Causal Compatibility Inequalities Admitting of Quantum Violations in the Triangle Scenario

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Quantum Networks IIP, Nov. 2016

References I

Todo (TC Fraser): Figure out how to get references at the end

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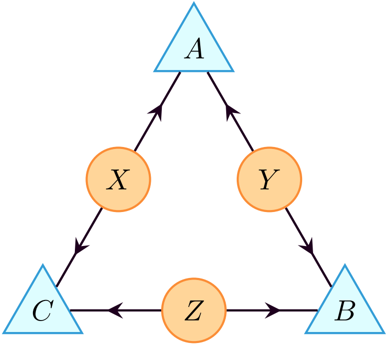
Introduction

1. Thank International Institute for Physics (IIP) for support
2. Thank Perimeter Institute for Theoretical Physics (PI) for support

Objective

- Derive causal compatibility/locality inequalities that distinguish quantum distributions from classical ones
- Specifically for the Triangle Scenario (TS)



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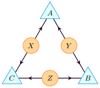

1. Research project's goal was to derive new causal compatibility inequalities that distinguish quantum correlations from classical correlations
2. More specifically, the ambition of the project is to obtain such inequalities that constraint compatibility with The Triangle Scenario
3. Triangle Scenario has be studied extensively before

Objective Cont'd

- In Branciard et al. (2012) it was noted that characterizing locality in TS remained an open problem and that identifying compatibility constraints in this configuration **seemed challenging**
- In Fritz (2012), Fritz demonstrated that TS is the **smallest** correlation scenario in which their exists quantum incompatible distributions (proof without inequalities)
- In Henson et al. (2014), TS was classified as an **interesting** causal structure: conditional independence relations are not a sufficient characterization of compatibility
- Several other authors (see Steudel and Ay (2010), Chaves et al. (2014), Wolfe et al. (2016), ...) have investigated TS without achieving research objective

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1. At the time of publishing Branciard et al. (2012), problem seems hard.
2. Important because its smallest correlation scenario with quantum non-locality.
3. Compatibility can not be determined from conditional independence relations (there are none)
4. It would be interesting to find quantum non-locality in the triangle scenario that does not rely on Bell's theorem.

This Talk

- Report the discovery of such inequalities
- Explain how these inequalities were obtained
- Discuss quantum distributions that violate compatibility inequalities
- Attempts at finding new quantum distributions different than those proposed by Fritz (2012)
- Briefly discuss symmetric inequalities

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- Attempts at finding new quantum distributions different than those proposed by Fritz (2012)
- Briefly discuss symmetric inequalities

1. The purpose of this talk is to present these new-found inequalities and explain how they were obtained
2. Additionally I will talk about my attempts at using these inequalities to find new incompatible quantum distributions
3. In doing so, will discuss how we obtained symmetric compatibility inequalities that have violations in the Triangle Scenario

Example Inequality

- Quick example/preview:

$$P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333) \leq$$
$$2P(020)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213)+$$
$$2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210)+$$
$$2P(123)P(310) + 2P(130)P(213) + 2P(132)P(311) + 2P(133)P(210)+$$
$$+ \cdots \text{ 324 more terms } \cdots +$$
$$P(320)P(323) + P(320)P(333) + P(323)P(330) + P(330)P(333)$$

- $P(abc)$ shorthand for $P_{ABC}(abc)$
- Four outcomes for each A, B, C
- Polynomial in P_{ABC}, marginals P_{AB}, P_{BC}, P_{AC}, P_A, P_B, P_C

- Quick example/preview:

$$\begin{gathered}P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333) \leq \\ 2P(020)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213) + \\ 2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210) + \\ 2P(123)P(310) + 2P(130)P(213) + 2P(132)P(311) + 2P(133)P(210) + \\ + \cdots \text{ 324 more terms } \cdots + \\ P(320)P(323) + P(320)P(333) + P(323)P(330) + P(330)P(333)\end{gathered}$$

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1. As a quick example or preview of what is to come, here is a an example inequality admits quantum violations in the triangle scenario
2. Some features of note: inequality is polynomial in P_{ABC} and its marginals

Notation

Question: Which marginal models $P^{\mathcal{M}}$ are compatible with a causal structure $\mathcal{G}$?

- Marginal model $P^{\mathcal{M}}$ is collection of probability distributions

$$P^{\mathcal{M}} = \{P_{V_1}, \ldots, P_{V_k}\}$$

- Marginal scenario $\mathcal{M} = \{V_1, \ldots, V_k\}$

$$V \in \mathcal{M}, V' \subseteq V \implies V' \in \mathcal{M}$$

- Joint random variables $\mathcal{J} = \bigcup_i V_i = \{v_1, \ldots, v_n\}$
- Causal Structure $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is a directed acyclic graph (DAG)
- Nodes classified into latent nodes $\mathcal{N}_L$ and observed nodes $\mathcal{N}_O$

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1. Before continuing, I will define exactly what I mean by causal compatibility
2. Causal compatibility refers to the compatibility between causal structures and marginal models
3. Marginal model is collection of probability distributions over sets of random variables
4. Marginal scenario refers to the those sets of random variables
5. The complete set of random variables are the joint random variables

Graph Theory [Optional Slide]

Let $n, m \in \mathcal{N}$ be nodes of the graph $\mathcal{G}$.

- parents of n: $\mathsf{Pa}_{\mathcal{G}}(n) \equiv \{m \mid m \to n\}$
- children of n: $\mathsf{Ch}_{\mathcal{G}}(n) \equiv \{m \mid n \to m\}$
- ancestry of n: $\mathsf{An}_{\mathcal{G}}(n) \equiv \bigcup_{i \in \mathbb{W}} \mathsf{Pa}^{i}_{\mathcal{G}}(n)$

$$\mathsf{Pa}^{0}_{\mathcal{G}}(n) = n \qquad \mathsf{Pa}^{i}_{\mathcal{G}}(n) \equiv \mathsf{Pa}_{\mathcal{G}}\Big(\mathsf{Pa}^{i-1}_{\mathcal{G}}(n)\Big)$$

Notation extends to sets of nodes $N \subseteq \mathcal{N}$,

- parents of N: $\mathsf{Pa}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Pa}_{\mathcal{G}}(n)$
- children of N: $\mathsf{Ch}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Ch}_{\mathcal{G}}(n)$
- ancestry of N: $\mathsf{An}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{An}_{\mathcal{G}}(n)$

An induced subgraph of $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ due to $N \subseteq \mathcal{N}$

$$\mathsf{Sub}_{\mathcal{G}}(N) = (N, \{e \in \mathcal{E} \mid e \subseteq N\})$$

Question: Which marginal models $P^{\mathcal{M}}$ are compatible with a causal structure $\mathcal{G}$?

Answer: $P^{\mathcal{M}}$ is compatible with $\mathcal{G}$ if there exists a set of casual parameters

$$\left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\}$$

Such that for each $V \in \mathcal{M}$, P_V can be recovered:

1. $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$
2. $P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$

Inequality: A casual compatibility inequality I is an inequality over $P^{\mathcal{M}}$ that is satisfied by all compatible $P^{\mathcal{M}}$

Deriving Inequalities

Two necessary components to compatibility:

1. Marginal problem: $\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$
   - Is the marginal model contextual or non-contextual?
   - 3 distinct ways to tackle this problem
     1. Convex hull, Polytope projection, Fourier-Motzkin
     2. Possibilistic Hardy Inequalities (Hypergraph transversals)
     3. Linear Program Feasibility/Infeasibility
2. Markov Separation: $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$
   - Much harder to determine since latent nodes $\mathcal{N}_O$ have unspecified behaviour
   - It is possible to turn Markov Separation problem into a Marginal problem (at least partially)

Inflation Technique

Developed by Wolfe, Spekkens, and Fritz Wolfe et al. (2016)

Definition

An inflation of a causal structure $\mathcal{G}$ is another causal structure $\mathcal{G}'$ such that:

$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n)$$

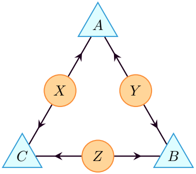
Where $\mathsf{AnSub}_{\mathcal{G}}(n)$ denotes the ancestral sub-graph of n in $\mathcal{G}$

$$\mathsf{AnSub}_{\mathcal{G}}(n) = \mathsf{Sub}_{\mathcal{G}}(\mathsf{An}_{\mathcal{G}}(n))$$

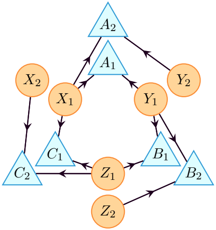
And '$\sim$' is a copy-index equivalence relation

$$A_1 \sim A_2 \sim A \not\sim B_1 \sim B_2 \sim B$$

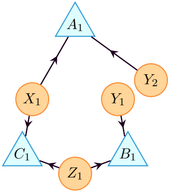
Some Inflations of the Triangle Scenario



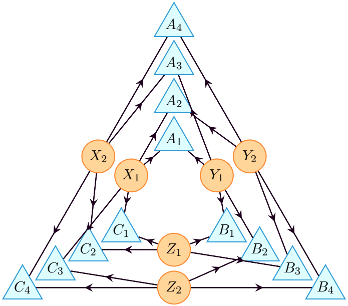
The Triangle Scenario



Spiral Inflation



Cut Inflation



Large Inflation

Inflation Lemma

If one has obtained $\mathcal{G}$, inflation $\mathcal{G}'$ and *compatible* marginal distribution P_N where $N \subseteq \mathcal{N}$, then:

1. There exists causal parameters $\left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\}$ such that

$$P_N = \prod_{n \in N} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

2. $\mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n) \implies \mathsf{Pa}_{\mathcal{G}'}(n') \sim \mathsf{Pa}_{\mathcal{G}}(n)$
3. Construct inflated causal parameters

$$\forall n' \in \mathcal{N}' : P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')} \equiv P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

4. Obtain *compatible* marginal distributions over any $N' \subseteq \mathcal{N}'$

$$P_{N'} = \prod_{n' \in N'} P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')}$$

Inflation Lemma Cont'd

- Inflation procedure holds for any $N \in \mathcal{N}, N' \in \mathcal{N}'$ where $N \sim N'$
- Define injectable sets of $\mathcal{G}'$ and images of the injectable of $\mathcal{G}$

$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : N \sim N'\}$$
$$\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : N \sim N'\}$$

- For $N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$ there is a *unique* $N \subseteq \mathcal{N}$ such that $N \sim N'$
- For $N \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G})$ there can *exist many* $N' \subseteq \mathcal{N}'$ such that $N \sim N'$

Inflation Lemma Cont'd

Lemma

The Inflation Lemma: (Wolfe et al., 2016, lemma 3) Given a particular inflation $\mathcal{G}'$ of $\mathcal{G}$, if a marginal model $\{P_N \mid N \in \mathrm{ImInj}_{\mathcal{G}}(\mathcal{G}')\}$ is compatible with $\mathcal{G}$ then all marginal models $\{P_{N'} \mid N' \in \mathrm{Inj}_{\mathcal{G}}(\mathcal{G}')\}$ are compatible with $\mathcal{G}'$ provided that $P_N = P_{N'}$ for all instances where $N \sim N'$.

Corollary

Any causal compatibility inequality I' constraining the injectable sets $\mathrm{Inj}_{\mathcal{G}}(\mathcal{G}')$ can be deflated into a causal compatibility inequality I constraining the images of the injectable sets $\mathrm{ImInj}_{\mathcal{G}}(\mathcal{G}')$.

d-Separation Polynomial

- Deflation only holds when inequality constrains probabilities $P_{N'}, N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$
- **Linear inequality** for $\mathcal{G}'$

$$
\begin{tikzcd}
P_{N'} \arrow[rr, "\substack{N'_1 \cup N'_2 = N \\ N'_1 \perp N'_2}" description] \arrow[d, "N' \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')" description] & & P_{N'_1} P_{N'_2} \arrow[d, "N'_1, N'_2 \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')" description] \\
?? & & P_{N_1} P_{N_2}
\end{tikzcd}
$$

- **Polynomial inequality** for $\mathcal{G}$!

Pre-injectable Sets

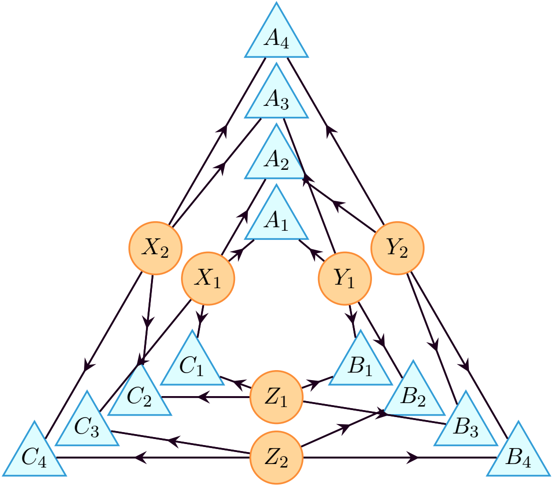
- d-separation relations + inflation = polynomial inequalities over $\mathcal{G}$
- Restrict focus to sets N' that are partitioned into N'_1, N'_2 d-separated by empty set $\emptyset$
- A pre-injectable set N':

$$N' = \coprod_i N'_i \quad \forall i : N'_i \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$\forall i, j : N'_i \perp N'_j \iff \mathsf{An}_{\mathcal{G}'}(N'_i) \cap \mathsf{An}_{\mathcal{G}'}\left(N'_j\right) = \emptyset$$

- Only need to consider maximal pre-injectable sets $\mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$

Pre-injectable Sets of Large Inflation



- Has 12 maximal pre-injectable sets (to follow)

Large Inflation Pre-injectable Sets

Maximal Pre-injectable Sets	**Ancestral Independences**
$\{A_1, B_1, C_1, A_4, B_4, C_4\}$	$\{A_1, B_1, C_1\} \perp \{A_4, B_4, C_4\}$
$\{A_1, B_2, C_3, A_4, B_3, C_2\}$	$\{A_1, B_2, C_3\} \perp \{A_4, B_3, C_2\}$
$\{A_2, B_3, C_1, A_3, B_2, C_4\}$	$\{A_2, B_3, C_1\} \perp \{A_3, B_2, C_4\}$
$\{A_2, B_4, C_3, A_3, B_1, C_2\}$	$\{A_2, B_4, C_3\} \perp \{A_3, B_1, C_2\}$
$\{A_1, B_3, C_4\}$	$\{A_1\} \perp \{B_3\} \perp \{C_4\}$
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$\{A_4, B_2, C_1\}$	$\{A_4\} \perp \{B_2\} \perp \{C_1\}$

Deriving Inequalities

- Inflation facilitates turning linear, inflated inequalities into polynomial deflated ones
- **Question:** How to derive compatibility inequalities for $\mathcal{G}'$?
- **Answer:** Use your favorite technique for deriving compatibility inequalities:
  - Entropic inequalities
  - Finite outcome inequalities
- Here we solve the marginal problem for $\mathcal{M} = \mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$
- First some notation and formalism

Outcomes and Events

Definition

Each variable v has finite set of outcomes O_v.
Each set of variables V has finite set of events $\mathcal{E}(V)$:

$$\mathcal{E}(V) \equiv \{s : V \to O_V \mid \forall v \in V, s(v) \in O_v\}$$

Definition

The set of events over the joint variables $\mathcal{E}(\mathcal{J})$ are termed the joint events.

Definition

The set of events over the marginal contexts are the marginal events

$$\mathcal{E}(\mathcal{M}) \equiv \coprod_{V \in \mathcal{M}} \mathcal{E}(V)$$

Distribution Vectors

Definition

The joint distribution vector $\mathcal{P}^{\mathcal{J}}$

$$\mathcal{P}^{\mathcal{J}}_j = P_{\mathcal{J}}(j) \quad \forall j \in \mathcal{E}(\mathcal{J})$$

Definition

The marginal distribution vector $\mathcal{P}^{\mathcal{M}}$

$$\mathcal{P}^{\mathcal{M}}_m = P_{\mathcal{D}(m)}(m) \quad \forall m \in \mathcal{E}(\mathcal{M}), \mathcal{D}(m) \in \mathcal{M}$$

Can now write complete marginal problem as matrix multiplication:

$$\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J} \setminus V} P_{\mathcal{J}} \iff \mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

Incidence Matrix

- Incidence matrix M is a bit-wise matrix
- Row-indexed by marginal events $m \in \mathcal{E}(\mathcal{M})$
- Column-indexed by joint events $j \in \mathcal{E}(\mathcal{J})$

$$M_{m,j} = \begin{cases} 1 & m = j|_{\mathcal{D}(m)} \\ 0 & \text{otherwise} \end{cases}$$

$$\#\text{Columns} = |\mathcal{E}(\mathcal{J})| = \prod_{v \in \mathcal{J}} |O_v|$$

$$\#\text{Rows} = |\mathcal{E}(\mathcal{M})| = \sum_{V \in \mathcal{M}} \prod_{v \in V} |O_v|$$

Example

Let $\mathcal{J}$ be 3 binary variables $\mathcal{J} = \{A, B, C\}$ and $\mathcal{M}$ be the marginal scenario $\mathcal{M} = \{\{A, B\}, \{B, C\}, \{A, C\}\}$. The incidence matrix becomes:

$$
M = \begin{array}{r c}
(A,B,C) \; = & \begin{array}{cccccccc} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \end{array} \\
\begin{array}{r}
(A{=}0,B{=}0) \\ (A{=}0,B{=}1) \\ (A{=}1,B{=}0) \\ (A{=}1,B{=}1) \\ (B{=}0,C{=}0) \\ (B{=}0,C{=}1) \\ (B{=}1,C{=}0) \\ (B{=}1,C{=}1) \\ (A{=}0,C{=}0) \\ (A{=}0,C{=}1) \\ (A{=}1,C{=}0) \\ (A{=}1,C{=}1)
\end{array} &
\begin{pmatrix}
\mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\
\mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\
\mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1}
\end{pmatrix}
\end{array}
$$

Marginal Linear Program

- Obtain inequalities from incidence matrix M and known incompatible distribution $\mathcal{P}^{\mathcal{M}}$

Marginal LP:

$$\begin{aligned} \text{minimize:} \quad & \emptyset \cdot x \\ \text{subject to:} \quad & x \succeq 0 \\ & M \cdot x = \mathcal{P}^{\mathcal{M}} \end{aligned}$$

Dual Marginal LP:

$$\begin{aligned} \text{minimize:} \quad & y \cdot \mathcal{P}^{\mathcal{M}} \\ \text{subject to:} \quad & y \cdot M \succeq 0 \end{aligned}$$

- y is an infeasibility certificate:

$$y \cdot M \cdot x = y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$$

- Infeasibility inequality: $y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$
- Most linear programing toolkits return certificates (*Mosek*, *Gurobi*, *CPLEX*, *cvxr*/*cvxopt*.)

- Obtain inequalities from incidence matrix M and known incompatible distribution $\mathcal{P}^{\mathcal{M}}$

Marginal LP:

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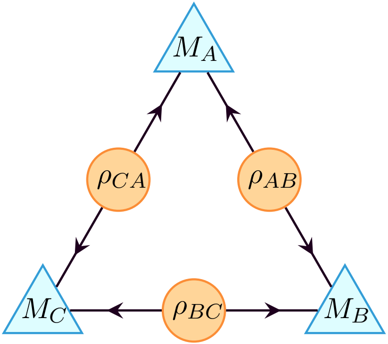
- Most linear programing toolkits return certificates (*Mosek*, *Gurobi*, *CPLEX*, *cvxr*/*cvxopt*.)

1. It is possible to derive non-contextuality inequalities from M and a known incompatible distribution $\mathcal{P}^{\mathcal{M}}$
2. Note the null objective; the minimization objective is trivially always zero
3. The primal value of the linear program is of no interest, all that matters is its *feasibility*.
4. If *feasible*, then there exists a vector x that is a valid joint distribution vector $\mathcal{P}^{\mathcal{J}}$.
5. Feasibility implies that $\mathcal{P}^{\mathcal{J}} = x$.
6. Infeasibility implies contextuality

Known Incompatible Distribution in The Triangle Scenario

Fritz (2012) provides quantum-accessible distribution incompatible with TS

$$P_{ABC}(abc) = \mathrm{Tr}[\Pi^{\intercal}\rho_{AB}\otimes\rho_{BC}\otimes\rho_{CA}\Pi M_{A,a}\otimes M_{B,b}\otimes M_{C,c}]$$



Fritz Distribution

The Fritz Distribution P_F:

- three-party $P_F = P_{ABC}$
- each party has 4 outcomes

$$P_F(000) = P_F(110) = P_F(301) = P_F(211) = P_F(122) = P_F(032) = P_F(233) = P_F(323) = \frac{1}{32}\left(2+\sqrt{2}\right)$$

$$P_F(010) = P_F(100) = P_F(311) = P_F(201) = P_F(132) = P_F(022) = P_F(223) = P_F(333) = \frac{1}{32}\left(2-\sqrt{2}\right)$$

- C's outcome acts as measurement "setting" for A, B; independent of ρ_{AB}
- Correlation coarse graining $\{0,1,2,3\} \to \{(0,3),(1,2)\}$

$$\langle AB|C=0,1,2\rangle = \frac{1}{\sqrt{2}} \quad \langle AB|C=3\rangle = -\frac{1}{\sqrt{2}}$$

- Gives CHSH violation

$$\langle AB|C=0\rangle + \langle AB|C=1\rangle + \langle AB|C=2\rangle - \langle AB|C=3\rangle = 2\sqrt{2} \not\leq 2$$

Quantum Implementation of Fritz Distribution

$$\rho_{AB} = \left|\Psi^+\right\rangle\left\langle\Psi^+\right| \quad \rho_{BC} = \rho_{CA} = \left|\Phi^+\right\rangle\left\langle\Phi^+\right|$$

$$M_A = \left\{|0\psi_0\rangle\langle 0\psi_0|, |0\psi_\pi\rangle\langle 0\psi_\pi|, \left|1\psi_{-\pi/2}\right\rangle\left\langle 1\psi_{-\pi/2}\right|, \left|1\psi_{\pi/2}\right\rangle\left\langle 1\psi_{\pi/2}\right|\right\}$$

$$M_B = \left\{\left|\psi_{\pi/4}0\right\rangle\left\langle\psi_{\pi/4}0\right|, \left|\psi_{5\pi/4}0\right\rangle\left\langle\psi_{5\pi/4}0\right|, \left|\psi_{3\pi/4}1\right\rangle\left\langle\psi_{3\pi/4}1\right|, \left|\psi_{-\pi/4}1\right\rangle\left\langle\psi_{-\pi/4}1\right|\right\}$$

$$M_C = \{|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\}$$

- Shorthand: $|\psi_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{ix}|1\rangle)$
- Maximally entangled Bell states:

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad \left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Notes of Fritz Distribution

- Proof contingent on perfect correlation between C and pseudo-settings
- Proof not robust to noise

Problem (2.17 in Fritz (2012))

Find an example of non-classical quantum correlations in TS together with a proof of its non-classicality which does not hinge on Bell's Theorem.

- "...would be helpful to have inequalities..."
- Possible to find inequalities violated by P_F using the Large inflation of the TS

Large Inflation Incidence

- Joint variables are all of the observable nodes $\mathcal{N}'_O = \mathcal{J}$

$$\mathcal{J} = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4\}$$

- Marginal scenario is composed of pre-injectable sets $\mathcal{M} = \mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$
- Inequalities Violated by Fritz distribution are inherently 4-outcome
- Incidence matrix M is very large ~ 2.25Gb
  - $\#$Columns $= 4^{12} = 16,777,216$
  - $\#$Rows $= 4 \times 4^6 + 8 \times 4^3 = 16,896$
  - $\#$Non-zero Entries $= 201,326,592$

Triangle Scenario Inequality

Causal Symmetry

- Desirable to find compatibility inequality I such that

$$\forall \varphi \in \mathsf{Perm}(A, B, C) : \varphi[I] = I$$

- Compatibility is independent of variable labels

$$\forall \varphi \in \mathsf{Perm}(\mathcal{N}) : I \overset{\mathsf{com}}{\to} \mathcal{G} \implies \varphi[I] \overset{\mathsf{com}}{\to} \varphi[\mathcal{G}]$$

- If $\varphi[\mathcal{G}] = \mathcal{G}$ then $\varphi[I] \overset{\mathsf{com}}{\to} \mathcal{G}$

Definition

The causal symmetry group of causal structure $\mathcal{G}$:

$$\mathsf{Aut}(\mathcal{G}) = \{\varphi \in \mathsf{Perm}(\mathcal{N}) \mid \varphi[\mathcal{G}] = \mathcal{G}\}$$

Strictly speaking, one needs to preserve observable nodes:

$$\mathsf{Aut}_{\mathcal{N}_O}(\mathcal{G}) = \{\varphi \in \mathsf{Aut}(\mathcal{G}) \mid \varphi[\mathcal{N}_O] = \mathcal{N}_O\}$$

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1. Fritz distribution is incompatible with Triangle scenario because party C plays the role of measurement settings for both A and B
2. In order to find quantum distributions different from P_F in the Triangle Scenario, it is therefore desirable to find a proof of its incompatibility (i.e. inequality) that is symmetric under exchange of parties
3. Surprisingly, it is possible to do so!
4. Here is how.
5. First, we will formally define the symmetry group in question
6. Causal symmetry group is the group of automorphisms on causal structure

Symmetric Incidence

Party Symmetric Inequality

$$2[P(001)P(333)]_3 + 2[P(010)P(323)]_3 + 6[P(000)P(323)]_3 + 6[P(000)P(333)]_1$$

$$\leq$$

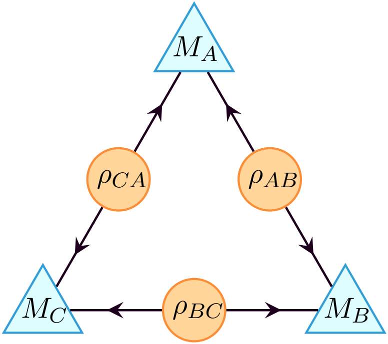
$$12[P(031)P(302)]_6 + 12[P(033)P(303)]_6 + 12[P(103)P(130)]_6 + 12[P(203)P(230)]_6 + 12[P(203)P(330)]_6 + 2[P(001)P(320)]_6 + 2[P(002)P(221)]_3 + 2[P(003)P(211)]_6 +$$
$$2[P(003)P(331)]_3 + 2[P(011)P(211)]_3 + 2[P(012)P(322)]_6 + 2[P(013)P(313)]_6 + 2[P(013)P(332)]_6 + 2[P(020)P(111)]_3 + 2[P(020)P(211)]_6 + 2[P(021)P(212)]_6 +$$
$$2[P(022)P(211)]_3 + 2[P(022)P(212)]_6 + 2[P(022)P(322)]_3 + 2[P(023)P(232)]_6 + 2[P(030)P(212)]_3 + 2[P(031)P(231)]_6 + 2[P(032)P(331)]_6 + 2[P(033)P(333)]_3 +$$
$$2[P(101)P(131)]_3 + 2[P(101)P(132)]_6 + 2[P(102)P(131)]_6 + 2[P(102)P(132)]_6 + 2[P(102)P(133)]_6 + 2[P(110)P(133)]_6 + 2[P(110)P(212)]_6 + 2[P(110)P(222)]_3 +$$
$$2[P(110)P(223)]_3 + 2[P(112)P(331)]_3 + 2[P(120)P(122)]_6 + 2[P(121)P(201)]_6 + 2[P(122)P(200)]_3 + 2[P(122)P(202)]_6 + 2[P(122)P(210)]_6 + 2[P(122)P(300)]_3 +$$
$$2[P(130)P(232)]_6 + 2[P(130)P(233)]_6 + 2[P(131)P(201)]_6 + 2[P(131)P(202)]_3 + 2[P(131)P(313)]_3 + 2[P(133)P(200)]_3 + 2[P(133)P(201)]_6 + 2[P(133)P(211)]_3 +$$
$$2[P(133)P(212)]_6 + 2[P(133)P(300)]_3 + 2[P(202)P(231)]_6 + 2[P(210)P(222)]_6 + 2[P(220)P(222)]_3 + 2[P(220)P(313)]_6 + 2[P(221)P(313)]_6 + 2[P(222)P(331)]_3 +$$
$$2[P(223)P(331)]_3 + 2[P(230)P(312)]_6 + 2[P(231)P(313)]_6 + 2[P(232)P(320)]_6 + 2[P(302)P(322)]_6 + 2[P(320)P(323)]_6 + 2[P(330)P(332)]_3 + 3[P(000)P(003)]_3 +$$
$$3[P(010)P(301)]_6 + 4[P(001)P(131)]_6 + 4[P(002)P(020)]_6 + 4[P(002)P(133)]_6 + 4[P(002)P(323)]_6 + 4[P(010)P(123)]_6 + 4[P(013)P(212)]_6 + 4[P(013)P(312)]_6 +$$
$$4[P(023)P(221)]_6 + 4[P(023)P(222)]_6 + 4[P(023)P(322)]_6 + 4[P(031)P(211)]_6 + 4[P(032)P(321)]_6 + 4[P(100)P(123)]_6 + 4[P(100)P(232)]_6 + 4[P(100)P(313)]_6 +$$
$$4[P(112)P(310)]_6 + 4[P(122)P(203)]_6 + 4[P(122)P(302)]_6 + 4[P(130)P(222)]_6 + 4[P(130)P(223)]_6 + 4[P(222)P(310)]_6 + 4[P(223)P(320)]_6 + 4[P(231)P(301)]_6 +$$
$$4[P(312)P(330)]_6 + 6[P(001)P(031)]_6 + 6[P(001)P(033)]_6 + 6[P(002)P(300)]_6 + 6[P(002)P(330)]_3 + 6[P(003)P(032)]_6 + 6[P(003)P(131)]_6 + 6[P(003)P(132)]_6 +$$
$$6[P(011)P(300)]_3 + 6[P(011)P(320)]_6 + 6[P(012)P(200)]_6 + 6[P(012)P(301)]_6 + 6[P(013)P(030)]_6 + 6[P(013)P(110)]_6 + 6[P(013)P(120)]_6 + 6[P(013)P(303)]_6 +$$
$$6[P(020)P(102)]_6 + 6[P(020)P(103)]_6 + 6[P(020)P(123)]_6 + 6[P(020)P(202)]_3 + 6[P(020)P(203)]_6 + 6[P(020)P(311)]_6 + 6[P(020)P(322)]_6 + 6[P(020)P(330)]_6 +$$
$$6[P(022)P(303)]_6 + 6[P(030)P(033)]_6 + 6[P(030)P(101)]_3 + 6[P(030)P(133)]_6 + 6[P(030)P(202)]_3 + 6[P(030)P(303)]_3 + 6[P(030)P(332)]_6 + 6[P(031)P(203)]_6 +$$
$$6[P(032)P(310)]_6 + 6[P(033)P(101)]_6 + 6[P(033)P(130)]_6 + 6[P(033)P(200)]_3 + 6[P(033)P(212)]_6 + 6[P(033)P(220)]_6 + 6[P(033)P(222)]_3 + 6[P(033)P(230)]_6 +$$
$$6[P(033)P(322)]_3 + 6[P(100)P(203)]_6 + 6[P(101)P(130)]_6 + 6[P(103)P(310)]_6 + 6[P(113)P(130)]_6 + 6[P(113)P(230)]_6 + 6[P(113)P(330)]_3 + 6[P(122)P(330)]_6 +$$
$$6[P(130)P(313)]_6 + 6[P(132)P(303)]_6 + 6[P(133)P(303)]_6 + 6[P(133)P(320)]_6 + 6[P(200)P(203)]_6 + 6[P(201)P(230)]_6 + 6[P(203)P(231)]_6 + 6[P(223)P(300)]_6 +$$
$$8[P(003)P(320)]_6 + 8[P(032)P(300)]_6$$

Numerical Optimization

Parameterizing Quantum Distributions

For our purposes, we need to parameterize the space of quantum-accessible distributions that are *realized* on the Triangle Scenario

$$P_{ABC}(abc) = \mathrm{Tr}[\Pi^{\intercal}\rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA}\Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$



Parameterizing Unitary Group

- Spengler, Huber and Heismayr Spengler et al. (2010) demonstrate a parameterization of $\mathcal{U}(d)$ where the parameters are organized in a $d \times d$-matrix of real values $\lambda_{n,m}$

$$U = \left[\prod_{m=1}^{d-1} \left(\prod_{n=m+1}^{d} R_{m,n} RP_{n,m} \right)\right] \cdot \left[\prod_{l=1}^{d} GP_l\right]$$

- Global Phase Terms: $GP_l = \exp(iP_l \lambda_{l,l})$
- Relative Phase Terms: $RP_{n,m} = \exp(iP_n \lambda_{n,m})$
- Rotation Terms: $R_{m,n} = \exp(i\sigma_{m,n} \lambda_{m,n})$
- Projection Operators: $P_l = |l\rangle\langle l|$
- Anti-symmetric σ-matrices: $\sigma_{m,n} = -i|m\rangle\langle n| + i|n\rangle\langle m|$
- Parameters $\lambda_{n,m} \in [0, 2\pi]$

Parameterizing Unitary Group Cont'd

- Each parameter $\lambda_{n,m}$ has physical interpretation
- Degeneracies are easily eliminated such as global phase

$$\forall l = 1, \ldots, d : \lambda_{l,l} = 0 \implies GP_l = 1$$

- Parameterize $U \in \mathcal{U}(d)$ up to global phase denoted $\tilde{U} \in \mathcal{U}(d)$
- Computationally efficient

$$\begin{aligned}
GP_l &= \mathbb{1} + P_l\left(e^{i\lambda_{l,l}} - 1\right) \\
RP_{n,m} &= \mathbb{1} + P_n\left(e^{i\lambda_{n,m}} - 1\right) \\
R_{m,n} &= \mathbb{1} + (|m\rangle\langle m| + |n\rangle\langle n|)(\cos\lambda_{n,m} - 1) \\
&\quad + (|m\rangle\langle n| - |n\rangle\langle m|)\sin\lambda_{n,m}
\end{aligned}$$

Parameterizing States

- Each latent resource $\rho \in (\rho_{AB}, \rho_{BC}, \rho_{CA})$ modeled as bipartite qubit state acting on $\mathcal{H}^{d/2} \otimes \mathcal{H}^{d/2}$
- $d \times d$ positive semi-definite (PSD) hermitian matrices with unitary trace
- Cholesky Parametrization allows one to write any hermitian PSD as $\rho = T^\dagger T$
- For $d = 4$:

$$T = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_2 + i\lambda_3 & \lambda_4 & 0 & 0 \\ \lambda_5 + i\lambda_6 & \lambda_7 + i\lambda_8 & \lambda_9 & 0 \\ \lambda_{10} + i\lambda_{11} & \lambda_{12} + i\lambda_{13} & \lambda_{14} + i\lambda_{15} & \lambda_{16} \end{bmatrix}$$

- d^2 real-valued parameters
- Normalized $\rho = T^\dagger T / \mathrm{Tr}\left(T^\dagger T\right)$ adds degeneracy

Parameterizing States Cont'd

- SHH parameterization Spengler et al. (2010) exploits spectral decomposition; for rank $k \leq d$ density matrix

$$\rho = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i| \qquad p_i \geq 0, \sum_i p_i = 1$$

- Orthonormal k-element sub-basis $\{|\psi_i\rangle\}$ of $\mathcal{H}^d$ can be transformed into computational basis $\{|i\rangle\}$ by unitary $U \in \mathcal{U}(d)$ such that $|\psi_i\rangle = U|i\rangle$
- Freedom to choice k
- Parameterize ρ through $\{p_i\}$ and $\tilde{U}_k$

$$\tilde{U}_k = \prod_{m=1}^{k} \left(\prod_{n=m+1}^{d} R_{m,n} RP_{n,m} \right)$$

- $d^2 - (d-k)^2 - k + (k-1) = 2dk - k^2 - 1$ real-valued parameters (no-degeneracy)

Parameterizing POVMs

- Each party (A, B, C) is assigned a projective-operator valued measure (POVM) (M_A, M_B, M_C)

$$\forall|\psi\rangle \in \mathcal{H}^d : \langle\psi|M_\chi|\psi\rangle \geq 0 \quad M_\chi = M_\chi^\dagger$$

- n-outcome measurement

$$M_\chi = \{M_{\chi,1}, \ldots, M_{\chi,n}\} \quad \sum_{i=1}^{n} M_{\chi,i} = \mathbb{1}$$

- For $n = 2$ outcomes, a parameterization exists by constraining the eigenvalues of $M_{\chi,i}$; for $n > 2$ not aware of anything
- Warrants consideration of projective-valued measures (PVMs) (for $n = d$ this is without loss of generality)

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1. Naimark's Dilation Theorem

Parameterizing PVMs

- Each party (A, B, C) is assigned n-outcome (M_A, M_B, M_C) such that,

$$M_{\chi,i}M_{\chi,j} = \delta_{ij}M_{\chi,i} \quad M_{\chi,i} = |m_{\chi,i}\rangle\langle m_{\chi,i}|$$

- Inspired by Pál and Vértesi (2010), parameterizing PVMs means parameterizing a n-element sub-basis $\{|m_{\chi,i}\rangle\}$
- Use unitary transformation again

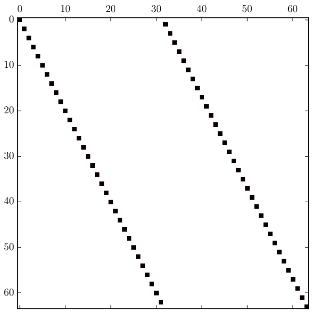
$$\{|m_{\chi,1}\rangle, \dots, |m_{\chi,n}\rangle\} = \{U|1\rangle, \dots, U|n\rangle\}$$

- Global phase and remaining basis irrelevant: $\tilde{U}_n$ requires $2dn - n^2 - 1$ real-valued parameters
- PVMs are computationally more efficient

$$P_{ABC}(abc) = \langle m_{A,a}m_{B,b}m_{C,c}|\Pi^{\mathsf{T}}\rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA}\Pi|m_{A,a}m_{B,b}m_{C,c}\rangle$$

Network Permutation Matrix

- States and measurements in the Triangle Scenario are not aligned
- Without Π, P_{ABC} would be separable
- Required to align B's measurement over $\mathrm{Tr}_{A,C}(\rho_{AB} \otimes \rho_{BC})$
- Π is a $2^6 \times 2^6$ matrix
- Shifts one qubit to the left



$$\Pi \equiv \sum_{|q_i\rangle \in \{|0\rangle, |1\rangle\}} |q_2 q_3 q_4 q_5 q_6 q_1\rangle\langle q_1 q_2 q_3 q_4 q_5 q_6|$$

Maximally Violating Distributions

Conclusions

- New causal compatibility inequalities have been found for the TS
- Inflation technique capable of producing inequalities with quantum/classical witnesses
- Proof of non-classicality is robust to noise
- Fritz witnessable by party symmetric inequalities
- Maximally violating distributions are different than Fritz but also similar
- Further research is necessary

Post-doc Opportunities At Perimeter