \leq H(A_0, B_0) + H(A_0, B_1) + H(A_1, B_0) \end{aligned} \tag{20}$$

and its permutations [7, 10].

The entropic inequalities associated with the Triangle scenario are given by

$$\begin{aligned}
&I(A:B) + I(A:C) \leq H(A)\\
\text{and}\quad &I(A:B) + I(A:C) + I(B:C)\\
&\quad \leq H(A,B) - I(A:B:C)\\
\text{and}\quad &I(A:B) + I(A:C) + I(B:C)\\
&\quad \leq \frac{H(A)+H(B)+H(C)}{2} - I(A:B:C)
\end{aligned} \tag{21}$$

and their permutations [7, 8, 11].

Note that bipartite mutual information may be understood as $I(A:B) \equiv H(A) + H(B) - H(A,B)$ and tripartite mutual information is defined as $I(A:B:C) \equiv H(A) + H(B) + H(C) - H(A,B) - H(A,C) - H(B,C) + H(A,B,C)$. It is straightforward to demonstrate the the distributions given in Eqs. (8,19) satisfy Eqs. (20,21) respectively.

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