

# Causal Compatibility Inequalities Admitting of Quantum Violations in the Triangle Scenario

T.C. Fraser<sup>1,2</sup>

<sup>1</sup>Perimeter Institute for Theoretical Physics  
Ontario, Canada

<sup>2</sup>University of Waterloo  
Ontario, Canada

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## Quantum Mechanics can do stuff that Classical Mechanics can't!

- Quantum Information/Communication
  - Quantum Key Distribution (QKD)
- Quantum Algorithms/Computation
  - Shor's Algorithm, Quantum Simulated Annealing, String Matching, ...
  - Polynomial or even exponential advantage
  - <http://math.nist.gov/quantum/zoo/>
- Quantum Foundations
  - Derivation of Bell inequalities
  - Abandoning local hidden variable theories (LHVs)

**Important to determine *precisely* where quantum/classical differ, but how?**

Given a set of observations, are they *compatible* with a given classical model?

## Causal Structure $\mathcal{G}$

(classical model)

$$\mathcal{G} = (\mathcal{N}, \mathcal{E})$$

$\mathcal{E}$  : causal influence

$\mathcal{N}$  : random variables

$\mathcal{N}_O$  : observable variables

$\mathcal{N}_L$  : latent variables

$$\mathcal{N} = \mathcal{N}_O \cup \mathcal{N}_L$$

## Marginal Model $P^{\mathcal{M}}$

(set of observations)

$\mathcal{M}$  : marginal scenario

$$\mathcal{M} = \{V \mid V \subseteq \mathcal{N}_O\}$$

$$P^{\mathcal{M}} = \{P_V \mid V \in \mathcal{M}\}$$

**Question:** Can your model  $\mathcal{G}$  explain the observation  $P^{\mathcal{M}}$ ?

**Question:** Is  $P^{\mathcal{M}}$  **compatible** with  $\mathcal{G}$ ?

**Answer:** If  $P^{\mathcal{M}}$  is to be compatible with  $\mathcal{G}$  then:

- 1  $P^{\mathcal{M}}$  must admit a **joint distribution**  $P_{\mathcal{N}}$ :

$$P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$$

- 2  $P_{\mathcal{N}}$  must obey **Markov conditions** (independence given parents):

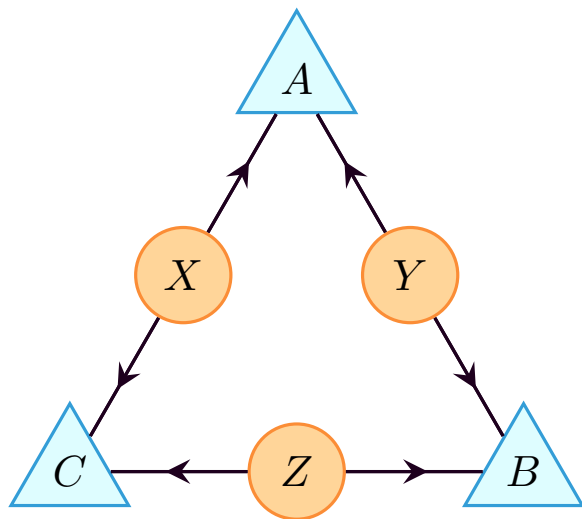
$$P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\text{Pa}_{\mathcal{G}}(n)}$$

**Task:** Determine if  $P^{\mathcal{M}}$  is **compatible** or **incompatible** with  $\mathcal{G}$

**Tool:** A **casual compatibility inequality**  $I$  is an inequality satisfied by all compatible  $P^{\mathcal{M}}$

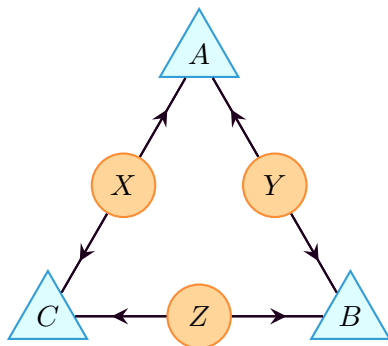
- Inequalities  $I$  are **extremely useful**:
- If  $P^{\mathcal{M}}$  satisfies *all* such constraints  $I$ , then it is *compatible* with  $\mathcal{G}$
- If  $P^{\mathcal{M}}$  violates *any* constraint  $I$ , then it is *incompatible* with  $\mathcal{G}$
- Inequalities act as *measures* of non-classicality
- $I$  identifies *fundamentally* new resources or quantum advantages!

# Case Study: The Triangle Scenario



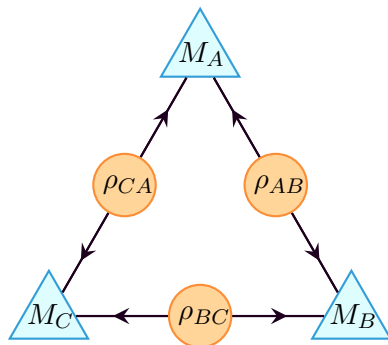
# Classical (Compatible) Distributions for Triangle Scenario

$$P_{ABC} = \sum_{X,Y,Z} P_{A|X,Y} P_{B|Y,Z} P_{C|Z,X} P_X P_Y P_Z$$



# Quantum Distributions for Triangle Scenario

$$P_{ABC} = \text{Tr}[\Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_A \otimes M_B \otimes M_C]$$





# Why Triangle Scenario?

- The TS could be home to **brand new** entanglement resources which could lead to new technologies and fundamental physics, albeit very difficult to find
- In **Branciard\_2012** it was noted that characterizing quantum non-classicality in TS remained an **open problem** and that identifying compatibility inequalities **seemed challenging**
- In **Fritz\_2012** Fritz demonstrated that TS is the **smallest** correlation scenario in which quantum non-classicality is manifest (proof without inequalities)
- In **Henson\_2014** TS was classified as an **interesting** causal structure: conditional independence relations are not a sufficient characterization of compatibility (there are none)
- Several other authors (see **Steudel\_2010 Chaves\_2014 Inflation ...**) have been unable to find inequalities with quantum violations

# Deriving Inequalities

- Two necessary components to compatibility: joint distribution, Markov separability
- It is possible to cast Markov separability problem into a joint distribution problem (at least partially)!
- **Inflation Technique??** developed by E. Wolfe, R. Spekkens, T. Fritz provides tools for solving compatibility problem
- **Definite Extension Procedure** developed by myself, E. Wolfe and others allows one to derive inequalities for **very large** causal structures

$$P_{ABC}(000)P_C(3)$$

$$\leq$$

$$\begin{aligned} & 2P_{ABC}(203)P_C(0) + 2P_{ABC}(303)P_C(0) + P_{ABC}(003)P_C(0) + P_{ABC}(013)P_C(0) + \\ & P_{ABC}(020)P_C(2) + P_{ABC}(022)P_C(2) + P_{ABC}(023)P_C(0) + P_{ABC}(023)P_C(2) + \\ & P_{ABC}(030)P_C(2) + P_{ABC}(031)P_C(2) + P_{ABC}(032)P_C(2) + P_{ABC}(033)P_C(0) + \\ & P_{ABC}(033)P_C(2) + P_{ABC}(103)P_C(0) + P_{ABC}(113)P_C(0) + P_{ABC}(123)P_C(0) + \\ & P_{ABC}(133)P_C(0) + P_{ABC}(200)P_C(0) + P_{ABC}(200)P_C(1) + P_{ABC}(200)P_C(2) + \\ & P_{ABC}(200)P_C(3) + P_{ABC}(201)P_C(0) + P_{ABC}(201)P_C(1) + P_{ABC}(201)P_C(2) + \\ & P_{ABC}(201)P_C(3) + P_{ABC}(203)P_C(1) + P_{ABC}(203)P_C(2) + P_{ABC}(203)P_C(3) + \\ & P_{ABC}(213)P_C(0) + P_{ABC}(223)P_C(0) + P_{ABC}(300)P_C(0) + P_{ABC}(300)P_C(1) + \\ & P_{ABC}(300)P_C(2) + P_{ABC}(300)P_C(3) + P_{ABC}(301)P_C(0) + P_{ABC}(301)P_C(1) + \\ & P_{ABC}(301)P_C(2) + P_{ABC}(301)P_C(3) + P_{ABC}(302)P_C(1) + P_{ABC}(303)P_C(1) + \\ & P_{ABC}(303)P_C(2) + P_{ABC}(303)P_C(3) + P_{ABC}(313)P_C(0) \end{aligned}$$

# Causal Compatibility Inequalities (Simplified)

$$\begin{aligned} &P_{ABC}(000)P_C(3) + P_{ABC}(021)P_C(2) + P_{ABC}(202) + P_{ABC}(302) + \\ &\quad P_{ABC}(233)P_C(0) + P_{AC}(33)P_C(0) \\ &\quad \leq \\ &P_{ABC}(303)P_C(0) + P_{ABC}(313)P_C(0) + P_{ABC}(302)P_C(1) + \\ &P_{AB}(02)P_C(2) + P_{AB}(03)P_C(2) + P_{AB}(20) + P_{AB}(30) + P_C(3)P_C(0) \end{aligned}$$

$$\begin{aligned} &P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333) \leq \\ &2P(020)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213) + \\ &2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210) + \\ &2P(123)P(310) + 2P(130)P(213) + 2P(132)P(311) + 2P(133)P(210) + \\ &\quad + \cdots \text{ 324 more terms } \cdots + \\ &P(320)P(323) + P(320)P(333) + P(323)P(330) + P(330)P(333) \end{aligned}$$

**Note:**  $P(abc)$  shorthand for  $P_{ABC}(abc)$

# Certificate for Fritz Distribution (Full)

$$P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333)$$

2

$2P(202)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213) + 2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210) + 2P(123)P(310) + 2P(130)P(213) + 2P(132)P(311) + 2P(133)P(210) + 2P(133)P(310) + P(000)P(023) + P(000)P(013) + P(000)P(023) + P(000)P(033) + P(000)P(103) + P(000)P(113) + P(000)P(123) + P(000)P(133) + P(000)P(203) + P(000)P(210) + P(000)P(213) + P(000)P(223) + P(000)P(010) + P(000)P(020) + P(003)P(020) + P(003)P(030) + P(003)P(100) + P(003)P(110) + P(003)P(120) + P(003)P(130) + P(003)P(200) + P(003)P(210) + P(003)P(220) + P(003)P(230) + P(003)P(300) + P(003)P(310) + P(003)P(320) + P(003)P(330) + P(010)P(013) + P(010)P(023) + P(010)P(033) + P(010)P(103) + P(010)P(113) + P(010)P(123) + P(010)P(133) + P(010)P(203) + P(010)P(213) + P(010)P(223) + P(010)P(230) + P(010)P(300) + P(010)P(310) + P(010)P(320) + P(010)P(330) + P(013)P(030) + P(013)P(100) + P(013)P(110) + P(013)P(120) + P(013)P(130) + P(013)P(200) + P(013)P(210) + P(013)P(220) + P(013)P(230) + P(013)P(300) + P(013)P(310) + P(013)P(320) + P(013)P(330) + P(020)P(023) + P(020)P(033) + P(020)P(103) + P(020)P(113) + P(020)P(123) + P(020)P(133) + P(020)P(203) + P(020)P(210) + P(020)P(211) + P(020)P(212) + P(020)P(223) + P(020)P(233) + P(020)P(310) + P(020)P(311) + P(020)P(312) + P(020)P(313) + P(021)P(210) + P(021)P(211) + P(021)P(212) + P(021)P(213) + P(021)P(310) + P(021)P(311) + P(021)P(312) + P(021)P(313) + P(022)P(210) + P(022)P(211) + P(022)P(212) + P(022)P(213) + P(022)P(310) + P(022)P(311) + P(022)P(312) + P(022)P(313) + P(023)P(030) + P(023)P(100) + P(023)P(110) + P(023)P(120) + P(023)P(130) + P(023)P(200) + P(023)P(210) + P(023)P(212) + P(023)P(213) + P(023)P(220) + P(023)P(230) + P(023)P(300) + P(023)P(311) + P(023)P(312) + P(023)P(313) + P(023)P(320) + P(023)P(330) + P(030)P(033) + P(030)P(103) + P(030)P(113) + P(030)P(123) + P(030)P(133) + P(030)P(203) + P(030)P(210) + P(030)P(211) + P(030)P(212) + P(030)P(223) + P(030)P(233) + P(030)P(310) + P(030)P(311) + P(030)P(312) + P(030)P(313) + P(031)P(210) + P(031)P(211) + P(031)P(212) + P(031)P(213) + P(031)P(310) + P(031)P(311) + P(031)P(312) + P(031)P(313) + P(032)P(210) + P(032)P(211) + P(032)P(212) + P(032)P(213) + P(032)P(310) + P(032)P(311) + P(032)P(312) + P(032)P(313) + P(033)P(100) + P(033)P(110) + P(033)P(120) + P(033)P(130) + P(033)P(200) + P(033)P(210) + P(033)P(212) + P(033)P(213) + P(033)P(220) + P(033)P(230) + P(033)P(300) + P(033)P(310) + P(033)P(311) + P(033)P(312) + P(033)P(313) + P(100)P(103) + P(100)P(113) + P(100)P(123) + P(100)P(133) + P(100)P(203) + P(100)P(213) + P(100)P(223) + P(103)P(110) + P(103)P(120) + P(103)P(130) + P(103)P(200) + P(103)P(210) + P(103)P(220) + P(103)P(230) + P(103)P(300) + P(103)P(310) + P(103)P(320) + P(103)P(330) + P(110)P(113) + P(110)P(123) + P(110)P(133) + P(110)P(203) + P(110)P(213) + P(110)P(223) + P(113)P(120) + P(113)P(130) + P(113)P(200) + P(113)P(210) + P(113)P(220) + P(113)P(230) + P(113)P(300) + P(113)P(310) + P(113)P(320) + P(113)P(330) + P(120)P(123) + P(120)P(133) + P(120)P(203) + P(120)P(210) + P(120)P(211) + P(120)P(212) + P(120)P(223) + P(120)P(233) + P(120)P(310) + P(120)P(311) + P(120)P(312) + P(120)P(313) + P(121)P(210) + P(121)P(211) + P(121)P(212) + P(121)P(213) + P(121)P(310) + P(121)P(311) + P(121)P(312) + P(121)P(313) + P(122)P(210) + P(122)P(212) + P(122)P(213) + P(122)P(310) + P(122)P(311) + P(122)P(312) + P(122)P(313) + P(123)P(130) + P(123)P(200) + P(123)P(211) + P(123)P(212) + P(123)P(213) + P(123)P(220) + P(123)P(230) + P(123)P(300) + P(123)P(310) + P(123)P(312) + P(123)P(313) + P(123)P(320) + P(123)P(330) + P(130)P(133) + P(130)P(203) + P(130)P(210) + P(130)P(211) + P(130)P(212) + P(130)P(223) + P(130)P(233) + P(130)P(310) + P(130)P(311) + P(130)P(312) + P(130)P(313) + P(131)P(210) + P(131)P(211) + P(131)P(212) + P(131)P(213) + P(131)P(310) + P(131)P(311) + P(131)P(312) + P(131)P(313) + P(132)P(201) + P(132)P(210) + P(132)P(211) + P(132)P(212) + P(132)P(213) + P(132)P(301) + P(132)P(310) + P(132)P(312) + P(132)P(313) + P(133)P(200) + P(133)P(211) + P(133)P(212) + P(133)P(213) + P(133)P(220) + P(133)P(230) + P(133)P(300) + P(133)P(310) + P(133)P(312) + P(133)P(313) + P(133)P(320) + P(133)P(330) + P(200)P(203) + P(200)P(213) + P(200)P(223) + P(200)P(233) + P(200)P(303) + P(200)P(313) + P(200)P(323) + P(200)P(333) + P(203)P(210) + P(203)P(220) + P(203)P(230) + P(203)P(300) + P(203)P(310) + P(203)P(320) + P(203)P(330) + P(210)P(213) + P(210)P(223) + P(210)P(233) + P(210)P(303) + P(210)P(313) + P(210)P(323) + P(210)P(333) + P(213)P(220) + P(213)P(230) + P(213)P(300) + P(213)P(310) + P(213)P(320) + P(213)P(330) + P(220)P(223) + P(220)P(233) + P(220)P(303) + P(220)P(313) + P(220)P(323) + P(223)P(230) + P(223)P(3$

# Party Symmetric Inequality

$$2[P(001)P(333)]_3 + 2[P(010)P(323)]_3 + 6[P(000)P(323)]_3 + 6[P(000)P(333)]_1$$

$$\leq$$

$$12[P(031)P(302)]_6 + 12[P(033)P(303)]_6 + 12[P(103)P(130)]_6 + 12[P(203)P(230)]_6 +$$

$$+ \cdots \text{ 126 more terms } \cdots +$$

$$6[P(101)P(130)]_6 + 6[P(103)P(310)]_6 + 6[P(113)P(130)]_6 + 6[P(113)P(230)]_6 +$$

$$6[P(113)P(330)]_3 + 6[P(122)P(330)]_6 + 6[P(130)P(313)]_6 + 6[P(132)P(303)]_6 +$$

$$6[P(133)P(303)]_6 + 6[P(133)P(320)]_6 + 6[P(200)P(203)]_6 + 6[P(201)P(230)]_6 +$$

$$6[P(203)P(231)]_6 + 6[P(223)P(300)]_6 + 8[P(003)P(320)]_6 + 8[P(032)P(300)]_6$$

**Note:**  $[P(113)P(330)]_3$  shorthand sum over permutations:

$$P(113)P(330) + P(131)P(303) + P(311)P(033)$$

# Party Symmetric Inequality (Full)

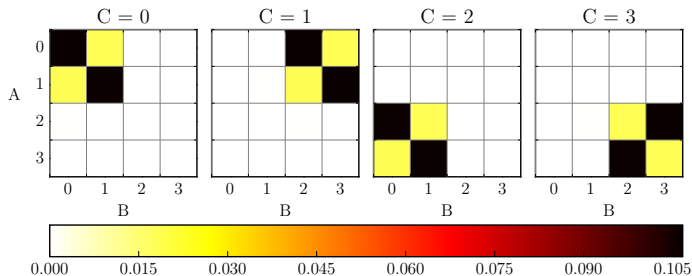
$$2[P(001)P(333)]_3 + 2[P(010)P(323)]_3 + 6[P(000)P(323)]_3 + 6[P(000)P(333)]_1$$

$\leq$

$$\begin{aligned} & 12[P(031)P(302)]_6 + 12[P(033)P(303)]_6 + 12[P(103)P(130)]_6 + 12[P(203)P(230)]_6 + 12[P(203)P(330)]_6 + 2[P(001)P(320)]_6 + 2[P(002)P(221)]_3 + 2[P(003)P(211)]_6 + \\ & 2[P(003)P(331)]_3 + 2[P(011)P(211)]_3 + 2[P(012)P(322)]_6 + 2[P(013)P(313)]_6 + 2[P(013)P(332)]_6 + 2[P(020)P(111)]_3 + 2[P(020)P(211)]_6 + 2[P(021)P(212)]_6 + \\ & 2[P(022)P(211)]_3 + 2[P(022)P(212)]_6 + 2[P(022)P(322)]_3 + 2[P(023)P(232)]_6 + 2[P(030)P(212)]_3 + 2[P(031)P(231)]_6 + 2[P(032)P(331)]_6 + 2[P(033)P(333)]_3 + \\ & 2[P(101)P(131)]_3 + 2[P(101)P(132)]_6 + 2[P(102)P(131)]_6 + 2[P(102)P(132)]_6 + 2[P(102)P(133)]_6 + 2[P(110)P(133)]_6 + 2[P(110)P(212)]_6 + 2[P(110)P(222)]_3 + \\ & 2[P(110)P(223)]_3 + 2[P(112)P(331)]_3 + 2[P(120)P(122)]_6 + 2[P(121)P(201)]_6 + 2[P(122)P(200)]_3 + 2[P(122)P(202)]_6 + 2[P(122)P(210)]_6 + 2[P(122)P(300)]_3 + \\ & 2[P(130)P(232)]_6 + 2[P(130)P(233)]_6 + 2[P(131)P(201)]_6 + 2[P(131)P(202)]_3 + 2[P(131)P(313)]_3 + 2[P(133)P(200)]_3 + 2[P(133)P(201)]_6 + 2[P(133)P(211)]_3 + \\ & 2[P(133)P(212)]_6 + 2[P(133)P(300)]_3 + 2[P(202)P(231)]_6 + 2[P(210)P(222)]_6 + 2[P(220)P(222)]_3 + 2[P(220)P(313)]_6 + 2[P(221)P(313)]_6 + 2[P(222)P(331)]_3 + \\ & 2[P(223)P(331)]_3 + 2[P(230)P(312)]_6 + 2[P(231)P(313)]_6 + 2[P(232)P(320)]_6 + 2[P(302)P(322)]_6 + 2[P(320)P(323)]_6 + 2[P(330)P(332)]_3 + 3[P(000)P(003)]_3 + \\ & 3[P(010)P(301)]_6 + 4[P(001)P(131)]_6 + 4[P(002)P(020)]_6 + 4[P(002)P(133)]_6 + 4[P(002)P(323)]_6 + 4[P(010)P(123)]_6 + 4[P(013)P(212)]_6 + 4[P(013)P(312)]_6 + \\ & 4[P(023)P(221)]_6 + 4[P(023)P(222)]_6 + 4[P(023)P(322)]_6 + 4[P(031)P(211)]_6 + 4[P(032)P(321)]_6 + 4[P(100)P(123)]_6 + 4[P(100)P(232)]_6 + 4[P(100)P(313)]_6 + \\ & 4[P(112)P(310)]_6 + 4[P(122)P(203)]_6 + 4[P(122)P(302)]_6 + 4[P(130)P(222)]_6 + 4[P(130)P(223)]_6 + 4[P(222)P(310)]_6 + 4[P(223)P(320)]_6 + 4[P(231)P(301)]_6 + \\ & 4[P(312)P(330)]_6 + 6[P(001)P(031)]_6 + 6[P(001)P(033)]_6 + 6[P(002)P(300)]_6 + 6[P(002)P(330)]_3 + 6[P(003)P(032)]_6 + 6[P(003)P(131)]_6 + 6[P(003)P(132)]_6 + \\ & 6[P(011)P(300)]_3 + 6[P(011)P(320)]_6 + 6[P(012)P(200)]_6 + 6[P(012)P(301)]_6 + 6[P(013)P(030)]_6 + 6[P(013)P(110)]_6 + 6[P(013)P(120)]_6 + 6[P(013)P(303)]_6 + \\ & 6[P(020)P(102)]_6 + 6[P(020)P(103)]_6 + 6[P(020)P(123)]_6 + 6[P(020)P(202)]_6 + 6[P(020)P(203)]_6 + 6[P(020)P(311)]_6 + 6[P(020)P(322)]_6 + 6[P(020)P(330)]_6 + \\ & 6[P(022)P(303)]_6 + 6[P(030)P(033)]_6 + 6[P(030)P(101)]_3 + 6[P(030)P(133)]_6 + 6[P(030)P(202)]_3 + 6[P(030)P(303)]_3 + 6[P(030)P(332)]_6 + 6[P(031)P(203)]_6 + \\ & 6[P(032)P(310)]_6 + 6[P(033)P(101)]_6 + 6[P(033)P(130)]_6 + 6[P(033)P(200)]_3 + 6[P(033)P(212)]_6 + 6[P(033)P(220)]_6 + 6[P(033)P(222)]_3 + 6[P(033)P(230)]_6 + \\ & 6[P(033)P(322)]_3 + 6[P(100)P(203)]_6 + 6[P(101)P(130)]_6 + 6[P(103)P(310)]_6 + 6[P(103)P(130)]_6 + 6[P(113)P(230)]_6 + 6[P(113)P(330)]_3 + 6[P(122)P(330)]_6 + \\ & 6[P(130)P(313)]_6 + 6[P(132)P(303)]_6 + 6[P(133)P(303)]_6 + 6[P(133)P(320)]_6 + 6[P(200)P(203)]_6 + 6[P(201)P(230)]_6 + 6[P(203)P(231)]_6 + 6[P(223)P(300)]_6 + \\ & 8[P(003)P(320)]_6 + 8[P(032)P(300)]_6 \end{aligned}$$



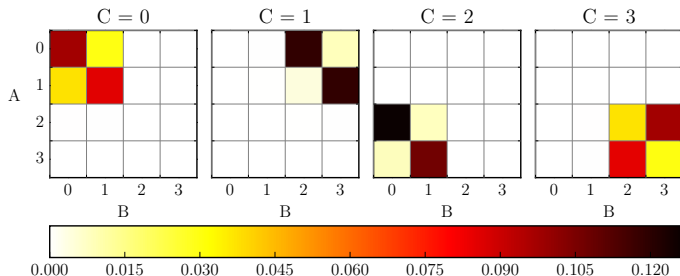
Fritz Distribution



$$\text{Yellow} = \frac{1}{32} (2 - \sqrt{2}) \quad \text{Black} = \frac{1}{32} (2 + \sqrt{2})$$

# Maximal Violation

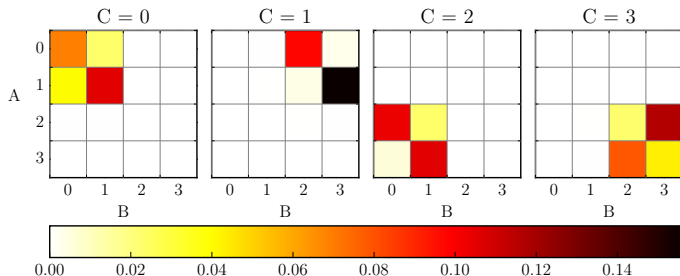
Maximum Violation of  $I_0$  ( $V_F = 1.501$ )



Relative Violation: 
$$V_F = \frac{\min_P \{I(P)\}}{I(P_F)}$$

# Maximal Violation Symmetric

Maximum Violation of  $I_2$  ( $V_F = 2.61$ )

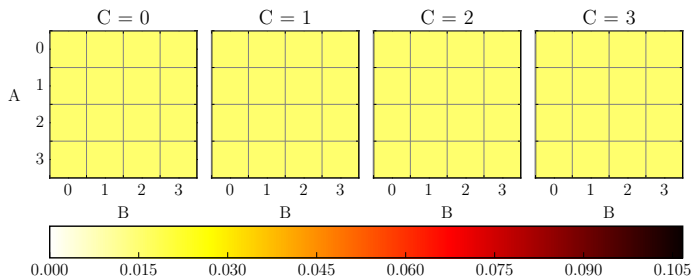


Relative Violation: 
$$V_F = \frac{\min_P \{I(P)\}}{I(P_F)}$$

# Uniform Noise

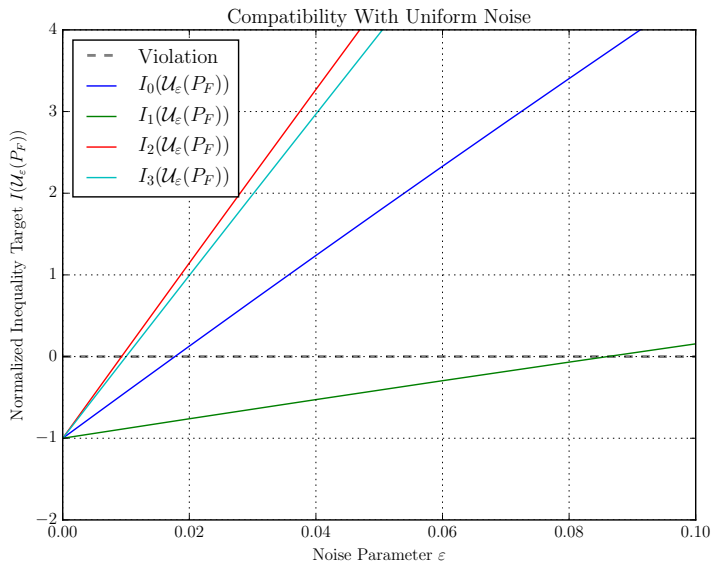
$$\mathcal{U}_\varepsilon(P) = (1 - \varepsilon)P + \varepsilon\mathcal{U}$$

Uniform Distribution



$$\text{Yellow square} = \frac{1}{64}$$

# Robust to Noise Zoomed



- Quantum/Classical difference can be observed in the triangle scenario
- Successful demonstration of the inflation technique
- Developed new algorithmic tools to find such inequalities
- New measures of non-classicality which lead to new quantum mechanical resources



## Appendix A: Optional Slides



**Question:** Which marginal models  $P^{\mathcal{M}}$  are **compatible** with a causal structure  $\mathcal{G}$ ?

- **Marginal model**  $P^{\mathcal{M}}$  is collection of probability distributions

$$P^{\mathcal{M}} = \{P_{V_1}, \dots, P_{V_k}\}$$

- **Marginal scenario**  $\mathcal{M} = \{V_1, \dots, V_k\}$

$$V \in \mathcal{M}, V' \subseteq V \implies V' \in \mathcal{M}$$

- **Joint random variables**  $\mathcal{J} = \bigcup_i V_i = \{v_1, \dots, v_n\}$
- **Causal Structure**  $\mathcal{G} = (\mathcal{N}, \mathcal{E})$  is a directed acyclic graph (DAG)
- Nodes classified into **latent nodes**  $\mathcal{N}_L$  and **observed nodes**  $\mathcal{N}_O$

Let  $n, m \in \mathcal{N}$  be nodes of the graph  $\mathcal{G}$ .

- **parents of  $n$** :  $\text{Pa}_{\mathcal{G}}(n) \equiv \{m \mid m \rightarrow n\}$
- **children of  $n$** :  $\text{Ch}_{\mathcal{G}}(n) \equiv \{m \mid n \rightarrow m\}$
- **ancestry of  $n$** :  $\text{An}_{\mathcal{G}}(n) \equiv \bigcup_{i \in \mathbb{W}} \text{Pa}_{\mathcal{G}}^i(n)$

$$\text{Pa}_{\mathcal{G}}^0(n) = n \quad \text{Pa}_{\mathcal{G}}^i(n) \equiv \text{Pa}_{\mathcal{G}}(\text{Pa}_{\mathcal{G}}^{i-1}(n))$$

Notation extends to sets of nodes  $N \subseteq \mathcal{N}$ ,

- **parents of  $N$** :  $\text{Pa}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \text{Pa}_{\mathcal{G}}(n)$
- **children of  $N$** :  $\text{Ch}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \text{Ch}_{\mathcal{G}}(n)$
- **ancestry of  $N$** :  $\text{An}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \text{An}_{\mathcal{G}}(n)$

An **induced subgraph** of  $\mathcal{G} = (\mathcal{N}, \mathcal{E})$  due to  $N \subseteq \mathcal{N}$

$$\text{Sub}_{\mathcal{G}}(N) = (N, \{e \in \mathcal{E} \mid e \subseteq N\})$$

**Question:** Which marginal models  $P^{\mathcal{M}}$  are **compatible** with a causal structure  $\mathcal{G}$ ?

**Answer:**  $P^{\mathcal{M}}$  is compatible with  $\mathcal{G}$  if there exists a set of **casual parameters**

$$\left\{ P_{n|\text{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N} \right\}$$

Such that for each  $V \in \mathcal{M}$ ,  $P_V$  can be recovered:

1  $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\text{Pa}_{\mathcal{G}}(n)}$

2  $P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$

**Inequality:** A **casual compatibility inequality**  $I$  is an inequality over  $P^{\mathcal{M}}$  that is satisfied by all compatible  $P^{\mathcal{M}}$

Two necessary components to compatibility:

- 1 **Marginal problem:**  $\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$ 
  - Is the marginal model contextual or non-contextual?
  - 3 distinct ways to tackle this problem
    - 1 Convex hull, Polytope projection, Fourier-Motzkin
    - 2 Possibilistic Hardy Inequalities (Hypergraph transversals)
    - 3 Linear Program Feasibility/Infeasibility
- 2 **Markov Separation:**  $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\text{Pa}_{\mathcal{G}}(n)}$ 
  - Much harder to determine since latent nodes  $\mathcal{N}_O$  have unspecified behaviour
  - It is possible to turn Markov Separation problem into a Marginal problem (at least partially)

Developed by Wolfe, Spekkens, and Fritz **Inflation**

## Definition

An **inflation** of a causal structure  $\mathcal{G}$  is another causal structure  $\mathcal{G}'$  such that:

$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \text{AnSub}_{\mathcal{G}'}(n') \sim \text{AnSub}_{\mathcal{G}}(n)$$

Where  $\text{AnSub}_{\mathcal{G}}(n)$  denotes the ancestral sub-graph of  $n$  in  $\mathcal{G}$

$$\text{AnSub}_{\mathcal{G}}(n) = \text{Sub}_{\mathcal{G}}(\text{An}_{\mathcal{G}}(n))$$

And ' $\sim$ ' is a **copy-index** equivalence relation

$$A_1 \sim A_2 \sim A \not\sim B_1 \sim B_2 \sim B$$

# Inflation Lemma

If one has obtained  $\mathcal{G}$ , inflation  $\mathcal{G}'$  and *compatible* marginal distribution  $P_N$  where  $N \subseteq \mathcal{N}$ , then:

- 1 There exists causal parameters  $\{P_{n|\text{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\}$  such that

$$P_N = \prod_{n \in N} P_{n|\text{Pa}_{\mathcal{G}}(n)}$$

- 2  $\text{AnSub}_{\mathcal{G}'}(n') \sim \text{AnSub}_{\mathcal{G}}(n) \implies \text{Pa}_{\mathcal{G}'}(n') \sim \text{Pa}_{\mathcal{G}}(n)$

- 3 Construct **inflated causal parameters**

$$\forall n' \in \mathcal{N}' : P_{n'|\text{Pa}_{\mathcal{G}'}(n')} \equiv P_{n|\text{Pa}_{\mathcal{G}}(n)}$$

- 4 Obtain *compatible* marginal distributions over any  $N' \subseteq \mathcal{N}'$

$$P_{N'} = \prod_{n' \in N'} P_{n'|\text{Pa}_{\mathcal{G}'}(n')}$$

- Inflation procedure holds for any  $N \in \mathcal{N}, N' \in \mathcal{N}'$  where  $\text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')$
- Define **injectable sets of  $\mathcal{G}'$**  and **images of the injectable of  $\mathcal{G}$**

$$\text{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : \text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')\}$$

$$\text{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : \text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')\}$$

- For  $N' \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')$  there is a *unique*  $N \subseteq \mathcal{N}$  such that  $\text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')$
- For  $N \in \text{ImInj}_{\mathcal{G}}(\mathcal{G})$  there can *exist many*  $N' \subseteq \mathcal{N}'$  such that  $\text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')$

## Lemma

***The Inflation Lemma: Inflation** Given a particular inflation  $\mathcal{G}'$  of  $\mathcal{G}$ , if a marginal model  $\{P_N \mid N \in \text{ImInj}_{\mathcal{G}}(\mathcal{G}')\}$  is compatible with  $\mathcal{G}$  then all marginal models  $\{P_{N'} \mid N' \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')\}$  are compatible with  $\mathcal{G}'$  provided that  $P_N = P_{N'}$  for all instances where  $N \sim N'$ .*

## Corollary

*Any causal compatibility inequality  $I'$  constraining the injectable sets  $\text{Inj}_{\mathcal{G}}(\mathcal{G}')$  can be **deflated** into a causal compatibility inequality  $I$  constraining the images of the injectable sets  $\text{ImInj}_{\mathcal{G}}(\mathcal{G}')$ .*



- $d$ -separation relations + inflation = polynomial inequalities over  $\mathcal{G}$
- Restrict focus to sets  $N'$  that are partitioned into  $N'_1, N'_2$   $d$ -separated by empty set  $\emptyset$
- A **pre-injectable set**  $N'$ :

$$N' = \coprod_i N'_i \quad \forall i : N'_i \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')$$

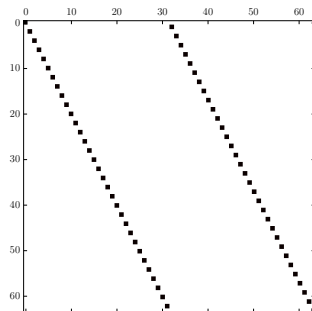
$$\forall i, j : N'_i \perp N'_j \iff \text{An}_{\mathcal{G}'}(N'_i) \cap \text{An}_{\mathcal{G}'}(N'_j) = \emptyset$$

- Only need to consider **maximal pre-injectable sets**  $\text{PreInj}_{\mathcal{G}}(\mathcal{G}')$

# Network Permutation Matrix

- States and measurements in the Triangle Scenario are not aligned
- Without  $\Pi$ ,  $P_{ABC}$  would be separable
- Required to align  $B$ 's measurement over  $\text{Tr}_{A,C}(\rho_{AB} \otimes \rho_{BC})$
- $\Pi$  is a  $2^6 \times 2^6$  matrix
- Shifts one qubit to the left

$$\Pi \equiv \sum_{|q_i\rangle \in \{|0\rangle, |1\rangle\}} |q_2 q_3 q_4 q_5 q_6 q_1\rangle \langle q_1 q_2 q_3 q_4 q_5 q_6|$$



# Outcomes and Events

## Definition

Each variable  $v$  has finite set of **outcomes**  $O_v$ .

Each set of variables  $V$  has finite set of **events**  $\mathcal{E}(V)$ :

$$\mathcal{E}(V) \equiv \{s : V \rightarrow O_V \mid \forall v \in V, s(v) \in O_v\}$$

## Definition

The set of events over the joint variables  $\mathcal{E}(\mathcal{J})$  are termed the **joint events**.

## Definition

The set of events over the marginal contexts are the **marginal events**

$$\mathcal{E}(\mathcal{M}) \equiv \coprod_{V \in \mathcal{M}} \mathcal{E}(V)$$

## Definition

The **joint distribution vector**  $\mathcal{P}^{\mathcal{J}}$

$$\mathcal{P}_j^{\mathcal{J}} = P_{\mathcal{J}}(j) \quad \forall j \in \mathcal{E}(\mathcal{J})$$

## Definition

The **marginal distribution vector**  $\mathcal{P}^{\mathcal{M}}$

$$\mathcal{P}_m^{\mathcal{M}} = P_{\mathcal{D}(m)}(m) \quad \forall m \in \mathcal{E}(\mathcal{M}), \mathcal{D}(m) \in \mathcal{M}$$

Can now write complete marginal problem as matrix multiplication:

$$\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J} \setminus V} P_{\mathcal{J}} \iff \mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

# Incidence Matrix

- **Incidence matrix**  $M$  is a bit-wise matrix
- Row-indexed by marginal events  $m \in \mathcal{E}(\mathcal{M})$
- Column-indexed by joint events  $j \in \mathcal{E}(\mathcal{J})$

$$M_{m,j} = \begin{cases} 1 & m = j|_{\mathcal{D}(m)} \\ 0 & \text{otherwise} \end{cases}$$

$$\# \text{Columns} = |\mathcal{E}(\mathcal{J})| = \prod_{v \in \mathcal{J}} |O_v|$$

$$\# \text{Rows} = |\mathcal{E}(\mathcal{M})| = \sum_{V \in \mathcal{M}} \prod_{v \in V} |O_v|$$

- Desirable to find compatibility inequality  $I$  such that

$$\forall \varphi \in \text{Perm}(A, B, C) : \varphi[I] = I$$

- Compatibility is independent of variable labels  
 $I, \mathcal{G} \rightarrow \varphi[I], \varphi[\mathcal{G}]$
- Need  $\varphi[\mathcal{G}] = \mathcal{G}$  to find new  $\varphi[I]$

## Definition

The **causal symmetry group** of causal structure  $\mathcal{G}$ :

$$\text{Aut}(\mathcal{G}) = \{\varphi \in \text{Perm}(\mathcal{N}) \mid \varphi[\mathcal{G}] = \mathcal{G}\}$$

Strictly speaking, one needs to preserve observable nodes:

$$\text{Aut}_{\mathcal{N}_O}(\mathcal{G}) = \{\varphi \in \text{Aut}(\mathcal{G}) \mid \varphi[\mathcal{N}_O] = \mathcal{N}_O\}$$

- Causal symmetry group for  $\mathcal{G}'$  is no good!
- Not possible to deflate inequality if it's not in terms of injectable sets

## Definition

The **restricted causal symmetry group**  $\Phi$  of  $\mathcal{G}'$ :

$$\Phi = \text{Aut}_{\text{PreInj}_{\mathcal{G}}}(\mathcal{G}')$$

# Restricted Causal Symmetry of Large Inflation

- $\Phi$  for the large inflation is an order 48 group with 4 generators

$\varphi_1$	$\varphi_2$	$\varphi_3$	$\varphi_4$
$A_1 \rightarrow A_4$	$A_1 \rightarrow A_1$	$A_1 \rightarrow C_1$	$A_1 \rightarrow A_1$
$A_2 \rightarrow A_3$	$A_2 \rightarrow A_3$	$A_2 \rightarrow C_2$	$A_2 \rightarrow A_2$
$A_3 \rightarrow A_2$	$A_3 \rightarrow A_2$	$A_3 \rightarrow C_3$	$A_3 \rightarrow A_3$
$A_4 \rightarrow A_1$	$A_4 \rightarrow A_4$	$A_4 \rightarrow C_4$	$A_4 \rightarrow A_4$
$B_1 \rightarrow B_4$	$B_1 \rightarrow C_1$	$B_1 \rightarrow A_1$	$B_1 \rightarrow B_2$
$B_2 \rightarrow B_3$	$B_2 \rightarrow C_3$	$B_2 \rightarrow A_2$	$B_2 \rightarrow B_1$
$B_3 \rightarrow B_2$	$B_3 \rightarrow C_2$	$B_3 \rightarrow A_3$	$B_3 \rightarrow B_4$
$B_4 \rightarrow B_1$	$B_4 \rightarrow C_4$	$B_4 \rightarrow A_4$	$B_4 \rightarrow B_3$
$C_1 \rightarrow C_4$	$C_1 \rightarrow B_1$	$C_1 \rightarrow B_1$	$C_1 \rightarrow C_3$
$C_2 \rightarrow C_3$	$C_2 \rightarrow B_3$	$C_2 \rightarrow B_2$	$C_2 \rightarrow C_4$
$C_3 \rightarrow C_2$	$C_3 \rightarrow B_2$	$C_3 \rightarrow B_3$	$C_3 \rightarrow C_1$
$C_4 \rightarrow C_1$	$C_4 \rightarrow B_4$	$C_4 \rightarrow B_4$	$C_4 \rightarrow C_2$



# Symmetric Incidence

- Group orbits through repeated action of  $\varphi \in \Phi$  on  $m \in \mathcal{E}(\mathcal{M})$  and  $j \in \mathcal{E}(\mathcal{J})$

$$\Phi[m] \equiv \{\varphi[m] \mid \varphi \in \Phi\}$$

$$\Phi[j] \equiv \{\varphi[j] \mid \varphi \in \Phi\}$$

- Construct **symmetric incidence matrix**  $\Phi[M]$

$$\Phi[M]_{\Phi[m], \Phi[j]} = \sum_{m' \in \Phi[m]} \sum_{j' \in \Phi[j]} M_{m', j'}$$

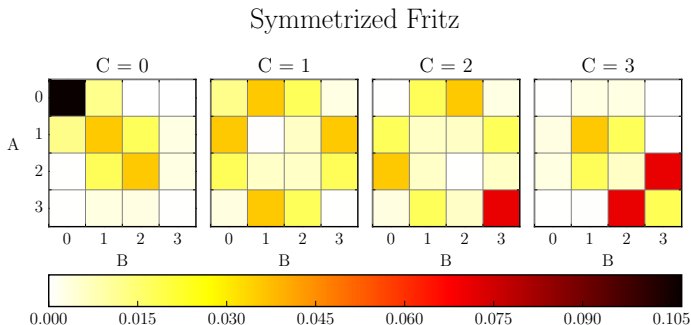
$$\Phi[M] = \Lambda_{\Phi[\mathcal{E}(\mathcal{M})]} \cdot M \cdot \Lambda_{\Phi[\mathcal{E}(\mathcal{J})]}$$

- $\Phi[M]$  not a bit-wise matrix like  $M$
- For large inflation  $M$  is  $16,896 \times 16,777,216$
- For large inflation  $\Phi[M]$  is  $450 \times 358,120$

# Symmetric Inequalities

- Certificate inequalities are curated/constructed to witness violation of particular distribution
- Above inequality not the best candidate to search for non-locality different than Fritz distribution
- Perhaps one could obtain inequality that does not assign special preference to  $C$
- Desirable to find compatibility inequality  $I$  such that

$$\forall \varphi \in \text{Perm}(A, B, C) : \varphi[I] = I$$



- More non-local yet not quantum-accessible
- Quantum-accessible distributions form non-context set

- Each party  $(A, B, C)$  is assigned a **projective-operator valued measure (POVM)**  $(M_A, M_B, M_C)$

$$\forall |\psi\rangle \in \mathcal{H}^d : \langle \psi | M_\chi | \psi \rangle \geq 0 \quad M_\chi = M_\chi^\dagger$$

- $n$ -outcome measurement

$$M_\chi = \{M_{\chi,1}, \dots, M_{\chi,n}\} \quad \sum_{i=1}^n M_{\chi,i} = \mathbb{1}$$

- For  $n = 2$  outcomes, a parameterization exists by constraining the eigenvalues of  $M_{\chi,i}$ ; for  $n > 2$  not aware of anything
- Warrants consideration of **projective-valued measures (PVMs)**

# Cholesky Parameterization of States

- Each latent resource  $\rho \in (\rho_{AB}, \rho_{BC}, \rho_{CA})$  modeled as bipartite qubit state acting on  $\mathcal{H}^{d/2} \otimes \mathcal{H}^{d/2}$
- $d \times d$  positive semi-definite (PSD) hermitian matrices with unitary trace
- **Cholesky Parametrization** allows one to write any hermitian PSD as  $\rho = T^\dagger T$
- For  $d = 4$ :

$$T = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_2 + i\lambda_3 & \lambda_4 & 0 & 0 \\ \lambda_5 + i\lambda_6 & \lambda_7 + i\lambda_8 & \lambda_9 & 0 \\ \lambda_{10} + i\lambda_{11} & \lambda_{12} + i\lambda_{13} & \lambda_{14} + i\lambda_{15} & \lambda_{16} \end{bmatrix}$$

- $d^2$  real-valued parameters
- Normalized  $\rho = T^\dagger T / \text{Tr}(T^\dagger T)$  adds degeneracy

## Appendix B: Local Minima Concerns

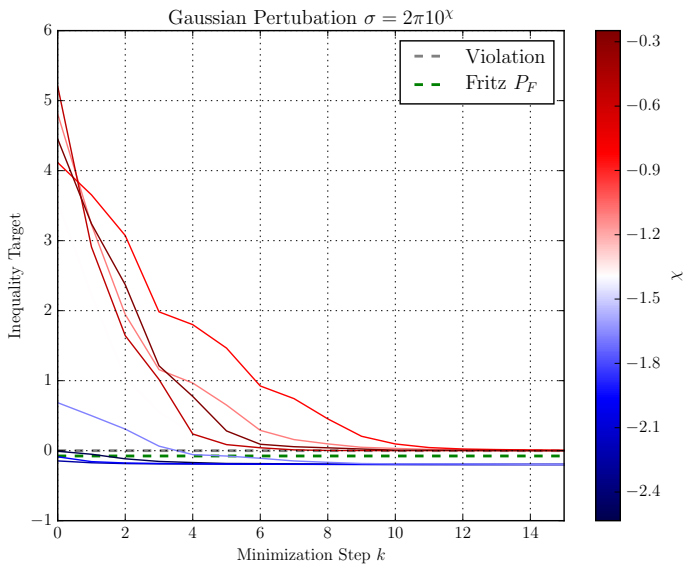
- Finding global minimum is tricky
- Difficult to converge to violation
- Noisy seed (Gaussian noise):

$$\lambda_{(0)} = \lambda_{(F)} + \delta\lambda \quad \delta\lambda_i \sim \mathcal{N}(\mu = 0, \sigma^2 = (2\pi 10^x)^2)$$

- Uniform seed:

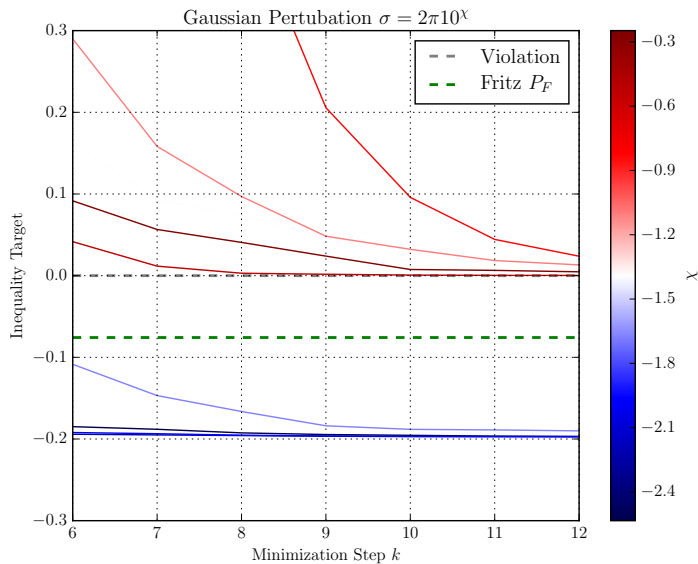
$$\lambda_{(0),i} \sim \mathcal{U}([0, 2\pi])$$

# Fritz Local Minima

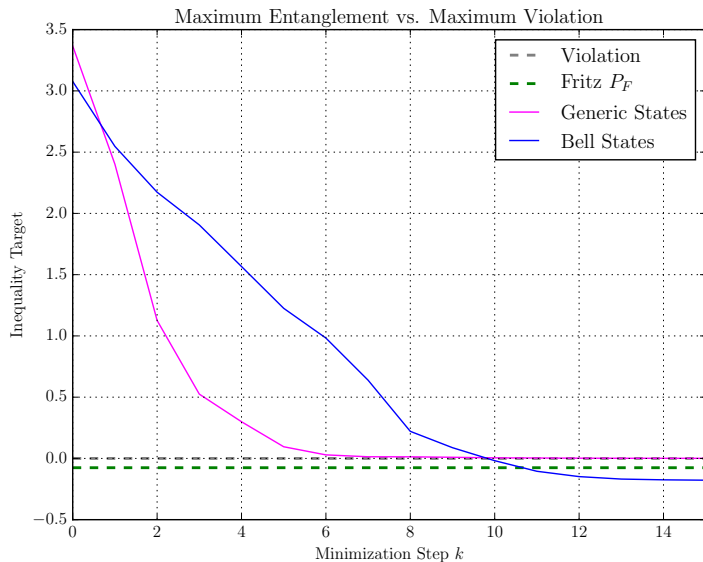




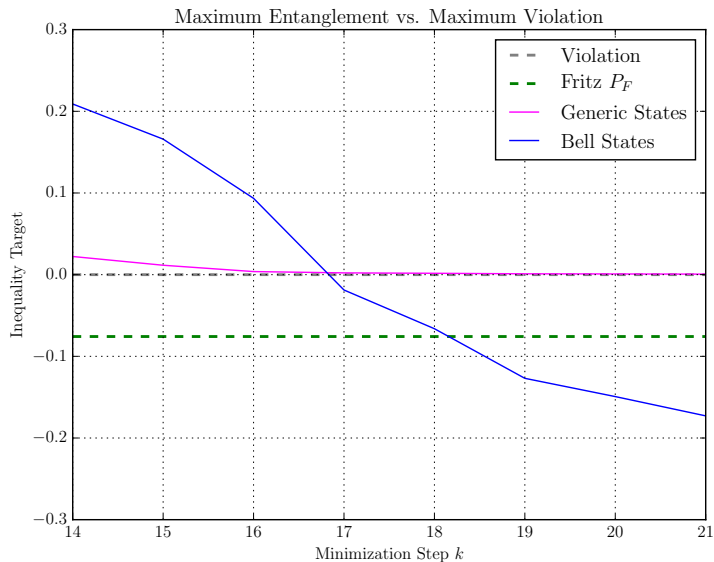
# Fritz Local Minima Zoomed



# Max Entangled vs. Max Violating (???)

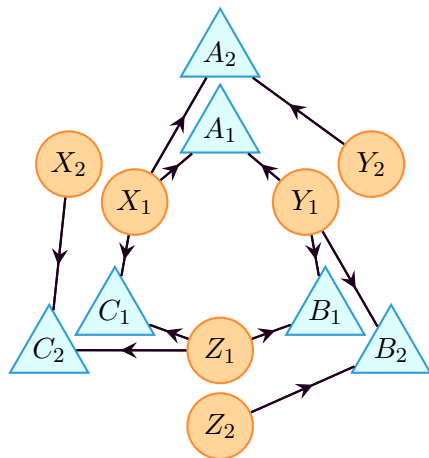
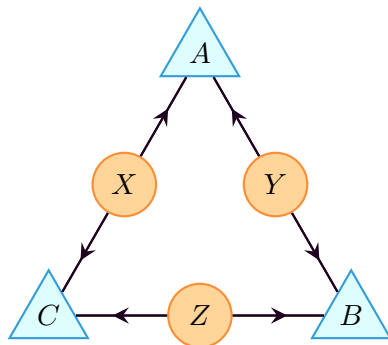


# Max Entangled vs. Max Violating (???)



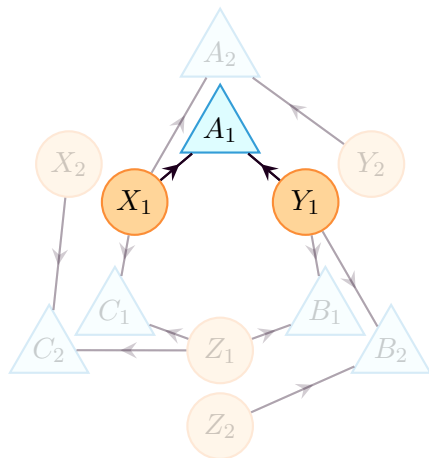
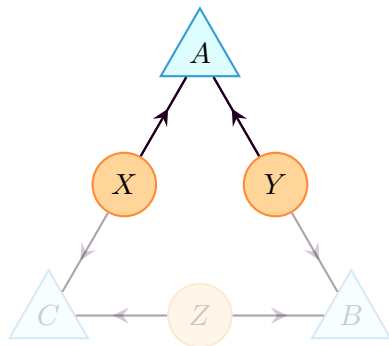
## Section C: Inflation Technique

# Demonstrating Inflation Technique



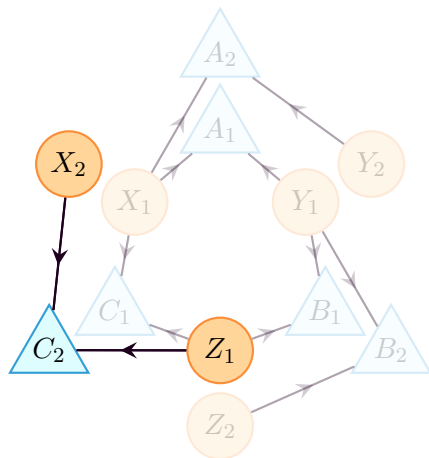
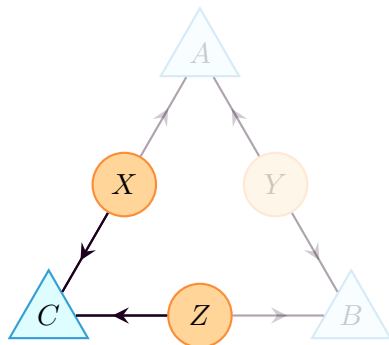
$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \text{AnSub}_{\mathcal{G}'}(n') \sim \text{AnSub}_{\mathcal{G}}(n)$$

# Demonstrating Inflation Technique



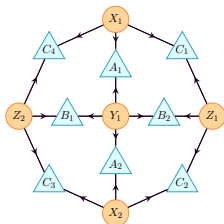
$$\text{AnSub}_G(A) \sim \text{AnSub}_{G'}(A_1)$$

# Demonstrating Inflation Technique

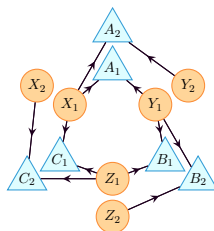


$$\text{AnSub}_G(C) \sim \text{AnSub}_{G'}(C_2)$$

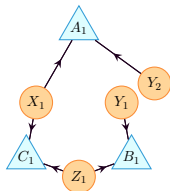
# Some Inflations of the Triangle Scenario



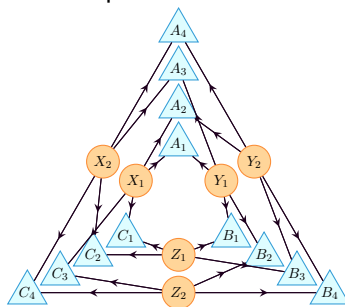
Wagon Wheel Inflation



Spiral Inflation



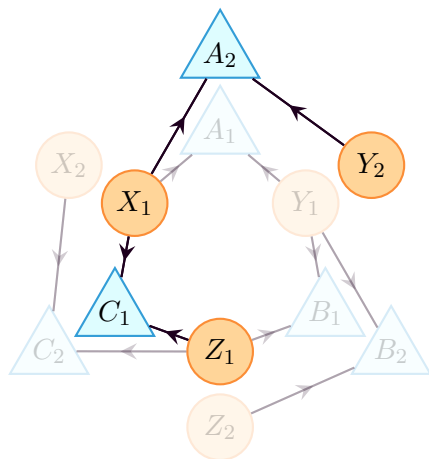
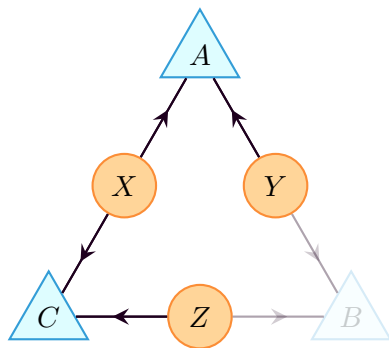
Cut Inflation



Web Inflation

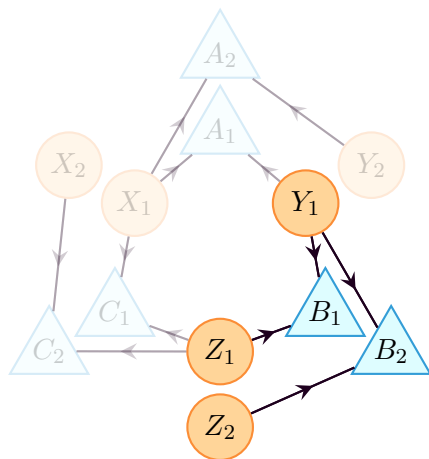
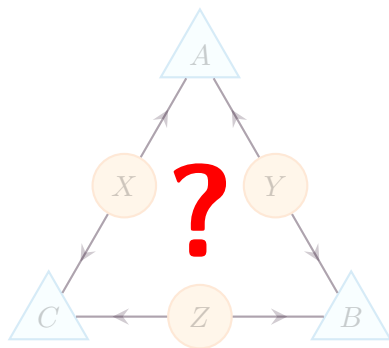


# What are Injectable Sets?



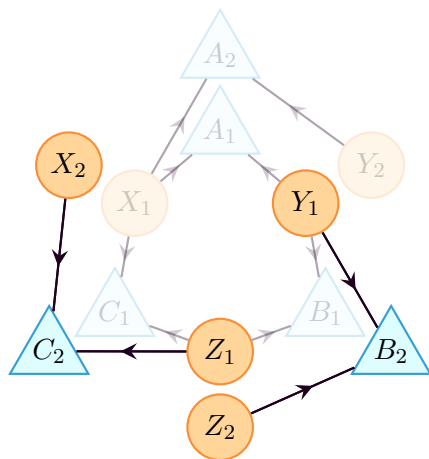
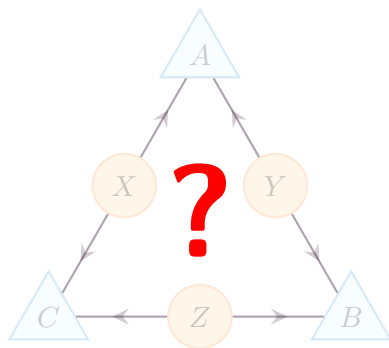
$$\text{AnSub}_{\mathcal{G}}(A, C) \sim \text{AnSub}_{\mathcal{G}'}(A_2, C_1)$$

# What are Injectable Sets?



$?? \not\sim \text{AnSub}_{G'}(B_1, B_2)$

# What are Injectable Sets?



??  $\not\sim \text{AnSub}_{G'}(B_2, C_2)$

The **injectable sets** in  $\mathcal{G}'$ :

$$\text{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : \text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')\}$$

The **images of the injectable sets** in  $\mathcal{G}$ :

$$\text{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : \text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')\}$$

**What makes injectable sets useful?**

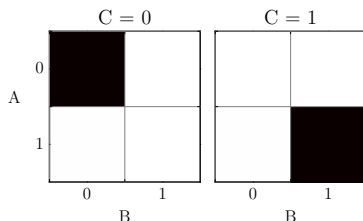
## Lemma (Inflation Lemma)

Given  $\mathcal{G} = (\mathcal{N}, \mathcal{E})$  and inflation  $\mathcal{G}' = (\mathcal{N}', \mathcal{E}')$ :

$$\begin{array}{ccc} \underbrace{\{P_N \mid N \in \text{ImInj}_{\mathcal{G}}(\mathcal{G}')\}}_{\text{compatible with } \mathcal{G}} & \longrightarrow & \{P_{n|\text{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\} \\ & & \downarrow \text{define} \\ \underbrace{\{P_{N'} \mid N' \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')\}}_{\text{compatible with } \mathcal{G}'} & \longleftarrow & \{P_{n'|\text{Pa}_{\mathcal{G}'}(n')} \mid n' \in \mathcal{N}'\} \end{array}$$

# Perfect Correlation Is Incompatible

Perfect Correlation



$$\blacksquare = \frac{1}{2}$$

$$P_{ABC}(abc) = \frac{[000] + [111]}{2}$$

$$P_{ABC}(abc) = \begin{cases} \frac{1}{2} & a = b = c \\ 0 & \text{otherwise} \end{cases}$$

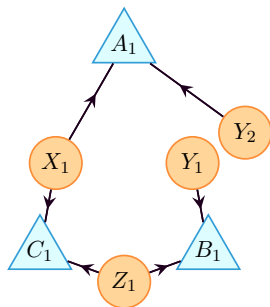
Compatibility Inequality

$$P_A(0)P_B(1) \leq P_{BC}(10) + P_{AC}(01)$$

Witnesses Perfect Correlation

$$\left(\frac{1}{2}\right)^2 \not\leq 0 + 0$$

# Deriving Compatibility Inequalities



$$\mathcal{M} = \{\{A_1, B_1\}, \{B_1, C_1\}, \{A_1, C_1\}\}$$

$$P^{\mathcal{M}} = \{P_{A_1 B_1}, P_{B_1 C_1}, P_{A_1 C_1}\}$$

Compatibility requires:  $\exists P_{\mathcal{J}} = P_{A_1 B_1 C_1}$

$$P_{A_1 B_1} = \sum_{C_1} P_{\mathcal{J}} \quad P_{B_1 C_1} = \sum_{A_1} P_{\mathcal{J}} \quad P_{A_1 C_1} = \sum_{B_1} P_{\mathcal{J}}$$

# Deriving Compatibility Inequalities Cont'd

$$\underbrace{P_{A_1 B_1} = \sum_{C_1} P_{\mathcal{J}} \quad P_{B_1 C_1} = \sum_{A_1} P_{\mathcal{J}} \quad P_{A_1 C_1} = \sum_{B_1} P_{\mathcal{J}}}$$

$$\underbrace{\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J} \setminus V} P_{\mathcal{J}}}$$

$$\mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

$$\mathcal{P}^{\mathcal{M}} = \begin{pmatrix} P_{A_1 B_1}(00) \\ P_{A_1 B_1}(01) \\ P_{A_1 B_1}(10) \\ P_{A_1 B_1}(11) \\ \hline P_{B_1 C_1}(00) \\ P_{B_1 C_1}(01) \\ P_{B_1 C_1}(10) \\ P_{B_1 C_1}(11) \\ \hline P_{A_1 C_1}(00) \\ P_{A_1 C_1}(01) \\ P_{A_1 C_1}(10) \\ P_{A_1 C_1}(11) \end{pmatrix} \quad \mathcal{P}^{\mathcal{J}} = \begin{pmatrix} P_{A_1 B_1 C_1}(000) \\ P_{A_1 B_1 C_1}(001) \\ P_{A_1 B_1 C_1}(010) \\ P_{A_1 B_1 C_1}(011) \\ P_{A_1 B_1 C_1}(100) \\ P_{A_1 B_1 C_1}(101) \\ P_{A_1 B_1 C_1}(110) \\ P_{A_1 B_1 C_1}(111) \end{pmatrix}$$



# Incidence Example

$$M = \begin{array}{l} (A_1, B_1, C_1) = \\ (A_1=0, B_1=0) \\ (A_1=0, B_1=1) \\ (A_1=1, B_1=0) \\ (A_1=1, B_1=1) \\ (B_1=0, C_1=0) \\ (B_1=0, C_1=1) \\ (B_1=1, C_1=0) \\ (B_1=1, C_1=1) \\ (A_1=0, C_1=0) \\ (A_1=0, C_1=1) \\ (A_1=1, C_1=0) \\ (A_1=1, C_1=1) \end{array} \begin{pmatrix} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix}$$

$$\mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

# Marginal Linear Program

## Marginal LP:

minimize:  $\emptyset \cdot x$

subject to:  $\mathcal{P}^{\mathcal{J}} \succeq 0$

$$M \cdot \mathcal{P}^{\mathcal{J}} = \mathcal{P}^{\mathcal{M}}$$

## Dual Marginal LP:

minimize:  $y \cdot \mathcal{P}^{\mathcal{M}}$

subject to:  $y \cdot M \succeq 0$

- If  $\mathcal{P}^{\mathcal{J}}$  exists, then:

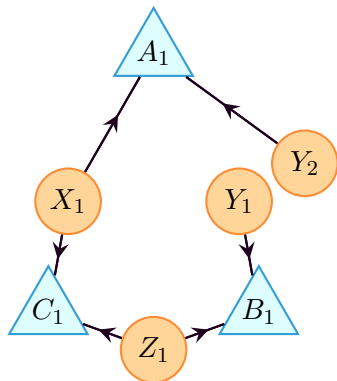
$$y \cdot M \cdot \mathcal{P}^{\mathcal{J}} = y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$$

- If not, then  $y$  is an **infeasibility certificate** which generates **infeasibility inequality**:

$$y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$$

- Most linear programming toolkits return certificates (*Mosek*, *Gurobi*, *CPLEX*, *cvxr/cvxopt*.)

# Deflating Inequalities



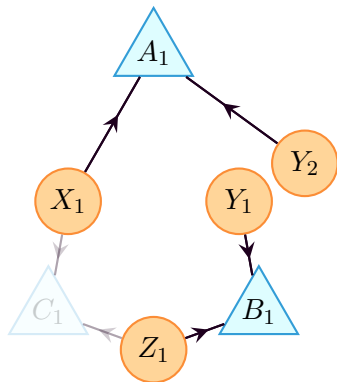
$$\text{Inj}_{\mathcal{G}}(\mathcal{G}') = \left\{ \begin{array}{l} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{array} \right\}$$

$$\{A_1, B_1\} \notin \text{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

**Can not deflate inequality!**

# Deflating Inequalities



$$\text{Inj}_{\mathcal{G}'} = \left\{ \begin{array}{l} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{array} \right\}$$

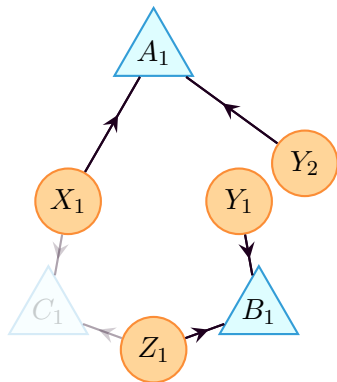
$$\{A_1, B_1\} \notin \text{Inj}_{\mathcal{G}'}$$

$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

**However!**

$$\text{AnSub}_{\mathcal{G}'}(A_1) \cap \text{AnSub}_{\mathcal{G}'}(B_1) = \emptyset \iff A_1 \perp B_1$$

# Deflating Inequalities



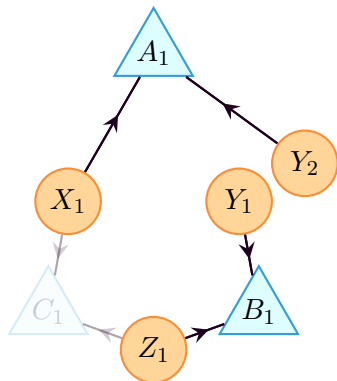
$$\text{Inj}_{\mathcal{G}}(\mathcal{G}') = \left\{ \begin{array}{l} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{array} \right\}$$

$$\{A_1, B_1\} \notin \text{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

$$P_{A_1}(0)P_{B_1}(1) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

# Deflating Inequalities



$$\text{Inj}_{\mathcal{G}'} = \left\{ \begin{array}{l} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{array} \right\}$$

$$\{A_1, B_1\} \notin \text{Inj}_{\mathcal{G}'}(\mathcal{G}')$$

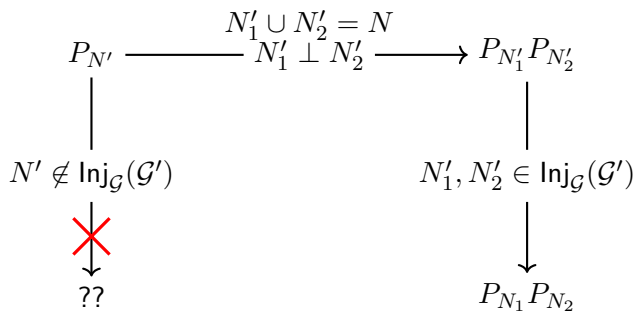
$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

$$P_{A_1}(0)P_{B_1}(1) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

$$P_A(0)P_B(1) \leq P_{BC}(10) + P_{AC}(01)$$

# Inflation Gives Polynomial Inequalities

- Deflation only holds when inequality constrains probabilities  
 $P_{N'}, N' \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')$
- **Linear inequality** for  $\mathcal{G}'$



- **Polynomial inequality** for  $\mathcal{G}$ !

A **pre-injectable set**  $N'$  is:

$$N' = \coprod_i N'_i \quad \forall i : N'_i \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')$$

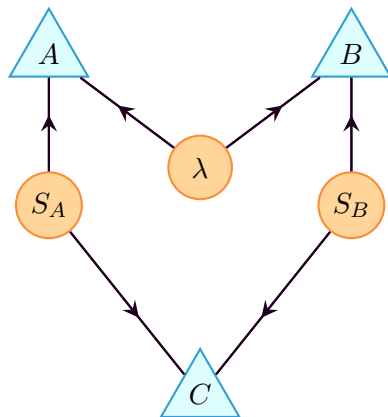
$$\forall i, j : N'_i \perp N'_j \iff \text{An}_{\mathcal{G}'}(N'_i) \cap \text{An}_{\mathcal{G}'}(N'_j) = \emptyset$$

Only need to consider **maximal pre-injectable sets** denoted  $\text{PreInj}_{\mathcal{G}}(\mathcal{G}')$

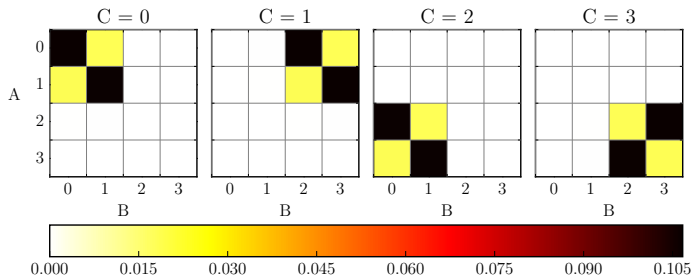


## Appendix D: Quantum Non-classicality From Inflation

# Triangle Scenario As Bell Scenario

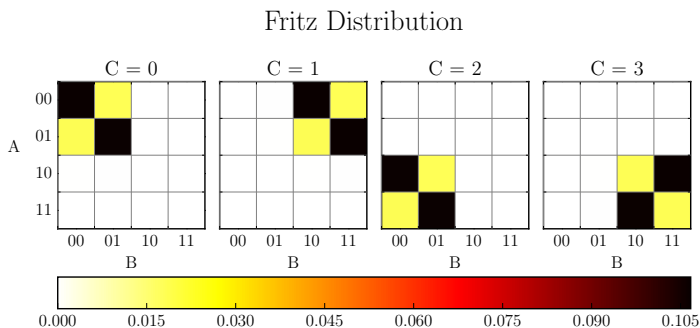


Fritz Distribution



$$\text{Yellow} = \frac{1}{32} (2 - \sqrt{2}) \quad \text{Black} = \frac{1}{32} (2 + \sqrt{2})$$

# Fritz Distribution Bit-Measurements



$$\text{Yellow} = \frac{1}{32} (2 - \sqrt{2}) \quad \text{Black} = \frac{1}{32} (2 + \sqrt{2})$$

# Quantum Implementation of Fritz Distribution

## ■ States:

$$\rho_{AB} = |\Phi^+\rangle\langle\Phi^+| \quad \rho_{BC} = \rho_{CA} = \frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

## ■ Measurements:

$$M_A = \{|0\psi_1\rangle\langle 0\psi_1|, |0\psi_5\rangle\langle 0\psi_5|, |1\psi_3\rangle\langle 1\psi_3|, |1\psi_7\rangle\langle 1\psi_7|\}$$

$$M_B = \{|\psi_6 0\rangle\langle \psi_6 0|, |\psi_2 0\rangle\langle \psi_2 0|, |\psi_0 1\rangle\langle \psi_0 1|, |\psi_4 1\rangle\langle \psi_4 1|\}$$

$$M_C = \{|00\rangle\langle 00|, |10\rangle\langle 10|, |01\rangle\langle 01|, |11\rangle\langle 11|\}$$

## ■ Shorthand: $|\psi_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{in/4}|1\rangle)$

# Fritz Distribution Violating CHSH

- $C$ 's outcome acts as measurement “setting” for  $A$ ,  $B$ ;  
independent of  $\rho_{AB}$
- Correlation between right bits

$$\langle A_r B_r \rangle = P_{A_r B_r}(00) + P_{A_r B_r}(11) - P_{A_r B_r}(01) - P_{A_r B_r}(10)$$

$$\langle A_r B_r | C = 0, 1, 2 \rangle = \frac{1}{\sqrt{2}} \quad \langle A_r B_r | C = 3 \rangle = -\frac{1}{\sqrt{2}}$$

- Gives CHSH violation

$$\begin{aligned} & \langle A_r B_r | C = 0 \rangle + \langle A_r B_r | C = 1 \rangle + \langle A_r B_r | C = 2 \rangle - \langle A_r B_r | C = 3 \rangle \\ &= 3 \left( \frac{1}{\sqrt{2}} \right) - \left( -\frac{1}{\sqrt{2}} \right) \\ &= 2\sqrt{2} \not\leq 2 \end{aligned}$$

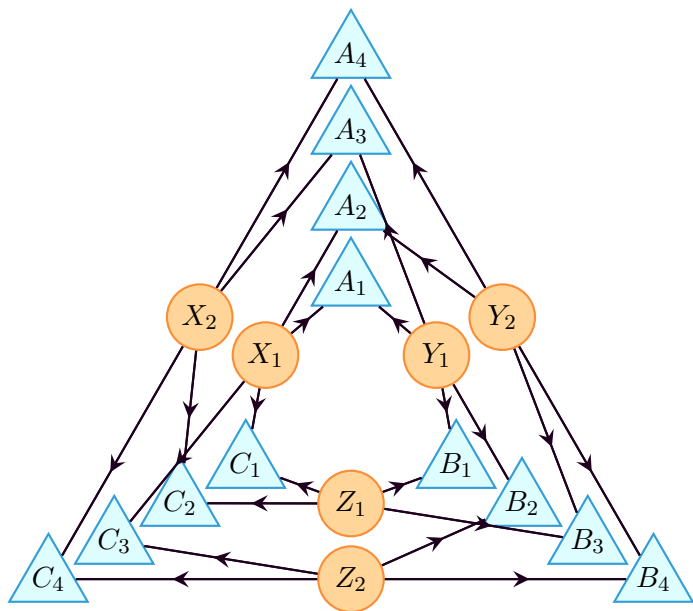
- Incompatibility proof contingent on perfect correlation between  $C$  and pseudo-settings
- Proof not robust to noise

## Problem (2.17 in **Fritz\_2012**)

*Find an example of non-classical quantum correlations in  $TS$  together with a proof of its non-classicality which does not hinge on Bell's Theorem.*

- “...would be helpful to have inequalities...”

# Large Inflation





# Large Inflation Pre-injectable Sets

## Maximal Pre-injectable Sets

$\{A_1, B_1, C_1, A_4, B_4, C_4\}$

$\{A_1, B_2, C_3, A_4, B_3, C_2\}$

$\{A_2, B_3, C_1, A_3, B_2, C_4\}$

$\{A_2, B_4, C_3, A_3, B_1, C_2\}$

$\{A_1, B_3, C_4\}$

$\{A_1, B_4, C_2\}$

$\{A_2, B_1, C_4\}$

$\{A_2, B_2, C_2\}$

$\{A_3, B_3, C_3\}$

$\{A_3, B_4, C_1\}$

$\{A_4, B_1, C_3\}$

$\{A_4, B_2, C_1\}$

## Ancestral Independences

$\{A_1, B_1, C_1\} \perp \{A_4, B_4, C_4\}$

$\{A_1, B_2, C_3\} \perp \{A_4, B_3, C_2\}$

$\{A_2, B_3, C_1\} \perp \{A_3, B_2, C_4\}$

$\{A_2, B_4, C_3\} \perp \{A_3, B_1, C_2\}$

$\{A_1\} \perp \{B_3\} \perp \{C_4\}$

$\{A_1\} \perp \{B_4\} \perp \{C_2\}$

$\{A_2\} \perp \{B_1\} \perp \{C_4\}$

$\{A_2\} \perp \{B_2\} \perp \{C_2\}$

$\{A_3\} \perp \{B_3\} \perp \{C_3\}$

$\{A_3\} \perp \{B_4\} \perp \{C_1\}$

$\{A_4\} \perp \{B_1\} \perp \{C_3\}$

$\{A_4\} \perp \{B_2\} \perp \{C_1\}$

- Joint variables are all of the observable nodes  $\mathcal{N}'_O = \mathcal{J}$

$$\mathcal{J} = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4\}$$

- Marginal scenario is composed of pre-injectable sets  
 $\mathcal{M} = \text{PreInj}_{\mathcal{G}}(\mathcal{G}')$
- Inequalities violated by Fritz distribution are inherently 4-outcome
- Incidence matrix  $M$  is **very large**  $\sim 2.25\text{Gb}$ 
  - #Columns =  $4^{12} = 16,777,216$
  - #Rows =  $4 \times 4^6 + 8 \times 4^3 = 16,896$
  - #Non-zero Entries = 201,326,592

## Appendix F: Maximal Violations & Noise

Minimize objective function  $f(\lambda) \in \mathbb{R}$ :

- 1 Real-valued parameters  $\lambda = (\lambda_0, \dots, \lambda_n)$
- 2 Quantum states/measurements  $\rho_{AB}, \rho_{BC}, \rho_{CA}, M_A, M_B, M_C$

$$P_{ABC}(abc) = \text{Tr}[\Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$

- 3 Distribution  $P_{ABC}$
- 4 Plug into inequality  $I$  in homogeneous form  $I(P_{ABC}) \geq 0$
- 5 Output is objective value  $I(P_{ABC})$

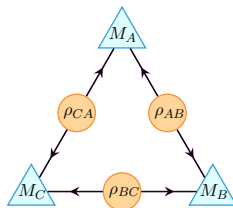
- Numerical minimization of  $f(\lambda)$

$$f(\lambda_{(k+1)}) = \lambda_{(k)} - \gamma_{(k)} \nabla f(\lambda_{(k)})$$

- Non-convex, non-linear, smooth/continuous
- Gradient Descent, BFGS Method, Nelder-Mead simplex method
- Stochastic methods: simulated annealing, basin-hopping

# Quantum Model On Triangle Scenario

$$P_{ABC}(abc) = \text{Tr}[\Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$



- Each latent resource  $\rho \in (\rho_{AB}, \rho_{BC}, \rho_{CA})$  modeled as bipartite **qubit** state acting on  $\mathcal{H}^2 \otimes \mathcal{H}^2$
- Each party  $(A, B, C)$  is assigned 4-outcome PVM set  $(M_A, M_B, M_C)$
- Parameterized using unitary transformations  $U \in \mathcal{U}(4)$

# Parameterizing Unitary Group

- **Spengler\_2010\_Unitary** parameterization of  $\mathcal{U}(d)$

$$\lambda = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,d} \\ \vdots & \ddots & \vdots \\ \lambda_{d,1} & \cdots & \lambda_{d,d} \end{pmatrix}$$

$$U = \left[ \prod_{m=1}^{d-1} \left( \prod_{n=m+1}^d R_{m,n} R P_{n,m} \right) \right] \cdot \left[ \prod_{l=1}^d G P_l \right]$$

- Global Phase Terms:  $G P_l = \exp(i P_l \lambda_{l,l})$
- Relative Phase Terms:  $R P_{n,m} = \exp(i P_n \lambda_{n,m})$
- Rotation Terms:  $R_{m,n} = \exp(i \sigma_{m,n} \lambda_{m,n})$
- Projection Operators:  $P_l = |l\rangle\langle l|$
- Anti-symmetric  $\sigma$ -matrices:  $\sigma_{m,n} = -i|m\rangle\langle n| + i|n\rangle\langle m|$
- Parameters  $\lambda_{n,m} \in [0, 2\pi]$

# Parameterizing Unitary Group Cont'd

- Each parameter  $\lambda_{n,m}$  has physical interpretation
- Degeneracies are easily eliminated such as global phase

$$\forall l = 1, \dots, d : \lambda_{l,l} = 0 \implies GP_l = \mathbb{1}$$

- Parameterize  $U \in \mathcal{U}(d)$  up to global phase denoted  $\tilde{U} \in \mathcal{U}(d)$

$$\tilde{U} = \prod_{m=1}^d \left( \prod_{n=m+1}^d R_{m,n} R P_{n,m} \right)$$

- Computationally efficient (no matrix exponentials)

$$GP_l = \mathbb{1} + P_l \left( e^{i\lambda_{l,l}} - 1 \right)$$

$$RP_{n,m} = \mathbb{1} + P_n \left( e^{i\lambda_{n,m}} - 1 \right)$$

$$\begin{aligned} R_{m,n} = \mathbb{1} &+ (|m\rangle\langle m| + |n\rangle\langle n|)(\cos \lambda_{n,m} - 1) \\ &+ (|m\rangle\langle n| - |n\rangle\langle m|) \sin \lambda_{n,m} \end{aligned}$$



# Parameterizing PVMs

- Each part is assigned  $d$  element **projective-valued measures (PVMs)** acting on  $\mathcal{H}^d$ ,

$$M = \{M_1, \dots, M_d\} \quad \sum_{i=1}^d M_i = \mathbb{1}$$

$$\forall i : \forall |\psi\rangle \in \mathcal{H}^d : \langle \psi | M_i | \psi \rangle \geq 0 \quad M_i = M_i^\dagger$$

$$M_i M_j = \delta_{ij} M_i \quad M_i = |m_i\rangle\langle m_i|$$

- Unitary transform maps  $M$  to computational basis

$$\{|m_1\rangle, \dots, |m_d\rangle\} = \{U|1\rangle, \dots, U|d\rangle\}$$

- Global phase irrelevant:  $\tilde{U}$  requires  $d(d-1)$  real-valued parameters
- PVMs are computationally more efficient than POVMs

$$P_{ABC}(abc) = \langle m_{A,a} m_{B,b} m_{C,c} | \Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi | m_{A,a} m_{B,b} m_{C,c} \rangle$$

# Parameterizing States

- Exploit spectral decomposition

$$\rho = \sum_{i=1}^d p_i |\psi_i\rangle\langle\psi_i| \quad p_i \geq 0, \sum_i p_i = 1$$

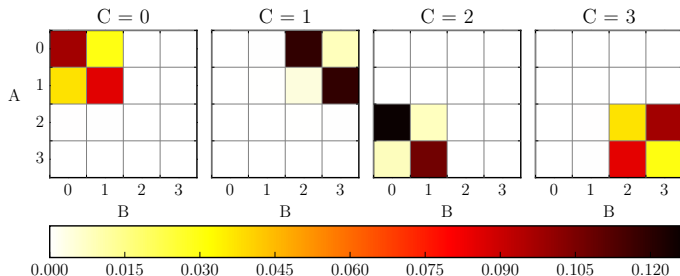
- Unitary transform maps  $\{|\psi_i\rangle\}$  to computational basis

$$\{|\psi_1\rangle, \dots, |\psi_d\rangle\} = \{U|1\rangle, \dots, U|d\rangle\}$$

- Global phase irrelevant again  $\tilde{U}$  requires  $d(d-1)$  real-valued parameters
- Eigenvalues  $\{p_i\}$  parameterized using hyper-spherical coordinates:  $d-1$  real-valued parameters

# Maximal Violation

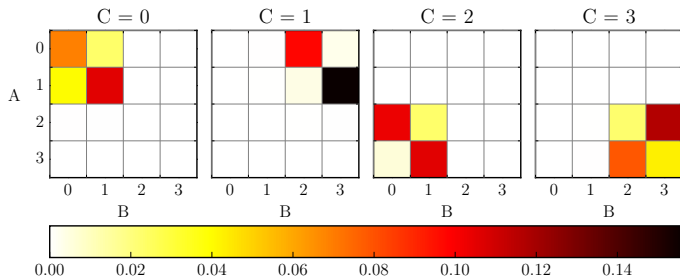
Maximum Violation of  $I_0$  ( $V_F = 1.501$ )



Relative Violation: 
$$V_F = \frac{\min_P \{I(P)\}}{I(P_F)}$$

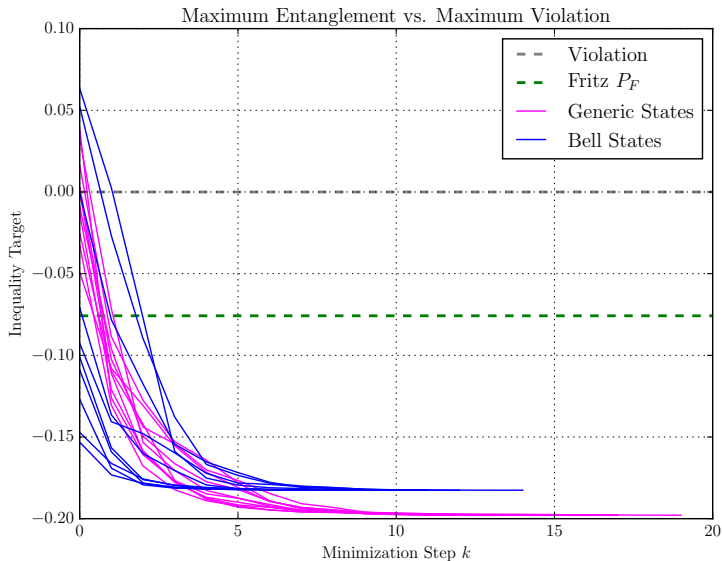
# Maximal Violation Symmetric

Maximum Violation of  $I_2$  ( $V_F = 2.61$ )



Relative Violation: 
$$V_F = \frac{\min_P \{I(P)\}}{I(P_F)}$$

# Max Entangled vs. Max Violating

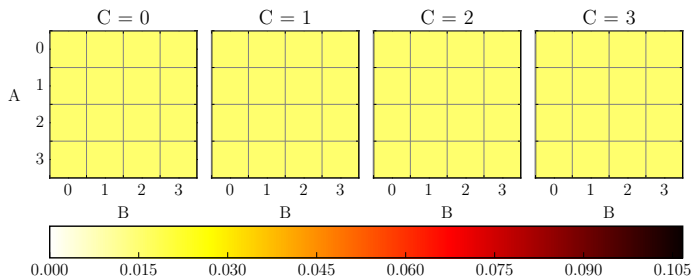


# Maximally Violating Distributions

- Able to out-perform violation provided by Fritz distribution
- Maximally-violating states are not maximally-entangled; similar to detection loop-hole example of **Methot\_2006**
- Both symmetric and asymmetric inequalities exhibit same qualitative features

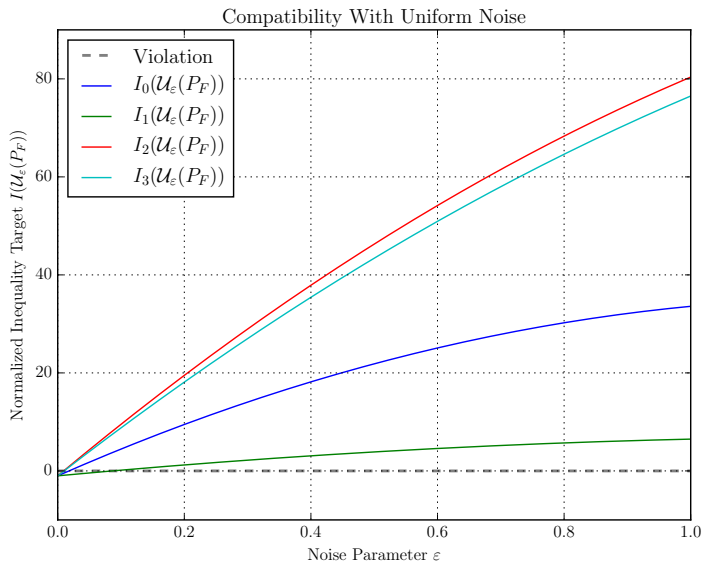
$$\mathcal{U}_\varepsilon(P) = (1 - \varepsilon)P + \varepsilon\mathcal{U}$$

Uniform Distribution



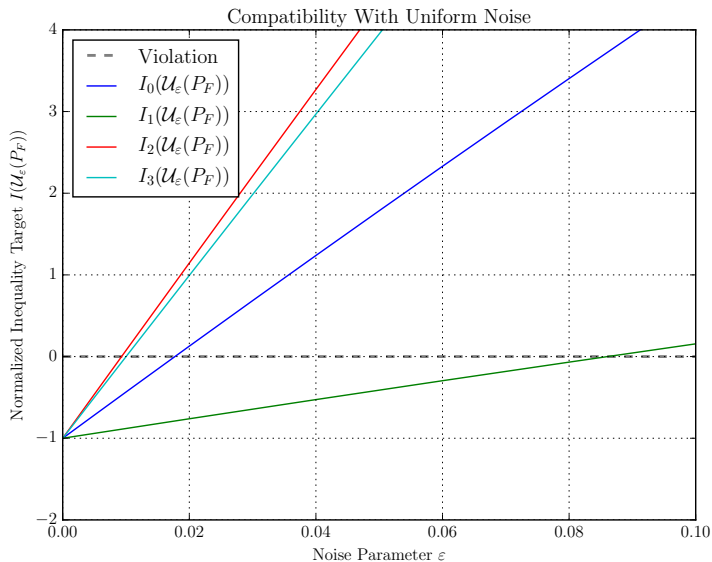
$$\text{Yellow square} = \frac{1}{64}$$

# Robust to Noise



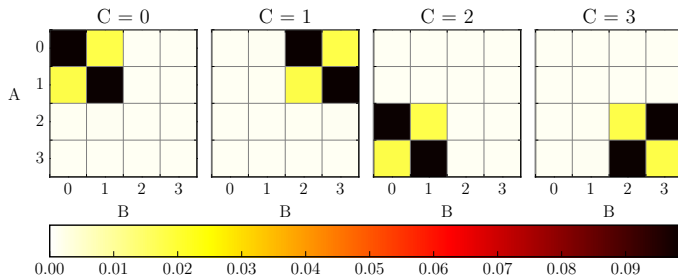


# Robust to Noise Zoomed



# Noisy Non-locality

Noisy Yet Non-Local Fritz Distribution



$$\square = 0.00133$$

- 1 **Inflation technique** capable of producing polynomial inequalities with quantum/classical witnesses
- 2 Fritz witness-able by **party-symmetric inequalities**
- 3 Maximally violating distributions require **non-maximally entangled states**

- 1 Do these new non-local distributions suggest **new quantum resources** in the triangle scenario?
- 2 Can any non-local quantum correlations in the triangle scenario **satisfy CHSH inequalities** (under Fritz type coarse graining)?
- 3 Which inequalities are **most robust** to noise? Facets?