Causal Compatibility Inequalities Admitting of Quantum Violations in the Triangle Scenario

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Quantum Networks Workshop, IIP, Nov. 2016

Triangle Inequalities

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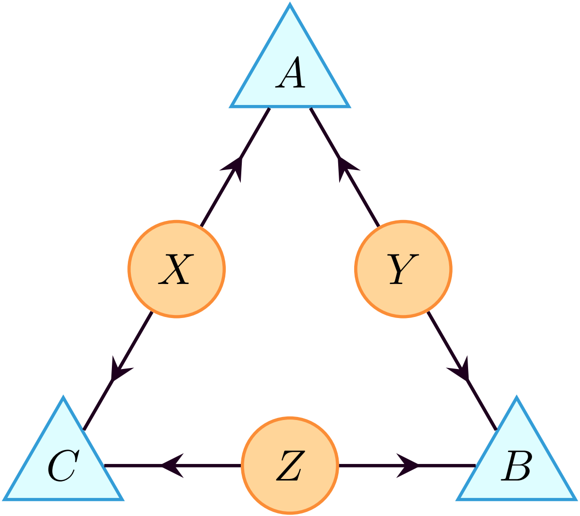
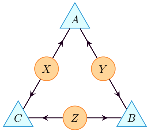
1. Thank International Institute for Physics (IIP) for support
2. Thank Perimeter Institute for Theoretical Physics (PI) for support

The Triangle Scenario (TS)





1. Research project's goal was to derive new causal compatibility inequalities that distinguish quantum correlations from classical correlations
2. Triangle Scenario has be studied extensively before
3. In **Branciard_2012** it was noted that characterizing locality in TS remained an open problem and that identifying compatibility constraints in this configuration **seemed challenging**
4. In **Fritz_2012** Fritz demonstrated that TS is the **smallest** correlation scenario in which their exists quantum incompatible distributions (proof without inequalities)
5. In **Henson_2014** TS was classified as an **interesting** causal structure: conditional independence relations are not a sufficient characterization of compatibility (there are none)
6. Several other authors (see **Steudel_2010 Chaves_2014 Inflation** ...) have investigated TS without achieving research objective

This Talk

1. Inflation technique provides polynomial inequalities
2. Inequalities witnessing non-local quantum correlations in the Triangle Scenario
3. Discuss search for new non-local quantum correlations

1. Inflation technique provides polynomial inequalities
2. Inequalities witnessing non-local quantum correlations in the Triangle Scenario
3. Discuss search for new non-local quantum correlations

1. This talk is going to have three components
2. Demonstrating how something called the inflation technique allows one derive polynomial compatibility constraints for and causal structure
3. Using the inflation technique to find inequalities that demonstrate non-local quantum distributions on the Triangle scenario

Causal Compatibility

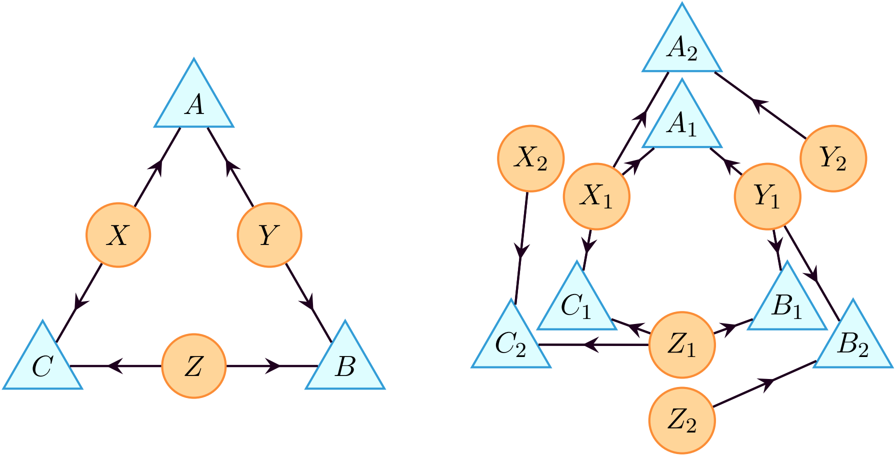
Triangle Scenario

$$P_{ABC} = \int_{XYZ} P_{A|X,Y} P_{B|Y,Z} P_{C|Z,X} P_X P_Y P_Z$$

General Setting

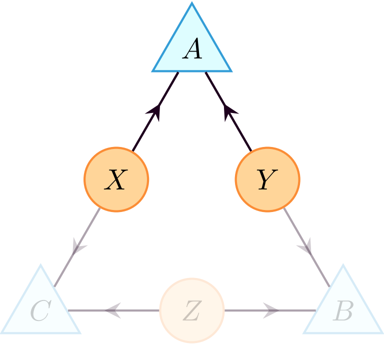
$$P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}} \qquad P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

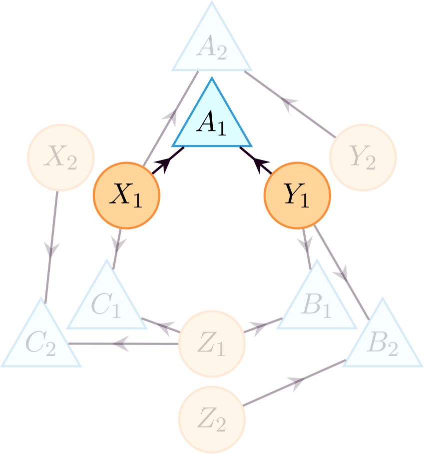
Demonstrating Inflation Technique



$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n)$$

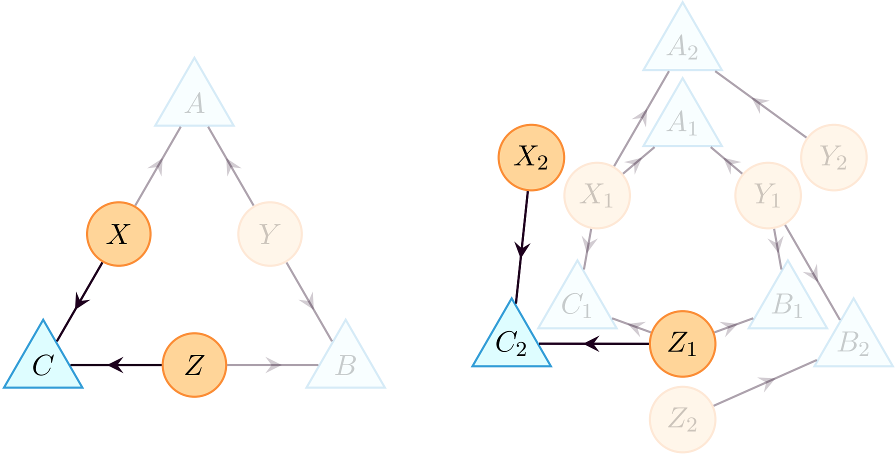
Demonstrating Inflation Technique





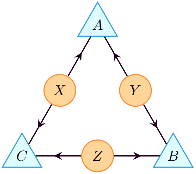
$$\mathsf{AnSub}_{\mathcal{G}}(A) \sim \mathsf{AnSub}_{\mathcal{G}'}(A_1)$$

Demonstrating Inflation Technique

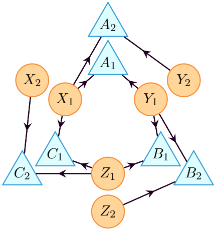


$$\mathsf{AnSub}_{\mathcal{G}}(C) \sim \mathsf{AnSub}_{\mathcal{G}'}(C_2)$$

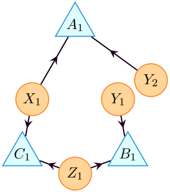
Some Inflations of the Triangle Scenario



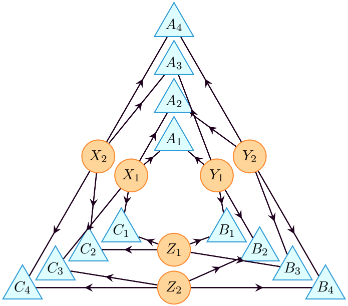
The Triangle Scenario



Spiral Inflation

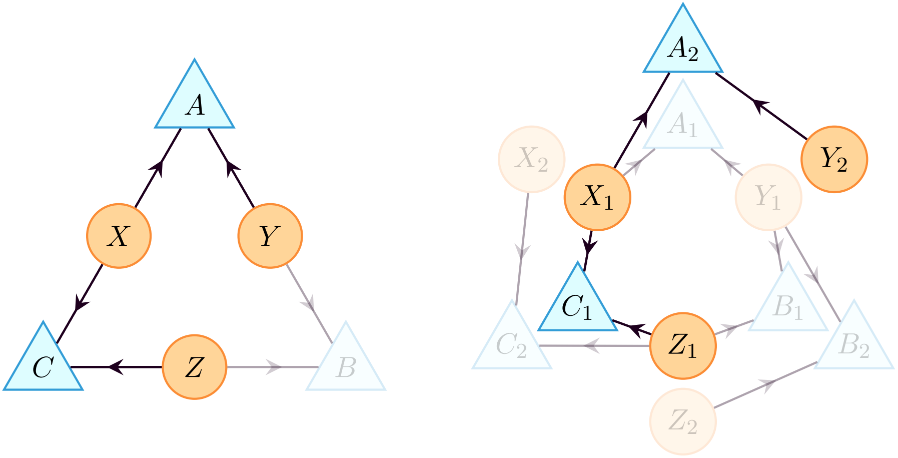


Cut Inflation



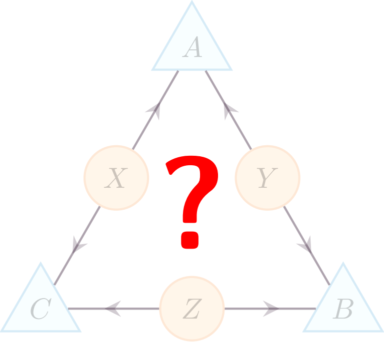
Large Inflation

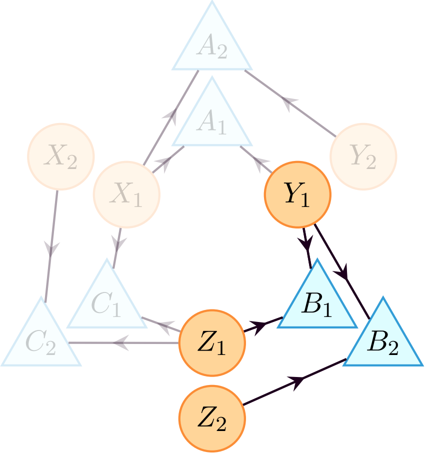
What are Injectable Sets?



$$\mathsf{AnSub}_{\mathcal{G}}(A, C) \sim \mathsf{AnSub}_{\mathcal{G}'}(A_2, C_1)$$

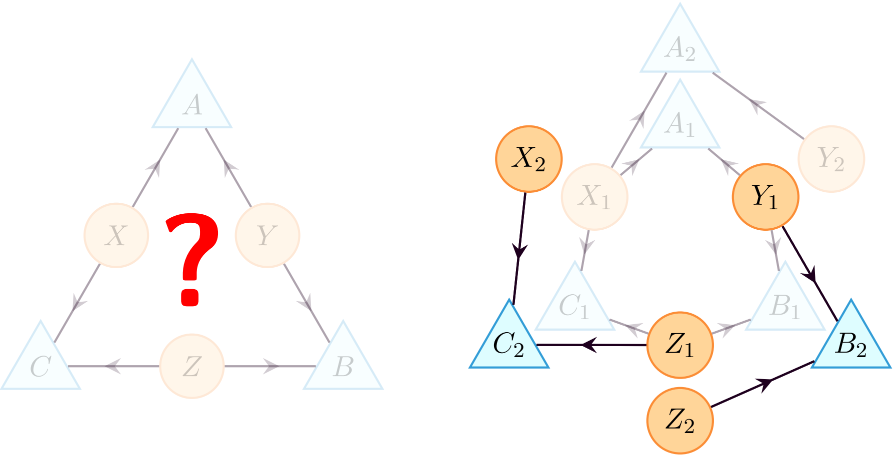
What are Injectable Sets?





$$?? \not\sim \mathsf{AnSub}_{\mathcal{G}'}(B_1, B_2)$$

What are Injectable Sets?



$$?? \not\sim \mathsf{AnSub}_{\mathcal{G}'}(B_2, C_2)$$

Injectable Sets Defined

The injectable sets in $\mathcal{G}'$:

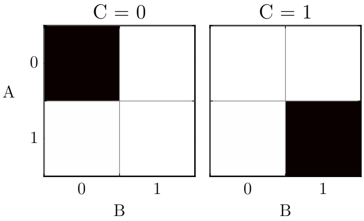
$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')\}$$

The images of the injectable sets in $\mathcal{G}$:

$$\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : \mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')\}$$

Perfect Correlation Is Incompatible

Perfect Correlation

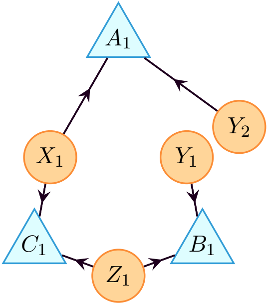


$$\blacksquare = \frac{1}{2}$$

$$P_{ABC}(abc) = \frac{[000] + [111]}{2}$$

$$P_{ABC}(abc) = \begin{cases} \frac{1}{2} & a = b = c \\ 0 & \text{otherwise} \end{cases}$$

Deriving Compatibility Inequalities



$$\mathcal{M} = \{\{A_1, B_1\}, \{B_1, C_1\}, \{A_1, C_1\}\}$$

$$P^{\mathcal{M}} = \{P_{A_1 B_1}, P_{B_1 C_1}, P_{A_1 C_1}\}$$

Compatibility requires: $\exists P_{\mathcal{J}} = P_{A_1 B_1 C_1}$

$$P_{A_1 B_1} = \sum_{C_1} P_{\mathcal{J}} \qquad P_{B_1 C_1} = \sum_{A_1} P_{\mathcal{J}} \qquad P_{A_1 C_1} = \sum_{B_1} P_{\mathcal{J}}$$

Deriving Compatibility Inequalities Cont'd

$$\underbrace{\underbrace{P_{A_1B_1} = \sum\nolimits_{C_1} P_{\mathcal{J}} \qquad P_{B_1C_1} = \sum\nolimits_{A_1} P_{\mathcal{J}} \qquad P_{A_1C_1} = \sum\nolimits_{B_1} P_{\mathcal{J}}}\quad \forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J}\setminus V} P_{\mathcal{J}}}$$

$$\mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

$$\mathcal{P}^{\mathcal{M}} = \left(\begin{array}{c} P_{A_1B_1}(00) \\ P_{A_1B_1}(01) \\ P_{A_1B_1}(10) \\ P_{A_1B_1}(11) \\ \hline P_{B_1C_1}(00) \\ P_{B_1C_1}(01) \\ P_{B_1C_1}(10) \\ P_{B_1C_1}(11) \\ \hline P_{A_1C_1}(00) \\ P_{A_1C_1}(01) \\ P_{A_1C_1}(10) \\ P_{A_1C_1}(11) \end{array}\right) \qquad \mathcal{P}^{\mathcal{J}} = \begin{pmatrix} P_{A_1B_1C_1}(000) \\ P_{A_1B_1C_1}(001) \\ P_{A_1B_1C_1}(010) \\ P_{A_1B_1C_1}(011) \\ P_{A_1B_1C_1}(100) \\ P_{A_1B_1C_1}(101) \\ P_{A_1B_1C_1}(110) \\ P_{A_1B_1C_1}(111) \end{pmatrix}$$

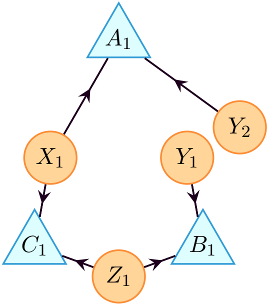
Incidence Example

$$M = \begin{array}{c} (A_1{,}B_1{,}C_1) = \\ (A_1{=}0{,}B_1{=}0) \\ (A_1{=}0{,}B_1{=}1) \\ (A_1{=}1{,}B_1{=}0) \\ (A_1{=}1{,}B_1{=}1) \\ (B_1{=}0{,}C_1{=}0) \\ (B_1{=}0{,}C_1{=}1) \\ (B_1{=}1{,}C_1{=}0) \\ (B_1{=}1{,}C_1{=}1) \\ (A_1{=}0{,}C_1{=}0) \\ (A_1{=}0{,}C_1{=}1) \\ (A_1{=}1{,}C_1{=}0) \\ (A_1{=}1{,}C_1{=}1) \end{array} \begin{array}{c} \begin{array}{cccccccc} (0{,}0{,}0) & (0{,}0{,}1) & (0{,}1{,}0) & (0{,}1{,}1) & (1{,}0{,}0) & (1{,}0{,}1) & (1{,}1{,}0) & (1{,}1{,}1) \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

$$\mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

Cut Inflation Inequalities

$$P_{A_1B_1}(00) \leq P_{B_1C_1}(00) + P_{A_1C_1}(01)$$



$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') = \{\{A_1, C_1\}, \{B_1, C_1\}, \{A_1\}, \{B_1\}, \{C_1\}\}$$

$$\{A_1, B_1\} \neq \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

Example Inequality

- Quick example/preview:

$$
\begin{aligned}
&P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333) \leq \\
&2P(020)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213) + \\
&2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210) + \\
&2P(123)P(310) + 2P(130)P(213) + 2P(132)P(311) + 2P(133)P(210) + \\
&\qquad + \cdots \text{ 324 more terms } \cdots + \\
&P(320)P(323) + P(320)P(333) + P(323)P(330) + P(330)P(333)
\end{aligned}
$$

- $P(abc)$ shorthand for $P_{ABC}(abc)$
- Four outcomes for each A, B, C
- Polynomial in P_{ABC}, marginals P_{AB}, P_{BC}, P_{AC}, P_A, P_B, P_C

Triangle Inequalities

└─Example Inequality

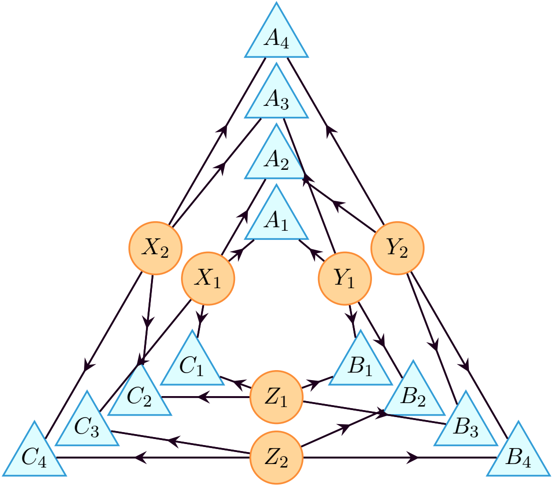
- Quick example/preview:

$$\begin{gathered}
P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333) \leq \\
2P(020)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213) + \\
2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210) + \\
2P(123)P(310) + 2P(130)P(213) + 2P(132)P(311) + 2P(133)P(210) + \\
+ \cdots \text{ 324 more terms } \cdots + \\
P(320)P(323) + P(320)P(333) + P(323)P(330) + P(330)P(333)
\end{gathered}$$

- $P(abc)$ shorthand for $P_{ABC}(abc)$
- Four outcomes for each A, B, C
- Polynomial in P_{ABC}, marginals P_{AB}, P_{BC}, P_{AC}, P_A, P_B, P_C

1. As a quick example or preview of what is to come, here is a an example inequality admits quantum violations in the triangle scenario
2. Some features of note: inequality is polynomial in P_{ABC} and its marginals

Pre-injectable Sets of Large Inflation



- Has 12 maximal pre-injectable sets (to follow)

Large Inflation Pre-injectable Sets

Maximal Pre-injectable Sets	**Ancestral Independences**
$\{A_1, B_1, C_1, A_4, B_4, C_4\}$	$\{A_1, B_1, C_1\} \perp \{A_4, B_4, C_4\}$
$\{A_1, B_2, C_3, A_4, B_3, C_2\}$	$\{A_1, B_2, C_3\} \perp \{A_4, B_3, C_2\}$
$\{A_2, B_3, C_1, A_3, B_2, C_4\}$	$\{A_2, B_3, C_1\} \perp \{A_3, B_2, C_4\}$
$\{A_2, B_4, C_3, A_3, B_1, C_2\}$	$\{A_2, B_4, C_3\} \perp \{A_3, B_1, C_2\}$
$\{A_1, B_3, C_4\}$	$\{A_1\} \perp \{B_3\} \perp \{C_4\}$
$\{A_1, B_4, C_2\}$	$\{A_1\} \perp \{B_4\} \perp \{C_2\}$
$\{A_2, B_1, C_4\}$	$\{A_2\} \perp \{B_1\} \perp \{C_4\}$
$\{A_2, B_2, C_2\}$	$\{A_2\} \perp \{B_2\} \perp \{C_2\}$
$\{A_3, B_3, C_3\}$	$\{A_3\} \perp \{B_3\} \perp \{C_3\}$
$\{A_3, B_4, C_1\}$	$\{A_3\} \perp \{B_4\} \perp \{C_1\}$
$\{A_4, B_1, C_3\}$	$\{A_4\} \perp \{B_1\} \perp \{C_3\}$
$\{A_4, B_2, C_1\}$	$\{A_4\} \perp \{B_2\} \perp \{C_1\}$

Deriving Inequalities

- Inflation facilitates turning linear, inflated inequalities into polynomial deflated ones
- **Question:** How to derive compatibility inequalities for $\mathcal{G}'$?
- **Answer:** Use your favorite technique for deriving compatibility inequalities:
  - Entropic inequalities
  - Finite outcome inequalities
- Here we solve the marginal problem for $\mathcal{M} = \mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$
- First some notation and formalism

Outcomes and Events

Definition

Each variable v has finite set of outcomes O_v.
Each set of variables V has finite set of events $\mathcal{E}(V)$:

$$\mathcal{E}(V) \equiv \{s : V \to O_V \mid \forall v \in V, s(v) \in O_v\}$$

Definition

The set of events over the joint variables $\mathcal{E}(\mathcal{J})$ are termed the joint events.

Definition

The set of events over the marginal contexts are the marginal events

$$\mathcal{E}(\mathcal{M}) \equiv \coprod_{V \in \mathcal{M}} \mathcal{E}(V)$$

Distribution Vectors

Definition

The joint distribution vector $\mathcal{P}^{\mathcal{J}}$

$$\mathcal{P}^{\mathcal{J}}_j = P_{\mathcal{J}}(j) \quad \forall j \in \mathcal{E}(\mathcal{J})$$

Definition

The marginal distribution vector $\mathcal{P}^{\mathcal{M}}$

$$\mathcal{P}^{\mathcal{M}}_m = P_{\mathcal{D}(m)}(m) \quad \forall m \in \mathcal{E}(\mathcal{M}), \mathcal{D}(m) \in \mathcal{M}$$

Can now write complete marginal problem as matrix multiplication:

$$\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J} \setminus V} P_{\mathcal{J}} \iff \mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

Incidence Matrix

- Incidence matrix M is a bit-wise matrix
- Row-indexed by marginal events $m \in \mathcal{E}(\mathcal{M})$
- Column-indexed by joint events $j \in \mathcal{E}(\mathcal{J})$

$$M_{m,j} = \begin{cases} 1 & m = j|_{\mathcal{D}(m)} \\ 0 & \text{otherwise} \end{cases}$$

$$\#\text{Columns} = |\mathcal{E}(\mathcal{J})| = \prod_{v \in \mathcal{J}} |O_v|$$

$$\#\text{Rows} = |\mathcal{E}(\mathcal{M})| = \sum_{V \in \mathcal{M}} \prod_{v \in V} |O_v|$$

Example

Let $\mathcal{J}$ be 3 binary variables $\mathcal{J} = \{A, B, C\}$ and $\mathcal{M}$ be the marginal scenario $\mathcal{M} = \{\{A, B\}, \{B, C\}, \{A, C\}\}$. The incidence matrix becomes:

Marginal Linear Program

- Obtain inequalities from incidence matrix M and known incompatible distribution $\mathcal{P}^{\mathcal{M}}$

Marginal LP:

$$\begin{aligned} \text{minimize:} \quad & \emptyset \cdot x \\ \text{subject to:} \quad & x \succeq 0 \\ & M \cdot x = \mathcal{P}^{\mathcal{M}} \end{aligned}$$

Dual Marginal LP:

$$\begin{aligned} \text{minimize:} \quad & y \cdot \mathcal{P}^{\mathcal{M}} \\ \text{subject to:} \quad & y \cdot M \succeq 0 \end{aligned}$$

- y is an infeasibility certificate:

$$y \cdot M \cdot x = y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$$

- Infeasibility inequality: $y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$
- Most linear programing toolkits return certificates (*Mosek*, *Gurobi*, *CPLEX*, *cvxr*/*cvxopt*.)

Triangle Inequalities

Marginal Linear Program

- Obtain inequalities from incidence matrix M and known incompatible distribution $\mathcal{P}^{\mathcal{M}}$

Marginal LP:

$$\begin{aligned} \text{minimize:}\quad & \emptyset \cdot x \\ \text{subject to:}\quad & x \succeq 0 \\ & M \cdot x = \mathcal{P}^{\mathcal{M}} \end{aligned}$$

Dual Marginal LP:

$$\begin{aligned} \text{minimize:}\quad & y \cdot \mathcal{P}^{\mathcal{M}} \\ \text{subject to:}\quad & y \cdot M \succeq 0 \end{aligned}$$

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$$y \cdot M \cdot x = y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$$

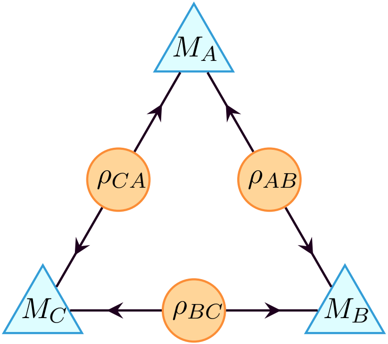
- Infeasibility inequality: $y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$
- Most linear programing toolkits return certificates (*Mosek*, *Gurobi*, *CPLEX*, *cvxr*/*cvxopt*.)

1. It is possible to derive non-contextuality inequalities from M and a known incompatible distribution $\mathcal{P}^{\mathcal{M}}$
2. Note the null objective; the minimization objective is trivially always zero
3. The primal value of the linear program is of no interest, all that matters is its *feasibility*.
4. If *feasible*, then there exists a vector x that is a valid joint distribution vector $\mathcal{P}^{\mathcal{J}}$.
5. Feasibility implies that $\mathcal{P}^{\mathcal{J}} = x$.
6. Infeasibility implies contextuality

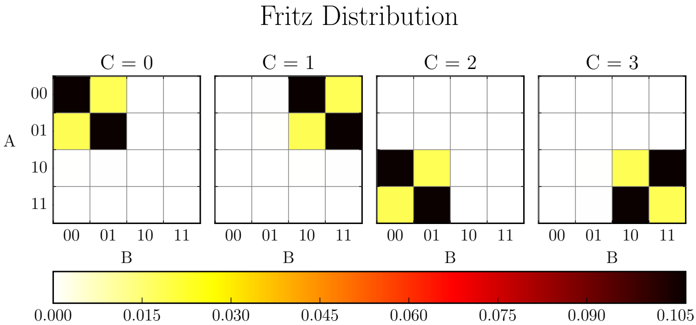
Known Incompatible Distribution in The Triangle Scenario

Fritz_2012 provides quantum-accessible distribution incompatible with TS

$$P_{ABC}(abc) = \mathrm{Tr}[\Pi^{\intercal} \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$



Fritz Distribution



$$\square = \frac{1}{32}\left(2-\sqrt{2}\right) \qquad \blacksquare = \frac{1}{32}\left(2+\sqrt{2}\right)$$

Fritz Distribution Violating CHSH

- C's outcome acts as measurement "setting" for A, B; independent of ρ_{AB}
- Correlation between right bits

$$\langle A_r B_r \rangle = P_{A_r B_r}(00) + P_{A_r B_r}(11) - P_{A_r B_r}(01) - P_{A_r B_r}(10)$$

$$\langle A_r B_r | C = 0, 1, 2 \rangle = \frac{1}{\sqrt{2}} \quad \langle A_r B_r | C = 3 \rangle = -\frac{1}{\sqrt{2}}$$

- Gives CHSH violation

$$\begin{aligned} &\langle A_r B_r | C = 0 \rangle + \langle A_r B_r | C = 1 \rangle + \langle A_r B_r | C = 2 \rangle - \langle A_r B_r | C = 3 \rangle \\ &= 3\left(\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) \\ &= 2\sqrt{2} \not\leq 2 \end{aligned}$$

Notes of Fritz Distribution

- Proof contingent on perfect correlation between C and pseudo-settings
- Proof not robust to noise

Problem (2.17 in Fritz_2012)

Find an example of non-classical quantum correlations in TS together with a proof of its non-classicality which does not hinge on Bell's Theorem.

- "...would be helpful to have inequalities..."
- Possible to find inequalities violated by P_F using the Large inflation of the TS

Large Inflation Incidence

- Joint variables are all of the observable nodes $\mathcal{N}'_O = \mathcal{J}$

$$\mathcal{J} = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4\}$$

- Marginal scenario is composed of pre-injectable sets $\mathcal{M} = \mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$
- Inequalities violated by Fritz distribution are inherently 4-outcome
- Incidence matrix M is **very large** $\sim 2.25\text{Gb}$
  - $\#$Columns $= 4^{12} = 16,777,216$
  - $\#$Rows $= 4 \times 4^6 + 8 \times 4^3 = 16,896$
  - $\#$Non-zero Entries $= 201,326,592$

Triangle Scenario Inequality

Causal Symmetry

- Desirable to find compatibility inequality I such that

$$\forall \varphi \in \mathsf{Perm}(A, B, C) : \varphi[I] = I$$

- Compatibility is independent of variable labels $I, \mathcal{G} \to \varphi[I], \varphi[\mathcal{G}]$
- Need $\varphi[\mathcal{G}] = \mathcal{G}$ to find new $\varphi[I]$

Definition

The causal symmetry group of causal structure $\mathcal{G}$:

$$\mathsf{Aut}(\mathcal{G}) = \{\varphi \in \mathsf{Perm}(\mathcal{N}) \mid \varphi[\mathcal{G}] = \mathcal{G}\}$$

Strictly speaking, one needs to preserve observable nodes:

$$\mathsf{Aut}_{\mathcal{N}_O}(\mathcal{G}) = \{\varphi \in \mathsf{Aut}(\mathcal{G}) \mid \varphi[\mathcal{N}_O] = \mathcal{N}_O\}$$

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1. Fritz distribution is incompatible with Triangle scenario because party C plays the role of measurement settings for both A and B
2. In order to find quantum distributions different from P_F in the Triangle Scenario, it is therefore desirable to find a proof of its incompatibility (i.e. inequality) that is symmetric under exchange of parties
3. Surprisingly, it is possible to do so!
4. Here is how.
5. First, we will formally define the symmetry group in question
6. Causal symmetry group is the group of automorphisms on causal structure

Causal Symmetry and Inflation

- Causal symmetry group for $\mathcal{G}'$ is no good!
- Not possible to deflate inequality if it's not in terms of injectable sets

Definition

The restricted causal symmetry group Φ of $\mathcal{G}'$:

$$\Phi = \mathsf{Aut}_{\mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')}(\mathcal{G}')$$

Restricted Causal Symmetry of Large Inflation

- Φ for the large inflation is an order 48 group with 4 generators

$$
\begin{array}{cccc}
\varphi 1 & \varphi 2 & \varphi 3 & \varphi 4 \\
A_1 \to A_4 & A_1 \to A_1 & A_1 \to C_1 & A_1 \to A_1 \\
A_2 \to A_3 & A_2 \to A_3 & A_2 \to C_2 & A_2 \to A_2 \\
A_3 \to A_2 & A_3 \to A_2 & A_3 \to C_3 & A_3 \to A_3 \\
A_4 \to A_1 & A_4 \to A_4 & A_4 \to C_4 & A_4 \to A_4 \\
B_1 \to B_4 & B_1 \to C_1 & B_1 \to A_1 & B_1 \to B_2 \\
B_2 \to B_3 & B_2 \to C_3 & B_2 \to A_2 & B_2 \to B_1 \\
B_3 \to B_2 & B_3 \to C_2 & B_3 \to A_3 & B_3 \to B_4 \\
B_4 \to B_1 & B_4 \to C_4 & B_4 \to A_4 & B_4 \to B_3 \\
C_1 \to C_4 & C_1 \to B_1 & C_1 \to B_1 & C_1 \to C_3 \\
C_2 \to C_3 & C_2 \to B_3 & C_2 \to B_2 & C_2 \to C_4 \\
C_3 \to C_2 & C_3 \to B_2 & C_3 \to B_3 & C_3 \to C_1 \\
C_4 \to C_1 & C_4 \to B_4 & C_4 \to B_4 & C_4 \to C_2
\end{array}
$$

Symmetric Incidence

- Group orbits through repeated action of $\varphi \in \Phi$ on $m \in \mathcal{E}(\mathcal{M})$ and $j \in \mathcal{E}(\mathcal{J})$

$$\Phi[m] \equiv \{\varphi[m] \mid \varphi \in \Phi\}$$
$$\Phi[j] \equiv \{\varphi[j] \mid \varphi \in \Phi\}$$

- Construct symmetric incidence matrix $\Phi[M]$

$$\Phi[M]_{\Phi[m],\Phi[j]} = \sum_{m' \in \Phi[m]} \sum_{j' \in \Phi[j]} M_{m',j'}$$

$$\Phi[M] = \Lambda_{\Phi[\mathcal{E}(\mathcal{M})]} \cdot M \cdot \Lambda_{\Phi[\mathcal{E}(\mathcal{J})]}$$

- $\Phi[M]$ not a bit-wise matrix like M
- For large inflation M is $16,896 \times 16,777,216$
- For large inflation $\Phi[M]$ is $450 \times 358,120$

Party Symmetric Inequality

$$2[P(001)P(333)]_3 + 2[P(010)P(323)]_3 + 6[P(000)P(323)]_3 + 6[P(000)P(333)]_1$$

$$\leq$$

$$\begin{gathered}
12[P(031)P(302)]_6 + 12[P(033)P(303)]_6 + 12[P(103)P(130)]_6 + 12[P(203)P(230)]_6 + 12[P(203)P(330)]_6 + 2[P(001)P(320)]_6 + 2[P(002)P(221)]_3 + 2[P(003)P(211)]_6 + \\
2[P(003)P(331)]_3 + 2[P(011)P(211)]_3 + 2[P(012)P(322)]_6 + 2[P(013)P(313)]_6 + 2[P(013)P(332)]_6 + 2[P(020)P(111)]_3 + 2[P(020)P(211)]_6 + 2[P(021)P(212)]_6 + \\
2[P(022)P(211)]_3 + 2[P(022)P(212)]_6 + 2[P(022)P(322)]_3 + 2[P(023)P(232)]_6 + 2[P(030)P(212)]_3 + 2[P(031)P(231)]_6 + 2[P(032)P(331)]_6 + 2[P(033)P(333)]_3 + \\
2[P(101)P(131)]_3 + 2[P(101)P(132)]_6 + 2[P(102)P(131)]_6 + 2[P(102)P(132)]_6 + 2[P(102)P(133)]_6 + 2[P(110)P(133)]_6 + 2[P(110)P(212)]_6 + 2[P(110)P(222)]_3 + \\
2[P(110)P(223)]_3 + 2[P(112)P(331)]_3 + 2[P(120)P(122)]_6 + 2[P(121)P(201)]_6 + 2[P(122)P(200)]_3 + 2[P(122)P(202)]_6 + 2[P(122)P(210)]_6 + 2[P(122)P(300)]_3 + \\
2[P(130)P(232)]_6 + 2[P(130)P(233)]_6 + 2[P(131)P(201)]_6 + 2[P(131)P(202)]_3 + 2[P(131)P(313)]_3 + 2[P(133)P(200)]_3 + 2[P(133)P(201)]_6 + 2[P(133)P(211)]_3 + \\
2[P(133)P(212)]_6 + 2[P(133)P(300)]_3 + 2[P(202)P(231)]_6 + 2[P(210)P(222)]_6 + 2[P(220)P(222)]_3 + 2[P(220)P(313)]_6 + 2[P(221)P(313)]_6 + 2[P(222)P(331)]_3 + \\
2[P(223)P(331)]_3 + 2[P(230)P(312)]_6 + 2[P(231)P(313)]_6 + 2[P(232)P(320)]_6 + 2[P(302)P(322)]_6 + 2[P(320)P(323)]_6 + 2[P(330)P(332)]_3 + 3[P(000)P(003)]_3 + \\
3[P(010)P(301)]_6 + 4[P(001)P(131)]_6 + 4[P(002)P(020)]_6 + 4[P(002)P(133)]_6 + 4[P(002)P(323)]_6 + 4[P(010)P(123)]_6 + 4[P(013)P(212)]_6 + 4[P(013)P(312)]_6 + \\
4[P(023)P(221)]_6 + 4[P(023)P(222)]_6 + 4[P(023)P(322)]_6 + 4[P(031)P(211)]_6 + 4[P(032)P(321)]_6 + 4[P(100)P(123)]_6 + 4[P(100)P(232)]_6 + 4[P(100)P(313)]_6 + \\
4[P(112)P(310)]_6 + 4[P(122)P(203)]_6 + 4[P(122)P(302)]_6 + 4[P(130)P(222)]_6 + 4[P(130)P(223)]_6 + 4[P(222)P(310)]_6 + 4[P(223)P(320)]_6 + 4[P(231)P(301)]_6 + \\
4[P(312)P(330)]_6 + 6[P(001)P(031)]_6 + 6[P(001)P(033)]_6 + 6[P(002)P(300)]_6 + 6[P(002)P(330)]_3 + 6[P(003)P(032)]_6 + 6[P(003)P(131)]_6 + 6[P(003)P(132)]_6 + \\
6[P(011)P(300)]_3 + 6[P(011)P(320)]_6 + 6[P(012)P(200)]_6 + 6[P(012)P(301)]_6 + 6[P(013)P(030)]_6 + 6[P(013)P(110)]_6 + 6[P(013)P(120)]_6 + 6[P(013)P(303)]_6 + \\
6[P(020)P(102)]_6 + 6[P(020)P(103)]_6 + 6[P(020)P(123)]_6 + 6[P(020)P(202)]_3 + 6[P(020)P(203)]_6 + 6[P(020)P(311)]_6 + 6[P(020)P(322)]_6 + 6[P(020)P(330)]_6 + \\
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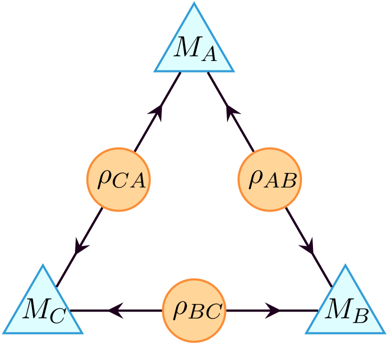
6[P(033)P(322)]_3 + 6[P(100)P(203)]_6 + 6[P(101)P(130)]_6 + 6[P(103)P(310)]_6 + 6[P(113)P(130)]_6 + 6[P(113)P(230)]_6 + 6[P(113)P(330)]_3 + 6[P(122)P(330)]_6 + \\
6[P(130)P(313)]_6 + 6[P(132)P(303)]_6 + 6[P(133)P(303)]_6 + 6[P(133)P(320)]_6 + 6[P(200)P(203)]_6 + 6[P(201)P(230)]_6 + 6[P(203)P(231)]_6 + 6[P(223)P(300)]_6 + \\
8[P(003)P(320)]_6 + 8[P(032)P(300)]_6
\end{gathered}$$

Parameterizing Quantum Distributions

For our purposes, we need to parameterize the space of quantum-accessible distributions that are *realized* on the Triangle Scenario

$$P_{ABC}(abc) = \mathrm{Tr}[\Pi^{\intercal} \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$



Numerical Optimization

- Attempt to find new non-classical distributions
- Objective function $f(\lambda) \in \mathbb{R}$
  1. Real-valued parameters $\lambda = (\lambda_0, \dots, \lambda_n)$
  2. Quantum states/measurements $\rho_{AB}, \rho_{BC}, \rho_{CA}, M_A, M_B, M_C$
  3. Distribution P_{ABC}
  4. Plug into inequality I in homogeneous form $I(P_{ABC}) \geq 0$
  5. Output is objective value $I(P_{ABC})$
- Numerical minimization of $f(\lambda)$

$$f(\lambda_{(k+1)}) = \lambda_{(k)} - \gamma_{(k)} \nabla f(\lambda_{(k)})$$

- Non-convex, non-linear, smooth/continuous
- Gradient Descent, BFGS Method, Nelder-Mead simplex method
- Stochastic methods: simulated annealing, basin-hopping

Parameterizing Unitary Group

- Spengler, Huber and Heismayr **Spengler_2010_Unitary** demonstrate a parameterization of $\mathcal{U}(d)$ where the parameters are organized in a $d \times d$-matrix of real values $\lambda_{n,m}$

$$U = \left[\prod_{m=1}^{d-1} \left(\prod_{n=m+1}^{d} R_{m,n} RP_{n,m} \right) \right] \cdot \left[\prod_{l=1}^{d} GP_l \right]$$

- Global Phase Terms: $GP_l = \exp(iP_l \lambda_{l,l})$
- Relative Phase Terms: $RP_{n,m} = \exp(iP_n \lambda_{n,m})$
- Rotation Terms: $R_{m,n} = \exp(i\sigma_{m,n} \lambda_{m,n})$
- Projection Operators: $P_l = |l\rangle\langle l|$
- Anti-symmetric σ-matrices: $\sigma_{m,n} = -i|m\rangle\langle n| + i|n\rangle\langle m|$
- Parameters $\lambda_{n,m} \in [0, 2\pi]$

- Each parameter $\lambda_{n,m}$ has physical interpretation
- Degeneracies are easily eliminated such as global phase

$$\forall l = 1, \ldots, d : \lambda_{l,l} = 0 \implies GP_l = \mathbb{1}$$

- Parameterize $U \in \mathcal{U}(d)$ up to global phase denoted $\tilde{U} \in \mathcal{U}(d)$
- Computationally efficient

$$\begin{aligned} GP_l &= \mathbb{1} + P_l\left(e^{i\lambda_{l,l}} - 1\right) \\ RP_{n,m} &= \mathbb{1} + P_n\left(e^{i\lambda_{n,m}} - 1\right) \\ R_{m,n} &= \mathbb{1} + (|m\rangle\langle m| + |n\rangle\langle n|)(\cos\lambda_{n,m} - 1) \\ &\quad + (|m\rangle\langle n| - |n\rangle\langle m|)\sin\lambda_{n,m} \end{aligned}$$

Parameterizing States

- Each latent resource $\rho \in (\rho_{AB}, \rho_{BC}, \rho_{CA})$ modeled as bipartite qubit state acting on $\mathcal{H}^{d/2} \otimes \mathcal{H}^{d/2}$
- $d \times d$ positive semi-definite (PSD) hermitian matrices with unitary trace
- Cholesky Parametrization allows one to write any hermitian PSD as $\rho = T^\dagger T$
- For $d = 4$:

$$T = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_2 + i\lambda_3 & \lambda_4 & 0 & 0 \\ \lambda_5 + i\lambda_6 & \lambda_7 + i\lambda_8 & \lambda_9 & 0 \\ \lambda_{10} + i\lambda_{11} & \lambda_{12} + i\lambda_{13} & \lambda_{14} + i\lambda_{15} & \lambda_{16} \end{bmatrix}$$

- d^2 real-valued parameters
- Normalized $\rho = T^\dagger T / \mathrm{Tr}\left(T^\dagger T\right)$ adds degeneracy

Parameterizing States Cont'd

- SHH parameterization **Spengler_2010_Unitary** exploits spectral decomposition; for rank $k \leq d$ density matrix

$$\rho = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i| \qquad p_i \geq 0, \sum_i p_i = 1$$

- Orthonormal k-element sub-basis $\{|\psi_i\rangle\}$ of $\mathcal{H}^d$ can be transformed into computational basis $\{|i\rangle\}$ by unitary $U \in \mathcal{U}(d)$ such that $|\psi_i\rangle = U|i\rangle$
- Freedom to choice k
- Parameterize ρ through $\{p_i\}$ and $\tilde{U}_k$

$$\tilde{U}_k = \prod_{m=1}^{k} \left(\prod_{n=m+1}^{d} R_{m,n} RP_{n,m} \right)$$

- real-value parameters $d^2 - (d-k)^2 - k$ for $\tilde{U}_k$, $k-1$ for $\{p_i\}$ (no degeneracy)

Parameterizing POVMs

- Each party (A, B, C) is assigned a projective-operator valued measure (POVM) (M_A, M_B, M_C)

$$\forall |\psi\rangle \in \mathcal{H}^d : \langle \psi | M_\chi | \psi \rangle \geq 0 \quad M_\chi = M_\chi^\dagger$$

- n-outcome measurement

$$M_\chi = \{M_{\chi,1}, \ldots, M_{\chi,n}\} \quad \sum_{i=1}^{n} M_{\chi,i} = \mathbb{1}$$

- For $n = 2$ outcomes, a parameterization exists by constraining the eigenvalues of $M_{\chi,i}$; for $n > 2$ not aware of anything
- Warrants consideration of projective-valued measures (PVMs) (for $n = d$ this is without loss of generality)

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- Warrants consideration of projective-valued measures (PVMs) (for $n = d$ this is without loss of generality)

1. Naimark's Dilation Theorem

Parameterizing PVMs

- Each party (A, B, C) is assigned n-outcome (M_A, M_B, M_C) such that,

$$M_{\chi,i}M_{\chi,j} = \delta_{ij}M_{\chi,i} \quad M_{\chi,i} = |m_{\chi,i}\rangle\langle m_{\chi,i}|$$

- Inspired by **Pal_2010** parameterizing PVMs means parameterizing a n-element sub-basis $\{|m_{\chi,i}\rangle\}$
- Use unitary transformation again

$$\{|m_{\chi,1}\rangle, \dots, |m_{\chi,n}\rangle\} = \{U|1\rangle, \dots, U|n\rangle\}$$

- Global phase and remaining basis irrelevant: $\tilde{U}_n$ requires $n(2d - n - 1)$ real-valued parameters
- PVMs are computationally more efficient

$$P_{ABC}(abc) = \langle m_{A,a}m_{B,b}m_{C,c}|\Pi^{\intercal}\rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA}\Pi|m_{A,a}m_{B,b}m_{C,c}\rangle$$

Results

- Full rank ρ, M gives 81 free parameters

$$|\lambda| = 3 \cdot (12 + 3) + 3 \cdot 12 = 81$$

- Parameterization of quantum distributions still degenerate
- Noisy seed (Gaussian noise):

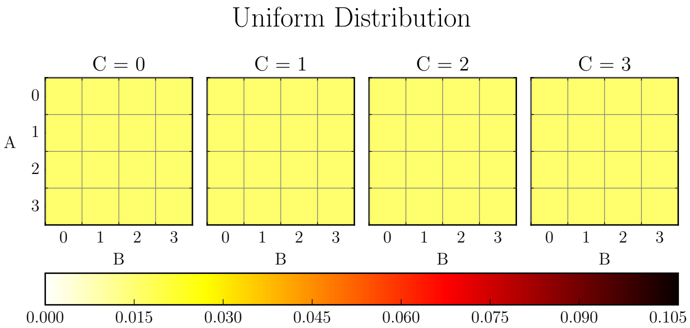
$$\lambda_{(0)} = \lambda_{(F)} + \delta\lambda \qquad \delta\lambda_i \sim \mathcal{N}\Big(\mu = 0, \sigma^2 = (2\pi 10^{\chi})^2\Big)$$

- Uniform seed:

$$\lambda_{(0),i} \sim \mathcal{U}([0, 2\pi])$$

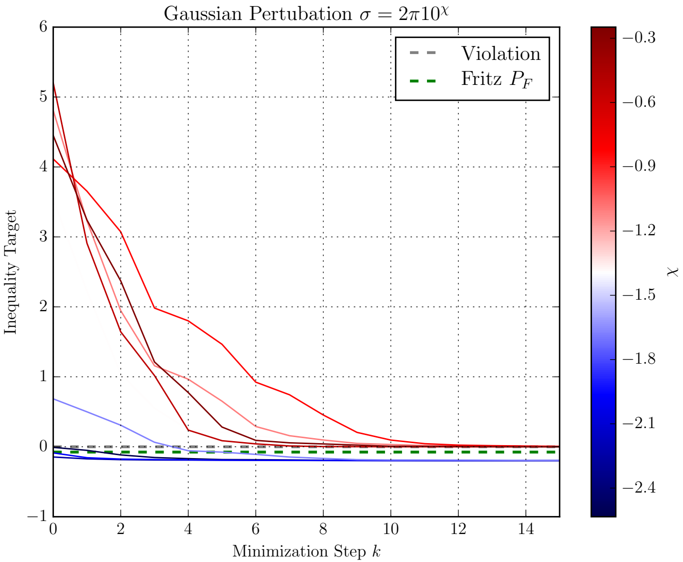
Uniform Noise

$$\mathcal{U}_\varepsilon(P) = (1+\varepsilon)P + \varepsilon\mathcal{U}$$

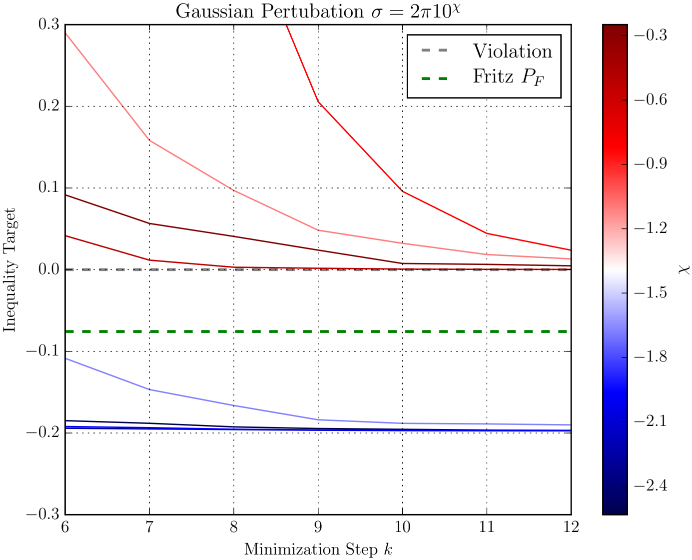


$$= \frac{1}{64}$$

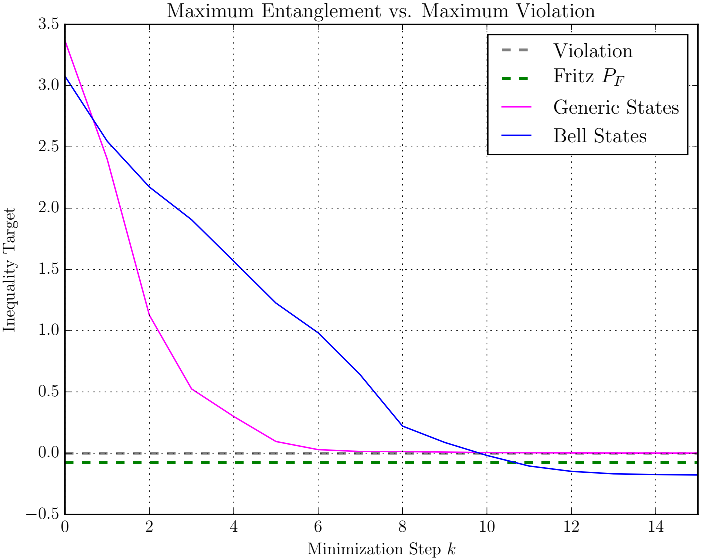
Fritz Local Minima



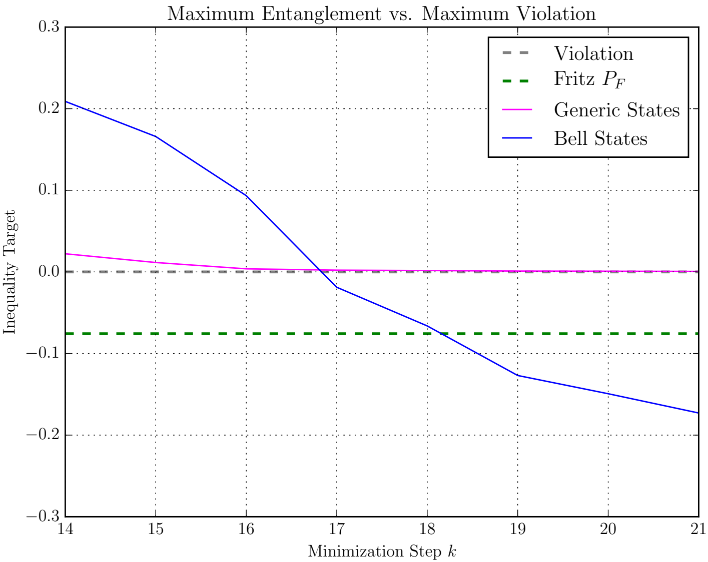
Fritz Local Minima Zoomed



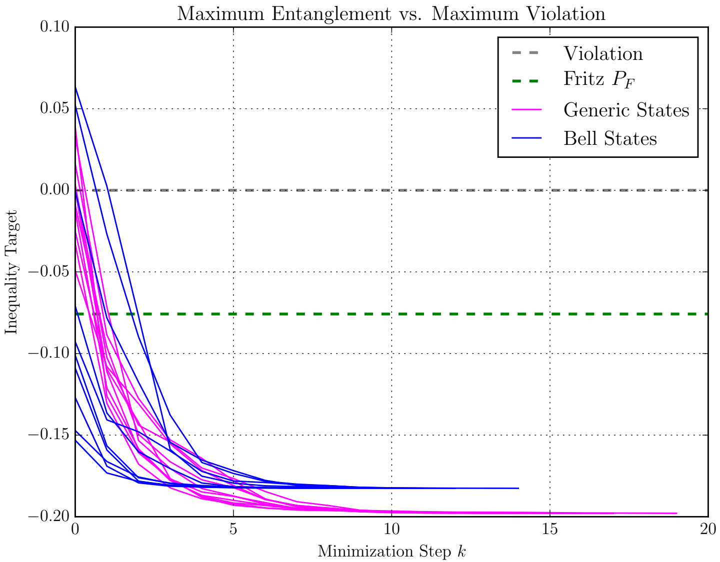
Max Entangled vs. Max Violating (???) [Optional]



Max Entangled vs. Max Violating (???) [Optional]



Max Entangled vs. Max Violating



Maximally Violating Distributions

- Able to out-perform violation provided by Fritz distribution
- Maximally-violating states are not maximally-entangled; similar to detection loop-hole example of **Methot_2006**
- Violation very sensitive to the initial parameters $\lambda_{(0)}$
- Both symmetric and asymmetric inequalities exhibit same qualitative features

Conclusions

- New causal compatibility inequalities have been found for the TS
- Inflation technique capable of producing inequalities with quantum/classical witnesses
- Proof of non-classicality is robust to noise
- Fritz witness-able by party symmetric inequalities
- Maximally violating distributions are different than Fritz but also similar
- Further research is necessary

Post-doc Opportunities At Perimeter

References I

Notation [Optional]

Question: Which marginal models $P^{\mathcal{M}}$ are compatible with a causal structure $\mathcal{G}$?

- Marginal model $P^{\mathcal{M}}$ is collection of probability distributions

$$P^{\mathcal{M}} = \{P_{V_1}, \dots, P_{V_k}\}$$

- Marginal scenario $\mathcal{M} = \{V_1, \dots, V_k\}$

$$V \in \mathcal{M}, V' \subseteq V \implies V' \in \mathcal{M}$$

- Joint random variables $\mathcal{J} = \bigcup_i V_i = \{v_1, \dots, v_n\}$
- Causal Structure $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is a directed acyclic graph (DAG)
- Nodes classified into latent nodes $\mathcal{N}_L$ and observed nodes $\mathcal{N}_O$

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1. Before continuing, I will define exactly what I mean by causal compatibility
2. Causal compatibility refers to the compatibility between causal structures and marginal models
3. Marginal model is collection of probability distributions over sets of random variables
4. Marginal scenario refers to the those sets of random variables
5. The complete set of random variables are the joint random variables

Let $n, m \in \mathcal{N}$ be nodes of the graph $\mathcal{G}$.

- parents of n: $\mathsf{Pa}_{\mathcal{G}}(n) \equiv \{m \mid m \to n\}$
- children of n: $\mathsf{Ch}_{\mathcal{G}}(n) \equiv \{m \mid n \to m\}$
- ancestry of n: $\mathsf{An}_{\mathcal{G}}(n) \equiv \bigcup_{i \in \mathbb{W}} \mathsf{Pa}^{i}_{\mathcal{G}}(n)$

$$\mathsf{Pa}^{0}_{\mathcal{G}}(n) = n \qquad \mathsf{Pa}^{i}_{\mathcal{G}}(n) \equiv \mathsf{Pa}_{\mathcal{G}}\Big(\mathsf{Pa}^{i-1}_{\mathcal{G}}(n)\Big)$$

Notation extends to sets of nodes $N \subseteq \mathcal{N}$,

- parents of N: $\mathsf{Pa}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Pa}_{\mathcal{G}}(n)$
- children of N: $\mathsf{Ch}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Ch}_{\mathcal{G}}(n)$
- ancestry of N: $\mathsf{An}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{An}_{\mathcal{G}}(n)$

An induced subgraph of $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ due to $N \subseteq \mathcal{N}$

$$\mathsf{Sub}_{\mathcal{G}}(N) = (N, \{e \in \mathcal{E} \mid e \subseteq N\})$$

Causal Compatibility [Optional]

Question: Which marginal models $P^{\mathcal{M}}$ are compatible with a causal structure $\mathcal{G}$?

Answer: $P^{\mathcal{M}}$ is compatible with $\mathcal{G}$ if there exists a set of casual parameters

$$\left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\}$$

Such that for each $V \in \mathcal{M}$, P_V can be recovered:

1. $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$
2. $P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$

Inequality: A casual compatibility inequality I is an inequality over $P^{\mathcal{M}}$ that is satisfied by all compatible $P^{\mathcal{M}}$

Deriving Inequalities [Optional]

Two necessary components to compatibility:

1. Marginal problem: $\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{N}\setminus V} P_\mathcal{N}$
   - Is the marginal model contextual or non-contextual?
   - 3 distinct ways to tackle this problem
     1. Convex hull, Polytope projection, Fourier-Motzkin
     2. Possibilistic Hardy Inequalities (Hypergraph transversals)
     3. Linear Program Feasibility/Infeasibility
2. Markov Separation: $P_\mathcal{N} = \prod_{n\in\mathcal{N}} P_{n|\mathsf{Pa}_\mathcal{G}(n)}$
   - Much harder to determine since latent nodes $\mathcal{N}_O$ have unspecified behaviour
   - It is possible to turn Markov Separation problem into a Marginal problem (at least partially)

Inflation Technique [Optional]

Developed by Wolfe, Spekkens, and Fritz **Inflation**

Definition

An inflation of a causal structure $\mathcal{G}$ is another causal structure $\mathcal{G}'$ such that:

$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n)$$

Where $\mathsf{AnSub}_{\mathcal{G}}(n)$ denotes the ancestral sub-graph of n in $\mathcal{G}$

$$\mathsf{AnSub}_{\mathcal{G}}(n) = \mathsf{Sub}_{\mathcal{G}}(\mathsf{An}_{\mathcal{G}}(n))$$

And '$\sim$' is a copy-index equivalence relation

$$A_1 \sim A_2 \sim A \not\sim B_1 \sim B_2 \sim B$$

Inflation Lemma [Optional]

If one has obtained $\mathcal{G}$, inflation $\mathcal{G}'$ and *compatible* marginal distribution P_N where $N \subseteq \mathcal{N}$, then:

1. There exists causal parameters $\left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\}$ such that

$$P_N = \prod_{n \in N} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

2. $\mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n) \implies \mathsf{Pa}_{\mathcal{G}'}(n') \sim \mathsf{Pa}_{\mathcal{G}}(n)$
3. Construct inflated causal parameters

$$\forall n' \in \mathcal{N}' : P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')} \equiv P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

4. Obtain *compatible* marginal distributions over any $N' \subseteq \mathcal{N}'$

$$P_{N'} = \prod_{n' \in N'} P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')}$$

Inflation Lemma Cont'd [Optional]

- Inflation procedure holds for any $N \in \mathcal{N}, N' \in \mathcal{N}'$ where $\mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')$
- Define injectable sets of $\mathcal{G}'$ and images of the injectable of $\mathcal{G}$

$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')\}$$
$$\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : \mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')\}$$

- For $N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$ there is a *unique* $N \subseteq \mathcal{N}$ such that $\mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')$
- For $N \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G})$ there can *exist many* $N' \subseteq \mathcal{N}'$ such that $\mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')$

Inflation Lemma Cont'd [Optional]

Lemma

The Inflation Lemma: **Inflation** *Given a particular inflation $\mathcal{G}'$ of $\mathcal{G}$, if a marginal model $\{P_N \mid N \in \mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}')\}$ is compatible with $\mathcal{G}$ then all marginal models $\{P_{N'} \mid N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')\}$ are compatible with $\mathcal{G}'$ provided that $P_N = P_{N'}$ for all instances where $N \sim N'$.*

Corollary

Any causal compatibility inequality I' constraining the injectable sets $\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$ can be deflated into a causal compatibility inequality I constraining the images of the injectable sets $\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}')$.

d-Separation Polynomial [Optional]

- Deflation only holds when inequality constrains probabilities $P_{N'}, N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$
- **Linear inequality** for $\mathcal{G}'$

$$
\begin{tikzcd}
P_{N'} \arrow[rr, "N'_1 \cup N'_2 = N", "N'_1 \perp N'_2" description] \arrow[d, dash] & & P_{N'_1} P_{N'_2} \arrow[d, dash] \\
N' \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \arrow[d, "\times" marking] & & N'_1, N'_2 \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \arrow[d] \\
?? & & P_{N_1} P_{N_2}
\end{tikzcd}
$$

- **Polynomial inequality** for $\mathcal{G}$!

Pre-injectable Sets [Optional]

- d-separation relations + inflation = polynomial inequalities over $\mathcal{G}$
- Restrict focus to sets N' that are partitioned into N'_1, N'_2 d-separated by empty set $\emptyset$
- A pre-injectable set N':

$$N' = \coprod_i N'_i \quad \forall i : N'_i \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$\forall i, j : N'_i \perp N'_j \iff \mathsf{An}_{\mathcal{G}'}(N'_i) \cap \mathsf{An}_{\mathcal{G}'}\left(N'_j\right) = \emptyset$$

- Only need to consider maximal pre-injectable sets $\mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$

Quantum Implementation of Fritz Distribution [Optional]

- States:

$$\rho_{AB} = \left|\Psi^+\right\rangle\left\langle\Psi^+\right| \quad \rho_{BC} = \rho_{CA} = \left|\Phi^+\right\rangle\left\langle\Phi^+\right|$$

- Maximally entangled Bell states:

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad \left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- Measurements:

$$M_A = \left\{\left|1\psi_{\pi/2}\right\rangle\left\langle 1\psi_{\pi/2}\right|, \left|1\psi_{-\pi/2}\right\rangle\left\langle 1\psi_{-\pi/2}\right|, |0\psi_0\rangle\langle 0\psi_0|, |0\psi_\pi\rangle\langle 0\psi_\pi|\right\}$$

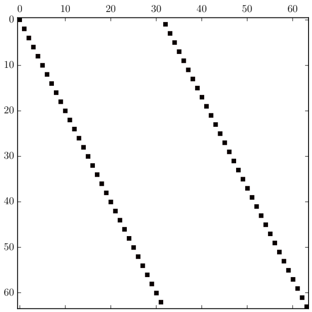
$$M_B = \left\{\left|\psi_{\pi/4}0\right\rangle\left\langle\psi_{\pi/4}0\right|, \left|\psi_{5\pi/4}0\right\rangle\left\langle\psi_{5\pi/4}0\right|, \left|\psi_{3\pi/4}1\right\rangle\left\langle\psi_{3\pi/4}1\right|, \left|\psi_{-\pi/4}1\right\rangle\left\langle\psi_{-\pi/4}1\right|\right\}$$

$$M_C = \{|01\rangle\langle 01|, |11\rangle\langle 11|, |00\rangle\langle 00|, |10\rangle\langle 10|\}$$

- Shorthand: $|\psi_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{ix}|1\rangle)$

Network Permutation Matrix [Optional]

- States and measurements in the Triangle Scenario are not aligned
- Without Π, P_{ABC} would be separable
- Required to align B's measurement over $\mathrm{Tr}_{A,C}(\rho_{AB} \otimes \rho_{BC})$
- Π is a $2^6 \times 2^6$ matrix
- Shifts one qubit to the left



$$\Pi \equiv \sum_{|q_i\rangle \in \{|0\rangle, |1\rangle\}} |q_2 q_3 q_4 q_5 q_6 q_1\rangle\langle q_1 q_2 q_3 q_4 q_5 q_6|$$