Inflation Technique

A Brief Introduction

2016

Graph Theory Notation

Let $n, m \in \mathcal{N}$ be nodes of the graph $\mathcal{G}$.

parents of n: $\mathsf{Pa}_{\mathcal{G}}(n) \equiv \{m \mid m \to n\}$

children of n: $\mathsf{Ch}_{\mathcal{G}}(n) \equiv \{m \mid n \to m\}$

ancestry of n: $\mathsf{An}_{\mathcal{G}}(n) \equiv \bigcup_{i \in \mathbb{W}} \mathsf{Pa}^{i}_{\mathcal{G}}(n)$

$$\mathsf{Pa}^{0}_{\mathcal{G}}(n) = n \qquad \mathsf{Pa}^{i}_{\mathcal{G}}(n) \equiv \mathsf{Pa}_{\mathcal{G}}\left(\mathsf{Pa}^{i-1}_{\mathcal{G}}(n)\right)$$

Notation extends to sets of nodes $N \subseteq \mathcal{N}$,

parents of N: $\mathsf{Pa}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Pa}_{\mathcal{G}}(n)$

children of N: $\mathsf{Ch}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Ch}_{\mathcal{G}}(n)$

ancestry of N: $\mathsf{An}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{An}_{\mathcal{G}}(n)$

An induced subgraph of $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ due to $N \subseteq \mathcal{N}$

$$\mathsf{Sub}_{\mathcal{G}}(N) = (N, \{e \in \mathcal{E} \mid e \subseteq N\})$$

Inflation Technique

Definition

An inflation of a causal structure $\mathcal{G}$ is another causal structure $\mathcal{G}'$ such that:

$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n)$$

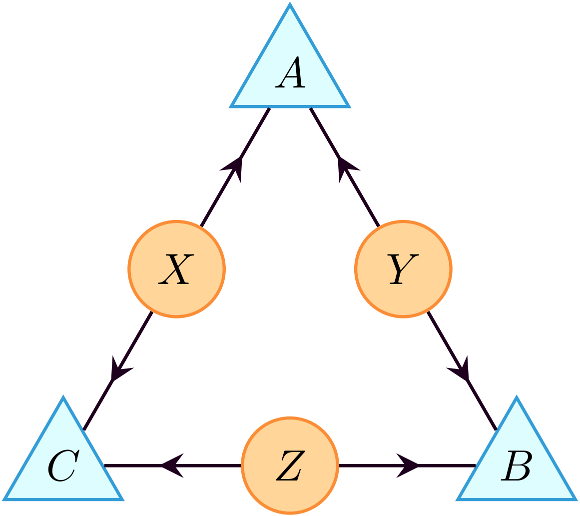
Where $\mathsf{AnSub}_{\mathcal{G}}(n)$ denotes the ancestral sub-graph of n in $\mathcal{G}$

$$\mathsf{AnSub}_{\mathcal{G}}(n) = \mathsf{Sub}_{\mathcal{G}}(\mathsf{An}_{\mathcal{G}}(n))$$

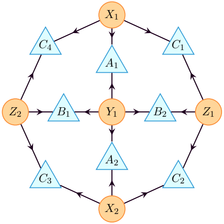
And '$\sim$' is a copy-index equivalence relation

$$A_1 \sim A_2 \sim A \nsim B_1 \sim B_2 \sim B$$

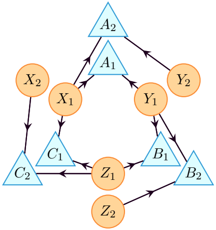
Case Study: The Triangle Scenario



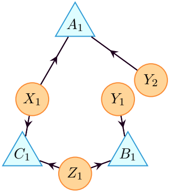
Some Inflations of the Triangle Scenario



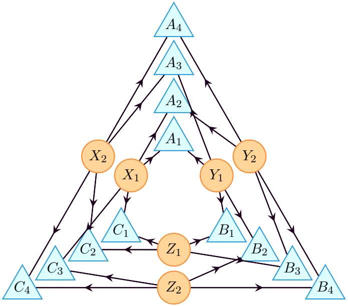
Wagon Wheel Inflation



Spiral Inflation

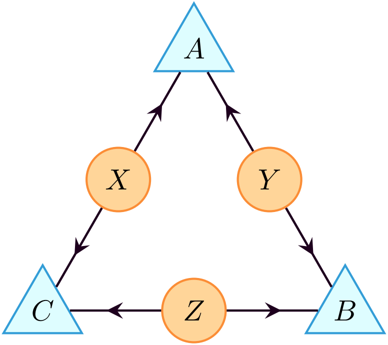


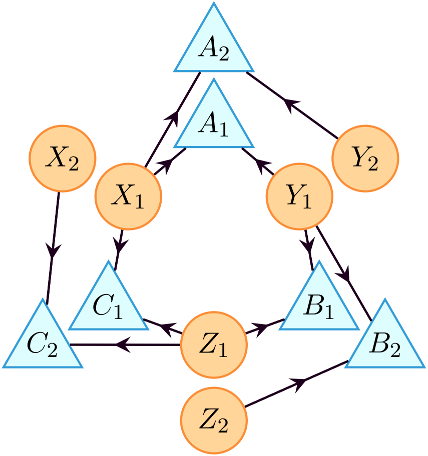
Cut Inflation



Web Inflation

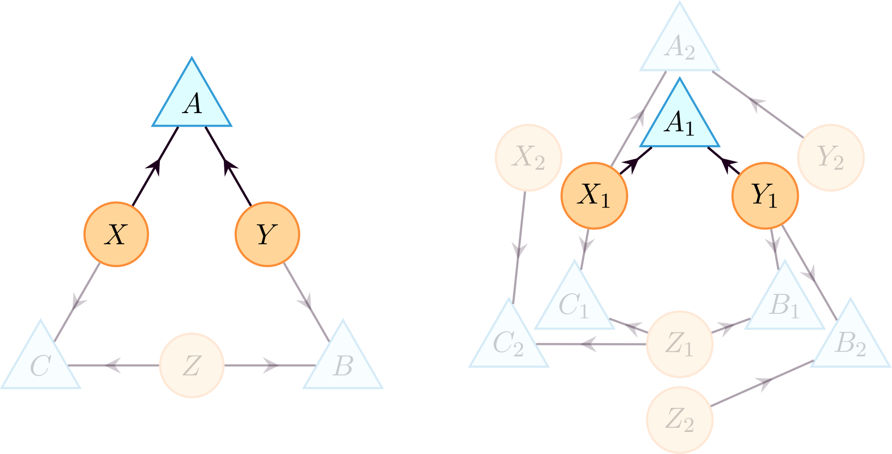
Demonstrating Inflation Technique





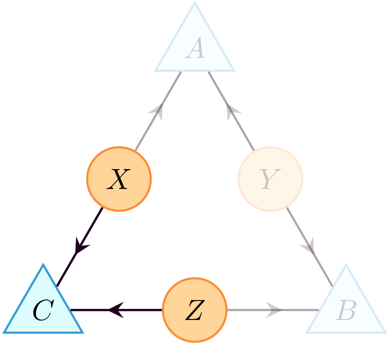
$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n)$$

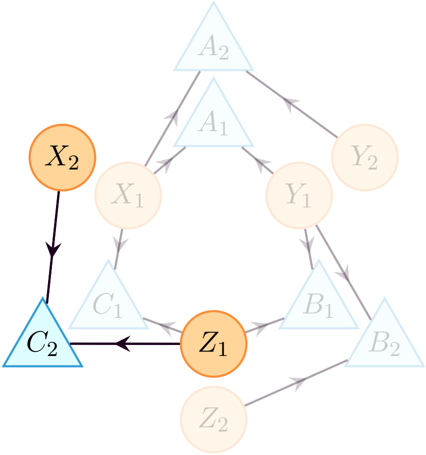
Demonstrating Inflation Technique



$$\mathsf{AnSub}_{\mathcal{G}}(A) \sim \mathsf{AnSub}_{\mathcal{G}'}(A_1)$$

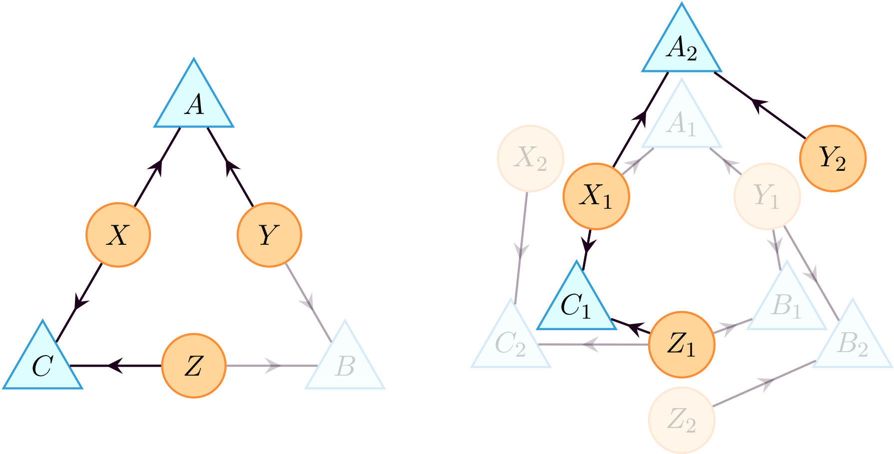
Demonstrating Inflation Technique





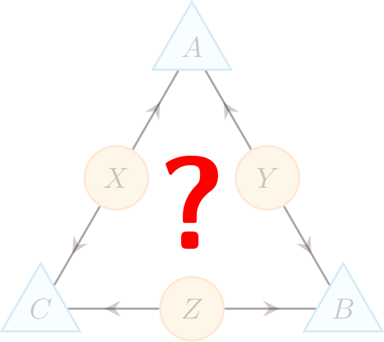
$$\mathsf{AnSub}_{\mathcal{G}}(C) \sim \mathsf{AnSub}_{\mathcal{G}'}(C_2)$$

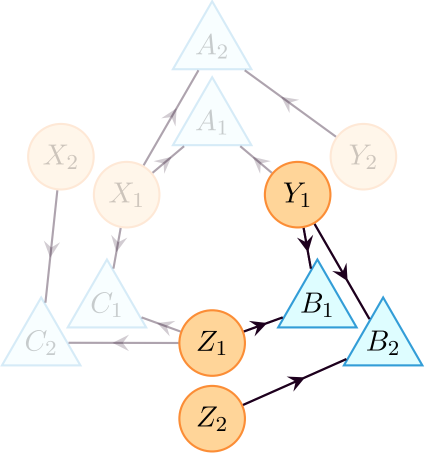
What are Injectable Sets?



$$\mathsf{AnSub}_{\mathcal{G}}(A, C) \sim \mathsf{AnSub}_{\mathcal{G}'}(A_2, C_1)$$

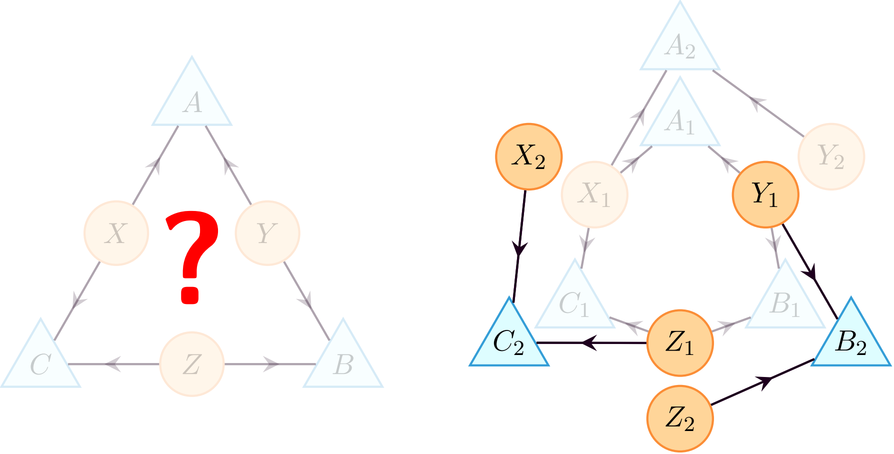
What are Injectable Sets?





$$?? \not\sim \mathsf{AnSub}_{\mathcal{G}'}(B_1, B_2)$$

What are Injectable Sets?



$$?? \not\sim \mathsf{AnSub}_{\mathcal{G}'}(B_2, C_2)$$

Injectable Sets Defined

The injectable sets in $\mathcal{G}'$:

$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : N \sim N'\}$$

The images of the injectable sets in $\mathcal{G}$:

$$\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : N \sim N'\}$$

What makes injectable sets useful?

Inflation Lemma

Lemma (Inflation Lemma)

Given $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ *and inflation* $\mathcal{G}' = (\mathcal{N}', \mathcal{E}')$*:*

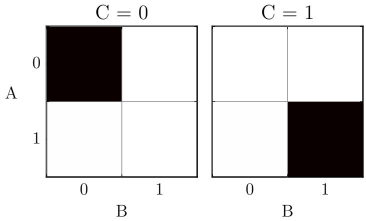
$$
\begin{tikzcd}
\underbrace{\{P_N \mid N \in \mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}')\}}_{\textit{compatible with } \mathcal{G}} \arrow[r] & \left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\} \arrow[d, "\textit{define}"] \\
\underbrace{\{P_{N'} \mid N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')\}}_{\textit{compatible with } \mathcal{G}'} & \left\{P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')} \mid n' \in \mathcal{N}'\right\} \arrow[l]
\end{tikzcd}
$$

Corollary

All inequalities I' *constraining* $\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$ *can be deflated into inequalities* I *constraining* $\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}')$ *by dropping copy-indices.*

Perfect Correlation Is Incompatible

Perfect Correlation



$$\blacksquare = \frac{1}{2}$$

$$P_{ABC}(abc) = \frac{[000] + [111]}{2}$$

$$P_{ABC}(abc) = \begin{cases} \frac{1}{2} & a = b = c \\ 0 & \text{otherwise} \end{cases}$$

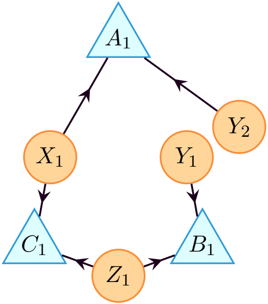
Compatibility Inequality

$$P_A(0)P_B(1) \leq P_{BC}(10) + P_{AC}(01)$$

Witnesses Perfect Correlation

$$\left(\frac{1}{2}\right)^2 \not\leq 0 + 0$$

Deriving Compatibility Inequalities



$$\mathcal{M} = \{\{A_1, B_1\}, \{B_1, C_1\}, \{A_1, C_1\}\}$$

$$P^{\mathcal{M}} = \{P_{A_1B_1}, P_{B_1C_1}, P_{A_1C_1}\}$$

Compatibility requires: $\exists P_{\mathcal{J}} = P_{A_1B_1C_1}$

$$P_{A_1B_1} = \sum\nolimits_{C_1} P_{\mathcal{J}} \qquad P_{B_1C_1} = \sum\nolimits_{A_1} P_{\mathcal{J}} \qquad P_{A_1C_1} = \sum\nolimits_{B_1} P_{\mathcal{J}}$$

Deriving Compatibility Inequalities Cont'd

$$\underbrace{P_{A_1B_1} = \sum_{C_1} P_{\mathcal{J}} \qquad P_{B_1C_1} = \sum_{A_1} P_{\mathcal{J}} \qquad P_{A_1C_1} = \sum_{B_1} P_{\mathcal{J}}}$$

$$\underbrace{\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J}\backslash V} P_{\mathcal{J}}}$$

$$\mathcal{P}^{\mathcal{M}} = M\mathcal{P}^{\mathcal{J}}$$

$$\mathcal{P}^{\mathcal{M}} = \left(\begin{array}{c} P_{A_1B_1}(00) \\ P_{A_1B_1}(01) \\ P_{A_1B_1}(10) \\ P_{A_1B_1}(11) \\ \hline P_{B_1C_1}(00) \\ P_{B_1C_1}(01) \\ P_{B_1C_1}(10) \\ P_{B_1C_1}(11) \\ \hline P_{A_1C_1}(00) \\ P_{A_1C_1}(01) \\ P_{A_1C_1}(10) \\ P_{A_1C_1}(11) \end{array}\right) \qquad \mathcal{P}^{\mathcal{J}} = \begin{pmatrix} P_{A_1B_1C_1}(000) \\ P_{A_1B_1C_1}(001) \\ P_{A_1B_1C_1}(010) \\ P_{A_1B_1C_1}(011) \\ P_{A_1B_1C_1}(100) \\ P_{A_1B_1C_1}(101) \\ P_{A_1B_1C_1}(110) \\ P_{A_1B_1C_1}(111) \end{pmatrix}$$

Incidence Example

$$
M = \begin{array}{rc}
(A_1,B_1,C_1) = & \begin{array}{cccccccc} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \end{array} \\
\begin{array}{r}
(A_1{=}0,B_1{=}0) \\
(A_1{=}0,B_1{=}1) \\
(A_1{=}1,B_1{=}0) \\
(A_1{=}1,B_1{=}1) \\
(B_1{=}0,C_1{=}0) \\
(B_1{=}0,C_1{=}1) \\
(B_1{=}1,C_1{=}0) \\
(B_1{=}1,C_1{=}1) \\
(A_1{=}0,C_1{=}0) \\
(A_1{=}0,C_1{=}1) \\
(A_1{=}1,C_1{=}0) \\
(A_1{=}1,C_1{=}1)
\end{array} &
\begin{pmatrix}
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\
\mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1}
\end{pmatrix}
\end{array}
$$

$$\mathcal{P}^{\mathcal{M}} = M\mathcal{P}^{\mathcal{J}}$$

Finding Inequalities

Linear Program:

minimize: $\emptyset x$

subject to: $\mathcal{P}^{\mathcal{J}} \succeq 0$

$M\mathcal{P}^{\mathcal{J}} = \mathcal{P}^{\mathcal{M}}$

Dual Linear Program:

minimize: $y\mathcal{P}^{\mathcal{M}}$

subject to: $yM \succeq 0$

Alternatively,

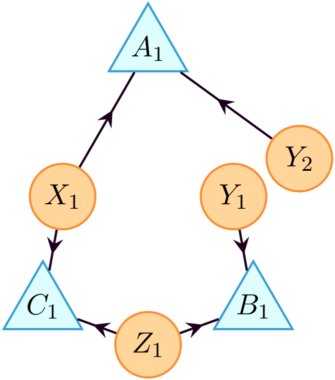
Linear Quantifier Elimination

Fourier-Motzkin

Polytope Description

etc.

Deflating Inequalities



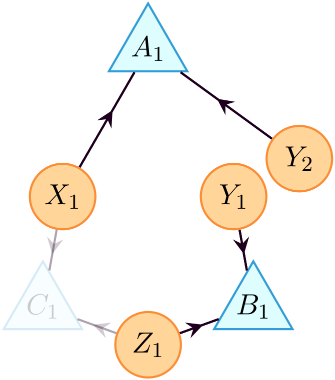
$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') = \left\{ \begin{matrix} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{matrix} \right\}$$

$$\{A_1, B_1\} \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

Can not deflate inequality!

Deflating Inequalities



$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') = \begin{Bmatrix} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{Bmatrix}$$

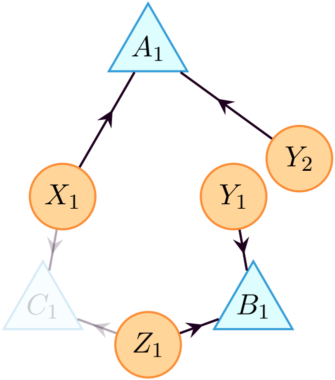
$$\{A_1, B_1\} \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1B_1}(01) \leq P_{B_1C_1}(10) + P_{A_1C_1}(01)$$

However!

$$\mathsf{AnSub}_{\mathcal{G}'}(A_1) \cap \mathsf{AnSub}_{\mathcal{G}'}(B_1) = \emptyset \iff A_1 \perp B_1$$

Deflating Inequalities



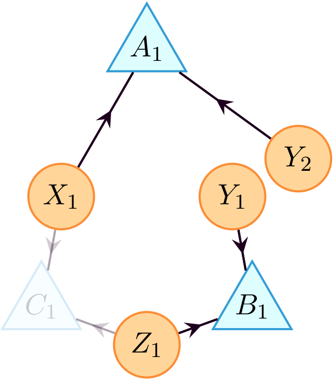
$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') = \left\{ \begin{matrix} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{matrix} \right\}$$

$$\{A_1, B_1\} \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

$$P_{A_1}(0) P_{B_1}(1) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

Deflating Inequalities



$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') = \left\{ \begin{array}{c} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{array} \right\}$$

$$\{A_1, B_1\} \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1B_1}(01) \leq P_{B_1C_1}(10) + P_{A_1C_1}(01)$$

$$P_{A_1}(0)P_{B_1}(1) \leq P_{B_1C_1}(10) + P_{A_1C_1}(01)$$

$$P_A(0)P_B(1) \leq P_{BC}(10) + P_{AC}(01)$$

Inflation Produces Polynomial Inequalities

Deflation demands inequality constrains injectable probabilities $P_{N'}, N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$

Linear inequality for $\mathcal{G}'$

$$
\begin{tikzcd}
P_{N'} \arrow[r, "N'_1 \cup N'_2 = N", "N'_1 \perp N'_2" description] \arrow[d, dash, "N' \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')" description] & P_{N'_1} P_{N'_2} \arrow[d, "{N'_1, N'_2 \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')}" description] \\
\text{??} \arrow[u, phantom, "\times" marking] & P_{N_1} P_{N_2}
\end{tikzcd}
$$

Polynomial inequality for $\mathcal{G}$!

Expressible Sets

d-separation relations on $\mathcal{G}'$

$+$

inequalities for $\mathcal{G}'$

$=$

polynomial inequalities for $\mathcal{G}$