

Causal Compatibility Inequalities Admitting of Quantum Violations in the Triangle Structure

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It has long been recognized that certain quantum correlations are incompatible with particular assumption about classical causal structure. Given a causal structure of unknown classicality, the presence of such correlations certifies the non-classical nature of the causal structure in a device independent fashion. In scenarios where all parties share a common resource, these non-classical correlations are also known as non-local correlations. Any constraint satisfied by all correlations which *are* classically compatible with a given causal structure defines a causal compatibility criterion. Such criteria were recently derived for the Triangle structure [[arXiv:1609.00672](https://arxiv.org/abs/1609.00672)] in the form of polynomial inequalities, begging the question: do any of those inequalities admit violation by quantum correlations? Numerical investigation suggests not, and we further conjecture that the set of correlations admitted by the classical Triangle structure is equivalent to the set of correlations admitted by its quantum generalization whenever the three observable variables are binary. Our main contribution in this work, however, is the derivation of new causal compatibility inequalities for the Triangle structure which *do* admit quantum violation. This provides the first robust-to-noise witness of quantum correlations in the Triangle structure. We conclude by considering the possibility of quantum resources potentially qualitatively different from those known previously.

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I. INTRODUCTION

In recent decades, the technical utility of quantum mechanics has become abundantly clear. In the realm of computation, quantum algorithms — such as Shor’s algorithm [1] and numerous others [2] — scale exponentially better than their classical counterparts. In the realm of secure communication, quantum protocols — a popular example being Quantum Key Distribution [3] — are able to provide privacy even against hypothetical adversaries with unlimited computational power, a desideratum which classical protocols are unable to fulfill. Throughout history, every aspect of quantum phenomena which fails to be emulated by classical resources has led to a practical exploitation of the non-classical feature in order to solve a some computational or communicational problem [4]. Motivated by past successes, a primal objective of modern quantum information theory is the discovery of new situations wherein quantum mechanics offers an advantage, and certification that the quantum advantage is genuine.

From a foundational prospective, the most robust demonstrations of quantum phenomena with no classical emulation have involved Bell inequalities [5, 6]. Originally, Bell inequalities were derived as a way to show that no hidden variable theory could ever account for quantum mechanics; in this sense Bell inequalities are a response to the famous EPR paradox [7]. The enumeration of Bell inequalities has since become a widespread systematic way of demonstrating the *non-classicality* of a given observation. More recently it has been appreciated that Bell inequalities can be understood as consequences of **causal inference** [8]. Causal inference is concerned with classifying observations into those which can and cannot be explained by a hypothesized causal structure. The abstract nature of causal inference is responsible for its presence in numerous scientific fields including machine learning and biology [9, 10]. Causal compatibility inequalities, such as Bell and Instrumental inequalities [11–13], characterize the space of observations that are compatible with a hypothesized causal structure, albeit the characterization offered by practically derivable inequalities is often only an approximation. To derive the traditional Bell inequalities from causal inference one starts with a (classical) causal structure known as the Bell structure, as depicted in Fig. 1. The fundamental Bell structure involves non-communicating parties making measurements on some hidden shared resource λ , where the measurement outcomes (A, B) are presumed to be stochastic functions of the local choices of measurement settings (S_A, S_B) and the shared resource λ . Quantum non-classicality in the Bell structure has been thoroughly studied since Bell’s original work [6]. More complex structures, however, such as the correlation scenarios proposed by Fritz [14, 15], are much less understood. Here we investigate one particular causal structure: the Triangle structure (Fig. 2).

The Triangle structure (Fig. 2) is a causal structure comprised of 3 parties labeled A, B, C arranged in a triangular configuration while pair-wise sharing hidden/latent variables X, Y, Z . It has been extensively studied previously (see [16, Fig. 1], [17, Fig. 6], [18, Fig. 8], [19, Fig. 8, App. E], [14, Fig. 3], [20, Fig. 4], [21, Fig. 1], ...). An overview of some milestone results is provided in Section IV. Identifying causal compatibility inequalities for this configuration has been seen as particularly challenging [18]. Further identifying causal compatibility inequalities of such high resolution that such inequalities can be violated by quantum-accessible distributions has remained out-of-reach for the Triangle structure.

This work is the first, therefore, to find causal compatibility inequalities for the Triangle structure that are able to be violated by quantum-accessible distributions. This accomplishment was made possible through the combination of two previous developments. First, the insight of Fritz [14] regarding the ability to re-interpret the Bell structure as a portion of the Triangle structure. Second, the new framework for solving causal inference problems developed by Wolfe *et al.* [21] called **The Inflation Technique**. Ultimately, this work serves as a validation that the Inflation Technique is sensitive enough to offer new insights into quantum non-classicality [22]. Moreover, these new inequalities offer an avenue for recognizing previously unknown forms of non-classicality. The authors’ attempts to find such novel resources were met with only partial success, suggesting clear refinements for future exploration.

The manuscript is organized into the following sections: Section II recalls important notions from causal inference theory and sets up the notation to be used. Section III offers a summary of the popular Bell structure and associated inequalities. Section IV discusses the Triangle structure and provides an overview of existing research; identifying its stark differences from the Bell structure and motivating why the Triangle structure is worth studying. Section V defines and discusses a singularly-quantum correlation first conceived of by Fritz [14], which we term the **Fritz distribution**. The Fritz distribution first was proven to be non-classical without the use of inequalities. Here we offer a direct proof using inequalities, with the advantage that our proof is robust to noise. A sample of such inequalities are presented in Section VII, specifically Ineqs. (11,13,9). In Section VII-B we quantify the robustness to noise of the Fritz distribution, and we numerically optimize our derived inequalities over quantum-accessible distributions (using qubits) in an effort to seek out new new flavors of non-classicality in the Triangle structure. Additionally, the Fritz distribution is subjected to noise in order to measure the robustness of the derived inequalities. Section IX concludes with proposed avenues for further research.

Appendix A briefly summarizes the Inflation Technique in the specific context of this work. Although the summary presented here is designed to be self-standing, a much more pedagogical introduction is offered by the original work [21]. Appendix A-2 demonstrates how the Inflation Technique was used to derive the causal compatibility inequalities for

the Triangle structure which admit violation by quantum-accessible distributions.

II. CAUSAL COMPATIBILITY

The task of causal inference is to determine the set of potentially observable probability distributions compatible with some hypothesis about causal relationships [9]. If an observed distribution can be explained by the hypothesized causal mechanism, then the distribution is said to be *compatible* with said causal mechanism. In order to define compatibility rigorously, we first need to formally define the notion of a causal hypothesis.

A hypothesis of causal mechanism is formally referred to as a **causal structure** and can be represented as a **directed acyclic graph**. A directed graph \mathcal{G} is an ordered tuple $(\mathcal{N}, \mathcal{E})$ of respectively *nodes* and *edges* where each edge $e \in \mathcal{E}$ connects a *pair* of nodes $n, m \in \mathcal{N}$ with a directed arrow $e = \{n \rightarrow m\}$. A directed graph is *acyclic* if there are no paths following the directions of the edges starting from and returning to the same node. The nodes \mathcal{N} of a causal structure represent random variables while the edges \mathcal{E} represent a casual influence from one variable to another pursuant to the prescribed direction.

Henceforth, we will utilize a number of familial notions from graph theory and denote them accordingly. The **parents of a node** $n \in \mathcal{N}$ are all nodes which point directly into n , i.e. $\text{Pa}_{\mathcal{G}}(n) \equiv \{m \mid m \rightarrow n\}$. Similarly defined are the **children of a node** $\text{Ch}_{\mathcal{G}}(n) \equiv \{m \mid n \rightarrow m\}$. Recursively defined are the **ancestors of a node** $\text{An}_{\mathcal{G}}(n) \equiv \bigcup_{i \in \mathbb{N}} \text{Pa}_{\mathcal{G}}^i(n)$ where $\text{Pa}_{\mathcal{G}}^i(n) \equiv \text{Pa}_{\mathcal{G}}(\text{Pa}_{\mathcal{G}}^{i-1}(n))$ and $\text{Pa}_{\mathcal{G}}^0(n) = n$ and the **descendants of a node** $\text{De}_{\mathcal{G}}(n) \equiv \bigcup_{i \in \mathbb{N}} \text{Ch}_{\mathcal{G}}^i(n)$ where $\text{Ch}_{\mathcal{G}}^i(n) \equiv \text{Ch}_{\mathcal{G}}(\text{Ch}_{\mathcal{G}}^{i-1}(n))$ and $\text{Ch}_{\mathcal{G}}^0(n) = n$. Finally, we extend this notation to a subset of nodes $N \subseteq \mathcal{N}$ by performing a union over elements. As an example, the parents of the nodes $N \subseteq \mathcal{N}$ are denoted $\text{Pa}_{\mathcal{G}}(N) = \bigcup_{n \in N} \text{Pa}_{\mathcal{G}}(n)$.

Let us now formalize the notion of compatibility between a causal structure \mathcal{G} and a probability distribution $P_{\mathcal{N}}$ defined over the nodes of \mathcal{G} . Intuitively, the absence of certain causal connections within \mathcal{G} should limit the set of possible observations that can be made on \mathcal{G} . Specifically, the causal structure \mathcal{G} hypothesizes that if each variable $n \in \mathcal{N}$ is causally influenced by its parents $\text{Pa}_{\mathcal{G}}(n)$. Therefore, if each variable is conditioned on its parentage, the probability distribution $P_{\mathcal{N}}$ should factorize accordingly:

$$P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\text{Pa}_{\mathcal{G}}(n)} = P_{n_1|\text{Pa}_{\mathcal{G}}(n_1)} \times \cdots \times P_{n_k|\text{Pa}_{\mathcal{G}}(n_k)} \quad (1)$$

If a given distribution $P_{\mathcal{N}}$ defined over *all* of the nodes \mathcal{N} of \mathcal{G} satisfies **Eq. (1)**, then $P_{\mathcal{N}}$ is said to be **compatible with \mathcal{G}** . The conditional distributions in **Eq. (1)** (i.e. $\{P_{n|\text{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\}$) are referred to as a set of **causal parameters for \mathcal{G}** . If a distribution $P_{\mathcal{N}}$ cannot be factorized according to **Eq. (1)**, $P_{\mathcal{N}}$ is said to be **incompatible with \mathcal{G}** . When one is specified with a joint distribution $P_{\mathcal{N}}$ defined over all nodes of a causal structure \mathcal{G} , it is possible to completely determine whether or not $P_{\mathcal{N}}$ is compatible with \mathcal{G} by computing the causal parameters induced by $P_{\mathcal{N}}$ and checking the equality of **Eq. (1)**.

A challenge, however, is presented when one is supplied with a partial observation, i.e. a joint distribution $P_{\mathcal{N}_O}$ where $\mathcal{N}_O \subset \mathcal{N}$ is some subset of variables referred to as the **observable nodes \mathcal{N}_O** . In such cases, $P_{\mathcal{N}_O}$ does not induce a unique set of causal parameters for \mathcal{G} and **Eq. (1)** can not be verified by direction calculation. Instead, compatibility between \mathcal{G} and $P_{\mathcal{N}_O}$ depends on the existence or non-existence of a set of causal parameters for \mathcal{G} such that $P_{\mathcal{N}_O} = \prod_{n \in \mathcal{N}_O} P_{n|\text{Pa}_{\mathcal{G}}(n)}$. The complementary, unobservable nodes are termed **latent nodes $\mathcal{N}_L = \mathcal{N} \setminus \mathcal{N}_O$** , and should be understood as hidden random variables that are unknowable either by some fundamental process or cannot be measured due to other limitations.

There are several approaches to tackling the compatibility problem when dealing with latent variables; there are two common approaches worth mentioning here. The first is to recognize that many equality constraints are implied by the causal structure, including conditional independence relations and so-called Verma constraints among others; see Refs. [23, 24] for thorough treatments. The failure to satisfy an equality constraint immediately disqualifies $P_{\mathcal{N}_O}$ from being compatible with \mathcal{G} . Equality constraints are easily derived given a causal structure, and checking equality-constraints satisfaction is the minimalistic algorithm which powers the overwhelming majority of practical causal inference hypothesis testing in the fields of machine learning and artificial intelligence. In quantum theory, however, we require strong, more sensitive, causal inference techniques. This is because the equality constraints satisfied by classical correlations (compatible with some classical causal structure) are also all satisfied by quantum correlations (generated by the quantumified causal structure in which latent variables are replaced with shared entanglement) [19]. Our focus, therefore, is on deriving *inequality* constraints (over $P_{\mathcal{N}_O}$ and its marginals) implied by a causal structure, which we term **causal compatibility inequalities**¹. For some causal structures, the equality constraints

¹ We refer to these inequalities as “causal compatibility inequalities” instead of Bell inequalities for two reasons. Firstly, “Bell inequalities”

associated with it are sufficient to perfectly characterize the distributions genuinely compatible with it; for others, however, inequality constraints are also important. Causal structures for which inequality constraints are relevant have been termed *interesting* [19], and such structures include the Instrumental structure, the Bell scenario, and the Triangle scenario studied here, among infinitely many others. Herein we use the Inflation Technique [21] to find causal compatibility inequalities; Appendix A discusses the Inflation Technique as applied in this work.

probabilistic inequalities defined over $P_{\mathcal{N}_O}$ and its marginals. We term these constraints **causal compatibility inequalities**². Appendix A provides a brief discussion of existing techniques used to derive these inequalities.

If a probability distribution $P_{\mathcal{N}_O}$ happens to *violate* any causal compatibility inequality, then that distribution is deemed *incompatible*. Conversely, however, a singular inequality can only be used to prove that a given distribution is *incompatible*; a single inequality cannot certify compatibility. A *complete characterization* of compatibility consists of a complete set of all valid causal compatibility inequalities such that satisfaction of the entire set certifies compatibility. Currently however, it is unknown how to obtain a complete characterization for any given causal structure, including the Triangle structure.

From the perspective of identifying quantum non-classicality, a causal structure \mathcal{G} adopts the role of a classical hypothesis. Therefore, non-classicality becomes synonymous with incompatibility: if a distribution $P_{\mathcal{N}_O}$ is incompatible, then it is non-classical. Henceforth, we will use these two terms interchangeably. From a resource standpoint, if the non-classicality of $P_{\mathcal{N}_O}$ with \mathcal{G} can be witnessed by an inequality I , but nevertheless $P_{\mathcal{N}_O}$ can be implemented using quantum states and measurements while otherwise respecting the causal relations of \mathcal{G} , then the causal compatibility inequality I represents a task or game where quantum resources out-perform classical resources relative to \mathcal{G} .

III. BELL STRUCTURE

This section aims to define the Bell causal structure and to review some of the traditional witnesses used to assess the classicality (or lack thereof) of distributions relative to it. The purpose of this section is to equip readers with the pertinent background and also to draw comparisons between the advancements made toward understanding non-classicality of the Bell structure versus analogous results obtained for Triangle structure later in this work.

The bipartite Bell structure (Fig. 1) refers to an iconic causal structure involving two distant parties who observe the outcomes of local measurements as random variables $A(B)$ determined by their individual measurement settings $S_A(S_B)$, and where the parties are also presumed to be commonly informed by some shared latent resource λ [6]. The resulting observable correlations hence naturally form a conditional probability distribution $P_{AB|S_AS_B}$. Subject to the notions of compatibility presented in Section II, this distribution is compatible with Fig. 1 if and only if there exists some functional specification of λ and causal parameters $P_{A|S_A,\lambda}$, $P_{B|S_B,\lambda}$ such that

$$P \text{ is classically compatible with Fig. 1} \quad \text{iff} \quad P_{AB|S_AS_B} = \sum_{\lambda} P_{A|S_A,\lambda} P_{B|S_B,\lambda}. \quad (2)$$

Any observed distribution $P_{AB|S_AS_B}$ which fails to be explained by the classical causal hypothesis of the Bell structure as defined by Eq. (2) is appropriately termed non-classical. Often, distributions incompatible with the Bell structure are referred to as non-local because the Bell structure markedly lacks causal influence from one party to another. In particular, Bell [5] demonstrated that there exists distributions that are non-classical yet are attainable via local measurements on a shared quantum resource.

Let us now contrast now the classical definition of compatibility with its quantum generalization. Quantum correlation arise from a quantumified version of the causal structure, in which latent variables are replaced with quantum systems. Concretely, a distribution is accessible with quantum resources and measurements for the quantumified Bell structure if and only if there exists a bipartite quantum state ρ_{AB} of arbitrary dimension, as well as Hilbert-space-localized measurement operators $M_{A|S_A}, M_{B|S_B}$ with indices conditional upon the measurement settings, such that

$$P \text{ is a quantum distribution of Fig. 1} \quad \text{iff} \quad P_{AB|S_AS_B} = \text{Tr}[\rho_{AB} M_{A|S_A} \otimes M_{B|S_B}]. \quad (3)$$

The non-classicality of quantum distributions for the Bell structure can be demonstrated through the use of Bell inequalities (such as the CHSH inequality Ineq. (4) reproduced below) which constrain the correlations between binary variables A and B for classical distributions [25].

$$\langle AB|S_A = 0, S_B = 0 \rangle + \langle AB|S_A = 0, S_B = 1 \rangle + \langle AB|S_A = 1, S_B = 0 \rangle - \langle AB|S_A = 1, S_B = 1 \rangle \leq 2 \quad (4)$$

usually are associated specifically with the Bell structure. Secondly, the inequalities derived in this work are fundamentally distinct from a typical Bell inequality in that these inequalities are *polynomial* over $P_{\mathcal{N}_O}$ instead of *linear*.

² We refer to these inequalities as “causal compatibility inequalities” instead of Bell inequalities for two reasons. Firstly, “Bell inequalities” usually are associated specifically with the Bell structure. Secondly, the inequalities derived in this work are fundamentally distinct from a typical Bell inequality in that these inequalities are *polynomial* over $P_{\mathcal{N}_O}$ instead of *linear*.



FIG. 1: The Bell structure consisting of two observers A, B together with measurement settings S_A and S_B respectively. The shared hidden variable is labeled λ .



FIG. 2: The casual structure of the Triangle structure. Three variables A, B, C are observable, while X, Y, Z are latent variables.

Non-trivial Bell inequalities such as the CHSH inequality are capable of witnessing the non-classical nature of distributions; the inequalities presented in Section VII are also of this type. Equality constraints, as previously mentioned, never have that sort of high-resolution discernment sensitivity.

IV. TRIANGLE STRUCTURE

As was mentioned in Sections I and II, the **Triangle structure** (Fig. 2) is a causal structure \mathcal{G} consisting of 3 observable variables A, B, C arranged in a triangular configuration while pair-wise sharing latent variables X, Y, Z . Following the definition of causal compatibility from Section II, a distribution $P_{\mathcal{N}_O} = P_{ABC}$ is compatible with the Triangle structure if and only if there exists a choice of causal parameters $\{P_{A|X,Y}, P_{B|Y,Z}, P_{C|Z,X}, P_X, P_Y, P_Z\}$ such that P_{ABC} is a marginalization of P_{ABCXYZ} over X, Y, Z ³, i.e.

$$P \text{ is classically compatible with Fig. 2 iff } P_{ABC} = \sum_{X,Y,Z} P_{A|X,Y} P_{B|Y,Z} P_{C|Z,X} P_X P_Y P_Z. \quad (5)$$

By contrast, the quantum distribution of the Triangle structure are defined with substantially greater freedom, namely

$$P \text{ is a quantum distribution of Fig. 1 iff } P_{ABC}(abc) = \text{Tr}[\Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}], \quad (6)$$

where $\rho_{AB}, \rho_{BC}, \rho_{CA}$ are bipartite density matrices, M_A, M_B, M_C are generic measurements sets, and Π is a permutation matrix to align the states and measurements appropriately. The Triangle structure serves as an excellent test case for furthering our understanding of quantum nonclassicality in network causal structures. It maintains superficial simplicity (only three observable variables) while introducing many challenging features not found in the study of the Bell structure. For example, the spaces of both classical and quantum distributions on the Triangle structure are non-convex Fritz [14], Wolfe *et al.* [21], unlike for the Bell structure. The convexity of the Bell structure's distributions is arguably responsible for the wealth of knowledge about it, including its complete characterization of classicality [6]. Importantly, Fritz [14] explicitly demonstrated existence of (at least) one incompatible-but-quantum distribution for the Triangle structure, so its known to possess quantum nonclassicality. It seems reasonable to assume that quantum non-classicality in the Triangle structure should be translatable into a computational advantage for certain computational circuits [26]; novel instances of nonclassicality are expected to correspond to novel information-theoretical quantum advantages. A fundamental limitation of Fritz's proof of non-classicality, however, is that it does not involve causal compatibility inequalities, and hence does not advance our repertoire of inequality constraints for the Triangle structure. Some inequality constraints for the Triangle structure have been derived in previous works. For example, Steudel and Ay [16] derived an inequality distinguishing the distributions compatible with the Triangle structure from those compatible with structures in which all the observable variables share a common latent ancestor. Henson *et al.* [19] derived a family of entropic inequalities for the Triangle structure, which was the expanded somewhat by Weilenmann and Colbeck [20]. Recently, Wolfe *et al.* [21] derived a variety of new polynomial causal compatibility inequalities for the Triangle structure. The inequalities of Ref. [21] are especially sensitive, in that they expose a

³ Here the summation \sum is used to denote a statistical marginalization over the latent variables X, Y, Z with unspecified support.



FIG. 3: The Triangle structure re-imagined to mimic the Bell structure. The measurement settings S_A, S_B are latent nodes unlike the Bell structure (Fig. 1).

distribution which had previously evaded detection as incompatible with the Triangle scenario as assessed by all formerly known constraints. Remarkably, *none* of the existing causal compatibility inequalities for the Triangle structure were known to admit violation by quantum distribution. That is to say, it was unknown if any inequality known for the Triangle structure might be useful for distinguishing quantum distributions from their classical counterparts. In Section VI we report on attempts to utilize these inequalities to search for incompatible distributions that are also quantum-accessible. The subsequent failure of these approaches effectively motivates the remainder of this work. In summary, the Triangle structure is a desirable case study because it is known to admit nonclassical distributions using quantum resources, but yet no inequality heretofore could separate its classical distributions from its quantum distributions. This failure represents a gap in our understanding of quantum non-classicality and prompts the discovery of *new* inequalities.

V. FRITZ DISTRIBUTION

As first realized by Fritz [14], one may construct quantum distributions incompatible with the Triangle structure by recasting quantum distributions incompatible with the familiar Bell structure into a settings-free tripartite format. To explain, imagine rearranging the Triangle into the configuration depicted in Fig. 3 so that it more closely resembles the Bell structure of Fig. 1. Evidently, under the correct relabeling, large portions of the Triangle structure resemble the Bell structure. The crucial distinction is that S_A, S_B are random variable representing the recorded measurement *settings* in the Bell structure whereas those S_A, S_B are *latent* variables in the Triangle structure, which get reported as *auxiliary outcomes* for Alice and Bob.

The analysis of nonclassicality changes, however, when S_A, S_B are not freely chosen by the observers but rather by a process outside of the individual party's control. Relaxing the assumption of measurement independence opens up a possible loophole, namely the possibility that the auxiliary outcomes S_A, S_B of Alice and Bob might be manipulated via dependence on their shared latent variable λ . We close this loophole by having the third party in the Triangle structure, Charlie, *also* report the latent variables S_A, S_B as a multivariate outcome. In this manner, the perfect correlation of C 's record of S_A, S_B with the records of S_A reported by A and of S_B reported by B testifies to the independence of S_A, S_B from λ . Consequently, any distribution over A, B, S_A, S_B that is incompatible with Bell structure is also incompatible with the Triangle structure provided that C is *perfectly correlated* with S_A, S_B [14].

The exemplifying quantum distribution corresponding to a recasting of a nonclassical Bell structure distribution into the Triangle structure is the **Fritz distribution** [14]. In the Fritz distribution, denoted P_F , each of the variables A, B, C is taken to have 4 possible outcomes $\{0, 1, 2, 3\}$. It is convenient to represent the outcomes using two binary digits, $\{00, 01, 10, 11\}$. Explicitly, P_F can be written as:

$$\begin{aligned} P_F(000) = P_F(110) = P_F(021) = P_F(131) = P_F(202) = P_F(312) = P_F(233) = P_F(323) &= \frac{1}{32}(2 + \sqrt{2}) \\ P_F(010) = P_F(100) = P_F(031) = P_F(121) = P_F(212) = P_F(302) = P_F(223) = P_F(333) &= \frac{1}{32}(2 - \sqrt{2}) \end{aligned} \quad (7)$$

Here the notation $P_F(abc) = P_{ABC}(abc) = P(A \mapsto a, B \mapsto b, C \mapsto c)$ is used as shorthand. The Fritz distribution is quantum-accessible in the sense that P_F can be implemented using a set of quantum states $\rho_{AB}, \rho_{BC}, \rho_{CA}$ and measurements M_A, M_B, M_C realized on Fig. 2 using Eq. (6). The Fritz distribution may be visualized as a $4 \times 4 \times 4$

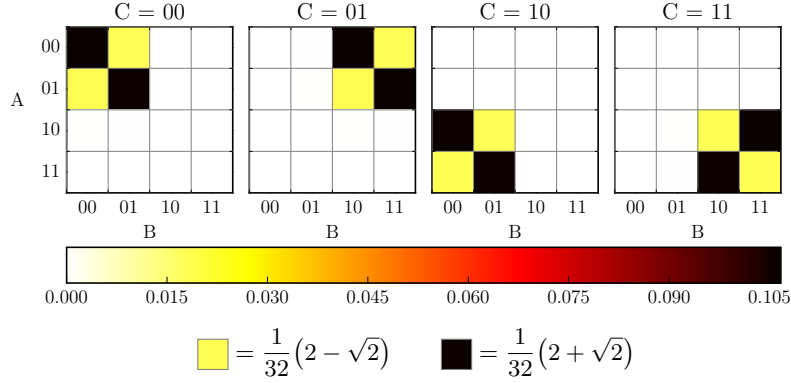


FIG. 4: The Fritz distribution visualized using a $4 \times 4 \times 4$ grid. The 4 outcomes of A, B, C are written in binary as a doublet of bits to illustrate that certain bits act as measurement pseudo-settings.

grid of possible outcomes as depicted in Fig. 4. From this diagram, it can be seen that each of C 's outcomes restricts the possible outcomes for A, B into a 2×2 block. When expressing the outcomes as pairs of binary digits, it can be seen that the left-hand bits for A and B (respectively denoted A_l, B_l) are fixed by (i.e. perfectly correlated with) the outcome of C . Pursuant to the embedding of Fig. 3 the left-hand bits A_l and B_l emulate the measurement settings S_A and S_B , whereas the right-hand bits A_r and B_r emulate the outcomes which would be obtained by A and B back in the Bell structure of Fig. 1. Critically, C 's bits are perfectly correlated with the left-bits of A, B ; $C_l = A_l$ and $C_r = B_l$. Consequently, it is possible to define the correlation $\langle A_r B_r \rangle$ between the right-hand bits of A and B :

$$\langle A_r B_r \rangle = P_{A_r B_r}(00) + P_{A_r B_r}(11) - P_{A_r B_r}(01) - P_{A_r B_r}(10)$$

Provided that C is perfectly correlated with A_l and B_l , any Bell inequality for the Bell structure defined over A, B, S_A, S_B can be directly converted to an inequality for the Triangle structure by performing the surjective relabeling:

$$A_r \leftarrow A, \quad B_r \leftarrow B, \quad A_l \leftarrow S_A, \quad B_l \leftarrow S_B, \quad C_l \leftarrow S_A, \quad C_r \leftarrow S_B.$$

As an example, the famous CHSH inequality [25] per Ineq. (4) is translated into a constraint on the correlation between the right-bits of A and B ,

$$\langle A_r B_r | C = 00 \rangle + \langle A_r B_r | C = 01 \rangle + \langle A_r B_r | C = 10 \rangle - \langle A_r B_r | C = 11 \rangle \leq 2 \quad (8)$$

Every compatible distribution P_{ABC} for which C is perfectly correlated with A_l, B_l must satisfy Ineq. (8). Substituting P_F into Ineq. (8) yields the traditional maximal quantum violation [27]: $3(1/\sqrt{2}) - (-1/\sqrt{2}) = 2\sqrt{2} \not\leq 2$.

It is important to understand the domain in which Fritz's proof of incompatibility is valid. The proof relies on the perfect correlation between C 's outcomes and the measurement pseudo-settings (left-bits) of A and B . For example, if one combines Eq. (7) with slight uniform noise, what can we say with confidence regarding if the resulting modified distribution is classical or not? At what point does the resultant distribution transition from incompatibility to compatibility? This question is partially answered in Section VII-B.

Plainly P_F is a valid but *manufactured* example. The phenomenology associated with Bell non-locality or Bell incompatibility are well understood; examining these distributions embedded in the Triangle structure offers no additional perspective onto the types of nonclassical resources made accessible by quantum mechanics. The goal, therefore, is to find incompatible quantum distributions that are qualitatively different than those previously considered for the Bell structure [28]. Recognizing this, Fritz [14] presented the following problem [14, Problem 2.17]:

Fritz's Problem: Find an example of non-classical quantum correlations in the Triangle structure together with a proof of its non-classicality which does not hinge on Bell's theorem.

Fritz's problem is concerned with how to find and recognize nonclassical quantum distributions specifically for the Triangle structure. The original proof of nonclassicality essentially recycled Bell's theorem, and was limited by the requirement of perfect correlations[14]. Fritz's problem, as originally stated, does not require that the *type* of non-classicality be novel to the Triangle structure; rather, only that the proof should avoid Bell's theorem. Section VI delineates our initial, failed attempts at resolving Fritz's problem; Section VII reports our eventual success, via the discovery of new causal compatibility inequalities.

Though not explicit, we read in the spirit of Fritz’s problem a desideratum for the discovery of a truly novel form of nonclassicality for the Triangle structure. Such a discovery would presumably lead to an understanding of new advantages of quantum resources in network structures; this related problem has attracted attention and conjecture [28], but remains open. We attempt to make progress on this problem by leveraging the new causal compatibility inequalities derived herein, but this effort was plagued by instabilities in our numerical optimization which we have not yet overcome. [Section IX](#) discusses these challenges, suggesting concrete targets for future research. [Section IX](#) also considers how to best contrast non-classicality across differing causal structures.

VI. PRELIMINARY RESEARCH

As a preliminary search for quantum incompatibility in the Triangle structure, we performed numerical optimizations using bipartite qubit density matrices and 2-outcome POVM measurements against the previously-published compatibility inequalities of Ref. [21], as well as against the entropic inequalities of Henson *et al.* [19] (which had also been investigated numerically independently by Weilenmann and Colbeck [20]). This preliminary investigation did not yield interesting solutions, as none of those inequalities were violated by any 2-outcome coarse-graining of the Fritz distribution. These early results suggest that there is no quantum-classical gap in the Triangle structure for *two-outcome* measurements, i.e. [Eq. \(5\)](#) and [Eq. \(6\)](#) define identical sets of binary-outcome tripartite distributions.

Operating under the assumptions that all quantum *non*-classical distributions in the Triangle structure must utilize more than two outcomes in at least one variable, this work expands the gamut of inequalities to optimize against to include inequalities referencing strictly more than two outcomes.

VII. TRIANGLE STRUCTURE INEQUALITIES

[Section II](#) introduced the notion of causal compatibility inequalities, and [Section V](#) discussed the Fritz distribution (P_F) together with the initial inequality-free proof of its incompatibility with the Triangle structure. Heretofore, there were no known causal compatibility inequalities for the Triangle structure that were capable of witnessing the non-classicality of any quantum distributions [21]. By leveraging the incompatibility of the Fritz distribution [14] and tools provided by the Inflation Technique [21], we have obtained numerous causal compatibility inequalities for the Triangle structure that are violated by the Fritz distribution. A representative trio of these P_F -incompatibility-witnessing inequalities are presented here: $I_{\text{WagonWheel}}$ per [Ineq. \(10\)](#), notable for its simplicity; I_{Web} per [Ineq. \(11\)](#), which best witnesses the non-classicality of P_F in the presence of noise; and $I_{\text{SymmetricWeb}}$ per [Ineq. \(13\)](#), which is symmetric with respect to all permutations of the three parties.

The methods used to derive these inequalities are largely based on the Inflation Technique [21]. A summary of the Inflation Technique and other methods used will be deferred to [Appendix A](#) where unfamiliar readers will find a succinct yet sufficient presentation of the requisites. We emphasize that each of these causal compatibility inequalities independently provides a positive resolution of Fritz’s problem. The inequalities are derived without making use of Bell’s theorem; rather, they follow from the Inflation Technique’s broader perspective on non-classicality as a special case of causal inference.

A. The WagonWheel Inequality

The first causal compatibility inequality chosen for presentation is reported below:

$$\begin{aligned}
& I_{\text{WagonWheel}} : \\
& +P_{ABC}(000)P_C(3) - 2P_{ABC}(203)P_C(0) - 2P_{ABC}(303)P_C(0) - P_{ABC}(003)P_C(0) \\
& - P_{ABC}(013)P_C(0) - P_{ABC}(020)P_C(2) - P_{ABC}(022)P_C(2) - P_{ABC}(023)P_C(0) \\
& - P_{ABC}(023)P_C(2) - P_{ABC}(030)P_C(2) - P_{ABC}(031)P_C(2) - P_{ABC}(032)P_C(2) \\
& - P_{ABC}(033)P_C(0) - P_{ABC}(033)P_C(2) - P_{ABC}(103)P_C(0) - P_{ABC}(113)P_C(0) \\
& - P_{ABC}(123)P_C(0) - P_{ABC}(133)P_C(0) - P_{ABC}(200)P_C(0) - P_{ABC}(200)P_C(1) \\
& - P_{ABC}(200)P_C(2) - P_{ABC}(200)P_C(3) - P_{ABC}(201)P_C(0) - P_{ABC}(201)P_C(1) \\
& - P_{ABC}(201)P_C(2) - P_{ABC}(201)P_C(3) - P_{ABC}(203)P_C(1) - P_{ABC}(203)P_C(2) \\
& - P_{ABC}(203)P_C(3) - P_{ABC}(213)P_C(0) - P_{ABC}(223)P_C(0) - P_{ABC}(300)P_C(0) \\
& - P_{ABC}(300)P_C(1) - P_{ABC}(300)P_C(2) - P_{ABC}(300)P_C(3) - P_{ABC}(301)P_C(0) \\
& - P_{ABC}(301)P_C(1) - P_{ABC}(301)P_C(2) - P_{ABC}(301)P_C(3) - P_{ABC}(302)P_C(1) \\
& - P_{ABC}(303)P_C(1) - P_{ABC}(303)P_C(2) - P_{ABC}(303)P_C(3) - P_{ABC}(313)P_C(0) \leq 0
\end{aligned} \tag{9}$$

Ineq. (9) is termed the Wagon Wheel inequality and denoted $I_{\text{WagonWheel}}$ after the (identically-named) inflated structure of **Fig. 8b** used to derive it. To reiterate, a summary of the methods used to derive **Ineq. (9)** can be found in **Appendices A-1 to A-3**. Aside from increasing outcome cardinality, we are also forced to consider larger inflations than those analyzed in Ref. [21]. This is because we found the smaller inflations considered there — such as the Spiral inflation depicted in **Fig. 8a** — were simply *unable* to witness the incompatibility of the Fritz distribution, even when analyzed explicitly using four possible outcomes for every observable variable.

By construction, every distribution P_{ABC} that is compatible with the Triangle structure per **Eq. (5)** must satisfy $I_{\text{WagonWheel}}$. On the other hand, the Fritz distribution violates $I_{\text{WagonWheel}}$ with violation $0.0129 \not\leq 0^4$. Consequently we affirm that the Fritz distribution is incompatible with the Triangle structure, a fact previously only demonstrated without inequalities.

It is possible to simplify $I_{\text{WagonWheel}}$ by identifying sets of terms such as $P_{ABC}(300)P_C(0) + P_{ABC}(300)P_C(1) + P_{ABC}(300)P_C(2) + P_{ABC}(300)P_C(3)$ which can be simplified to $P_{ABC}(300)$ by recognizing that $P_C(0) + P_C(1) + P_C(2) + P_C(3) = 1$. Upon doing so, one obtains an equivalent representation for $I_{\text{WagonWheel}}$,

$$\begin{aligned}
& I_{\text{WagonWheel}} : \\
& +P_{ABC}(000)P_C(3) + P_{ABC}(021)P_C(2) + P_{ABC}(202) + P_{ABC}(302) + P_{ABC}(233)P_C(0) + P_{AC}(33)P_C(0) \\
& - P_{ABC}(303)P_C(0) - P_{ABC}(313)P_C(0) - P_{ABC}(302)P_C(1) - P_{AB}(02)P_C(2) - \\
& - P_{AB}(03)P_C(2) - P_{AB}(20) - P_{AB}(30) - P_C(3)P_C(0) \leq 0
\end{aligned} \tag{10}$$

B. The Web Inequality

We next present **Ineq. (11)**, another causal compatibility inequality for the Triangle structure that is violated by the Fritz distribution. The Web inflation shown in **Fig. 8c** was used to produce the eponymous I_{Web} per **Ineq. (11)**. The Web inflation being considerably larger than the Wagon-Wheel inflation of **Fig. 8b**, it is computationally more demanding to work with, but the Web inflation is capable of yielding strictly stronger inequalities. For brevity, we employ the shorthand $P(abc)$ in lieu of $P_{ABC}(abc)$ in presenting **Ineq. (11)**.

⁴ To aid verification: the first term in **Ineq. (9)** for the Fritz distribution is $P_{ABC}(000)P_C(3) = \frac{1}{128}(2 + \sqrt{2}) \simeq 0.0267$ while all of the remaining terms sum to $\simeq 0.0136$.

$$\begin{aligned}
& I_{\text{Web}} : \tag{11} \\
& +P(000)P(202) + P(000)P(212) + P(202)P(233) + P(302)P(312) \\
& -2P(123)P(210) - 2P(123)P(310) - 2P(130)P(213) - 2P(133)P(210) - 2P(133)P(310) - P(000)P(003) - P(000)P(013) - P(000)P(023) \\
& -P(000)P(033) - P(000)P(103) - P(000)P(113) - P(000)P(123) - P(000)P(133) - P(000)P(203) - P(000)P(213) - P(003)P(010) \\
& -P(003)P(020) - P(003)P(030) - P(003)P(100) - P(003)P(110) - P(003)P(120) - P(003)P(130) - P(003)P(200) - P(003)P(210) \\
& -P(003)P(220) - P(003)P(230) - P(003)P(300) - P(003)P(310) - P(003)P(320) - P(003)P(330) - P(010)P(013) - P(010)P(021) \\
& -P(010)P(023) - P(010)P(033) - P(010)P(100) - P(010)P(103) - P(010)P(113) - P(010)P(123) - P(010)P(133) - P(010)P(203) \\
& -P(010)P(213) - P(013)P(020) - P(013)P(030) - P(013)P(100) - P(013)P(110) - P(013)P(120) - P(013)P(130) - P(013)P(200) \\
& -P(013)P(210) - P(013)P(220) - P(013)P(230) - P(013)P(300) - P(013)P(310) - P(013)P(320) - P(013)P(330) - P(020)P(023) \\
& -P(020)P(033) - P(020)P(103) - P(020)P(113) - P(020)P(123) - P(020)P(133) - P(020)P(203) - P(020)P(210) - P(020)P(211) \\
& -P(020)P(212) - P(020)P(223) - P(020)P(233) - P(020)P(310) - P(020)P(311) - P(020)P(312) - P(020)P(313) - P(021)P(100) \\
& -P(021)P(121) - P(021)P(210) - P(021)P(211) - P(021)P(213) - P(021)P(310) - P(021)P(311) - P(021)P(313) - P(022)P(210) \\
& -P(022)P(211) - P(022)P(212) - P(022)P(213) - P(022)P(310) - P(022)P(311) - P(022)P(312) - P(022)P(313) - P(023)P(030) \\
& -P(023)P(100) - P(023)P(110) - P(023)P(120) - P(023)P(130) - P(023)P(200) - P(023)P(211) - P(023)P(212) - P(023)P(213) \\
& -P(023)P(220) - P(023)P(230) - P(023)P(300) - P(023)P(311) - P(023)P(312) - P(023)P(313) - P(023)P(320) - P(023)P(330) \\
& -P(030)P(033) - P(030)P(103) - P(030)P(113) - P(030)P(123) - P(030)P(133) - P(030)P(203) - P(030)P(210) - P(030)P(211) \\
& -P(030)P(212) - P(030)P(223) - P(030)P(233) - P(030)P(310) - P(030)P(311) - P(030)P(312) - P(030)P(313) - P(031)P(210) \\
& -P(031)P(211) - P(031)P(213) - P(031)P(223) - P(031)P(310) - P(031)P(311) - P(031)P(313) - P(032)P(210) - P(032)P(211) \\
& -P(032)P(212) - P(032)P(213) - P(032)P(310) - P(032)P(311) - P(032)P(312) - P(032)P(313) - P(033)P(100) - P(033)P(110) \\
& -P(033)P(120) - P(033)P(130) - P(033)P(200) - P(033)P(211) - P(033)P(212) - P(033)P(213) - P(033)P(220) - P(033)P(230) \\
& -P(033)P(300) - P(033)P(311) - P(033)P(312) - P(033)P(313) - P(033)P(320) - P(033)P(330) - P(100)P(103) - P(100)P(113) \\
& -P(100)P(123) - P(100)P(133) - P(100)P(203) - P(100)P(213) - P(103)P(110) - P(103)P(120) - P(103)P(130) - P(103)P(200) \\
& -P(103)P(210) - P(103)P(220) - P(103)P(230) - P(103)P(300) - P(103)P(310) - P(103)P(320) - P(103)P(330) - P(110)P(113) \\
& -P(110)P(123) - P(110)P(133) - P(110)P(203) - P(110)P(213) - P(113)P(120) - P(113)P(130) - P(113)P(200) - P(113)P(210) \\
& -P(113)P(220) - P(113)P(230) - P(113)P(300) - P(113)P(310) - P(113)P(320) - P(113)P(330) - P(120)P(123) - P(120)P(133) \\
& -P(120)P(203) - P(120)P(210) - P(120)P(211) - P(120)P(212) - P(120)P(223) - P(120)P(233) - P(120)P(310) - P(120)P(311) \\
& -P(120)P(312) - P(120)P(313) - P(121)P(131) - P(121)P(210) - P(121)P(211) - P(121)P(213) - P(121)P(310) - P(121)P(311) \\
& -P(121)P(312) - P(121)P(313) - P(122)P(210) - P(122)P(211) - P(122)P(212) - P(122)P(213) - P(122)P(310) - P(122)P(311) \\
& -P(122)P(312) - P(122)P(313) - P(123)P(130) - P(123)P(200) - P(123)P(211) - P(123)P(212) - P(123)P(213) - P(123)P(220) \\
& -P(123)P(230) - P(123)P(300) - P(123)P(311) - P(123)P(312) - P(123)P(313) - P(123)P(320) - P(123)P(330) - P(130)P(133) \\
& -P(130)P(203) - P(130)P(210) - P(130)P(211) - P(130)P(212) - P(130)P(223) - P(130)P(233) - P(130)P(310) - P(130)P(311) \\
& -P(130)P(312) - P(130)P(313) - P(131)P(210) - P(131)P(211) - P(131)P(213) - P(131)P(310) - P(131)P(311) - P(131)P(313) \\
& -P(132)P(210) - P(132)P(211) - P(132)P(212) - P(132)P(213) - P(132)P(310) - P(132)P(311) - P(132)P(312) - P(132)P(313) \\
& -P(133)P(200) - P(133)P(211) - P(133)P(212) - P(133)P(213) - P(133)P(220) - P(133)P(230) - P(133)P(300) - P(133)P(311) \\
& -P(133)P(312) - P(133)P(313) - P(133)P(320) - P(133)P(330) - P(200)P(203) - P(200)P(213) - P(200)P(223) - P(200)P(233) \\
& -P(200)P(303) - P(200)P(313) - P(200)P(323) - P(200)P(333) - P(203)P(210) - P(203)P(220) - P(203)P(230) - P(203)P(300) \\
& -P(203)P(310) - P(203)P(320) - P(203)P(330) - P(210)P(213) - P(210)P(223) - P(210)P(233) - P(210)P(303) - P(210)P(313) \\
& -P(210)P(323) - P(210)P(333) - P(212)P(212) - P(212)P(233) - P(213)P(220) - P(213)P(230) - P(213)P(300) - P(213)P(310) \\
& -P(213)P(320) - P(213)P(330) - P(220)P(223) - P(220)P(233) - P(220)P(303) - P(220)P(313) - P(220)P(323) - P(220)P(333) \\
& -P(223)P(230) - P(223)P(300) - P(223)P(310) - P(223)P(320) - P(223)P(330) - P(230)P(233) - P(230)P(303) - P(230)P(313) \\
& -P(230)P(323) - P(230)P(333) - P(233)P(300) - P(233)P(310) - P(233)P(320) - P(233)P(330) - P(233)P(333) - P(300)P(303) \\
& -P(300)P(313) - P(300)P(323) - P(300)P(333) - P(303)P(310) - P(303)P(320) - P(303)P(330) - P(310)P(313) - P(310)P(323) \\
& -P(310)P(333) - P(313)P(320) - P(313)P(330) - P(320)P(323) - P(320)P(333) - P(323)P(330) - P(330)P(333) \leq 0
\end{aligned}$$

As mentioned in [Section V](#), this original proof of the non-classicality of P_F required an idealistic condition to hold: perfect correlations of C with A_l, B_l . It is impossible to use Fritz's original argument to confirm non-classicality from an experimental point of view, because every laboratory-achievable distribution is subject to some amount of noise. One can minimize noise, e.g. by developing high-accuracy measurement channels, but *perfect* correlations are unattainable. Causal compatibility inequalities such as I_{Web} permit there to be noise within a set of observations before the ability to certify non-classicality breaks down.

Here we quantify how much statistical noise can be added to the Fritz distribution P_F before I_{Web} fails to witness

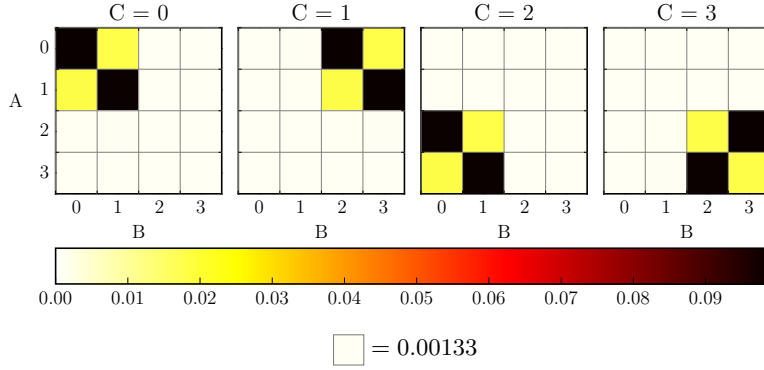


FIG. 5: $\mathcal{N}_{0.085}$: A noisy yet still non-classical variant of the Fritz distribution.

incompatibility; we define the ε -noisy Fritz distribution as

$$\begin{aligned} \mathcal{N}_\varepsilon &= (1 - \varepsilon)P_F + \varepsilon\mathcal{U}_{ABC} \\ \text{where} \\ \mathcal{U}_{ABC}(abc) &\equiv \frac{1}{64} \quad \forall a, b, c \in \{0, 1, 2, 3\}. \end{aligned} \tag{12}$$

As ε varies from 0 to 1, more and more noise is added to the Fritz distribution and \mathcal{N}_ε transitions from an incompatible distribution $\mathcal{N}_0 = P_F$ to a compatible distribution $\mathcal{N}_1 = \mathcal{U}_{ABC}$. We find that I_{Web} demonstrates that the Fritz distribution remains incompatible with the Triangle structure up to a noise parameter of $\varepsilon \simeq 0.085$; the associated distribution $\mathcal{N}_{0.085}$ is plotted in Fig. 5. Of course, there remains the possibility that another inequality will be able to withstand a larger degree of noise than I_{Web} ; an exhaustive search has not been conducted.

C. The Symmetric Web Inequality

In Section VIII we will consider numerically optimizing our inequalities over quantum strategies — seeking violations — towards the desired end-goal of discovering new forms of non-classicality in the Triangle structure. Of course, if some inequality achieves its optimal quantum violation on a distribution qualitatively similar to P_F , then such an inequality is unlikely to lead us to discover non-classicality of a type different than P_F . Indeed, we will show that the previous pair of $I_{\text{WagonWheel}}$ and I_{Web} , which were both specifically curated to demonstrate the incompatibility of P_F , apparently achieve maximum quantum violation on distribution very similar to P_F itself. To avoid the chasing our own tail in the numerical optimization we require an objective function which is *not* tailor-designed to witness P_F . Finding such an inequality proved extremely challenging: While we could generate thousands of inequalities using the Inflation Technique [21], it appeared that only a vanishingly-small fraction of those inequalities admitted quantum violation whatsoever. We therefore modified the Inflation Technique to give us an inequality which both witnesses the non-classicality of P_F but which is also symmetric with respect to any permutation of the variables A, B, C , resulting in Ineq. (13). The modification of the Inflation Techniques which forces symmetry is explained in Appendix A-4. As P_F is strongly assymmetric due to the special role of C , we can hope that a symmetric inequality (even one which does witness P_F) might achieve its *optimal* quantum violation on a distribution qualitative distinct from P_F . The details of our numerical finding are discussed in Section VIII.

$$I_{\text{SymmetricWeb}} : \quad (13)$$

$$\begin{aligned}
& +2[P(312)P(312)]_6 + 3[P(000)P(323)]_3 + 3[P(131)P(323)]_3 + [P(000)P(121)]_3 + [P(110)P(333)]_3 + [P(202)P(202)]_3 \\
& -12[P(002)P(031)]_6 - 12[P(010)P(330)]_6 - 12[P(032)P(210)]_6 - 12[P(101)P(331)]_6 - 12[P(122)P(321)]_6 - 12[P(130)P(312)]_6 \\
& -12[P(202)P(220)]_6 - 16[P(002)P(120)]_6 - 16[P(002)P(200)]_6 - 16[P(002)P(220)]_3 - 16[P(002)P(230)]_6 - 16[P(002)P(302)]_6 \\
& -16[P(002)P(303)]_6 - 16[P(002)P(330)]_3 - 16[P(003)P(032)]_6 - 16[P(003)P(033)]_6 - 16[P(003)P(231)]_6 - 16[P(003)P(300)]_6 \\
& -16[P(003)P(310)]_6 - 16[P(003)P(320)]_6 - 16[P(003)P(322)]_6 - 16[P(012)P(202)]_6 - 16[P(012)P(210)]_6 - 16[P(012)P(313)]_6 \\
& -16[P(013)P(120)]_6 - 16[P(013)P(130)]_6 - 16[P(013)P(203)]_6 - 16[P(013)P(220)]_6 - 16[P(020)P(101)]_3 - 16[P(020)P(103)]_6 \\
& -16[P(020)P(312)]_6 - 16[P(021)P(311)]_6 - 16[P(021)P(332)]_6 - 16[P(022)P(203)]_6 - 16[P(022)P(302)]_6 - 16[P(022)P(303)]_6 \\
& -16[P(022)P(311)]_3 - 16[P(022)P(313)]_6 - 16[P(023)P(103)]_6 - 16[P(023)P(222)]_6 - 16[P(023)P(232)]_6 - 16[P(030)P(201)]_6 \\
& -16[P(031)P(301)]_6 - 16[P(031)P(332)]_6 - 16[P(032)P(311)]_6 - 16[P(033)P(202)]_6 - 16[P(033)P(300)]_3 - 16[P(033)P(312)]_6 \\
& -16[P(033)P(331)]_6 - 16[P(033)P(333)]_3 - 16[P(102)P(130)]_6 - 16[P(102)P(220)]_6 - 16[P(102)P(230)]_6 - 16[P(102)P(310)]_6 \\
& -16[P(103)P(110)]_6 - 16[P(103)P(133)]_6 - 16[P(113)P(130)]_6 - 16[P(113)P(220)]_3 - 16[P(113)P(321)]_6 - 16[P(120)P(201)]_6 \\
& -16[P(120)P(223)]_6 - 16[P(120)P(303)]_6 - 16[P(120)P(313)]_6 - 16[P(123)P(230)]_6 - 16[P(123)P(302)]_6 - 16[P(123)P(330)]_6 \\
& -16[P(130)P(203)]_6 - 16[P(130)P(223)]_6 - 16[P(131)P(303)]_3 - 16[P(132)P(202)]_6 - 16[P(133)P(202)]_6 - 16[P(201)P(330)]_6 \\
& -16[P(203)P(330)]_6 - 16[P(210)P(213)]_6 - 16[P(222)P(303)]_3 - 16[P(223)P(310)]_6 - 16[P(231)P(301)]_6 - 16[P(232)P(330)]_6 \\
& -16[P(233)P(303)]_6 - 16[P(300)P(331)]_6 - 16[P(301)P(330)]_6 - 16[P(302)P(331)]_6 - 16[P(313)P(320)]_6 - 20[P(132)P(301)]_6 \\
& -20[P(133)P(301)]_6 - 24[P(033)P(310)]_6 - 24[P(102)P(231)]_6 - 24[P(103)P(331)]_6 - 24[P(130)P(303)]_6 - 2[P(010)P(031)]_6 \\
& -2[P(100)P(100)]_3 - 2[P(100)P(223)]_6 - 2[P(110)P(223)]_3 - 2[P(121)P(312)]_6 - 2[P(212)P(302)]_6 - 3[P(000)P(333)]_1 \\
& -3[P(031)P(312)]_6 - 3[P(223)P(333)]_3 - 3[P(302)P(312)]_6 - 4[P(000)P(022)]_3 - 4[P(000)P(212)]_3 - 4[P(000)P(301)]_6 \\
& -4[P(001)P(031)]_6 - 4[P(003)P(211)]_6 - 4[P(010)P(132)]_6 - 4[P(010)P(311)]_6 - 4[P(011)P(211)]_3 - 4[P(012)P(021)]_6 \\
& -4[P(022)P(122)]_3 - 4[P(022)P(222)]_3 - 4[P(022)P(232)]_6 - 4[P(031)P(233)]_6 - 4[P(100)P(122)]_3 - 4[P(100)P(230)]_6 \\
& -4[P(101)P(131)]_3 - 4[P(110)P(123)]_6 - 4[P(110)P(320)]_6 - 4[P(111)P(311)]_3 - 4[P(111)P(321)]_6 - 4[P(122)P(122)]_3 \\
& -4[P(122)P(220)]_6 - 4[P(122)P(232)]_6 - 4[P(123)P(131)]_6 - 4[P(132)P(332)]_6 - 4[P(201)P(201)]_6 - 4[P(201)P(311)]_6 \\
& -4[P(202)P(311)]_6 - 4[P(202)P(321)]_6 - 4[P(212)P(310)]_6 - 4[P(212)P(331)]_6 - 4[P(223)P(312)]_6 - 4[P(302)P(333)]_6 \\
& -4[P(311)P(323)]_6 - 5[P(010)P(333)]_3 - 5[P(031)P(100)]_6 - 5[P(223)P(223)]_3 - 5[P(233)P(333)]_3 - 8[P(000)P(030)]_3 \\
& -8[P(000)P(032)]_6 - 8[P(003)P(111)]_3 - 8[P(003)P(331)]_3 - 8[P(012)P(030)]_6 - 8[P(012)P(110)]_6 - 8[P(012)P(221)]_6 \\
& -8[P(013)P(021)]_6 - 8[P(013)P(022)]_6 - 8[P(013)P(311)]_6 - 8[P(020)P(100)]_6 - 8[P(020)P(110)]_6 - 8[P(020)P(112)]_6 \\
& -8[P(020)P(210)]_6 - 8[P(022)P(121)]_6 - 8[P(022)P(130)]_6 - 8[P(023)P(112)]_6 - 8[P(023)P(213)]_6 - 8[P(030)P(210)]_6 \\
& -8[P(030)P(311)]_6 - 8[P(032)P(111)]_6 - 8[P(033)P(121)]_6 - 8[P(033)P(131)]_6 - 8[P(100)P(132)]_6 - 8[P(102)P(321)]_6 \\
& -8[P(110)P(300)]_6 - 8[P(111)P(130)]_6 - 8[P(112)P(123)]_6 - 8[P(112)P(331)]_3 - 8[P(113)P(200)]_6 - 8[P(113)P(331)]_3 \\
& -8[P(120)P(220)]_6 - 8[P(120)P(302)]_6 - 8[P(120)P(310)]_6 - 8[P(121)P(211)]_6 - 8[P(121)P(301)]_6 - 8[P(121)P(303)]_3 \\
& -8[P(121)P(320)]_6 - 8[P(122)P(201)]_6 - 8[P(122)P(203)]_6 - 8[P(123)P(200)]_6 - 8[P(123)P(203)]_6 - 8[P(123)P(222)]_6 \\
& -8[P(123)P(313)]_6 - 8[P(130)P(212)]_6 - 8[P(131)P(310)]_6 - 8[P(132)P(310)]_6 - 8[P(132)P(312)]_6 - 8[P(132)P(322)]_6 \\
& -8[P(133)P(200)]_3 - 8[P(133)P(213)]_6 - 8[P(133)P(222)]_3 - 8[P(133)P(300)]_3 - 8[P(133)P(312)]_6 - 8[P(133)P(322)]_3 \\
& -8[P(201)P(313)]_6 - 8[P(210)P(320)]_6 - 8[P(211)P(221)]_6 - 8[P(211)P(320)]_6 - 8[P(213)P(322)]_6 - 8[P(220)P(223)]_3 \\
& -8[P(221)P(312)]_6 - 8[P(222)P(312)]_6 - 8[P(231)P(312)]_6 - 8[P(232)P(313)]_3 - 8[P(233)P(313)]_6 - 8[P(302)P(332)]_6 \\
& -8[P(311)P(321)]_6 - [P(000)P(121)]_3 - [P(000)P(323)]_3 - [P(110)P(333)]_3 - [P(121)P(333)]_3 - [P(131)P(323)]_3 \\
& -[P(312)P(312)]_6 \leq 0
\end{aligned}$$

Note that $P(abc)$ is shorthand for $P_{ABC}(abc)$ and $[P(113)P(330)]_3$ is shorthand for a sum the over permutations of A, B, C , i.e. $[P(113)P(330)]_3 \equiv P(113)P(330) + P(131)P(303) + P(311)P(033)$.

VIII. NUMERICAL OPTIMIZATION

In order to find quantum distributions that are more non-classical than the Fritz, we performed numerical optimizations against each inequality I by parameterizing the space of quantum-accessible probability distributions that can be realized on the Triangle structure [Fig. 2] and thus expressed as follows:

$$P_{ABC}(abc) = \text{Tr}[\Pi^\Gamma \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}] \quad (14)$$

where $\rho_{AB}, \rho_{BC}, \rho_{CA}$ are bipartite density matrices, M_A, M_B, M_C are generic measurements sets and Π is a permutation matrix that aligns the states and measurements appropriately. In order to parameterize all such distributions, we elect to parameterize the states and measurements separately. In order to qualify the scope of Eq. (14) and associated computational complexity of the parameterization, there are two restrictions that are made with justification.

Motivated by the fact that the Fritz distribution [Section V] only requires qubit states, the states ρ are taken to be bipartite *qubit* states which are more computationally feasible compared to n -dimensional states whereby the joint density matrix $\rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA}$ becomes an $n^6 \times n^6$ matrix. Additionally, we restrict our focus to projective-valued measures (PVMs) instead of projective-operator valued measures (POVMs) for three reasons. First, Section V demonstrates that PVMs are sufficient for generating incompatible quantum distributions in the Triangle structure. Second, although generating k -outcome POVM measurements is possible using rejection sampling techniques [29], a valid, unbiased parameterization was not found for $k > 2$. Finally, PVMs provide considerable computational advantage over POVMs as they permit Eq. (14) to be re-written as Eq. (15).

$$P_{ABC}(abc) = \langle m_{A,a} m_{B,b} m_{C,c} | \Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi | m_{A,a} m_{B,b} m_{C,c} \rangle \quad (15)$$

Although there are numerous techniques that can be used when parameterizing quantum states and measurements [4, 29–33], a single technique by Spengler *et al.* [34] was found to be most computationally suitable for our purposes. Spengler *et al.* [34] demonstrated that all $d \times d$ unitary matrices U can be parameterized without degeneracy as follows:

$$U = \left[\prod_{m=1}^{d-1} \left(\prod_{n=m+1}^d \exp(iP_n \lambda_{n,m}) \exp(i\sigma_{m,n} \lambda_{m,n}) \right) \right] \cdot \left[\prod_{l=1}^d \exp(iP_l \lambda_{l,l}) \right] \quad (16)$$

Where the real valued parameters $\lambda = \{\lambda_{n,m} | n, m \in 1, \dots, d\}$ have periodicities $\lambda_{m,n} \in [0, \frac{\pi}{2}]$ for $m < n$ and $\lambda_{m,n} \in [0, 2\pi]$ for $m \geq n$. Moreover, P_l are one-dimensional projective operators $P_l = |l\rangle\langle l|$ and the $\sigma_{m,n}$ are generalized anti-symmetric σ -matrices $\sigma_{m,n} = -i|m\rangle\langle n| + i|n\rangle\langle m|$ where $1 \leq m < n \leq d$. This parameterization has the useful feature that each of the real-valued parameters $\lambda_{n,m}$ has a direct and intuitive physical affect on each element of a computational basis $\{|1\rangle, \dots, |d\rangle\}$. Explicitly, $\exp(i\sigma_{m,n} \lambda_{m,n})$ applies a rotation to the sub-space spanned by $|m\rangle$ and $|n\rangle$ for $m < n$. Analogously, $\exp(iP_n \lambda_{n,m})$ generates the relative phase between $|m\rangle$ and $|n\rangle$ for $m > n$ and $\exp(iP_l \lambda_{l,l})$ fixes the global phase of $|l\rangle$. Finally, although not explicitly mentioned in [34], it is possible to remove the reliance on the computationally expensive matrix exponential operations in Eq. (16) [35].

By parameterizing unitary matrices, it becomes possible to parameterize d -dimensional density matrices and d -element PVMs by recognizing that any orthonormal basis $\{|\psi_j\rangle\}$ (where $1 \leq j \leq d$) can be transformed into the computational basis $\{|j\rangle\}$ by a unitary transformation U , i.e. $U|\psi_j\rangle = |j\rangle$. First consider a d -element PVM $M = \{|m_j\rangle\langle m_j| | 1 \leq j \leq d\}$. Since $\{|m_j\rangle\}$ forms an orthonormal basis, one can parameterize M by writing $M = \{U^\dagger |j\rangle\langle j| U | 1 \leq j \leq d\}$ and parameterizing U using Eq. (16). This method was inspired by the measurement seeding method for iterative optimization used by Pál and Vértesi [36]. Analogously, this argument can be extended to full-rank d -dimensional density matrices ρ by performing a spectral decomposition $\rho = \sum_{j=1}^d p_j |p_j\rangle\langle p_j|$ into eigenvalues $\{p_j\}$ and eigenstates $\{|p_j\rangle\}$. Since $\text{Tr}(\rho) = \sum_{j=1}^d p_j = 1$ the eigenvalues of ρ are parameterized without degeneracy using a tuple of $d - 1$ real-valued parameters with periodicity $[0, 2\pi]$ using hyper-spherical coordinates [30, 34]. Additionally, since ρ is Hermitian, the eigenstates $\{|p_j\rangle\}$ form an orthonormal basis and therefore the eigenstates are analogously parameterized using Eq. (16): $\rho = \sum_{j=1}^d p_j U^\dagger |j\rangle\langle j| U$. For our purposes, we have set $d = 4$ and fixed $\lambda_{l,l} = 0$ for $1 \leq l \leq d$ in Eq. (16) because the global phase contributions are irrelevant for Eq. (15).

Since the inequalities we seek to optimally violate are polynomial, and since the space of quantum-accessible distributions is non-convex, we employ a number of optimization methods consecutively in an attempt to avoid pitfalls associated with local minima or ill-conditioned convergence. Specifically, the BFGS method [37, p.142] and the Nelder-Mead simplex method [37, p.238] were used along with a method called Basin Hopping [38], which is a hybrid between simulated annealing and gradient descent-based methods.

Nevertheless, we strongly caution against misinterpreting the numerical optima present in Section VIII-A as if our findings represent genuine global-maxima of violation. Evidence for the unreliability of our numerical methods is the following: when supplied with randomly sampled initial parameters, *all* optimization methods consistently converged to *saturating* the target inequalities, instead of violating it! This, even though we *know* that all the inequalities we consider *do* admit quantum violation, namely by P_F . To achieve inequality violation, we found ourselves forced to seed the numerical method with initial parameters such that the initial point of the optimization was close to P_F .

The fact that the *final* points of optimization as returned by our numerical methods are invariably also close to P_F makes interpreting the results somewhat murky. Could it be that the inequality's global maximum is, in-fact, not far from P_F ? Or is this a limitation of the ill-conditioned nature of the optimization? The phenomena is likely due to a combination of both effects.

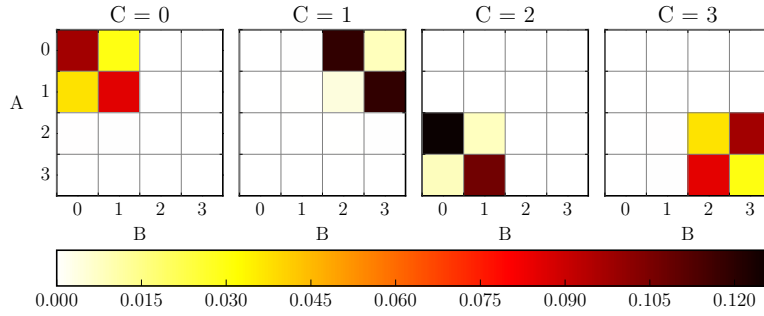


FIG. 6: A quantum probability distribution of the Triangle structure that maximizes violation of I_{Web} . Notice that this distribution has precisely the same possibilistic structure as the Fritz distribution, as might be expected, since I_{Web} was specifically generated to witness the non-classicality of P_F .

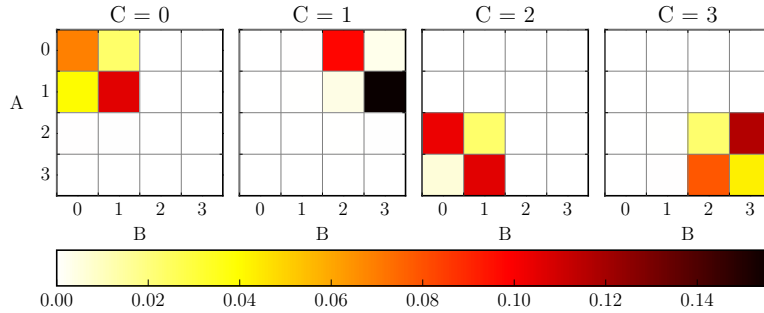


FIG. 7: A quantum probability distribution of the Triangle structure that maximizes violation of $I_{\text{SymmetricWeb}}$. Notice that this distribution has precisely the same (assymmetric!) possibilistic structure as the Fritz distribution, even though $I_{\text{SymmetricWeb}}$ itself is symmetric with respect to permutations of the variables A, B, C .

A. Numerical Optimization Results

Our best numerical optimization of $I_{\text{WagonWheel}}$ was found to be the Fritz distribution itself, visualized in Fig. 4. P_F is therefore either a local minimum of the parameter space or is in fact *the* maximally violating distribution of $I_{\text{WagonWheel}}$; it remains unclear to us if this behavior is related to the global optimality of the Fritz distribution, or if it's an artifact of the methods used to derive $I_{\text{WagonWheel}}$, though we suspect the former.

Our best numerical optimization of I_{Web} is visualized in Fig. 6. Almost immediately, it is evident that Fig. 6 closely resembles the Fritz distribution; in fact Fig. 6 and Fig. 4 share their **possibilistic structure**. A distribution's possibilistic structure is the unitization of its probabilities, i.e. the partitioning of events into disjoint subsets: one containing those events that occur with nonzero probability (possible), the other containing the zero-probability events (impossible). That Fig. 6 and Fig. 4 have the same possibilistic structure is not entirely unexpected; As was mentioned in Section VII, as I_{Web} was derived specifically to prove the incompatibility of the Fritz distribution with the Triangle structure. Consequently, it seems that I_{Web} best witnesses incompatibility of those distributions that closely resemble the P_F . It could be, however, that the sharing of possibilistic structure is an artifact of unreliable convergence, such that the true optimally-violating I_{Web} quantum distribution might have a different structure.

It is perhaps curious, however, that the distribution in Fig. 6 is not accessible when the bipartite states in Eq. (14) are restricted to maximally entangled states. This distributions certainly violate I_{Web} more than P_F , even it isn't the global maximum of I_{Web} . This finding of more-violation with less-entanglement resembles a feature of quantum mechanics originally presented by Methot and Scarani [39] demonstrating that entanglement and non-classicality are different resources.

Our best numerical optimization of $I_{\text{SymmetricWeb}}$ is visualized in Fig. 7. Counter-intuitively, this distribution — which violates $I_{\text{SymmetricWeb}}$ considerably better than P_F — *still* shares its possibilistic structure with the Fritz distribution! Since $I_{\text{SymmetricWeb}}$ is symmetric with respect to the permutation of parties, one might expect that the maximum violating distribution be symmetric as well. Our numerical optimization, however, failed to find *any* symmetric quantum distributions capable of violating $I_{\text{SymmetricWeb}}$, not even violating it suboptimally than P_F . The *only* violations of $I_{\text{SymmetricWeb}}$ appear to be by assymmetric distributions.

IX. CONCLUSIONS

In [Section IV](#), we elucidated that establishing or rejecting compatibility with the Triangle structure has been a challenging problem for nearly a decade [14, 16, 18, 19, 28]. Though some causal compatibility inequalities were known, those inequalities did not appear useful for the purpose of witnessing quantum non-classicality. Recently, Fritz [14] gave the the first example of quantum non-classicality in the Triangle structure, relying on an inequality-free proof that is not robust to any amount of noise.

In [Section VII](#) we presented the first examples of causal compatibility inequalities capable of having quantum violations in the sense that they are violated by quantum-accessible distributions. This result was made possible through the Inflation Technique [Appendix A](#) applied to the Fritz distribution.

In [Section VII-B](#) we demonstrated that a causal compatibility inequalities is robust to noise, directly revealing a critical departure from Fritz’s original proof of the incompatibility of the Fritz’s perfect-correlation example. In [Section VIII](#) we found quantum distributions quantitatively distinct from the Fritz’s recycled-Bell’s-theorem example, such that these variant distribution more strongly violate certain causal compatibility inequalities.

Despite these advancements, the distributions we discovered hew closely to the Fritz distribution, indicating that their non-classical nature remains some recycled version of the non-classicality found in the Bell structure. Currently, it remains speculative about how to recognize whether-or-not a given distribution for the Triangle structure exhibits non-classical correlations of a fundamentally novel qualitative nature than Bell-type non-classicality. We imagine that advantages from a resource perspective might be critical to establish such a novelty signature. Perhaps another signature would be if the distribution can arise in the Triangle structure only through the use of of entangled measurements, such as perhaps the conjecture in Ref. [28]. Ultimately, given any said classifier, it remains unclear whether or not such distributions even exist.

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Appendix A: Summary of Inflation Technique

1. Marginal Satisfiability and Inequalities

This section aims to explain how to solve the following decision problem: given a collection of probability distributions $\{P_{V_1}, \dots, P_{V_m}\}$ where each set of variables $V_i \subseteq \mathcal{J}$ is a subset of some complete set of variables \mathcal{J} , does there exist a joint distribution $P_{\mathcal{J}}$ such that each P_{V_i} can be obtained by marginalizing $P_{\mathcal{J}}$ over the variables not in V_i , i.e. $P_{V_i} = \sum_{\mathcal{J} \setminus V_i} P_{\mathcal{J}}$? Colloquially, this problem is referred to as **the marginal problem** [40]. Additionally, this section aims to accomplish something further. If such a joint distribution $P_{\mathcal{J}}$ exists, how does one find it? Moreover, if a joint distribution does not exist, find an inequality that $\{P_{V_1}, \dots, P_{V_m}\}$ fails which proves such a joint distribution does not exist. This is accomplished by illustrating how the marginal problem can be expressed as a linear program in which the solution to the marginal problem is encoded in the feasibility or infeasibility of said linear program. This section is presented prior to [Appendix A-2](#) as the marginal problem becomes an integral component of the Inflation Technique in subsequently deriving the inequalities presented in [Section VII](#). Moreover, the Marginal problem is presented here logically independent from the remainder of the manuscript both for procedural clarity and because the marginal problem has applications to numerous areas of mathematics including game theory [41], database theory, knowledge integration of expert system, and of course, quantum information theory [40].

To begin, several pieces of nomenclature will be introduced to facilitate discussions. First, the set $\mathcal{M} = \{V \mid V \subseteq \mathcal{J}\}$ of subsets of \mathcal{J} is referred to as the **marginal scenario** and each element $V \in \mathcal{M}$ is termed a **(marginal) context** of \mathcal{M} . The complete set of marginal distributions is referred to as the **marginal model** and is denoted with an superscript $P^{\mathcal{M}} \equiv \{P_V \mid V \in \mathcal{M}\}$. A marginal model acts as the most general description of a family of observations that can be made over \mathcal{J} . Strictly speaking, as defined by [40], a marginal scenario forms an *abstract simplicial complex* where it is required that all subsets of contexts are also contexts: $\forall V \in \mathcal{M}, V' \subset V : V' \in \mathcal{M}$. Throughout

this section, we exclusively consider (without loss of generality) the *maximal* marginal scenario; restricting our focus to the largest contexts in the marginal scenario. Additionally, all marginal scenarios are taken to be complete in the sense that the marginal scenario covers the complete set of observable variables, i.e. $\mathcal{J} = \bigcup_{V \in \mathcal{M}} V$. Finally, we henceforth assume that each variable $v \in \mathcal{J}$ has a finite cardinality.

The marginal problem asks: given a marginal model $P^{\mathcal{M}} = \{P_V \mid V \in \mathcal{M}\}$ marginal to the joint variables \mathcal{J} , does there exist a joint distribution $P_{\mathcal{J}}$ such that each context P_V can be obtained by marginalizing $P_{\mathcal{J}}$?

$$\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J} \setminus V} P_{\mathcal{J}} \quad (\text{A1})$$

A marginal model $P^{\mathcal{M}}$ is said to be *contextual* if it *does not* admit a joint distribution and *non-contextual* otherwise. Notice that Eq. (A1) is inherently a linear system of constraints which be solved rather efficiently using linear programs. In consideration of this, we will now endeavor to discuss how to cast the Eq. (A1) as a matrix multiplication equation so that it becomes possible to discuss existing methods for deriving constraints on the set of contextual marginal models.

To every discrete random variable v there corresponds a prescribed set of **outcomes** O_v . We also define the set of all **events over** v , denoted $\mathcal{S}(v)$,⁵ to be the set of all functions $s : \{v\} \rightarrow O_v$ each representing the event that a measurement on v was made where $s(v) \in O_v$ was observed. Evidently, $\mathcal{S}(v)$ and O_v have a one-to-one correspondence and this distinction can be confounding. There is rarely any harm in referring synonymously to either as outcomes. Nonetheless, a sheaf-theoretic treatment of contextuality [42] demands the distinction. Specifically for this work, the distinction becomes essential for our discussion and exploit of marginal symmetries in Appendix A-4. As a natural generalization we define the events over a set of random variables $V = \{v_1, \dots, v_n\}$ in a parallel manner,

$$\mathcal{S}(V) \equiv \{s : V \rightarrow O_V \mid \forall i : s(v_i) \in O_{v_i}\} \quad (\text{A2})$$

Each event s can be compactly represented as a set of mappings over each element of V , i.e. $s = \{v_i \mapsto s(v_i)\}_{i=1}^k$. The domain $\mathcal{D}(s)$ of an event s is the set of random variables it evaluates, i.e. if $s \in \mathcal{S}(V)$, then $\mathcal{D}(s) = V$. Under this framework, a probability distribution P_V can be considered as a map from $\mathcal{S}(V)$ to the unit interval $[0, 1]$. The marginal problem inherently depends on the concept of probabilistic marginalization. This concept can be understood at the level of events; one event $s \in \mathcal{S}(V)$ can be “marginalized” or restricted to a smaller event $s' \in \mathcal{S}(W)$ whenever $W \subseteq V$. For every $W \subseteq V$ and $s \in \mathcal{S}(V)$, the **restriction of s onto W** (denoted $s|_W \in \mathcal{S}(W)$) is the event in $\mathcal{S}(W)$ that *agrees* with each of s ’s assignments for variables in W : $\forall v \in W : s|_W(v) = s(v)$.

For every marginal scenario $\mathcal{M} = \{V_1, \dots, V_k\}$, it is useful to put special emphasis on the **joint events** $\mathcal{S}(\mathcal{J})$ which represent all possible global events over the entire set of joint variables. Similarly, we define the **context events** for a particular context $V \in \mathcal{M}$ as $\mathcal{S}(V)$. Finally, we elect to define the **marginal events** as the disjoint union over all context events and by an abuse of notation we will denote this union as $\mathcal{S}(\mathcal{M}) = \bigsqcup_{V \in \mathcal{M}} \mathcal{S}(V)$. Each marginal section $m \in \mathcal{S}(\mathcal{M})$ has a domain $\mathcal{D}(m) = V$ for some $V \in \mathcal{M}$. By construction each marginal event $m \in \mathcal{S}(\mathcal{M})$ is a restriction of some joint event $j \in \mathcal{S}(\mathcal{J})$.

The marginalization operation (Eq. (A1)) of the marginal problem is a linear operator. This linear operator maps a joint probability distribution $P_{\mathcal{J}} : \mathcal{S}(\mathcal{J}) \rightarrow [0, 1]$ into a marginal model $P^{\mathcal{M}} = \{P_V : \mathcal{S}(V) \rightarrow [0, 1] \mid V \in \mathcal{M}\}$. Since $\mathcal{S}(\mathcal{M})$ and $\mathcal{S}(\mathcal{J})$ are finite, the marginal problem can be represented as a $|\mathcal{S}(\mathcal{M})| \times |\mathcal{S}(\mathcal{J})|$ matrix.

Definition 1. The **incidence matrix** M for a marginal scenario $\mathcal{M} = \{V_1, \dots, V_k\}$ is a bit-wise matrix where the columns are indexed by *joint* events $j \in \mathcal{S}(\mathcal{J})$ and the rows are events by *marginal* events $m \in \mathcal{S}(\mathcal{M})$. The entries of M are populated whenever a marginal event m is a restriction of the joint event j .

$$M_{m,j} \equiv \begin{cases} 1 & m = j|_{\mathcal{D}(m)} \\ 0 & \text{otherwise} \end{cases}$$

The incidence matrix has $|\mathcal{S}(\mathcal{J})|$ columns, $|\mathcal{S}(\mathcal{M})| = \sum_i |\mathcal{S}(V_i)|$ rows and $k|\mathcal{S}(\mathcal{J})|$ non-zero entries.

To illustrate this concretely, consider the following example. Let \mathcal{J} be 3 binary variables $\mathcal{J} = \{A, B, C\}$ and \mathcal{M}

⁵ In the language of sheaf-theory, $\mathcal{S}(v)$ is the *sheaf of events* [42].

be the marginal scenario $\mathcal{M} = \{\{A, B\}, \{B, C\}, \{A, C\}\}$. The incidence matrix for \mathcal{M} is:

$$M = \begin{array}{c} \begin{array}{l} (A, B, C) \mapsto \\ (A \mapsto 0, B \mapsto 0) \\ (A \mapsto 0, B \mapsto 1) \\ (A \mapsto 1, B \mapsto 0) \\ (A \mapsto 1, B \mapsto 1) \\ (B \mapsto 0, C \mapsto 0) \\ (B \mapsto 0, C \mapsto 1) \\ (B \mapsto 1, C \mapsto 0) \\ (B \mapsto 1, C \mapsto 1) \\ (A \mapsto 0, C \mapsto 0) \\ (A \mapsto 0, C \mapsto 1) \\ (A \mapsto 1, C \mapsto 0) \\ (A \mapsto 1, C \mapsto 1) \end{array} \mapsto \begin{pmatrix} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix} \end{array} \quad (\text{A3})$$

The incidence matrix needs to act (to the right) on a vector representing the joint probability $P_{\mathcal{J}}$ distribution and outputs a vector representing the marginal model $P^{\mathcal{M}}$. The **joint distribution vector** $\mathcal{P}^{\mathcal{J}}$ for a probability distribution $P_{\mathcal{J}}$ is the vector indexed by joint events $j \in \mathcal{S}(\mathcal{J})$ whose entries are populated by the probabilities that $P_{\mathcal{J}}$ assigns to each joint event: $\mathcal{P}_j^{\mathcal{J}} = P_{\mathcal{J}}(j)$. Analogously, **marginal distribution vector** $\mathcal{P}^{\mathcal{M}}$ for a marginal model $P^{\mathcal{M}}$ is the vector whose entries are probabilities over the set of marginal outcomes $\mathcal{S}(\mathcal{M})$: $\mathcal{P}_m^{\mathcal{M}} = P_{\mathcal{D}(\mathcal{M})}(m)$.

By design, the marginal and joint distribution vectors are related via the incidence matrix M . Given a joint distribution vector $\mathcal{P}^{\mathcal{J}}$ one can obtain the marginal distribution vector $\mathcal{P}^{\mathcal{M}}$ by multiplying M by $\mathcal{P}^{\mathcal{J}}$.

$$\mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}} \quad (\text{A4})$$

As a quick remark, the particular ordering of the rows and columns of M carries *no importance*, but it *must* be consistent between M , $\mathcal{P}^{\mathcal{J}}$ and $\mathcal{P}^{\mathcal{M}}$. The marginal problem can now be rephrased in the language of the incidence matrix. Suppose one obtains a marginal distribution vector $\mathcal{P}^{\mathcal{M}}$; the marginal problem becomes equivalent to the question: does there exist a joint distribution vector $\mathcal{P}^{\mathcal{J}}$ such that Eq. (A4) holds? This question is naturally framed as the **marginal linear program**:

$$\begin{aligned} &\text{minimize: } \emptyset \cdot x \\ &\text{subject to: } x \succeq 0 \\ &M \cdot x = \mathcal{P}^{\mathcal{M}} \end{aligned}$$

If this “optimization”⁶ is *feasible*, then there exists a vector x than can satisfy Eq. (A4) and is a valid joint distribution vector. Therefore, feasibility of the marginal linear program not only implies that $\mathcal{P}^{\mathcal{J}}$ exists but returns $\mathcal{P}^{\mathcal{J}}$. Moreover if the marginal linear program is *infeasible*, then there *does not* exist a joint distribution $\mathcal{P}^{\mathcal{J}}$. To every linear program, there exists a dual linear program that characterizes the feasibility of the original [43]. Constructing the dual linear program is straightforward [44].

$$\begin{aligned} &\text{minimize: } y \cdot \mathcal{P}^{\mathcal{M}} \\ &\text{subject to: } y \cdot M \succeq 0 \end{aligned}$$

The dual marginal linear program not only answers the marginal problem for a specific marginal model $\mathcal{P}^{\mathcal{M}}$ but as a by-product provides an inequality that witnesses its contextuality. If this is not obvious, first notice that the dual problem is *never infeasible*; by choosing y to be trivially the null vector \emptyset of appropriate size, all constraints become satisfied. Secondly if the dual constraint $y \cdot M \succeq 0$ holds *and* the primal is feasible, then $y \cdot \mathcal{P}^{\mathcal{M}} = y \cdot M \cdot x \geq 0$. Therefore the *sign* of the dual objective $d \equiv \min(y \cdot \mathcal{P}^{\mathcal{M}})$ classifies a marginal model’s contextuality; if $d < 0$ then $y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$ is violated and therefore $\mathcal{P}^{\mathcal{M}}$ is contextual. Likewise if $d \geq 0$ (satisfying $y \cdot \mathcal{P}^{\mathcal{M}}$), then $\mathcal{P}^{\mathcal{M}}$ is non-contextual.⁷ This is manifestation of Farkas’s lemma [43]. An **infeasibility certificate** [46] is any vector y that satisfies the $y \cdot M \succeq 0$.

⁶ “Optimization” is presented in quotes here because the minimization objective is trivially always zero (\emptyset denotes the null vector of all zero entries). The primal value of the linear program is of no interest, all that matters is its *feasibility*.

⁷ Actually, if $d \geq 0$ then it is exactly $d = 0$ due to the existence of the trivial $y = \emptyset$. This observation is an instance of the *Complementary Slackness Property* of [45]. Moreover, if $d < 0$, then it is unbounded $d = -\infty$. This latter point becomes clear upon recognizing that for any y with $d < 0$, another $y' = \alpha y$ can be constructed (with $\alpha > 1$) such that $d' = \alpha d < d$. Since a more negative d' can always be found, it must be that d is unbounded. This is a demonstration of the fundamental *Unboundedness Property* of [45]; if the dual is unbounded, then the primal is infeasible.

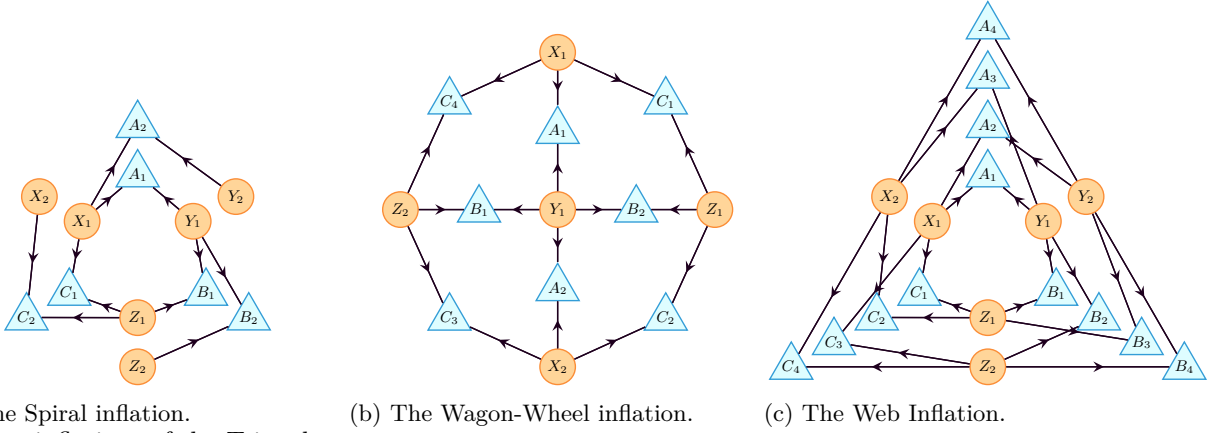


FIG. 8: Some inflations of the Triangle structure.

Most linear program software packages such as Mosek [47], Gurobi [48], CPLEX [49] and CVX/CVX OPT [50, 51] are capable of producing infeasibility certificates. Furthermore, for every y satisfying $y \cdot M \geq 0$ there corresponds a **certificate inequality** that constraints the set of non-contextual marginal models. If y is an infeasibility certificate, then $y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$ is satisfied by all contextual marginal models.

The marginal problem can sometimes take on a more general variant that does not begin with a specific marginal model [40, 52, 53]: *Given a marginal scenario \mathcal{M} , what is the set of all non-contextual marginal models?* Pitowsky [54] demonstrates that the set of non-contextual marginal models forms a *convex* polytope called the **marginal polytope**. The extremal rays of a marginal polytope directly correspond to the columns of M which further correspond to *deterministic* joint distributions $\mathcal{P}^{\mathcal{J}}$. Since all joint distributions $\mathcal{P}^{\mathcal{J}}$ are probability distributions, their entries must sum to unity $\sum_{j \in \mathcal{S}(\mathcal{J})} \mathcal{P}_j^{\mathcal{J}} = 1$. This normalization defines the convexity of the polytope; all non-contextual marginal models are convex mixtures of the deterministic marginal models pursuant to Eq. (A4). The marginal polytope is a beneficial tool for understanding contextuality. First, the *facets* of a marginal polytope correspond to a finite set of linear inequalities that are complete in the sense that all contextual distributions violate at least one facet inequality [6]. From the perspective of a marginal polytope, convex hull algorithms or linear quantifier elimination can be used to compute a representation of the complete set of linear inequalities and completely solve the marginal problem. A popular tool for linear quantifier elimination is **Fourier-Motzkin elimination** [21, 52, 55, 56]. Applying the Fourier-Motzkin procedure to completely solve the marginal problem is discussed in more detail in Fritz and Chaves [40]. An excellent survey of existing techniques for solving the marginal problem including Equality Set Projection [56] and Hardy-type hypergraph transversals can be found in Wolfe *et al.* [21]. In conclusion, there are a number of computational tools available to solve the marginal problem completely whenever no marginal model is provided.

Each of the above mentioned techniques suffers from computational complexity limitations. For example, the Fourier-Motzkin procedure is in the worst case doubly exponential in the number of initial inequalities [55]. For the purposes of this research, solving the marginal problem without reference to a marginal model was intractable. This will become apparent in Appendix A-3 when the Inflation Technique is applied to the Triangle structure, producing considerably large marginal models. Luckily, the Fritz distribution (Section V) allows one to avoid the complexity issues of the complete marginal problem and instead focus on the original problem of determining whether or not a particular marginal model admits a joint distribution or not.

2. Inflation Technique

The **Inflation Technique**, invented by Wolfe *et al.* [21] and inspired by the *do calculus* and *twin networks* of Pearl [9], is a family of causal inference techniques that can be used to determine if an observable probability distribution $P_{\mathcal{N}_O}$ is compatible or incompatible with a given causal structure \mathcal{G} . As a precursor, the Inflation Technique begins by *augmenting* a causal structure \mathcal{G} with additional copies of its nodes, producing an *inflated* causal structure \mathcal{G}' called an **inflation**, and then exposes how causal inference tasks on the inflation can be used to make inferences on the original causal structure. For reference, a few inflations of the Triangle structure are depicted in Fig. 8. Copies of nodes in the inflated causal structure are distinguished by an additional subscript called the **copy-index**. For example, node A of Fig. 2 has copies A_1, A_2, A_3, A_4 in the inflated causal structure in Fig. 8c. All such copies are deemed equivalent via a **copy-index equivalence** relation denoted ' \sim '. A copy-index is effectively arbitrary, so we

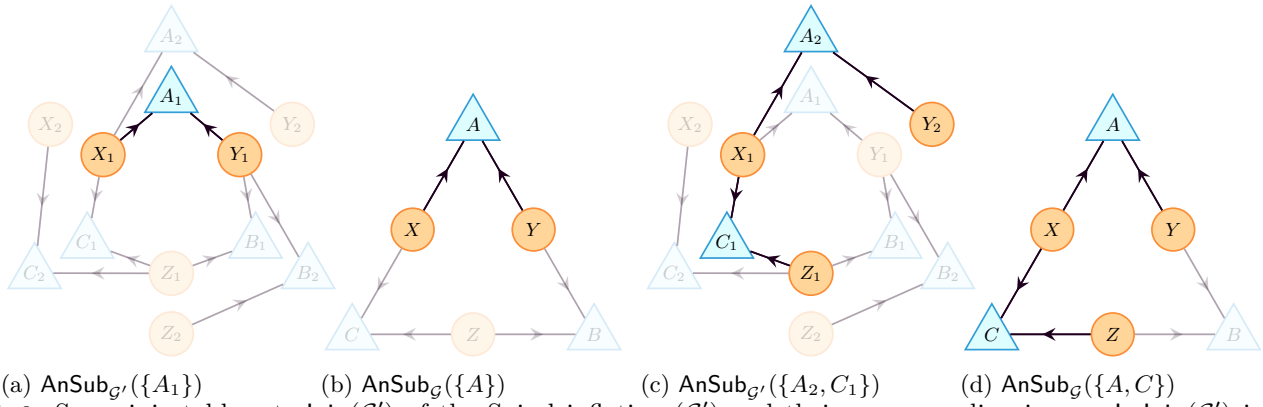


FIG. 9: Some injectable sets $\text{Inj}_{\mathcal{G}}(\mathcal{G}')$ of the Spiral inflation (\mathcal{G}') and their corresponding images $\text{ImInj}_{\mathcal{G}}(\mathcal{G}')$ in the Triangle structure (\mathcal{G}).

will refer to an arbitrary inflated copy of A as A' , i.e. $A \sim A_1 \sim A' \not\sim B \sim B_1 \sim B'$.⁸

In addition to the common graph-theoretic terminology and notation presented in Section II, two related concepts need introductions. First, an **induced subgraph** of \mathcal{G} for a subset of nodes $N \subseteq \mathcal{N}$ is the graph composed of nodes N and all edges e of the original graph that are contained in N : $\text{Sub}_{\mathcal{G}}(N) \equiv (N, \{e = \{n \rightarrow m\} \mid n, m \in N\})$. An **ancestral subgraph** of \mathcal{G} for a subset of nodes $N \subseteq \mathcal{N}$ is the induced subgraph due to the ancestry of N : $\text{AnSub}_{\mathcal{G}}(N) \equiv \text{Sub}_{\mathcal{G}}(\text{An}_{\mathcal{G}}(N))$.

The Inflation Technique begins with distribution P_N defined over some observable nodes N of \mathcal{G} and the a priori assumption that it is compatible with \mathcal{G} pursuant to the definitions provided in Section II. Regarding \mathcal{G} as a causal hypothesis, the observable correlation P_N can only be influenced by the ancestry of N in \mathcal{G} . Consequently, for any set of nodes N' of an inflation \mathcal{G}' where the ancestral subgraph $\text{AnSub}_{\mathcal{G}'}(N')$ happens to be homomorphic to the ancestral subgraph $\text{AnSub}_{\mathcal{G}}(N)$, one can conclude that the distribution $P_{N'}$ (induced by P_N ⁹) is compatible with \mathcal{G}' using the same latent explanations for P_N in \mathcal{G} . This observation is known as the **Inflation Lemma** [21, Lemma 3].

To formalize the Inflation Lemma, we define the **injectable sets of \mathcal{G}'** , denoted $\text{Inj}_{\mathcal{G}}(\mathcal{G}')$, as all sets of nodes in \mathcal{G}' whose ancestral subgraphs are homomorphic (via copy-index removal) to an ancestral subgraph in \mathcal{G} : $\text{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : \text{AnSub}_{\mathcal{G}'}(N') \sim \text{AnSub}_{\mathcal{G}}(N)\}$. Analogously defined are the **images of the injectable sets in \mathcal{G}** : $\text{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : \text{AnSub}_{\mathcal{G}'}(N') \sim \text{AnSub}_{\mathcal{G}}(N)\}$.

To illustrate these concepts, consider the Spiral inflation \mathcal{G}' depicted in Fig. 8a. The ancestral subgraph of $\{A_1\}$ in the Spiral inflation (denoted $\text{AnSub}_{\mathcal{G}'}(\{A_1\})$) is highlighted in Fig. 9a and is clearly homomorphic to the ancestral subgraph of its image $\{A\}$ in the Triangle structure (\mathcal{G}) (Fig. 9b). Additionally, the set $\{A_2, C_1\}$ is an injectable set of \mathcal{G}' because $\text{AnSub}_{\mathcal{G}'}(\{A_2, C_1\})$ (highlighted in Fig. 9c) is homomorphic (via copy-index removal) to the $\text{AnSub}_{\mathcal{G}}(\{A, C\})$ in the Triangle structure (\mathcal{G}) (Fig. 9d).

The injectable sets of an inflation are of *principle importance* to the Inflation Technique. Given a distribution P_{N_O} defined over the observable nodes of \mathcal{G} , one can compute a marginal model defined over the injectable sets $P^{\text{Inj}_{\mathcal{G}}(\mathcal{G}')} = \{P_{N'} \mid N' \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')\}$ and their images $P^{\text{ImInj}_{\mathcal{G}}(\mathcal{G}')} = \{P_N \mid N \in \text{ImInj}_{\mathcal{G}}(\mathcal{G}')\}$. The contrapositive of the Inflation Lemma casts the compatibility problem for $P^{\text{ImInj}_{\mathcal{G}}(\mathcal{G}')}$ (which exhibits equivalent compatibility as P_{N_O}) into a compatibility problem for $P^{\text{Inj}_{\mathcal{G}}(\mathcal{G}')}$ of any non-trivial inequalities. If $P^{\text{Inj}_{\mathcal{G}}(\mathcal{G}')}$ is found to be incompatible with \mathcal{G}' , then $P^{\text{ImInj}_{\mathcal{G}}(\mathcal{G}')}$ (and also P_{N_O}) must be incompatible with \mathcal{G} . Fortunately, the inflated causal structure \mathcal{G}' possesses its own *d-separation relations* which enforces conditional independence equality constraints on $P^{\text{Inj}_{\mathcal{G}}(\mathcal{G}')}$. A useful subset of d-separation relations are those corresponding to unconditional d-separations, known as **ancestrally separated sets** [9]. Two sets N'_1, N'_2 are ancestrally separated ($N'_1 \perp N'_2$) if they have distinct ancestry in \mathcal{G}' .

$$N'_1 \perp N'_2 \iff \text{An}_{\mathcal{G}'}(N'_1) \cap \text{An}_{\mathcal{G}'}(N'_2) = \emptyset$$

Ancestral independence implies an unconditional probabilistic independence — if $N'_1 \perp N'_2$ then $P_{N'_1 \cup N'_2} = P_{N'_1} P_{N'_2}$. If $N'_1, N'_2 \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')$, then the associated probabilistic constraint is applicable to $P^{\text{Inj}_{\mathcal{G}}(\mathcal{G}')}$. This notion generalizes to more than two ancestrally separated sets. A set $N' \subseteq \mathcal{N}'$ is an **as-expressible set** if it can be decomposed

⁸ Note that we preemptively generalize the notion of copy-index equivalence to other mathematical objects like sets, graphs, and groups by saying that $X \sim Y$ if and only if X is equivalent to Y upon removal of the copy-index.

⁹ The inflated distribution $P_{N'}$ assigns the same probability to all events of P_N whenever the events are equivalent under the removal of copy-indices.

Maximally AS-Expressible Sets for the Wagon-Wheel Inflation	
AnSepExpr $_{\mathcal{G}'}(\mathcal{G}')$	Ancestral Separations
$\{A_2, B_1, C_3, C_1\}$	$\{A_2, B_1, C_3\} \perp \{C_1\}$
$\{A_1, B_1, C_4, C_2\}$	$\{A_1, B_1, C_4\} \perp \{C_2\}$
$\{A_1, B_2, C_1, C_3\}$	$\{A_1, B_2, C_1\} \perp \{C_3\}$
$\{A_2, B_2, C_2, C_4\}$	$\{A_2, B_2, C_2\} \perp \{C_4\}$

TABLE I: The maximally as-expressible sets for the Wagon-Wheel Inflation.

Maximally AS-Expressible Sets for the Web Inflation	
AnSepExpr $_{\mathcal{G}'}(\mathcal{G}')$	Ancestral Separations
$\{A_1, B_1, C_1, A_4, B_4, C_4\}$	$\{A_1, B_1, C_1\} \perp \{A_4, B_4, C_4\}$
$\{A_1, B_2, C_3, A_4, B_3, C_2\}$	$\{A_1, B_2, C_3\} \perp \{A_4, B_3, C_2\}$
$\{A_2, B_3, C_1, A_3, B_2, C_4\}$	$\{A_2, B_3, C_1\} \perp \{A_3, B_2, C_4\}$
$\{A_2, B_4, C_3, A_3, B_1, C_2\}$	$\{A_2, B_4, C_3\} \perp \{A_3, B_1, C_2\}$
$\{A_1, B_3, C_4\}$	$\{A_1\} \perp \{B_3\} \perp \{C_4\}$
$\{A_1, B_4, C_2\}$	$\{A_1\} \perp \{B_4\} \perp \{C_2\}$
$\{A_2, B_1, C_4\}$	$\{A_2\} \perp \{B_1\} \perp \{C_4\}$
$\{A_2, B_2, C_2\}$	$\{A_2\} \perp \{B_2\} \perp \{C_2\}$
$\{A_3, B_3, C_3\}$	$\{A_3\} \perp \{B_3\} \perp \{C_3\}$
$\{A_3, B_4, C_1\}$	$\{A_3\} \perp \{B_4\} \perp \{C_1\}$
$\{A_4, B_1, C_3\}$	$\{A_4\} \perp \{B_1\} \perp \{C_3\}$
$\{A_4, B_2, C_1\}$	$\{A_4\} \perp \{B_2\} \perp \{C_1\}$

TABLE II: The maximally as-expressible sets for the Web Inflation.

into the disjoint union of injectable sets $N' = \coprod_i N'_i \mid N'_i \in \text{Inj}_{\mathcal{G}'}(\mathcal{G}')$ and all pairs N'_i, N'_j are ancestrally separated: $\forall i, j : N'_i \perp N'_j$. The analogous probabilistic constraint is: $P_{N'} = P_{\coprod_i N'_i} = \prod_i P_{N'_i}$ [9].¹⁰ Throughout this work, we let $\text{AnSepExpr}_{\mathcal{G}'}(\mathcal{G}')$ denote the set of all as-expressible sets.¹¹ Efficient algorithms for computing the injectable and as-expressible sets of an inflation can be found in [21]. Tables I and II tabulate the as-expressible sets and the associated ancestral separations for the Wagon-Wheel and Web inflations (respectively Figs. 8b and 8c).

Unlike $P^{\text{Inj}_{\mathcal{G}'}}(\mathcal{G}')$, which is non-contextual by construction, $P^{\text{Inj}_{\mathcal{G}'}}(\mathcal{G}')$ contains overlapping marginals meaning its contextuality remains unknown and must be determined using any of the techniques discussed in Appendix A-1. More importantly, the Inflation Technique introduces new constraints relating to the as-expressible sets. In practice, the equality constraints implied by $\text{AnSepExpr}_{\mathcal{G}'}(\mathcal{G}')$ permits one to construct a marginal model defined over the as-expressible sets $\text{AnSepExpr}_{\mathcal{G}'}(\mathcal{G}')$ resulting in greater resolution in classifying the compatibility of P_{N_O} . In summary, the Inflation Technique partially transforms the compatibility problem into a marginal problem, wherein one can solve the marginal problem (see Appendix A-1) to either determine the compatibility of an observable distribution P_{N_O} with \mathcal{G} or derive compatibility inequalities for \mathcal{G} — the latter of which is discussed in Appendix A-3.

3. Deriving Inequalities from the Inflation Technique

Appendix A-2 summarizes how the Inflation Technique [21] can cast the compatibility problem into the marginal problem by leveraging inflations of the Triangle structure. Explicitly, for a given inflation \mathcal{G}' of the Triangle structure, one constructs a marginal problem (see Appendix A-1) for the marginal scenario \mathcal{M} composed of the maximal as-expressible sets of \mathcal{G}' , i.e. $\mathcal{M} = \text{AnSepExpr}_{\mathcal{G}'}(\mathcal{G}')$.

The first inequality presented in Section VII was Ineq. (9). Ineq. (9) was derived using the Wagon-Wheel inflation in Fig. 8b. The Wagon-Wheel inflation possesses 4 copies of C (C_1, C_2, C_3, C_4) and 2 copies of A (A_1, A_2) and B (B_1, B_2) arranged in the shape of a Wagon-Wheel. The maximal as-expressible sets of the Wagon-Wheel inflation along with their ancestral dependences can be found in Table I. These maximal as-expressible sets define a marginal scenario where the joint variables \mathcal{J} are the set of observable nodes in the Wagon-Wheel inflation $\mathcal{J} = N'_O$:

$$\mathcal{M} = \text{AnSepExpr}_{\mathcal{G}'}(\mathcal{G}') = \{\{A_2, B_1, C_3, C_1\}, \{A_1, B_1, C_4, C_2\}, \{A_1, B_2, C_1, C_3\}, \{A_2, B_2, C_2, C_4\}\} \quad (\text{A5})$$

Given that the Fritz distribution is defined over 4-outcomes variables, the variables in the marginal scenario are assigned 4-outcomes as well. This marginal scenario \mathcal{M} then defines an incidence matrix M capable of accommodating the Fritz distribution that has $|\mathcal{S}(\mathcal{M})| = 4 \cdot 4^4 = 1024$ rows and $|\mathcal{S}(\mathcal{J})| = 4^8 = 65,536$ columns — this matrix is not reproduced here. The sheer size of the incidence matrix used here makes complete solutions of the marginal problem using tools such as linear-quantifier elimination intractable. To construct the marginal distribution vector $\mathcal{P}^{\mathcal{M}}$ for

¹⁰ In [21], this concept is generalized into terms which can be factorized via d-separation conditions and the corresponding inflated sets are termed *expressible sets*. This generalization was not required for this work in particular because the ancestral independence relations formed a generating set of d-separation conditions for all of the inflations considered in Fig. 8.

¹¹ Analogously to a marginal scenario \mathcal{M} , the as-expressible sets $\text{AnSepExpr}_{\mathcal{G}'}(\mathcal{G}')$ form an *abstract simplicial complex*. Therefore, in practice, it is completely sufficient to focus on the *maximal* as-expressible sets.

the Fritz distribution, one begins with $\mathcal{P}^{\mathcal{M}}$ defined in a *symbolic* form,

$$\mathcal{P}^{\mathcal{M}^\top} = \underbrace{(P_{A_2B_1C_3C_1}(0000), \dots, P_{A_1B_1C_4C_2}(3232), \dots, P_{A_2B_2C_2C_4}(3333))}_{1024 \text{ entries}} \quad (\text{A6})$$

Which can be factored into the ancestrally separated injectable sets using [Table I](#),

$$\mathcal{P}^{\mathcal{M}^\top} = (P_{A_2B_1C_3}(000)P_{C_1}(0), \dots, P_{A_1B_1C_4}(323)P_{C_2}(2), \dots, P_{A_2B_2C_2}(333)P_{C_4}(3)) \quad (\text{A7})$$

And penultimately, each probability distribution in [Eq. \(A7\)](#) is deflated by dropping copy-indices.

$$\mathcal{P}^{\mathcal{M}^\top} = (P_{ABC}(000)P_C(0), \dots, P_{ABC}(323)P_C(2), \dots, P_{ABC}(333)P_C(3)) \quad (\text{A8})$$

This step is permitted because all of the remaining distributions in [Eq. \(A7\)](#) are defined over the injectable sets of the Wagon-Wheel inflation. Finally, each of the elements of $\mathcal{P}^{\mathcal{M}}$ are replaced with numerical values pursuant to the Fritz distribution. For example, $P_{ABC}(323)P_C(2)$ is assigned the following numerical value:

$$P_{ABC}(323)P_C(2) = \frac{1}{32} (2 + \sqrt{2}) \cdot \frac{1}{4} \simeq 0.0267$$

The same is applied to all other entries of $\mathcal{P}^{\mathcal{M}}$. Finally, this numerical version of $\mathcal{P}^{\mathcal{M}}$ and M are subjected to linear programming software and an infeasibility certificate y was obtained, corresponding precisely to [Ineq. \(9\)](#) using [Eq. \(A8\)](#).

The remaining inequalities in [Section VII](#) were derived using the Web inflation of [Fig. 8c](#). The maximal as-expressible sets of the Wagon-Wheel inflation along with their ancestral dependences can be found in [Table II](#). Analogously to the Wagon-Wheel inflation, these as-expressible sets form a marginal scenario \mathcal{M} which defines an incidence matrix M that has $|\mathcal{S}(\mathcal{M})| = 4 \cdot 4^6 + 8 \cdot 4^3 = 16,896$ rows, $|\mathcal{J}(\mathcal{J})| = 4^{12} = 16,777,216$ columns and 201,326,592 non-zero entries.

4. Deriving Symmetric Causal Compatibility Inequalities

[Appendix A-2](#) detailed how to obtain causal compatibility inequalities for any causal structure by constructing a marginal problem (see [Appendix A-1](#)) and supplying an incompatible distribution to generate an infeasibility certificate. [Ineq. \(13\)](#) presented the causal compatibility inequality $I_{\text{SymmetricWeb}}$ which is symmetric under permutations of the variables A, B, C . This section aims to describe a general technique that can be used to derive $I_{\text{SymmetricWeb}}$ and other inequalities also exhibiting this symmetry. In brief, this is accomplished by grouping marginal events $m \in \mathcal{S}(\mathcal{M})$ of a marginal scenario \mathcal{M} into orbits under the action of variable permutations.

Exploiting symmetries of the marginal scenario is useful for a few distinct reasons. First, [Bancal et al. \[57\]](#) discuss computational advantages in considering symmetric versions of marginal polytopes mentioned in [Appendix A-1](#); the number of extremal points typically grows exponentially in \mathcal{J} , but only polynomial for the symmetric polytope. They also note a number of interesting inequalities (such as CHSH [\[25\]](#)) can be written in a way that is symmetric under the exchange of parties, demonstrating that non-trivial inequalities can be recovered from facets of a symmetric polytope. Secondly, numerical optimizations against symmetric inequalities lead to one of two interesting cases: either the extremal distribution is symmetric itself or it is not. The latter case generates a family of incompatible distributions obtained by applying symmetry operations on the extremal distribution.¹²

To clarify which symmetries we have in mind, first consider the marginal scenario $\mathcal{M} = \{\{A, B, C\}, \{C, D\}, \{A, D\}\}$ where each variable in $\mathcal{J} = \{A, B, C, D\}$ has binary outcomes $\{0, 1\}$. $I_{\mathcal{M}} \equiv \{P_{ABC}(010) \leq P_{CD}(00) + P_{AD}(01)\}$ is an inequality constraining the set of non-contextual marginal models $P^{\mathcal{M}}$. Now, the contextuality of a distribution P_{ABCD} should be *invariant* under those permutations of the variable labels in \mathcal{J} which map the marginal scenario to itself; but not all variable relabellings preserve \mathcal{M} . An example of a permutation which does *not* preserve \mathcal{M} is the $\varphi \in \text{Perm}(\mathcal{J})$ ¹³ which acts like $\varphi[\{a, b, c, d\}] = \{c, a, d, b\}$. Then, the action of φ on $I_{\mathcal{M}}$ is

$$\begin{aligned} \varphi[I_{\mathcal{M}}] &= \{P_{\varphi[abc]}(010) \leq P_{\varphi[cd]}(00) + P_{\varphi[ad]}(01)\} \\ &= \{P_{cad}(010) \leq P_{db}(00) + P_{cb}(01)\} \\ &= \{P_{acd}(100) \leq P_{bd}(00) + P_{bc}(10)\} \end{aligned} \quad (\text{A9})$$

¹² If the extremal distribution happens to be asymmetric, then one can conclude the space of accessible distributions is non-convex.

¹³ The permutation group $\text{Perm}(S)$ over a set S is the set of all bijective maps $\varphi : S \rightarrow S$.

which yields a vali-but-irrelevant inequality, as the resulting inequality no longer pertains to \mathcal{M}

Permutations φ that modify the marginal scenario have no application within the framework of the Inflation Technique ([Appendix A-2](#)) because the Inflation Lemma only holds when the inflated inequality constrains injectable sets. Therefore, the desired set of symmetries for our purposes is a subgroup of $\text{Perm}(\mathcal{J})$ that takes \mathcal{M} to \mathcal{M} .

The **variable permutation group** $\Phi(\mathcal{M})$ for a marginal scenario \mathcal{M} is the joint permutation subgroup that stabilizes the marginal scenario, $\Phi(\mathcal{M}) \equiv \{\varphi \in \text{Perm}(\mathcal{J}) \mid \forall V \in \mathcal{M} : \varphi[V] \in \mathcal{M}\}$. In general, the variable permutation group $\Phi(\mathcal{M})$ can be obtained using group stabilizer algorithms. After obtaining $\Phi(\mathcal{M})$, one can take known compatibility inequalities $I_{\mathcal{M}}$ can create a whole family of inequalities $\{\varphi[I_{\mathcal{M}}] \mid \varphi \in \Phi(\mathcal{M})\}$ that are valid for the same marginal scenario $\varphi[I_{\mathcal{M}}] = \varphi[I]_{\varphi[\mathcal{M}]} = \varphi[I]_{\mathcal{M}}$.

Although useful for reducing computational complexity [57], we divert our attention to finding *symmetric* inequalities, i.e. those where $\forall \varphi \in \Phi(\mathcal{M}) : \varphi[I_{\mathcal{M}}] = I_{\mathcal{M}}$. To generate inequalities that exhibit certain symmetries using the methods described in [Appendix A-1](#), it is sufficient to perform a change of basis on the incidence matrix M for a given marginal scenario \mathcal{M} . Through repeated action of $\varphi \in \Phi$ on marginal outcomes $m \in \mathcal{S}(\mathcal{M})$ and joint outcomes $j \in \mathcal{S}(\mathcal{J})$, one can define **group orbits** of Φ in $\mathcal{S}(\mathcal{M})$ and $\mathcal{S}(\mathcal{J})$; respectively $\text{Orb}_{\Phi}(m) \equiv \{\varphi[m] \mid \varphi \in \Phi\}$, $\text{Orb}_{\Phi}(j) \equiv \{\varphi[j] \mid \varphi \in \Phi\}$. The action of $\varphi \in \Phi$ on any outcome $f \in \mathcal{S}(V)$ (denoted $\omega[f]$) is defined as $(\varphi[f])(v) \equiv f(\varphi^{-1}[v])$ pursuant to intuitive action used in [Eq. \(A9\)](#). Using these group orbits, it is possible to contract the incidence matrix M of a marginal scenario into a symmetrized version. The **symmetric incidence matrix** M_{Φ} for a marginal scenario \mathcal{M} and the variable permutation group Φ is a contracted version of the incidence matrix M for \mathcal{M} . Each row of M_{Φ} corresponds to a marginal orbit $\text{Orb}_{\Phi}(m)$. Analogously each column of M_{Φ} corresponds to a joint orbit $\text{Orb}_{\Phi}(j)$. The entries of M_{Φ} are integers and correspond to summing over the rows and columns of M that belong to each orbit.

$$(M_{\Phi})_{\text{Orb}_{\Phi}(m), \text{Orb}_{\Phi}(j)} = \sum_{\substack{j' \in \text{Orb}_{\Phi}(j) \\ m' \in \text{Orb}_{\Phi}(m)}} M_{m', j'}$$

It is possible to analogously define a **symmetric joint distribution vector** $\mathcal{P}_{\Phi}^{\mathcal{J}}$ indexed by $\text{Orb}_{\Phi}(j)$, $(\mathcal{P}_{\Phi}^{\mathcal{J}})_{\text{Orb}_{\Phi}(j)} = \sum_{j' \in \text{Orb}_{\Phi}(j)} \mathcal{P}_{j'}^{\mathcal{J}}$ and a **symmetric marginal distribution vector** $\mathcal{P}_{\Phi}^{\mathcal{M}}$ indexed by $\text{Orb}_{\Phi}(m)$, $(\mathcal{P}_{\Phi}^{\mathcal{M}})_{\text{Orb}_{\Phi}(m)} = \sum_{m' \in \text{Orb}_{\Phi}(m)} \mathcal{P}_{m'}^{\mathcal{M}}$. Together, $M_{\Phi}, \mathcal{P}_{\Phi}^{\mathcal{J}}$ and $\mathcal{P}_{\Phi}^{\mathcal{M}}$ define a new, **symmetric marginal problem**: $\mathcal{P}_{\Phi}^{\mathcal{M}} = M_{\Phi} \cdot \mathcal{P}_{\Phi}^{\mathcal{J}}$. The symmetric marginal problem can be solved using the same computational methods discussed in [Appendix A-1](#) and will produce symmetric inequalities.

In the context of the Inflation Technique, a variable symmetry $\Phi(\mathcal{M}')$ over an inflated marginal scenario \mathcal{M}' does not always correspond to a variable symmetry under deflation $\Phi(\mathcal{M})$. In order to derive deflated inequalities that are symmetric under an exchange of parties, it is required that $\Phi(\mathcal{M}') \sim \Phi(\mathcal{M})$ are equivalent up to copy-index.

For the Triangle structure in particular, the variable permutation group is the set of permutations on A, B, C : $\Phi(\mathcal{M}) = \text{Perm}(A, B, C)$. For the Web inflation ([Fig. 8c](#)), we have obtained $\Phi(\text{AnSepExpr}_{\mathcal{G}}(\mathcal{G}'))$ — an order 48 group with the following 4 generators:

φ_1	φ_2	φ_3	φ_4
$A_1 \rightarrow A_4$	$A_1 \rightarrow A_1$	$A_1 \rightarrow C_1$	$A_1 \rightarrow A_1$
$A_2 \rightarrow A_3$	$A_2 \rightarrow A_3$	$A_2 \rightarrow C_2$	$A_2 \rightarrow A_2$
$A_3 \rightarrow A_2$	$A_3 \rightarrow A_2$	$A_3 \rightarrow C_3$	$A_3 \rightarrow A_3$
$A_4 \rightarrow A_1$	$A_4 \rightarrow A_4$	$A_4 \rightarrow C_4$	$A_4 \rightarrow A_4$
$B_1 \rightarrow B_4$	$B_1 \rightarrow C_1$	$B_1 \rightarrow A_1$	$B_1 \rightarrow B_2$
$B_2 \rightarrow B_3$	$B_2 \rightarrow C_3$	$B_2 \rightarrow A_2$	$B_2 \rightarrow B_1$
$B_3 \rightarrow B_2$	$B_3 \rightarrow C_2$	$B_3 \rightarrow A_3$	$B_3 \rightarrow B_4$
$B_4 \rightarrow B_1$	$B_4 \rightarrow C_4$	$B_4 \rightarrow A_4$	$B_4 \rightarrow B_3$
$C_1 \rightarrow C_4$	$C_1 \rightarrow B_1$	$C_1 \rightarrow B_1$	$C_1 \rightarrow C_3$
$C_2 \rightarrow C_3$	$C_2 \rightarrow B_3$	$C_2 \rightarrow B_2$	$C_2 \rightarrow C_4$
$C_3 \rightarrow C_2$	$C_3 \rightarrow B_2$	$C_3 \rightarrow B_3$	$C_3 \rightarrow C_1$
$C_4 \rightarrow C_1$	$C_4 \rightarrow B_4$	$C_4 \rightarrow B_4$	$C_4 \rightarrow C_2$

Notice that $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are all automorphisms of the web inflation. Moreover they stabilize the as-expressible sets of [Table II](#). Importantly,

$$\Phi(\text{AnSepExpr}_{\mathcal{G}}(\mathcal{G}')) \sim \text{Perm}(A, B, C)$$

To see this, φ_1 and φ_4 become the identity element in $\text{Perm}(A, B, C)$ upon removal of the copy-index, leaving φ_2 to generate reflections and φ_3 to generate rotations.

The symmetric incidence matrix M_Φ for the Web inflation is considerably smaller than M . The number of rows of M_Φ is a number of distinct orbits $\text{Orb}_\Phi(m)$ in $\mathcal{S}(\mathcal{M})$. Likewise the number of columns is the number of distinct orbits $\text{Orb}_\Phi(j)$ in $\mathcal{S}(\mathcal{J})$. For the Web inflation in particular, M_Φ is $450 \times 358,120$. Using the symmetric incidence matrix and linear programming methods, an infeasibility certificate was found that is capable of witnessing the Fritz distribution. The corresponding deflated inequality is presented in [Section VII](#) as [Ineq. \(13\)](#).

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