Causal Compatibility Inequalities Admitting of Quantum Violations in the Triangle Scenario

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Quantum Networks, 2016

References I

Todo (TC Fraser): Figure out how to get references at the end

[1] K. F. Pál and T. Vértesi.
Maximal violation of a bipartite three-setting, two-outcome bell inequality using infinite-dimensional quantum systems.
Phys. Rev. A, 82(2), aug 2010.

[2] C. Spengler, M. Huber, and B. C. Hiesmayr.
A composite parameterization of unitary groups, density matrices and subspaces.
2010.

[3] E. Wolfe, R. W. Spekkens, and T. Fritz.
The inflation technique for causal inference with latent variables, 2016.

Table of Contents

Introduction

1. Todo (TC Fraser):

Notation

- Complete set of random variables are the joint random variables

$$\mathcal{J} = \{v_1, \cdots, v_n\}$$

- A subset of $V \subset \mathcal{J}$ is a marginal context
- Marginal scenario $\mathcal{M}$

$$\mathcal{M} = \{V_1, \dots, V_k \mid V_i \subset \mathcal{J}\} \quad \mathcal{J} = \bigcup_i V_i$$

- Marginal scenario forms a *simplicial complex*

$$V \in \mathcal{M}, V' \subseteq V \implies V' \in \mathcal{M}$$

- Restrict focus to *maximal* marginal scenario where only the largest contexts are present

$$\forall V_i, V_j \in \mathcal{M} : V_i \not\subset V_j$$

Notation Cont'd

- Marginal model $P^{\mathcal{M}}$ is collection of probability distributions

$$P^{\mathcal{M}} = \{P_{V_1}, \ldots, P_{V_k}\}$$

- Causal Structure $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is a directed acyclic graph (DAG)
- Nodes classified into latent nodes $\mathcal{N}_L$ and observed nodes $\mathcal{N}_O$

Todo (TC Fraser): Insert generic causal structure

Graph Theory [Optional Slide]

Let $n, m \in \mathcal{N}$ be nodes of the graph $\mathcal{G}$.

- parents of n: $\mathsf{Pa}_{\mathcal{G}}(n) \equiv \{m \mid m \to n\}$
- children of n: $\mathsf{Ch}_{\mathcal{G}}(n) \equiv \{m \mid n \to m\}$
- ancestry of n: $\mathsf{An}_{\mathcal{G}}(n) \equiv \bigcup_{i \in \mathbb{W}} \mathsf{Pa}^{i}_{\mathcal{G}}(n)$

$$\mathsf{Pa}^{0}_{\mathcal{G}}(n) = n \qquad \mathsf{Pa}^{i}_{\mathcal{G}}(n) \equiv \mathsf{Pa}_{\mathcal{G}}\Big(\mathsf{Pa}^{i-1}_{\mathcal{G}}(n)\Big)$$

Notation extends to sets of nodes $N \subseteq \mathcal{N}$,

- parents of N: $\mathsf{Pa}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Pa}_{\mathcal{G}}(n)$
- children of N: $\mathsf{Ch}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Ch}_{\mathcal{G}}(n)$
- ancestry of N: $\mathsf{An}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{An}_{\mathcal{G}}(n)$

An induced subgraph of $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ due to $N \subseteq \mathcal{N}$

$$\mathsf{Sub}_{\mathcal{G}}(N) = (N, \{e \in \mathcal{E} \mid e \subseteq N\})$$

Causal Compatibility

Question: Is a marginal model $P^{\mathcal{M}}$ compatible with a causal structure $\mathcal{G}$?

$$\mathcal{M} = \{V_1, \ldots, V_k \mid V_i \subset \mathcal{N}_O\}$$

Answer: $P^{\mathcal{M}}$ is compatible with $\mathcal{G}$ if there exists a set of casual parameters

$$\left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\}$$

Such that for each $V \in \mathcal{M}$, P_V can be recovered:

1. $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$
2. $P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$

A casual compatibility inequality is an inequality over $P^{\mathcal{M}}$ that is satisfied by all compatible $P^{\mathcal{M}}$

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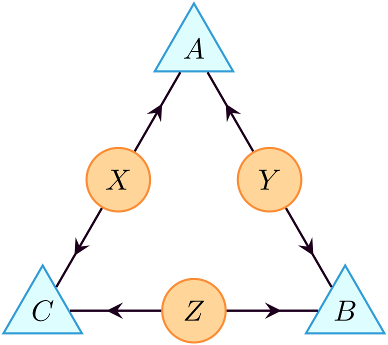
1.

Triangle Scenario



- Three parties $\mathcal{N}_O = \{A, B, C\}$
- Pair-wise sharing three latent variables $\mathcal{N}_L = \{X, Y, Z\}$
- Todo (TC Fraser): Inject info about existing work
- There exists *quantum-accessible* distributions P_{ABC} that are *incompatible* with the triangle scenario

Fritz Distribution

The Fritz Distribution P_F:

- three-party $P_F = P_{ABC}$
- each party has four outcomes

Inflation Technique

- Developed by Wolfe, Spekkens, and Fritz [3]

Definition

An inflation of a causal structure $\mathcal{G}$ is another causal structure $\mathcal{G}'$ such that:

$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n)$$

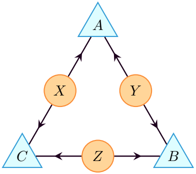
Where $\mathsf{AnSub}_{\mathcal{G}}(n)$ denotes the ancestral sub-graph of n in $\mathcal{G}$

$$\mathsf{AnSub}_{\mathcal{G}}(n) = \mathsf{Sub}_{\mathcal{G}}(\mathsf{An}_{\mathcal{G}}(n))$$

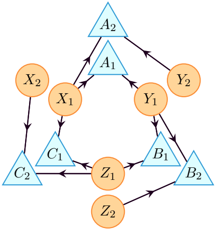
And '$\sim$' is a copy-index equivalence relation

$$A_1 \sim A_2 \sim A \not\sim B_1 \sim B_2 \sim B$$

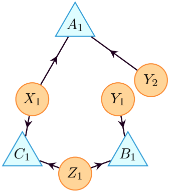
Inflations of the Triangle Scenario



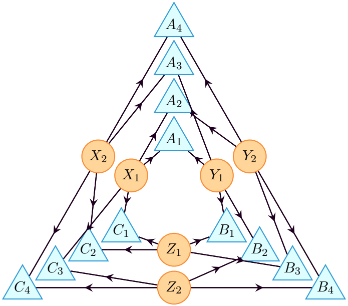
The Triangle Scenario



Spiral Inflation



Cut Inflation



Large Inflation

Inflation Lemma

If one has obtained $\mathcal{G}$, inflation $\mathcal{G}'$ and *compatible* marginal distribution P_N where $N \subseteq \mathcal{N}$, then:

1. There exists causal parameters $\left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\}$ such that

$$P_N = \prod_{n \in N} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

2. $\mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n) \implies \mathsf{Pa}_{\mathcal{G}'}(n') \sim \mathsf{Pa}_{\mathcal{G}}(n)$
3. Construct inflated causal parameters

$$\forall n' \in \mathcal{N}' : P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')} \equiv P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

4. Obtain *compatible* marginal distributions over any $N' \subseteq \mathcal{N}'$

$$P_{N'} = \prod_{n' \in N'} P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')}$$

Inflation Lemma Cont'd

- Inflation procedure holds for any $N \in \mathcal{N}, N' \in \mathcal{N}'$ where $N \sim N'$
- Define injectable sets of $\mathcal{G}'$ and images of the injectable of $\mathcal{G}$

$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : N \sim N'\}$$
$$\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : N \sim N'\}$$

- For $N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$ there is a *unique* $N \subseteq \mathcal{N}$ such that $N \sim N'$
- For $N \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G})$ there can *exist many* $N' \subseteq \mathcal{N}'$ such that $N \sim N'$

Inflation Lemma Cont'd

Lemma

The Inflation Lemma: [3, lemma 3] Given a particular inflation $\mathcal{G}'$ of $\mathcal{G}$, if a marginal model $\{P_N \mid N \in \mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}')\}$ is compatible with $\mathcal{G}$ then all marginal models $\{P_{N'} \mid N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')\}$ are compatible with $\mathcal{G}'$ provided that $P_N = P_{N'}$ for all instances where $N \sim N'$.

Corollary

Any causal compatibility inequality I' constraining the injectable sets $\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$ can be deflated into a causal compatibility inequality I constraining the images of the injectable sets $\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}')$.

Inflation Lemma Cont'd

Lemma

The Inflation Lemma: [3, lemma 3] *Given a particular inflation $\mathcal{G}'$ of $\mathcal{G}$, if a marginal model $\{P_N \mid N \in \mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}')\}$ is compatible with $\mathcal{G}$ then all marginal models $\{P_{N'} \mid N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')\}$ are compatible with $\mathcal{G}'$ provided that $P_N = P_{N'}$ for all instances where $N \sim N'$.*

Corollary

Any causal compatibility inequality I' constraining the injectable sets $\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$ can be deflated into a causal compatibility inequality I constraining the images of the injectable sets $\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}')$.

Inflation Pipeline

└Inflation Pipeline

1. Walk through how the inflation pipeline works

Outcomes and Events

- Set of outcomes O_v for each variable v
- Set of events for a set of variables V

$$\mathcal{E}(V) \equiv \{s : V \to O_V \mid \forall v \in V, s(v) \in O_v\}$$

Definition

The set of events over the joint variables $\mathcal{E}(\mathcal{J})$ are termed the joint events.

Definition

The set of events over the marginal contexts are the marginal events

$$\mathcal{E}(\mathcal{M}) \equiv \coprod_{V \in \mathcal{M}} \mathcal{E}(V)$$

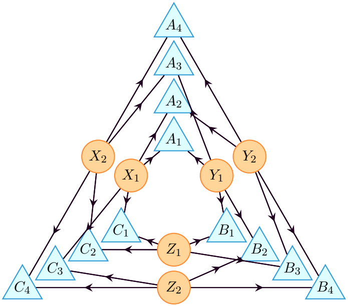
Incidence Matrix

- Incidence matrix M is a bit-wise matrix
- Row-indexed by marginal events $m \in \mathcal{E}(\mathcal{M})$
- Column-indexed by joint events $j \in \mathcal{E}(\mathcal{J})$

$$M_{m,j} = \begin{cases} 1 & m \text{ compatible with } j \\ 0 & \text{otherwise} \end{cases}$$

Application to Large Inflation

- Tackling Large inflation
- 12 pre-injectable sets (to follow)



Large Inflation Pre-injectable Sets

Maximal Pre-injectable Sets	**Ancestral Independences**
$\{A_1, B_1, C_1, A_4, B_4, C_4\}$	$\{A_1, B_1, C_1\} \perp \{A_4, B_4, C_4\}$
$\{A_1, B_2, C_3, A_4, B_3, C_2\}$	$\{A_1, B_2, C_3\} \perp \{A_4, B_3, C_2\}$
$\{A_2, B_3, C_1, A_3, B_2, C_4\}$	$\{A_2, B_3, C_1\} \perp \{A_3, B_2, C_4\}$
$\{A_2, B_4, C_3, A_3, B_1, C_2\}$	$\{A_2, B_4, C_3\} \perp \{A_3, B_1, C_2\}$
$\{A_1, B_3, C_4\}$	$\{A_1\} \perp \{B_3\} \perp \{C_4\}$
$\{A_1, B_4, C_2\}$	$\{A_1\} \perp \{B_4\} \perp \{C_2\}$
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$\{A_4, B_1, C_3\}$	$\{A_4\} \perp \{B_1\} \perp \{C_3\}$
$\{A_4, B_2, C_1\}$	$\{A_4\} \perp \{B_2\} \perp \{C_1\}$

Large Inflation Incidence

- Joint variables are all of the observable nodes $\mathcal{N}'_O = \mathcal{J}$

$$\mathcal{J} = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4\}$$

- Marginal scenario is composed of pre-injectable sets $\mathcal{M} = \mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$
- Inequalities Violated by Fritz distribution are inherently 4-outcome
- Incidence matrix M is very large ~ 2.25Gb

$$\#\text{Columns} = \prod_{v \in \mathcal{J}} O_v = 4^{12} = 16,777,216$$

$$\#\text{Rows} = \sum_{V \in \mathcal{M}} \prod_{v \in V} O_v = 4 \times 4^6 + 8 \times 4^3 = 16,896$$

Inequalities Found

Causal Symmetry

Symmetric Incidence

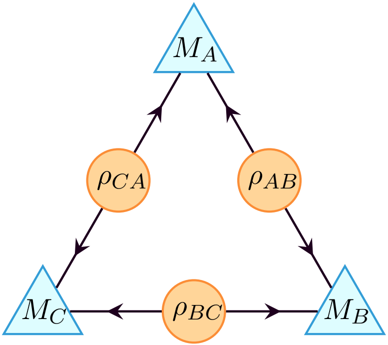
Symmetric Inequalities

Numerical Optimization

Parameterizing Quantum Distributions

For our purposes, we need to parameterize the space of quantum-accessible distributions that are *realized* on the Triangle Scenario

$$P_{ABC}(abc) = \mathrm{Tr}[\Pi^{\intercal} \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$



Parameterizing Unitary Group

- Spengler, Huber and Heismayr [2] demonstrate a parameterization of $\mathcal{U}(d)$ where the parameters are organized in a $d \times d$-matrix of real values $\lambda_{n,m}$

$$U = \left[\prod_{m=1}^{d-1} \left(\prod_{n=m+1}^{d} R_{m,n} RP_{n,m} \right) \right] \left[\prod_{l=1}^{d} GP_l \right]$$

- Global Phase Terms: $GP_l = \exp(iP_l \lambda_{l,l})$
- Relative Phase Terms: $RP_{m,n} = \exp(iP_n \lambda_{n,m})$
- Rotation Terms: $R_{n,m} = \exp(i\sigma_{m,n} \lambda_{m,n})$
- Projection Operators: $P_l = |l\rangle\langle l|$
- Anti-symmetric σ-matrices: $\sigma_{m,n} = -i|m\rangle\langle n| + i|n\rangle\langle m|$

Parameterizing Unitary Group Cont'd

- Each parameter $\lambda_{n,m}$ has physical interpretation
- Degeneracies are easily eliminated such as global phase

$$\forall l = 1, \ldots, d : \lambda_{l,l} = 0 \implies GP_l = 1$$

- Parameterize $U \in \mathcal{U}(d)$ up to global phase denoted $\tilde{U} \in \mathcal{U}(d)$
- Computationally efficient

$$\begin{aligned} GP_l &= \mathbb{1} + P_l \left(e^{i\lambda_{l,l}} - 1 \right) \\ RP_{m,n} &= \mathbb{1} + P_n \left(e^{i\lambda_{n,m}} - 1 \right) \\ R_{n,m} &= \mathbb{1} + (|m\rangle\langle m| + |n\rangle\langle n|)(\cos\lambda_{n,m} - 1) \\ &\quad + (|m\rangle\langle n| - |n\rangle\langle m|)\sin\lambda_{n,m} \end{aligned}$$

Parameterizing States

- Each latent resource $\rho \in (\rho_{AB}, \rho_{BC}, \rho_{CA})$ modeled as bipartite qubit state acting on $\mathcal{H}^{d/2} \otimes \mathcal{H}^{d/2}$
- $d \times d$ positive semi-definite (PSD) hermitian matrices with unitary trace
- Cholesky Parametrization allows one to write any hermitian PSD as $\rho = T^\dagger T$
- For $d = 4$:

$$T = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_2 + i\lambda_3 & \lambda_4 & 0 & 0 \\ \lambda_5 + i\lambda_6 & \lambda_7 + i\lambda_8 & \lambda_9 & 0 \\ \lambda_{10} + i\lambda_{11} & \lambda_{12} + i\lambda_{13} & \lambda_{14} + i\lambda_{15} & \lambda_{16} \end{bmatrix}$$

- d^2 real-valued parameters
- Normalized $\rho = T^\dagger T / \mathrm{Tr}\left(T^\dagger T\right)$ adds degeneracy

Parameterizing States Cont'd

- SHH parameterization [2] exploits spectral decomposition; for rank $k \leq d$ density matrix

$$\rho = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i| \qquad p_i \geq 0, \sum_i p_i = 1$$

- Orthonormal k-element sub-basis $\{|\psi_i\rangle\}$ of $\mathcal{H}^d$ can be transformed into computational basis $\{|i\rangle\}$ by unitary $U \in \mathcal{U}(d)$ such that $|\psi_i\rangle = U|i\rangle$
- Freedom to choice k
- Parameterize ρ through $\{p_i\}$ and $\tilde{U}_k$

$$\tilde{U}_k = \prod_{m=1}^{k} \left(\prod_{n=m+1}^{d} R_{m,n} R P_{n,m} \right)$$

- $d^2 - (d-k)^2 - k + (k-1) = 2dk - k^2 - 1$ real-valued parameters (no-degeneracy)

Parameterizing POVMs

- Each party (A, B, C) is assigned a projective-operator valued measure (POVM) (M_A, M_B, M_C)

$$\forall |\psi\rangle \in \mathcal{H}^d : \langle\psi|M_\chi|\psi\rangle \geq 0 \quad M_\chi = M_\chi^\dagger$$

- n-outcome measurement

$$M_\chi = \{M_{\chi,1}, \dots, M_{\chi,n}\} \quad \sum_{i=1}^{n} M_{\chi,i} = \mathbb{1}$$

- For $n = 2$ outcomes, a parameterization exists by constraining the eigenvalues of $M_{\chi,i}$; for $n > 2$ not aware of anything
- Warrants consideration of projective-valued measures (PVMs) (for $n = d$ this is without loss of generality)

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- Warrants consideration of projective-valued measures (PVMs) (for $n = d$ this is without loss of generality)

1. Naimark's Dilation Theorem

Parameterizing PVMs

- Each party (A, B, C) is assigned n-outcome (M_A, M_B, M_C) such that,

$$M_{\chi,i}M_{\chi,j} = \delta_{ij}M_{\chi,i} \quad M_{\chi,i} = |m_{\chi,i}\rangle\langle m_{\chi,i}|$$

- Inspired by [1], parameterizing PVMs means parameterizing a n-element sub-basis $\{|m_{\chi,i}\rangle\}$
- Use unitary transformation again

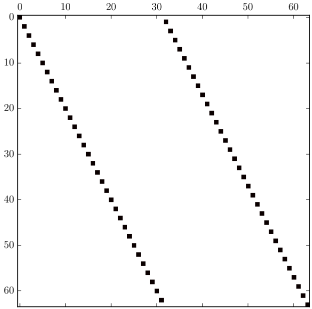
$$\{|m_{\chi,1}\rangle, \ldots, |m_{\chi,n}\rangle\} = \{U|1\rangle, \ldots, U|n\rangle\}$$

- Global phase and remaining basis irrelevant: $\tilde{U}_n$ requires $2dn - n^2 - 1$ real-valued parameters
- PVMs are computationally more efficient

$$P_{ABC}(abc) = \langle m_{A,a}m_{B,b}m_{C,c}|\Pi^\intercal \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA}\Pi|m_{A,a}m_{B,b}m_{C,c}\rangle$$

Network Permutation Matrix

- States and measurements in the Triangle Scenario are not aligned
- Without Π, P_{ABC} would be separable
- Required to align B's measurement over $\mathrm{Tr}_{A,C}(\rho_{AB} \otimes \rho_{BC})$
- Π is a $2^6 \times 2^6$ matrix
  - Shifts one qubit to the left



$$\Pi \equiv \sum_{|q_i\rangle \in \{|0\rangle, |1\rangle\}} |q_2 q_3 q_4 q_5 q_6 q_1\rangle\langle q_1 q_2 q_3 q_4 q_5 q_6|$$

Maximally Violating Distributions

Non-Trivial Inequalities For Large Inflation

Conclusions

Post-doc Opportunities At Perimeter