I. JUNE 1, 2020:

As of today, it appears that there is an error in the WagonWheel inequality that ended up in the final manuscript (i.e. $I_{\text{WagonWheelAlternativeProperSwap}}$). Specifically, the uniform distribution over all six bits $A_l, A_r, B_l, B_r, C_l, C_r$, which is causally compatible with the Triangle network, actually violates $I_{\text{WagonWheelAlternativeProperSwap}}$:

$$I_{\text{WagonWheelAlternativeProperSwap}}\left(\frac{1}{2^6}\right) = +\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 \cdots$$

$$\cdots - \left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8 = \frac{19}{64} \not\leq 0.$$

Therefore, $I_{\text{WagonWheelAlternativeProperSwap}}$ is clearly an invalid inequality and a mistake was introduced somewhere. The actually inequality that Elie proposed was $I_{\text{WagonWheelAlternative}}$. In order to make the inequality in the paper correspond to the Fritz distribution in the paper, wherein $C_lC_r = 11$ plays the distinguished role, I attempted to flip the bits for C_r and B_l and erroneously derived $I_{\text{WagonWheelAlternativeProperSwap}}$. The *correct* version would have been:

$$\begin{gathered} I_{\text{WagonWheelCorrectBitFlip}}: \\ -P_{A_lB_l}(11) \\ +P_{A_lB_lC_lC_r}(1111) \\ -P_{A_lB_l}(00)P_{C_lC_r}(11) \\ -P_{C_lC_r}(01)P_{C_lC_r}(10) \\ +P_{C_lC_r}(01)P_{A_lA_rB_lB_rC_lC_r}(100110) \\ +P_{C_lC_r}(01)P_{A_lA_rB_lB_rC_lC_r}(110010) \\ -P_{C_lC_r}(00)P_{A_lA_rB_lB_rC_lC_r}(101111) \\ -P_{C_lC_r}(00)P_{A_lA_rB_lB_rC_lC_r}(111011) \\ +P_{C_lC_r}(11)P_{A_lA_rB_lB_rC_lC_r}(000000) \\ +P_{C_lC_r}(11)P_{A_lA_rB_lB_rC_lC_r}(010100) \\ +P_{C_lC_r}(10)P_{A_lA_rB_lB_rC_lC_r}(001001) \\ +P_{C_lC_r}(10)P_{A_lA_rB_lB_rC_lC_r}(011101) \\ \leq 0 \end{gathered} \tag{1}$$

This inequality when evaluated at the Fritz distribution (P_F) reads term by term:

$$\begin{gathered} I_{\text{WagonWheelCorrectFlip}}(P_F): \\ -\frac{1}{4} + \frac{1}{4} - \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 \\ +\frac{1}{4}\left(\frac{1}{32}\left(2-\sqrt{2}\right)\right) + \frac{1}{4}\left(\frac{1}{32}\left(2-\sqrt{2}\right)\right) - \frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) - \frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) \\ +\frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) + \frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) + \frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) + \frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) \\ = -\frac{1}{16} \leq 0 \end{gathered} \tag{2}$$

Additionally, this inequality when evaluated at the uniform distribution reads term by term:

$$I_{\text{WagonWheelCorrectFlip}}\left(\frac{1}{2^6}\right) = -\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^4 \cdots$$

$$\cdots + \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8 = -\frac{19}{64} \leq 0$$

Upon closer inspection, $I_{\text{WagonWheelCorrectFlip}}$ is identical to the inequality found in the paper, with the direction of inequality reverse (i.e. it was off by an overall minus sign).

Therefore, it appears that either 1) I have isolated the wrong variables to bit-flip in Elie's $I_{\text{WagonWheelAlternative}}$, 2) $I_{\text{WagonWheelAlternative}}$ was transcribed from Elie incorrectly, 3) $I_{\text{WagonWheelAlternative}}$ is incapable of witnessing the Fritz distribution, or 4) $I_{\text{WagonWheelAlternative}}$ is not a valid inequality in the first place.

I suspect the resolution is of type 2) because the Fritz distribution is all about the agreement or disagreement between A_r and B_r in four different C_lC_r contexts, and $I_{\text{WagonWheelAlternative}}$ seems to have $A_r = B_r$ in two contexts and $A_r \neq B_r$ in the other two contexts, which is unlike the form of the Fritz distribution.

II. SEPTEMBER 18, 2017:

The current WagonWheel inequality we are using in the paper is the following:

$$
\begin{gathered}
I_{\text{WagonWheel}}: \\
+P_{ABC}(000)P_C(3) - 2P_{ABC}(203)P_C(0) - 2P_{ABC}(303)P_C(0) - P_{ABC}(003)P_C(0) \\
-P_{ABC}(013)P_C(0) - P_{ABC}(020)P_C(2) - P_{ABC}(022)P_C(2) - P_{ABC}(023)P_C(0) \\
-P_{ABC}(023)P_C(2) - P_{ABC}(030)P_C(2) - P_{ABC}(031)P_C(2) - P_{ABC}(032)P_C(2) \\
-P_{ABC}(033)P_C(0) - P_{ABC}(033)P_C(2) - P_{ABC}(103)P_C(0) - P_{ABC}(113)P_C(0) \\
-P_{ABC}(123)P_C(0) - P_{ABC}(133)P_C(0) - P_{ABC}(200)P_C(0) - P_{ABC}(200)P_C(1) \\
-P_{ABC}(200)P_C(2) - P_{ABC}(200)P_C(3) - P_{ABC}(201)P_C(0) - P_{ABC}(201)P_C(1) \\
-P_{ABC}(201)P_C(2) - P_{ABC}(201)P_C(3) - P_{ABC}(203)P_C(1) - P_{ABC}(203)P_C(2) \\
-P_{ABC}(203)P_C(3) - P_{ABC}(213)P_C(0) - P_{ABC}(223)P_C(0) - P_{ABC}(300)P_C(0) \\
-P_{ABC}(300)P_C(1) - P_{ABC}(300)P_C(2) - P_{ABC}(300)P_C(3) - P_{ABC}(301)P_C(0) \\
-P_{ABC}(301)P_C(1) - P_{ABC}(301)P_C(2) - P_{ABC}(301)P_C(3) - P_{ABC}(302)P_C(1) \\
-P_{ABC}(303)P_C(1) - P_{ABC}(303)P_C(2) - P_{ABC}(303)P_C(3) - P_{ABC}(313)P_C(0) \leq 0
\end{gathered} \tag{3}
$$

Which is currently simplified to:

$$
\begin{gathered}
I_{\text{WagonWheel}}: \\
+P_{ABC}(000)P_C(3) + P_{ABC}(021)P_C(2) + P_{ABC}(202) + P_{ABC}(302) + P_{ABC}(233)P_C(0) + P_{AC}(33)P_C(0) \\
-P_{ABC}(303)P_C(0) - P_{ABC}(313)P_C(0) - P_{ABC}(302)P_C(1) - P_{AB}(02)P_C(2) - \\
-P_{AB}(03)P_C(2) - P_{AB}(20) - P_{AB}(30) - P_C(3)P_C(0) \leq 0
\end{gathered} \tag{4}
$$

I went ahead and converted to the bit-notation and obtained the following:

$$
\begin{gathered}
I_{\text{WagonWheel}}: \\
-P_{A_lB_lB_r}(100) \\
-P_{C_lC_r}(11)P_{C_lC_r}(00) \\
+P_{A_lB_lB_rC_lC_r}(10010) \\
+P_{A_lA_rC_lC_r}(1111)P_{C_lC_r}(00) \\
-P_{A_lA_rB_lB_rC_lC_r}(110011)P_{C_lC_r}(00) \\
-P_{A_lA_rB_lB_rC_lC_r}(110010)P_{C_lC_r}(01) \\
+P_{A_lA_rB_lB_rC_lC_r}(001001)P_{C_lC_r}(10) \\
-P_{A_lA_rB_l}(001)P_{C_lC_r}(10) \\
+P_{A_lA_rB_lB_rC_lC_r}(000000)P_{C_lC_r}(11) \\
\leq 0
\end{gathered} \tag{5}
$$

The following inequality is the one that Elie Wolfe found, and was proposed as a replacement for the paper due to its algebraic clarity.

$$
\begin{gathered}
I_{\text{WagonWheelAlternative}}: \\
-P_{A_l B_l}(10) \\
+P_{A_l B_l C_l C_r}(1010) \\
-P_{A_l B_l}(01) P_{C_l C_r}(10) \\
-P_{C_l C_r}(00) P_{C_l C_r}(11) \\
+P_{C_l C_r}(00) P_{A_l A_r B_l B_r C_l C_r}(101111) \\
+P_{C_l C_r}(00) P_{A_l A_r B_l B_r C_l C_r}(111011) \\
-P_{C_l C_r}(01) P_{A_l A_r B_l B_r C_l C_r}(100110) \\
-P_{C_l C_r}(01) P_{A_l A_r B_l B_r C_l C_r}(110010) \\
+P_{C_l C_r}(10) P_{A_l A_r B_l B_r C_l C_r}(001001) \\
+P_{C_l C_r}(10) P_{A_l A_r B_l B_r C_l C_r}(011101) \\
+P_{C_l C_r}(11) P_{A_l A_r B_l B_r C_l C_r}(000000) \\
+P_{C_l C_r}(11) P_{A_l A_r B_l B_r C_l C_r}(010100) \\
\leq 0
\end{gathered}
\tag{6}
$$

There are some observations:

- $I_{\text{WagonWheel}}$ in bit-notation is actually shorter than $I_{\text{WagonWheelAlternative}}$

- The paper mentions the exact violation ($0.0129 \not\leq 0$) of $I_{\text{WagonWheel}}$, something I would have to calculate for $I_{\text{WagonWheelAlternative}}$

Swapping the bits C_r and B_l,

$$
\begin{gathered}
I_{\text{WagonWheelAlternativeProperSwap}}: \\
+P_{A_l B_l}(11) \\
-P_{A_l B_l C_l C_r}(1111) \\
+P_{A_l B_l}(00) P_{C_l C_r}(11) \\
+P_{C_l C_r}(01) P_{C_l C_r}(10) \\
-P_{C_l C_r}(11) P_{A_l A_r B_l B_r C_l C_r}(000000) \\
-P_{C_l C_r}(11) P_{A_l A_r B_l B_r C_l C_r}(010100) \\
-P_{C_l C_r}(10) P_{A_l A_r B_l B_r C_l C_r}(001001) \\
-P_{C_l C_r}(10) P_{A_l A_r B_l B_r C_l C_r}(011101) \\
-P_{C_l C_r}(01) P_{A_l A_r B_l B_r C_l C_r}(100110) \\
-P_{C_l C_r}(01) P_{A_l A_r B_l B_r C_l C_r}(110010) \\
+P_{C_l C_r}(00) P_{A_l A_r B_l B_r C_l C_r}(101111) \\
+P_{C_l C_r}(00) P_{A_l A_r B_l B_r C_l C_r}(111011) \\
\leq 0
\end{gathered}
\tag{7}
$$

The violation using $I_{\text{WagonWheelAlternativeProperSwap}}$,

$$
\begin{aligned}
+P_{A_lB_l}(11) &= +\frac{1}{4} \\
-P_{A_lB_lC_lC_r}(1111) &= -\frac{1}{4} \\
+P_{A_lB_l}(00)P_{C_lC_r}(11) &= +\left(\frac{1}{4}\right)^2 \\
+P_{C_lC_r}(01)P_{C_lC_r}(10) &= +\left(\frac{1}{4}\right)^2 \\
-P_{C_lC_r}(01)P_{A_lA_rB_lB_rC_lC_r}(100110) &= -\frac{1}{4}\left(\frac{1}{32}\left(2-\sqrt{2}\right)\right) \\
-P_{C_lC_r}(01)P_{A_lA_rB_lB_rC_lC_r}(110010) &= -\frac{1}{4}\left(\frac{1}{32}\left(2-\sqrt{2}\right)\right) \\
-P_{C_lC_r}(00)P_{A_lA_rB_lB_rC_lC_r}(101111) &= -\frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) \\
-P_{C_lC_r}(00)P_{A_lA_rB_lB_rC_lC_r}(111011) &= -\frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) \\
-P_{C_lC_r}(11)P_{A_lA_rB_lB_rC_lC_r}(000000) &= -\frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) \\
-P_{C_lC_r}(11)P_{A_lA_rB_lB_rC_lC_r}(010100) &= -\frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) \\
+P_{C_lC_r}(10)P_{A_lA_rB_lB_rC_lC_r}(001001) &= +\frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right) \\
+P_{C_lC_r}(10)P_{A_lA_rB_lB_rC_lC_r}(011101) &= +\frac{1}{4}\left(\frac{1}{32}\left(2+\sqrt{2}\right)\right)
\end{aligned}
$$

$$
0.0625 = \frac{1}{16} \not\leq 0
$$

If we restrict the domain of $I_{\text{WagonWheelAlternativeProperSwap}}$ to only consider distributions where $C_l = A_l, C_r = B_l$ and P_C is uniform, we obtain the following constraint,

$$
I_{\text{WagonWheelRestricted}} :
$$

$$
\begin{gathered}
+\frac{1}{2} - P_{A_rB_rC_lC_r}(0110) - P_{A_rB_rC_lC_r}(1010) - P_{A_rB_rC_lC_r}(0000) - P_{A_rB_rC_lC_r}(1100) \\
-P_{A_rB_rC_lC_r}(0001) - P_{A_rB_rC_lC_r}(1101) + P_{A_rB_rC_lC_r}(0111) + P_{A_rB_rC_lC_r}(1011) \leq 0
\end{gathered}
$$