Causal Compatibility Inequalities Admitting of Quantum Violations in the Triangle Scenario

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Quantum/Classical Discrepancies

Quantum Mechanics can do stuff that Classical Mechanics can't!

- Quantum Information/Communication
  - Quantum Key Distribution (QKD)
- Quantum Algorithms/Computation
  - Shor's Algorithm, Quantum Simulated Annealing, String Matching, ...
  - Polynomial or even exponential advantage
  - `http://math.nist.gov/quantum/zoo/`
- Quantum Foundations
  - Derivation of Bell inequalities
  - Abandoning local hidden variable theories (LHVs)

Important to determine *precisely* where quantum/classical differ, but how?

Notation & Jargon

Given a set of observations, are they *compatible* with a given classical model?

Causal Structure $\mathcal{G}$

(classical model)

$$\mathcal{G} = (\mathcal{N}, \mathcal{E})$$

$\mathcal{E}$: causal influence

$\mathcal{N}$: random variables

$\mathcal{N}_O$: observable variables

$\mathcal{N}_L$: latent variables

$$\mathcal{N} = \mathcal{N}_O \cup \mathcal{N}_L$$

Marginal Model $P^{\mathcal{M}}$

(set of observations)

$\mathcal{M}$: marginal scenario

$$\mathcal{M} = \{V \mid V \subseteq \mathcal{N}_O\}$$

$$P^{\mathcal{M}} = \{P_V \mid V \in \mathcal{M}\}$$

Causal Compatibility

Question: Can your model $\mathcal{G}$ explain the observation $P^{\mathcal{M}}$?
Question: Is $P^{\mathcal{M}}$ compatible with $\mathcal{G}$?

Answer: If $P^{\mathcal{M}}$ is to be compatible with $\mathcal{G}$ then:

1. $P^{\mathcal{M}}$ must admit a joint distribution $P_{\mathcal{N}}$:

$$P_V = \sum_{\mathcal{N}\setminus V} P_{\mathcal{N}}$$

2. $P_{\mathcal{N}}$ must obey Markov conditions (independence given parents):

$$P_{\mathcal{N}} = \prod_{n\in\mathcal{N}} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

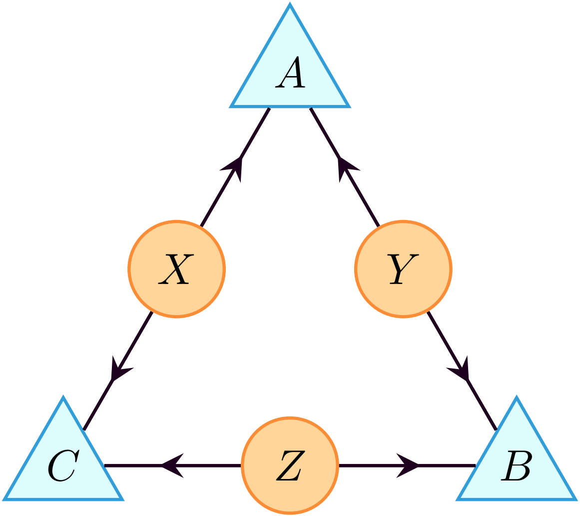
Inequalities

Task: Determine if $P^{\mathcal{M}}$ is compatible or incompatible with $\mathcal{G}$

Tool: A casual compatibility inequality I is an inequality satisfied by all compatible $P^{\mathcal{M}}$

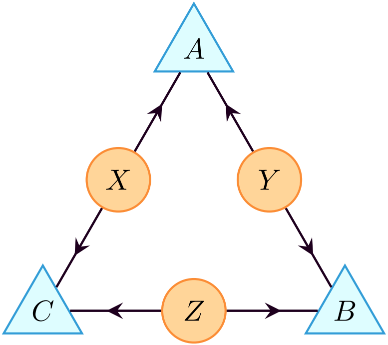
- Inequalities I are **extremely useful**:
- If $P^{\mathcal{M}}$ satisfies *all* such constraints I, then it is *compatible* with $\mathcal{G}$
- If $P^{\mathcal{M}}$ violates *any* constraint I, then it is *incompatible* with $\mathcal{G}$
- Inequalities act as *measures* of non-classicality
- I identifies *fundamentally* new resources or quantum advantages!

Case Study: The Triangle Scenario



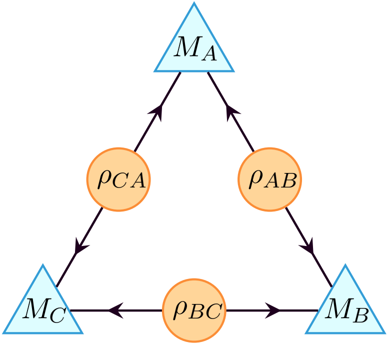
Classical (Compatible) Distributions for Triangle Scenario

$$P_{ABC} = \sum_{X,Y,Z} P_{A|X,Y} P_{B|Y,Z} P_{C|Z,X} P_X P_Y P_Z$$



Quantum Distributions for Triangle Scenario

$$P_{ABC} = \mathrm{Tr}[\Pi^{\intercal} \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_A \otimes M_B \otimes M_C]$$



Why Triangle Scenario?

- The TS could be home to **brand new** entanglement resources which could lead to new technologies and fundamental physics, albeit very difficult to find
- In **Branciard_2012** it was noted that characterizing quantum non-classicality in TS remained an **open problem** and that identifying compatibility inequalities **seemed challenging**
- In **Fritz_2012** Fritz demonstrated that TS is the **smallest** correlation scenario in which quantum non-classicality is manifest (proof without inequalities)
- In **Henson_2014** TS was classified as an **interesting** causal structure: conditional independence relations are not a sufficient characterization of compatibility (there are none)
- Several other authors (see **Steudel_2010 Chaves_2014 Inflation** ...) have been unable to find inequalities with quantum violations

Deriving Inequalities

- Two necessary components to compatibility: joint distribution, Markov separability
- It is possible to cast Markov separability problem into a joint distribution problem (at least partially)!
- Inflation Technique **Inflation** provides tools for solving compatibility problem
- Definite Extension Procedure developed by myself, E. Wolfe and others allows one to derive inequalities for **very large** causal structures

Causal Compatibility Inequalities

$$P_{ABC}(000)P_C(3)$$
$$\leq$$
$$2P_{ABC}(203)P_C(0) + 2P_{ABC}(303)P_C(0) + P_{ABC}(003)P_C(0) + P_{ABC}(013)P_C(0)+$$
$$P_{ABC}(020)P_C(2) + P_{ABC}(022)P_C(2) + P_{ABC}(023)P_C(0) + P_{ABC}(023)P_C(2)+$$
$$P_{ABC}(030)P_C(2) + P_{ABC}(031)P_C(2) + P_{ABC}(032)P_C(2) + P_{ABC}(033)P_C(0)+$$
$$P_{ABC}(033)P_C(2) + P_{ABC}(103)P_C(0) + P_{ABC}(113)P_C(0) + P_{ABC}(123)P_C(0)+$$
$$P_{ABC}(133)P_C(0) + P_{ABC}(200)P_C(0) + P_{ABC}(200)P_C(1) + P_{ABC}(200)P_C(2)+$$
$$P_{ABC}(200)P_C(3) + P_{ABC}(201)P_C(0) + P_{ABC}(201)P_C(1) + P_{ABC}(201)P_C(2)+$$
$$P_{ABC}(201)P_C(3) + P_{ABC}(203)P_C(1) + P_{ABC}(203)P_C(2) + P_{ABC}(203)P_C(3)+$$
$$P_{ABC}(213)P_C(0) + P_{ABC}(223)P_C(0) + P_{ABC}(300)P_C(0) + P_{ABC}(300)P_C(1)+$$
$$P_{ABC}(300)P_C(2) + P_{ABC}(300)P_C(3) + P_{ABC}(301)P_C(0) + P_{ABC}(301)P_C(1)+$$
$$P_{ABC}(301)P_C(2) + P_{ABC}(301)P_C(3) + P_{ABC}(302)P_C(1) + P_{ABC}(303)P_C(1)+$$
$$P_{ABC}(303)P_C(2) + P_{ABC}(303)P_C(3) + P_{ABC}(313)P_C(0)$$

Causal Compatibility Inequalities (Simplified)

$$P_{ABC}(000)P_C(3) + P_{ABC}(021)P_C(2) + P_{ABC}(202) + P_{ABC}(302) + P_{ABC}(233)P_C(0) + P_{AC}(33)P_C(0)$$

$$\leq$$

$$P_{ABC}(303)P_C(0) + P_{ABC}(313)P_C(0) + P_{ABC}(302)P_C(1) + P_{AB}(02)P_C(2) + P_{AB}(03)P_C(2) + P_{AB}(20) + P_{AB}(30) + P_C(3)P_C(0)$$

Certificate for Fritz Distribution

$$
\begin{aligned}
&P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333) \leq \\
&2P(020)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213)+ \\
&2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210)+ \\
&2P(123)P(310) + 2P(130)P(213) + 2P(132)P(311) + 2P(133)P(210)+ \\
&\qquad + \cdots \text{ 324 more terms } \cdots + \\
&P(320)P(323) + P(320)P(333) + P(323)P(330) + P(330)P(333)
\end{aligned}
$$

Note: $P(abc)$ shorthand for $P_{ABC}(abc)$

Certificate for Fritz Distribution (Full)

$$P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333)$$

$$\leq$$

$$\begin{gathered}
2P(020)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213) + 2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210) + 2P(123)P(310) + 2P(130)P(213)+ \\
2P(132)P(311) + 2P(133)P(210) + 2P(133)P(310) + P(000)P(003) + P(000)P(013) + P(000)P(023) + P(000)P(033) + P(000)P(103) + P(000)P(113) + P(000)P(123)+ \\
P(000)P(133) + P(000)P(203) + P(000)P(213) + P(000)P(223) + P(003)P(010) + P(003)P(020) + P(003)P(030) + P(003)P(100) + P(003)P(110) + P(003)P(120)+ \\
P(003)P(130) + P(003)P(200) + P(003)P(210) + P(003)P(220) + P(003)P(230) + P(003)P(300) + P(003)P(310) + P(003)P(320) + P(003)P(330) + P(010)P(013)+ \\
P(010)P(023) + P(010)P(033) + P(010)P(103) + P(010)P(113) + P(010)P(123) + P(010)P(133) + P(010)P(203) + P(010)P(213) + P(010)P(223) + P(013)P(020)+ \\
P(013)P(030) + P(013)P(100) + P(013)P(110) + P(013)P(120) + P(013)P(130) + P(013)P(200) + P(013)P(210) + P(013)P(220) + P(013)P(230) + P(013)P(300)+ \\
P(013)P(310) + P(013)P(320) + P(013)P(330) + P(020)P(023) + P(020)P(033) + P(020)P(103) + P(020)P(113) + P(020)P(123) + P(020)P(133) + P(020)P(203)+ \\
P(020)P(210) + P(020)P(211) + P(020)P(212) + P(020)P(223) + P(020)P(233) + P(020)P(310) + P(020)P(311) + P(020)P(312) + P(020)P(313) + P(021)P(210)+ \\
P(021)P(211) + P(021)P(212) + P(021)P(213) + P(021)P(310) + P(021)P(311) + P(021)P(312) + P(021)P(313) + P(022)P(210) + P(022)P(212) + P(022)P(213)+ \\
P(022)P(310) + P(022)P(311) + P(022)P(312) + P(022)P(313) + P(023)P(030) + P(023)P(100) + P(023)P(110) + P(023)P(120) + P(023)P(130) + P(023)P(200)+ \\
P(023)P(211) + P(023)P(212) + P(023)P(213) + P(023)P(220) + P(023)P(230) + P(023)P(300) + P(023)P(311) + P(023)P(312) + P(023)P(313) + P(023)P(320)+ \\
P(023)P(330) + P(030)P(033) + P(030)P(103) + P(030)P(113) + P(030)P(123) + P(030)P(133) + P(030)P(203) + P(030)P(210) + P(030)P(211) + P(030)P(212)+ \\
P(030)P(223) + P(030)P(233) + P(030)P(310) + P(030)P(311) + P(030)P(312) + P(030)P(313) + P(031)P(210) + P(031)P(211) + P(031)P(212) + P(031)P(213)+ \\
P(031)P(310) + P(031)P(311) + P(031)P(312) + P(031)P(313) + P(032)P(210) + P(032)P(212) + P(032)P(213) + P(032)P(310) + P(032)P(311) + P(032)P(312)+ \\
P(032)P(313) + P(033)P(100) + P(033)P(110) + P(033)P(120) + P(033)P(130) + P(033)P(200) + P(033)P(211) + P(033)P(212) + P(033)P(213) + P(033)P(220)+ \\
P(033)P(230) + P(033)P(300) + P(033)P(311) + P(033)P(312) + P(033)P(313) + P(033)P(320) + P(033)P(330) + P(100)P(103) + P(100)P(113) + P(100)P(123)+ \\
P(100)P(133) + P(100)P(203) + P(100)P(213) + P(100)P(223) + P(103)P(110) + P(103)P(120) + P(103)P(130) + P(103)P(200) + P(103)P(210) + P(103)P(220)+ \\
P(103)P(230) + P(103)P(300) + P(103)P(310) + P(103)P(320) + P(103)P(330) + P(110)P(113) + P(110)P(123) + P(110)P(133) + P(110)P(203) + P(110)P(213)+ \\
P(110)P(223) + P(113)P(120) + P(113)P(130) + P(113)P(200) + P(113)P(210) + P(113)P(220) + P(113)P(230) + P(113)P(300) + P(113)P(310) + P(113)P(320)+ \\
P(113)P(330) + P(120)P(123) + P(120)P(133) + P(120)P(203) + P(120)P(210) + P(120)P(211) + P(120)P(212) + P(120)P(223) + P(120)P(233) + P(120)P(310)+ \\
P(120)P(311) + P(120)P(312) + P(120)P(313) + P(121)P(210) + P(121)P(211) + P(121)P(212) + P(121)P(213) + P(121)P(310) + P(121)P(311) + P(121)P(312)+ \\
P(121)P(313) + P(122)P(210) + P(122)P(212) + P(122)P(213) + P(122)P(310) + P(122)P(311) + P(122)P(312) + P(122)P(313) + P(123)P(130) + P(123)P(200)+ \\
P(123)P(211) + P(123)P(212) + P(123)P(213) + P(123)P(220) + P(123)P(230) + P(123)P(300) + P(123)P(311) + P(123)P(312) + P(123)P(313) + P(123)P(320)+ \\
P(123)P(330) + P(130)P(133) + P(130)P(203) + P(130)P(210) + P(130)P(211) + P(130)P(212) + P(130)P(223) + P(130)P(233) + P(130)P(310) + P(130)P(311)+ \\
P(130)P(312) + P(130)P(313) + P(131)P(210) + P(131)P(211) + P(131)P(212) + P(131)P(213) + P(131)P(310) + P(131)P(311) + P(131)P(312) + P(131)P(313)+ \\
P(132)P(201) + P(132)P(210) + P(132)P(211) + P(132)P(212) + P(132)P(213) + P(132)P(301) + P(132)P(310) + P(132)P(312) + P(132)P(313) + P(133)P(200)+ \\
P(133)P(211) + P(133)P(212) + P(133)P(213) + P(133)P(220) + P(133)P(230) + P(133)P(300) + P(133)P(311) + P(133)P(312) + P(133)P(313) + P(133)P(320)+ \\
P(133)P(330) + P(200)P(203) + P(200)P(213) + P(200)P(223) + P(200)P(233) + P(200)P(303) + P(200)P(313) + P(200)P(323) + P(200)P(333) + P(203)P(210)+ \\
P(203)P(220) + P(203)P(230) + P(203)P(300) + P(203)P(310) + P(203)P(320) + P(203)P(330) + P(210)P(213) + P(210)P(223) + P(210)P(233) + P(210)P(303)+ \\
P(210)P(313) + P(210)P(323) + P(210)P(333) + P(213)P(220) + P(213)P(230) + P(213)P(300) + P(213)P(310) + P(213)P(320) + P(213)P(330) + P(220)P(223)+ \\
P(220)P(233) + P(220)P(303) + P(220)P(313) + P(220)P(323) + P(220)P(333) + P(223)P(230) + P(223)P(300) + P(223)P(310) + P(223)P(320) + P(223)P(330)+ \\
P(230)P(233) + P(230)P(303) + P(230)P(313) + P(230)P(323) + P(230)P(333) + P(233)P(300) + P(233)P(310) + P(233)P(320) + P(233)P(330) + P(300)P(303)+ \\
P(300)P(313) + P(300)P(323) + P(300)P(333) + P(303)P(310) + P(303)P(320) + P(303)P(330) + P(310)P(313) + P(310)P(323) + P(310)P(333) + P(313)P(320)+ \\
P(313)P(330) + P(320)P(323) + P(320)P(333) + P(323)P(330) + P(330)P(333)
\end{gathered}$$

Party Symmetric Inequality

$$2[P(001)P(333)]_3 + 2[P(010)P(323)]_3 + 6[P(000)P(323)]_3 + 6[P(000)P(333)]_1$$

$$\leq$$

$$12[P(031)P(302)]_6 + 12[P(033)P(303)]_6 + 12[P(103)P(130)]_6 + 12[P(203)P(230)]_6 +$$

$$+ \cdots \text{ 126 more terms } \cdots +$$

$$6[P(101)P(130)]_6 + 6[P(103)P(310)]_6 + 6[P(113)P(130)]_6 + 6[P(113)P(230)]_6 +$$

$$6[P(113)P(330)]_3 + 6[P(122)P(330)]_6 + 6[P(130)P(313)]_6 + 6[P(132)P(303)]_6 +$$

$$6[P(133)P(303)]_6 + 6[P(133)P(320)]_6 + 6[P(200)P(203)]_6 + 6[P(201)P(230)]_6 +$$

$$6[P(203)P(231)]_6 + 6[P(223)P(300)]_6 + 8[P(003)P(320)]_6 + 8[P(032)P(300)]_6$$

Note: $[P(113)P(330)]_3$ shorthand sum over permutations:

$$P(113)P(330) + P(131)P(303) + P(311)P(033)$$

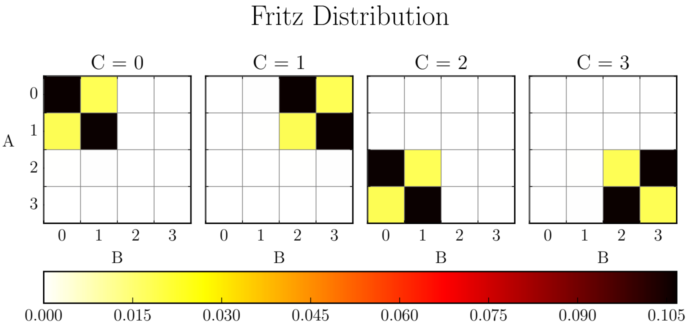
Party Symmetric Inequality (Full)

$$2[P(001)P(333)]_3 + 2[P(010)P(323)]_3 + 6[P(000)P(323)]_3 + 6[P(000)P(333)]_1$$

$$\leq$$

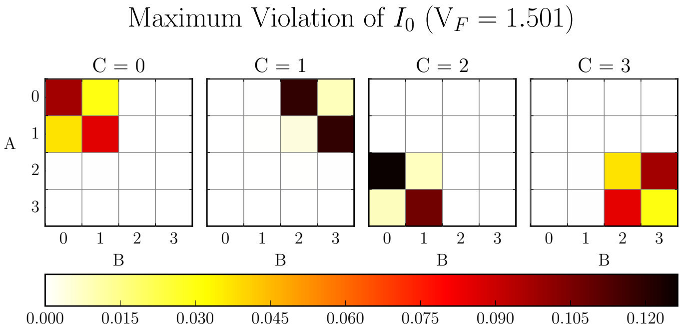
$$\begin{gathered}
12[P(031)P(302)]_6 + 12[P(033)P(303)]_6 + 12[P(103)P(130)]_6 + 12[P(203)P(230)]_6 + 12[P(203)P(330)]_6 + 2[P(001)P(320)]_6 + 2[P(002)P(221)]_3 + 2[P(003)P(211)]_6 + \\
2[P(003)P(331)]_3 + 2[P(011)P(211)]_3 + 2[P(012)P(322)]_6 + 2[P(013)P(313)]_6 + 2[P(013)P(332)]_6 + 2[P(020)P(111)]_3 + 2[P(020)P(211)]_6 + 2[P(021)P(212)]_6 + \\
2[P(022)P(211)]_3 + 2[P(022)P(212)]_6 + 2[P(022)P(322)]_3 + 2[P(023)P(232)]_6 + 2[P(030)P(212)]_3 + 2[P(031)P(231)]_6 + 2[P(032)P(331)]_6 + 2[P(033)P(333)]_3 + \\
2[P(101)P(131)]_3 + 2[P(101)P(132)]_6 + 2[P(102)P(131)]_6 + 2[P(102)P(132)]_6 + 2[P(102)P(133)]_6 + 2[P(110)P(133)]_6 + 2[P(110)P(212)]_6 + 2[P(110)P(222)]_3 + \\
2[P(110)P(223)]_3 + 2[P(112)P(331)]_3 + 2[P(120)P(122)]_6 + 2[P(121)P(201)]_6 + 2[P(122)P(200)]_3 + 2[P(122)P(202)]_6 + 2[P(122)P(210)]_6 + 2[P(122)P(300)]_3 + \\
2[P(130)P(232)]_6 + 2[P(130)P(233)]_6 + 2[P(131)P(201)]_6 + 2[P(131)P(202)]_3 + 2[P(131)P(313)]_3 + 2[P(133)P(200)]_3 + 2[P(133)P(201)]_6 + 2[P(133)P(211)]_3 + \\
2[P(133)P(212)]_6 + 2[P(133)P(300)]_3 + 2[P(202)P(231)]_6 + 2[P(210)P(222)]_6 + 2[P(220)P(222)]_3 + 2[P(220)P(313)]_6 + 2[P(221)P(313)]_6 + 2[P(222)P(331)]_3 + \\
2[P(223)P(331)]_3 + 2[P(230)P(312)]_6 + 2[P(231)P(313)]_6 + 2[P(232)P(320)]_6 + 2[P(302)P(322)]_6 + 2[P(320)P(323)]_6 + 2[P(330)P(332)]_3 + 3[P(000)P(003)]_3 + \\
3[P(010)P(301)]_6 + 4[P(001)P(131)]_6 + 4[P(002)P(020)]_6 + 4[P(002)P(133)]_6 + 4[P(002)P(323)]_6 + 4[P(010)P(123)]_6 + 4[P(013)P(212)]_6 + 4[P(013)P(312)]_6 + \\
4[P(023)P(221)]_6 + 4[P(023)P(222)]_6 + 4[P(023)P(322)]_6 + 4[P(031)P(211)]_6 + 4[P(032)P(321)]_6 + 4[P(100)P(123)]_6 + 4[P(100)P(232)]_6 + 4[P(100)P(313)]_6 + \\
4[P(112)P(310)]_6 + 4[P(122)P(203)]_6 + 4[P(122)P(302)]_6 + 4[P(130)P(222)]_6 + 4[P(130)P(223)]_6 + 4[P(222)P(310)]_6 + 4[P(223)P(320)]_6 + 4[P(231)P(301)]_6 + \\
4[P(312)P(330)]_6 + 6[P(001)P(031)]_6 + 6[P(001)P(033)]_6 + 6[P(002)P(300)]_6 + 6[P(002)P(330)]_3 + 6[P(003)P(032)]_6 + 6[P(003)P(131)]_6 + 6[P(003)P(132)]_6 + \\
6[P(011)P(300)]_3 + 6[P(011)P(320)]_6 + 6[P(012)P(200)]_6 + 6[P(012)P(301)]_6 + 6[P(013)P(030)]_6 + 6[P(013)P(110)]_6 + 6[P(013)P(120)]_6 + 6[P(013)P(303)]_6 + \\
6[P(020)P(102)]_6 + 6[P(020)P(103)]_6 + 6[P(020)P(123)]_6 + 6[P(020)P(202)]_3 + 6[P(020)P(203)]_6 + 6[P(020)P(311)]_6 + 6[P(020)P(322)]_6 + 6[P(020)P(330)]_6 + \\
6[P(022)P(303)]_6 + 6[P(030)P(033)]_6 + 6[P(030)P(101)]_3 + 6[P(030)P(133)]_6 + 6[P(030)P(202)]_3 + 6[P(030)P(303)]_3 + 6[P(030)P(332)]_6 + 6[P(031)P(203)]_6 + \\
6[P(032)P(310)]_6 + 6[P(033)P(101)]_6 + 6[P(033)P(130)]_6 + 6[P(033)P(200)]_3 + 6[P(033)P(212)]_6 + 6[P(033)P(220)]_6 + 6[P(033)P(222)]_3 + 6[P(033)P(230)]_6 + \\
6[P(033)P(322)]_3 + 6[P(100)P(203)]_6 + 6[P(101)P(130)]_6 + 6[P(103)P(310)]_6 + 6[P(113)P(130)]_6 + 6[P(113)P(230)]_6 + 6[P(113)P(330)]_3 + 6[P(122)P(330)]_6 + \\
6[P(130)P(313)]_6 + 6[P(132)P(303)]_6 + 6[P(133)P(303)]_6 + 6[P(133)P(320)]_6 + 6[P(200)P(203)]_6 + 6[P(201)P(230)]_6 + 6[P(203)P(231)]_6 + 6[P(223)P(300)]_6 + \\
8[P(003)P(320)]_6 + 8[P(032)P(300)]_6
\end{gathered}$$

Fritz Distribution



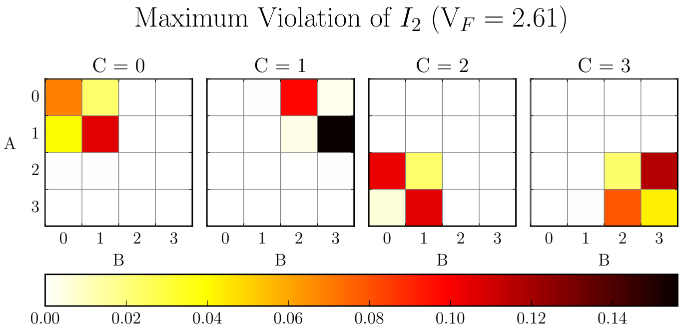
$$\square = \frac{1}{32}\left(2-\sqrt{2}\right) \qquad \blacksquare = \frac{1}{32}\left(2+\sqrt{2}\right)$$

Maximal Violation



Relative Violation: $$V_F = \frac{\min_P\{I(P)\}}{I(P_F)}$$

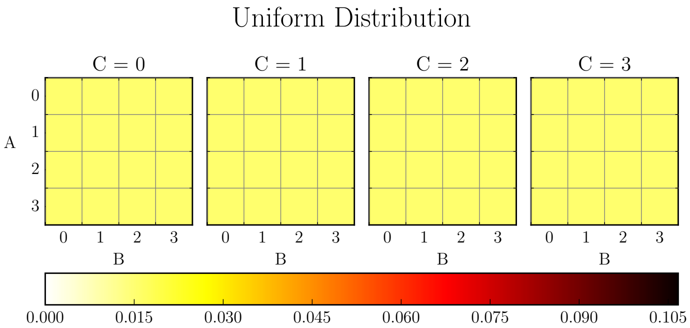
Maximal Violation Symmetric



Relative Violation: $$V_F = \frac{\min_P \{I(P)\}}{I(P_F)}$$

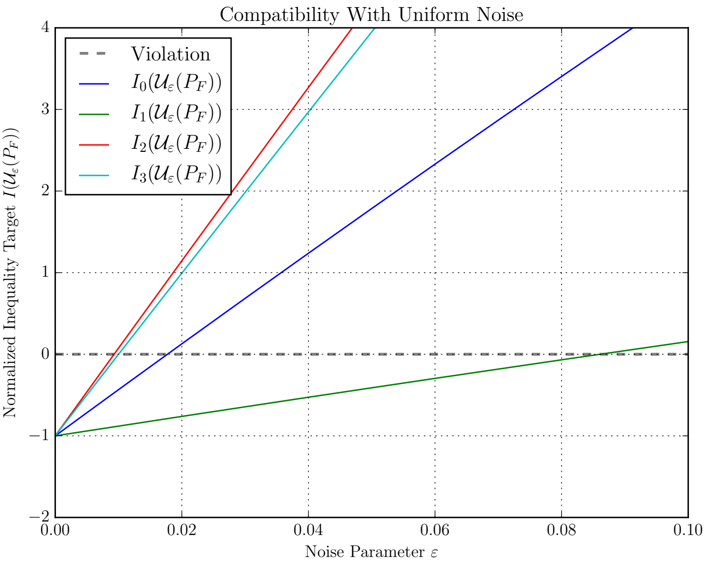
Uniform Noise

$$\mathcal{U}_\varepsilon(P) = (1-\varepsilon)P + \varepsilon\mathcal{U}$$



$$\square = \frac{1}{64}$$

Robust to Noise Zoomed



Conclusions

- Quantum/Classical difference can be observed in the triangle scenario
- Successful demonstration of the inflation technique
- Developed new algorithmic tools to find such inequalities
- New measures of non-classicality which lead to new quantum mechanical resources

References I

Appendix A: Optional Slides

Notation

Question: Which marginal models $P^{\mathcal{M}}$ are compatible with a causal structure $\mathcal{G}$?

- Marginal model $P^{\mathcal{M}}$ is collection of probability distributions

$$P^{\mathcal{M}} = \{P_{V_1}, \ldots, P_{V_k}\}$$

- Marginal scenario $\mathcal{M} = \{V_1, \ldots, V_k\}$

$$V \in \mathcal{M}, V' \subseteq V \implies V' \in \mathcal{M}$$

- Joint random variables $\mathcal{J} = \bigcup_i V_i = \{v_1, \ldots, v_n\}$
- Causal Structure $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is a directed acyclic graph (DAG)
- Nodes classified into latent nodes $\mathcal{N}_L$ and observed nodes $\mathcal{N}_O$

Graph Theory

Let $n, m \in \mathcal{N}$ be nodes of the graph $\mathcal{G}$.

- parents of n: $\mathsf{Pa}_{\mathcal{G}}(n) \equiv \{m \mid m \to n\}$
- children of n: $\mathsf{Ch}_{\mathcal{G}}(n) \equiv \{m \mid n \to m\}$
- ancestry of n: $\mathsf{An}_{\mathcal{G}}(n) \equiv \bigcup_{i \in \mathbb{W}} \mathsf{Pa}^{i}_{\mathcal{G}}(n)$

$$\mathsf{Pa}^{0}_{\mathcal{G}}(n) = n \qquad \mathsf{Pa}^{i}_{\mathcal{G}}(n) \equiv \mathsf{Pa}_{\mathcal{G}}\left(\mathsf{Pa}^{i-1}_{\mathcal{G}}(n)\right)$$

Notation extends to sets of nodes $N \subseteq \mathcal{N}$,

- parents of N: $\mathsf{Pa}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Pa}_{\mathcal{G}}(n)$
- children of N: $\mathsf{Ch}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{Ch}_{\mathcal{G}}(n)$
- ancestry of N: $\mathsf{An}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \mathsf{An}_{\mathcal{G}}(n)$

An induced subgraph of $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ due to $N \subseteq \mathcal{N}$

$$\mathsf{Sub}_{\mathcal{G}}(N) = (N, \{e \in \mathcal{E} \mid e \subseteq N\})$$

Causal Compatibility

Question: Which marginal models $P^{\mathcal{M}}$ are compatible with a causal structure $\mathcal{G}$?
Answer: $P^{\mathcal{M}}$ is compatible with $\mathcal{G}$ if there exists a set of casual parameters

$$\left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\}$$

Such that for each $V \in \mathcal{M}$, P_V can be recovered:

1. $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$
2. $P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$

Inequality: A casual compatibility inequality I is an inequality over $P^{\mathcal{M}}$ that is satisfied by all compatible $P^{\mathcal{M}}$

Deriving Inequalities

Two necessary components to compatibility:

1. Marginal problem: $\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$
   - Is the marginal model contextual or non-contextual?
   - 3 distinct ways to tackle this problem
     1. Convex hull, Polytope projection, Fourier-Motzkin
     2. Possibilistic Hardy Inequalities (Hypergraph transversals)
     3. Linear Program Feasibility/Infeasibility
2. Markov Separation: $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$
   - Much harder to determine since latent nodes $\mathcal{N}_O$ have unspecified behaviour
   - It is possible to turn Markov Separation problem into a Marginal problem (at least partially)

Inflation Technique

Developed by Wolfe, Spekkens, and Fritz **Inflation**

Definition

An inflation of a causal structure $\mathcal{G}$ is another causal structure $\mathcal{G}'$ such that:

$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n)$$

Where $\mathsf{AnSub}_{\mathcal{G}}(n)$ denotes the ancestral sub-graph of n in $\mathcal{G}$

$$\mathsf{AnSub}_{\mathcal{G}}(n) = \mathsf{Sub}_{\mathcal{G}}(\mathsf{An}_{\mathcal{G}}(n))$$

And '$\sim$' is a copy-index equivalence relation

$$A_1 \sim A_2 \sim A \not\sim B_1 \sim B_2 \sim B$$

Inflation Lemma

If one has obtained $\mathcal{G}$, inflation $\mathcal{G}'$ and *compatible* marginal distribution P_N where $N \subseteq \mathcal{N}$, then:

1. There exists causal parameters $\left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\}$ such that

$$P_N = \prod_{n \in N} P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

2. $\mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n) \implies \mathsf{Pa}_{\mathcal{G}'}(n') \sim \mathsf{Pa}_{\mathcal{G}}(n)$
3. Construct inflated causal parameters

$$\forall n' \in \mathcal{N}' : P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')} \equiv P_{n|\mathsf{Pa}_{\mathcal{G}}(n)}$$

4. Obtain *compatible* marginal distributions over any $N' \subseteq \mathcal{N}'$

$$P_{N'} = \prod_{n' \in N'} P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')}$$

Inflation Lemma Cont'd

- Inflation procedure holds for any $N \in \mathcal{N}, N' \in \mathcal{N}'$ where $\mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')$
- Define injectable sets of $\mathcal{G}'$ and images of the injectable of $\mathcal{G}$

$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')\}$$
$$\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : \mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')\}$$

- For $N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$ there is a *unique* $N \subseteq \mathcal{N}$ such that $\mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')$
- For $N \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G})$ there can *exist many* $N' \subseteq \mathcal{N}'$ such that $\mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')$

Inflation Lemma Cont'd

Lemma

The Inflation Lemma: **Inflation** *Given a particular inflation $\mathcal{G}'$ of $\mathcal{G}$, if a marginal model $\{P_N \mid N \in \mathrm{ImInj}_{\mathcal{G}}(\mathcal{G}')\}$ is compatible with $\mathcal{G}$ then all marginal models $\{P_{N'} \mid N' \in \mathrm{Inj}_{\mathcal{G}}(\mathcal{G}')\}$ are compatible with $\mathcal{G}'$ provided that $P_N = P_{N'}$ for all instances where $N \sim N'$.*

Corollary

Any causal compatibility inequality I' constraining the injectable sets $\mathrm{Inj}_{\mathcal{G}}(\mathcal{G}')$ can be deflated into a causal compatibility inequality I constraining the images of the injectable sets $\mathrm{ImInj}_{\mathcal{G}}(\mathcal{G}')$.

Pre-injectable Sets

- d-separation relations + inflation = polynomial inequalities over $\mathcal{G}$
- Restrict focus to sets N' that are partitioned into N'_1, N'_2 d-separated by empty set $\emptyset$
- A pre-injectable set N':

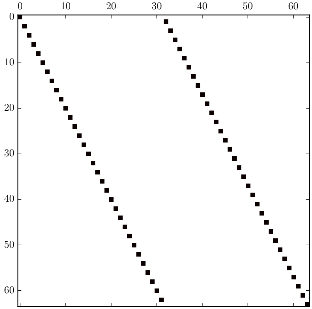
$$N' = \coprod_i N'_i \quad \forall i : N'_i \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$\forall i, j : N'_i \perp N'_j \iff \mathsf{An}_{\mathcal{G}'}(N'_i) \cap \mathsf{An}_{\mathcal{G}'}\left(N'_j\right) = \emptyset$$

- Only need to consider maximal pre-injectable sets $\mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$

Network Permutation Matrix

- States and measurements in the Triangle Scenario are not aligned
- Without Π, P_{ABC} would be separable
- Required to align B's measurement over $\mathrm{Tr}_{A,C}(\rho_{AB} \otimes \rho_{BC})$
- Π is a $2^6 \times 2^6$ matrix
- Shifts one qubit to the left



$$\Pi \equiv \sum_{|q_i\rangle \in \{|0\rangle,|1\rangle\}} |q_2 q_3 q_4 q_5 q_6 q_1\rangle\langle q_1 q_2 q_3 q_4 q_5 q_6|$$

Outcomes and Events

Definition

Each variable v has finite set of outcomes O_v.
Each set of variables V has finite set of events $\mathcal{E}(V)$:

$$\mathcal{E}(V) \equiv \{s : V \to O_V \mid \forall v \in V, s(v) \in O_v\}$$

Definition

The set of events over the joint variables $\mathcal{E}(\mathcal{J})$ are termed the joint events.

Definition

The set of events over the marginal contexts are the marginal events

$$\mathcal{E}(\mathcal{M}) \equiv \coprod_{V \in \mathcal{M}} \mathcal{E}(V)$$

Distribution Vectors

Definition

The joint distribution vector $\mathcal{P}^{\mathcal{J}}$

$$\mathcal{P}^{\mathcal{J}}_j = P_{\mathcal{J}}(j) \quad \forall j \in \mathcal{E}(\mathcal{J})$$

Definition

The marginal distribution vector $\mathcal{P}^{\mathcal{M}}$

$$\mathcal{P}^{\mathcal{M}}_m = P_{\mathcal{D}(m)}(m) \quad \forall m \in \mathcal{E}(\mathcal{M}), \mathcal{D}(m) \in \mathcal{M}$$

Can now write complete marginal problem as matrix multiplication:

$$\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J} \setminus V} P_{\mathcal{J}} \iff \mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

Incidence Matrix

- Incidence matrix M is a bit-wise matrix
- Row-indexed by marginal events $m \in \mathcal{E}(\mathcal{M})$
- Column-indexed by joint events $j \in \mathcal{E}(\mathcal{J})$

$$M_{m,j} = \begin{cases} 1 & m = j|_{\mathcal{D}(m)} \\ 0 & \text{otherwise} \end{cases}$$

$$\#\text{Columns} = |\mathcal{E}(\mathcal{J})| = \prod_{v \in \mathcal{J}} |O_v|$$

$$\#\text{Rows} = |\mathcal{E}(\mathcal{M})| = \sum_{V \in \mathcal{M}} \prod_{v \in V} |O_v|$$

Causal Symmetry

- Desirable to find compatibility inequality I such that

$$\forall \varphi \in \mathsf{Perm}(A, B, C) : \varphi[I] = I$$

- Compatibility is independent of variable labels
  $I, \mathcal{G} \to \varphi[I], \varphi[\mathcal{G}]$
- Need $\varphi[\mathcal{G}] = \mathcal{G}$ to find new $\varphi[I]$

Definition

The causal symmetry group of causal structure $\mathcal{G}$:

$$\mathsf{Aut}(\mathcal{G}) = \{\varphi \in \mathsf{Perm}(\mathcal{N}) \mid \varphi[\mathcal{G}] = \mathcal{G}\}$$

Strictly speaking, one needs to preserve observable nodes:

$$\mathsf{Aut}_{\mathcal{N}_O}(\mathcal{G}) = \{\varphi \in \mathsf{Aut}(\mathcal{G}) \mid \varphi[\mathcal{N}_O] = \mathcal{N}_O\}$$

Causal Symmetry and Inflation

- Causal symmetry group for $\mathcal{G}'$ is no good!
- Not possible to deflate inequality if it's not in terms of injectable sets

Definition

The restricted causal symmetry group Φ of $\mathcal{G}'$:

$$\Phi = \mathsf{Aut}_{\mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')}(\mathcal{G}')$$

Restricted Causal Symmetry of Large Inflation

- Φ for the large inflation is an order 48 group with 4 generators

$\varphi 1$	$\varphi 2$	$\varphi 3$	$\varphi 4$
$A_1 \to A_4$	$A_1 \to A_1$	$A_1 \to C_1$	$A_1 \to A_1$
$A_2 \to A_3$	$A_2 \to A_3$	$A_2 \to C_2$	$A_2 \to A_2$
$A_3 \to A_2$	$A_3 \to A_2$	$A_3 \to C_3$	$A_3 \to A_3$
$A_4 \to A_1$	$A_4 \to A_4$	$A_4 \to C_4$	$A_4 \to A_4$
$B_1 \to B_4$	$B_1 \to C_1$	$B_1 \to A_1$	$B_1 \to B_2$
$B_2 \to B_3$	$B_2 \to C_3$	$B_2 \to A_2$	$B_2 \to B_1$
$B_3 \to B_2$	$B_3 \to C_2$	$B_3 \to A_3$	$B_3 \to B_4$
$B_4 \to B_1$	$B_4 \to C_4$	$B_4 \to A_4$	$B_4 \to B_3$
$C_1 \to C_4$	$C_1 \to B_1$	$C_1 \to B_1$	$C_1 \to C_3$
$C_2 \to C_3$	$C_2 \to B_3$	$C_2 \to B_2$	$C_2 \to C_4$
$C_3 \to C_2$	$C_3 \to B_2$	$C_3 \to B_3$	$C_3 \to C_1$
$C_4 \to C_1$	$C_4 \to B_4$	$C_4 \to B_4$	$C_4 \to C_2$

Symmetric Incidence

- Group orbits through repeated action of $\varphi \in \Phi$ on $m \in \mathcal{E}(\mathcal{M})$ and $j \in \mathcal{E}(\mathcal{J})$

$$\Phi[m] \equiv \{\varphi[m] \mid \varphi \in \Phi\}$$
$$\Phi[j] \equiv \{\varphi[j] \mid \varphi \in \Phi\}$$

- Construct symmetric incidence matrix $\Phi[M]$

$$\Phi[M]_{\Phi[m],\Phi[j]} = \sum_{m' \in \Phi[m]} \sum_{j' \in \Phi[j]} M_{m',j'}$$

$$\Phi[M] = \Lambda_{\Phi[\mathcal{E}(\mathcal{M})]} \cdot M \cdot \Lambda_{\Phi[\mathcal{E}(\mathcal{J})]}$$

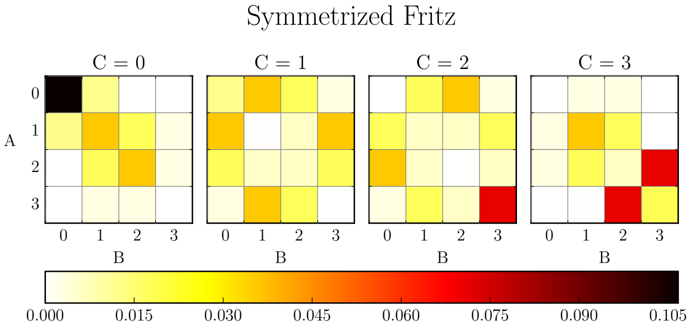
- $\Phi[M]$ not a bit-wise matrix like M
- For large inflation M is $16,896 \times 16,777,216$
- For large inflation $\Phi[M]$ is $450 \times 358,120$

Symmetric Inequalities

- Certificate inequalities are curated/constructed to witness violation of particular distribution
- Above inequality not the best candidate to search for non-locality different than Fritz distribution
- Perhaps one could obtain inequality that does not assign special preference to C
- Desirable to find compatibility inequality I such that

$$\forall \varphi \in \mathsf{Perm}(A, B, C) : \varphi[I] = I$$

Symmetrized Fritz



- More non-local yet not quantum-accessible
- Quantum-accessible distributions form non-context set

Parameterizing POVMs

- Each party (A, B, C) is assigned a projective-operator valued measure (POVM) (M_A, M_B, M_C)

$$\forall |\psi\rangle \in \mathcal{H}^d : \langle\psi|M_\chi|\psi\rangle \geq 0 \quad M_\chi = M_\chi^\dagger$$

- n-outcome measurement

$$M_\chi = \{M_{\chi,1}, \ldots, M_{\chi,n}\} \quad \sum_{i=1}^{n} M_{\chi,i} = \mathbb{1}$$

- For $n = 2$ outcomes, a parameterization exists by constraining the eigenvalues of $M_{\chi,i}$; for $n > 2$ not aware of anything
- Warrants consideration of projective-valued measures (PVMs)

Cholesky Parameterization of States

- Each latent resource $\rho \in (\rho_{AB}, \rho_{BC}, \rho_{CA})$ modeled as bipartite qubit state acting on $\mathcal{H}^{d/2} \otimes \mathcal{H}^{d/2}$
- $d \times d$ positive semi-definite (PSD) hermitian matrices with unitary trace
- Cholesky Parametrization allows one to write any hermitian PSD as $\rho = T^{\dagger}T$
- For $d = 4$:

$$T = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_2 + i\lambda_3 & \lambda_4 & 0 & 0 \\ \lambda_5 + i\lambda_6 & \lambda_7 + i\lambda_8 & \lambda_9 & 0 \\ \lambda_{10} + i\lambda_{11} & \lambda_{12} + i\lambda_{13} & \lambda_{14} + i\lambda_{15} & \lambda_{16} \end{bmatrix}$$

- d^2 real-valued parameters
- Normalized $\rho = T^{\dagger}T/\mathrm{Tr}\left(T^{\dagger}T\right)$ adds degeneracy

Appendix B: Local Minima Concerns

Results

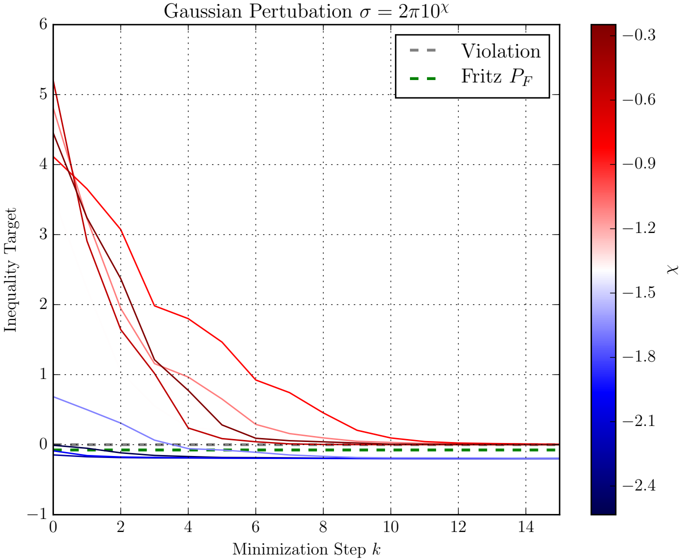
- Finding global minimum is tricky
- Difficult to converge to violation
- Noisy seed (Gaussian noise):

$$\lambda_{(0)} = \lambda_{(F)} + \delta\lambda \qquad \delta\lambda_i \sim \mathcal{N}\Big(\mu = 0, \sigma^2 = (2\pi 10^{\chi})^2\Big)$$

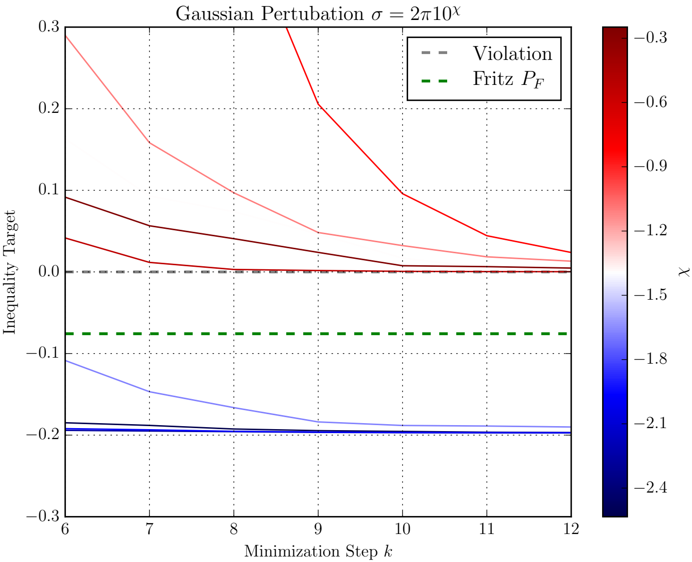
- Uniform seed:

$$\lambda_{(0),i} \sim \mathcal{U}([0, 2\pi])$$

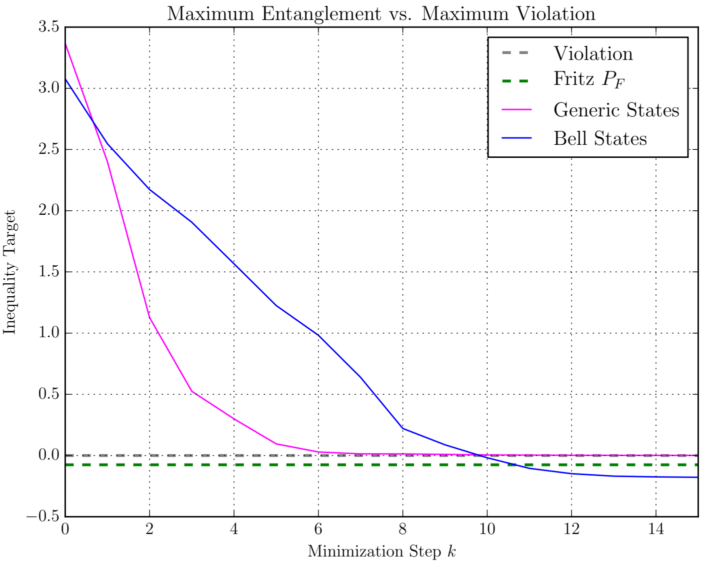
Fritz Local Minima



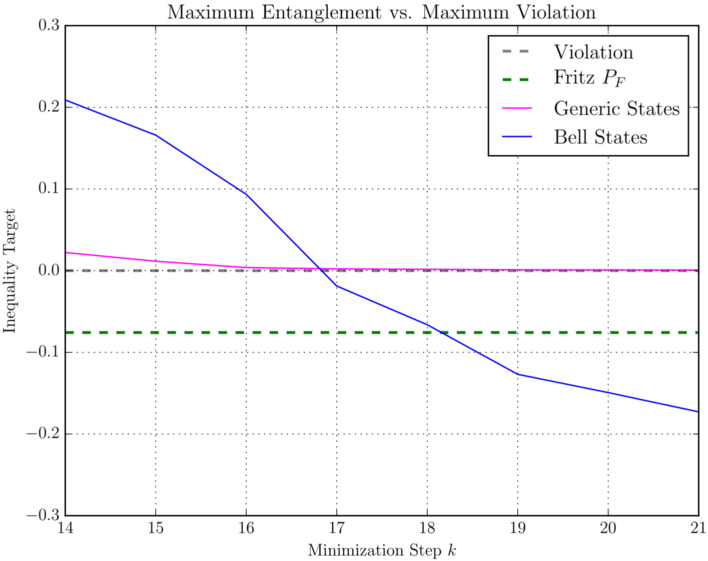
Fritz Local Minima Zoomed



Max Entangled vs. Max Violating (???)

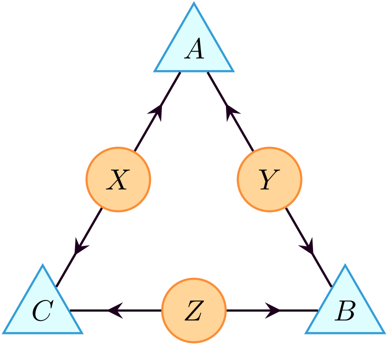


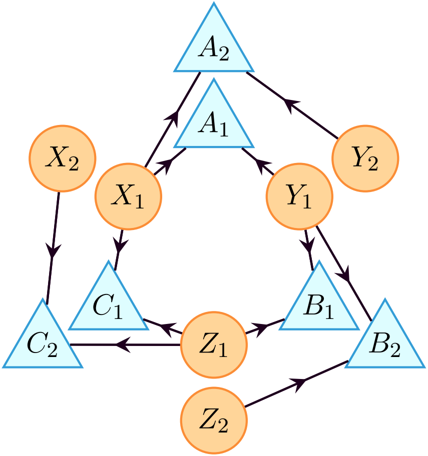
Max Entangled vs. Max Violating (???)



Section C: Inflation Technique

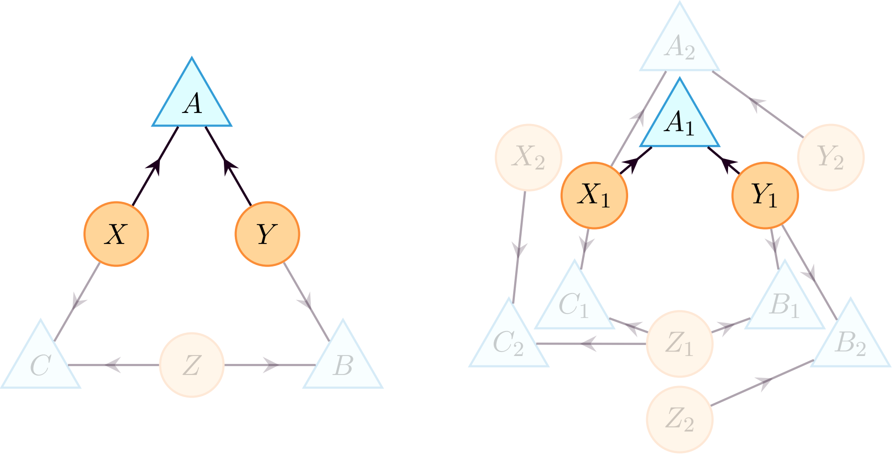
Demonstrating Inflation Technique





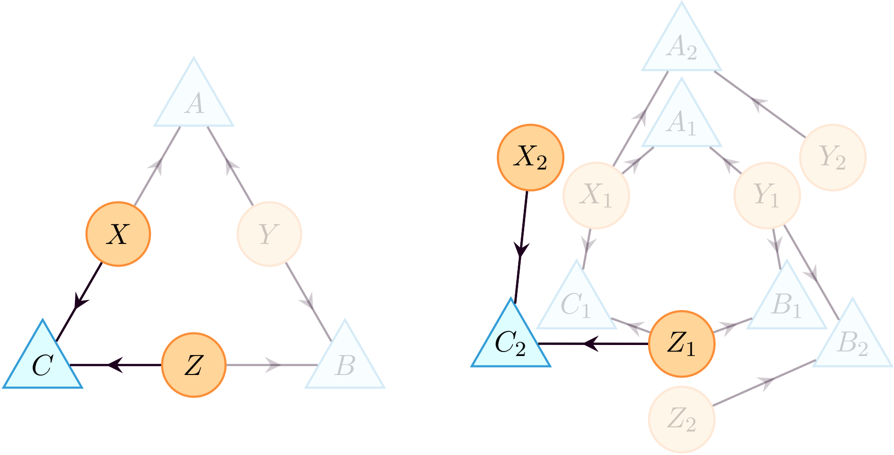
$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}'}(n') \sim \mathsf{AnSub}_{\mathcal{G}}(n)$$

Demonstrating Inflation Technique



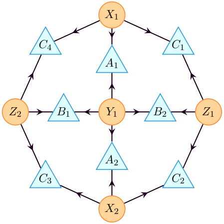
$$\mathsf{AnSub}_{\mathcal{G}}(A) \sim \mathsf{AnSub}_{\mathcal{G}'}(A_1)$$

Demonstrating Inflation Technique

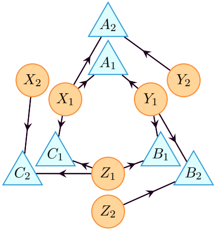


$$\mathsf{AnSub}_{\mathcal{G}}(C) \sim \mathsf{AnSub}_{\mathcal{G}'}(C_2)$$

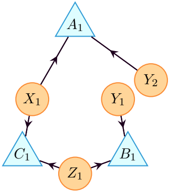
Some Inflations of the Triangle Scenario



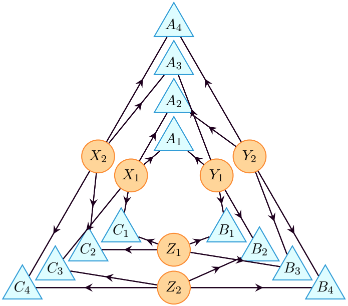
Wagon Wheel Inflation



Spiral Inflation

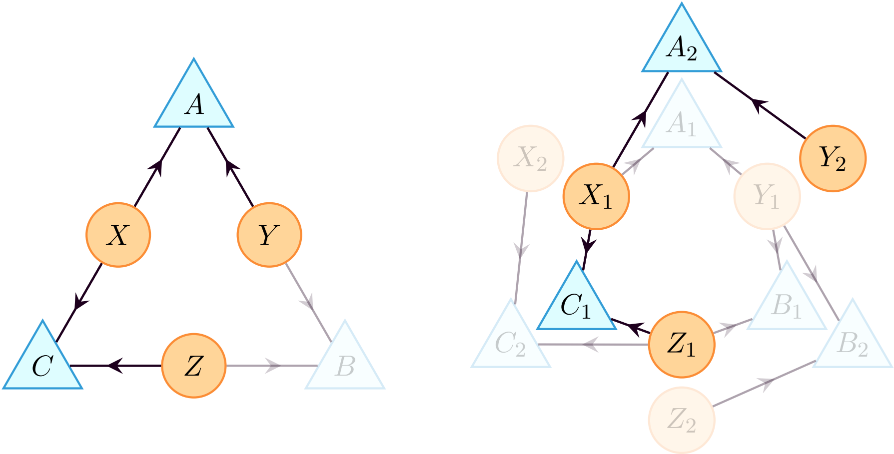


Cut Inflation



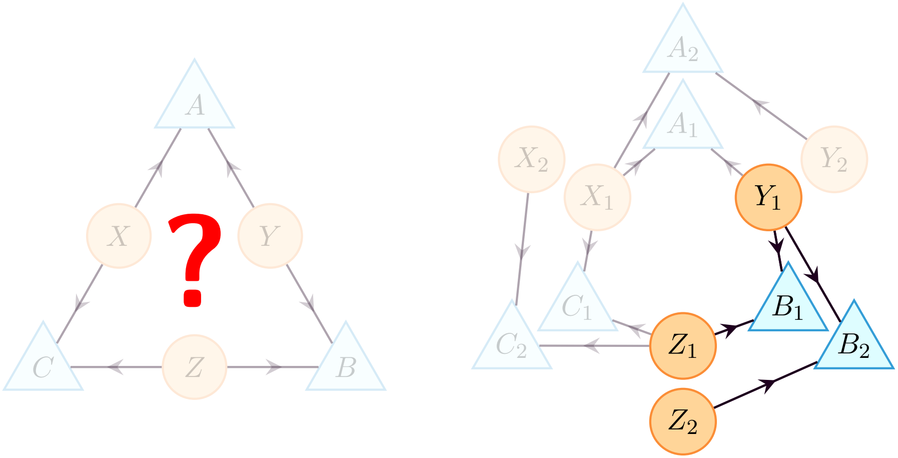
Web Inflation

What are Injectable Sets?



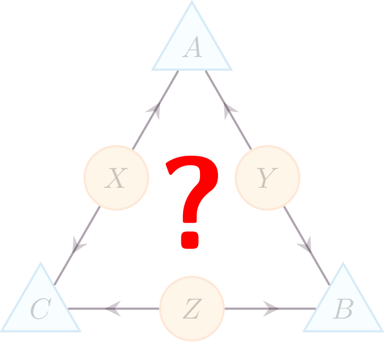
$$\mathsf{AnSub}_{\mathcal{G}}(A, C) \sim \mathsf{AnSub}_{\mathcal{G}'}(A_2, C_1)$$

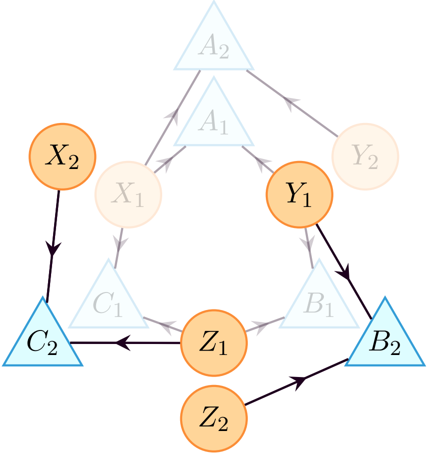
What are Injectable Sets?



$$?? \not\sim \mathsf{AnSub}_{\mathcal{G}'}(B_1, B_2)$$

What are Injectable Sets?





$$?? \not\sim \mathsf{AnSub}_{\mathcal{G}'}(B_2, C_2)$$

The injectable sets in $\mathcal{G}'$:

$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : \mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')\}$$

The images of the injectable sets in $\mathcal{G}$:

$$\mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : \mathsf{AnSub}_{\mathcal{G}}(N) \sim \mathsf{AnSub}_{\mathcal{G}'}(N')\}$$

What makes injectable sets useful?

Inflation Lemma

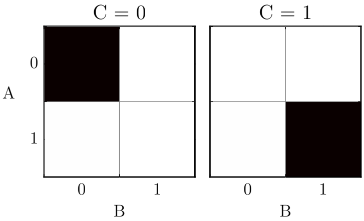
Lemma (Inflation Lemma)

Given $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ and inflation $\mathcal{G}' = (\mathcal{N}', \mathcal{E}')$:

$$\begin{tikzcd}
\underbrace{\{P_N \mid N \in \mathsf{ImInj}_{\mathcal{G}}(\mathcal{G}')\}}_{\text{compatible with } \mathcal{G}} \arrow[r] & \left\{P_{n|\mathsf{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\right\} \arrow[d, "\text{define}"] \\
\underbrace{\{P_{N'} \mid N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')\}}_{\text{compatible with } \mathcal{G}'} & \left\{P_{n'|\mathsf{Pa}_{\mathcal{G}'}(n')} \mid n' \in \mathcal{N}'\right\} \arrow[l]
\end{tikzcd}$$

Perfect Correlation Is Incompatible

Perfect Correlation



$$\blacksquare = \frac{1}{2}$$

$$P_{ABC}(abc) = \frac{[000] + [111]}{2}$$

$$P_{ABC}(abc) = \begin{cases} \frac{1}{2} & a = b = c \\ 0 & \text{otherwise} \end{cases}$$

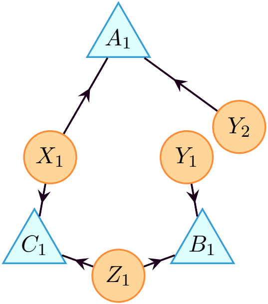
Compatibility Inequality

$$P_A(0)P_B(1) \leq P_{BC}(10) + P_{AC}(01)$$

Witnesses Perfect Correlation

$$\left(\frac{1}{2}\right)^2 \not\leq 0 + 0$$

Deriving Compatibility Inequalities



$$\mathcal{M} = \{\{A_1, B_1\}, \{B_1, C_1\}, \{A_1, C_1\}\}$$

$$P^{\mathcal{M}} = \{P_{A_1 B_1}, P_{B_1 C_1}, P_{A_1 C_1}\}$$

Compatibility requires: $\exists P_{\mathcal{J}} = P_{A_1 B_1 C_1}$

$$P_{A_1 B_1} = \sum_{C_1} P_{\mathcal{J}} \qquad P_{B_1 C_1} = \sum_{A_1} P_{\mathcal{J}} \qquad P_{A_1 C_1} = \sum_{B_1} P_{\mathcal{J}}$$

Deriving Compatibility Inequalities Cont'd

$$\underbrace{P_{A_1B_1} = \sum\nolimits_{C_1} P_{\mathcal{J}} \qquad P_{B_1C_1} = \sum\nolimits_{A_1} P_{\mathcal{J}} \qquad P_{A_1C_1} = \sum\nolimits_{B_1} P_{\mathcal{J}}}$$

$$\underbrace{\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J}\setminus V} P_{\mathcal{J}}}$$

$$\mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

$$\mathcal{P}^{\mathcal{M}} = \begin{pmatrix} P_{A_1B_1}(00) \\ P_{A_1B_1}(01) \\ P_{A_1B_1}(10) \\ P_{A_1B_1}(11) \\ \hline P_{B_1C_1}(00) \\ P_{B_1C_1}(01) \\ P_{B_1C_1}(10) \\ P_{B_1C_1}(11) \\ \hline P_{A_1C_1}(00) \\ P_{A_1C_1}(01) \\ P_{A_1C_1}(10) \\ P_{A_1C_1}(11) \end{pmatrix} \qquad \mathcal{P}^{\mathcal{J}} = \begin{pmatrix} P_{A_1B_1C_1}(000) \\ P_{A_1B_1C_1}(001) \\ P_{A_1B_1C_1}(010) \\ P_{A_1B_1C_1}(011) \\ P_{A_1B_1C_1}(100) \\ P_{A_1B_1C_1}(101) \\ P_{A_1B_1C_1}(110) \\ P_{A_1B_1C_1}(111) \end{pmatrix}$$

Incidence Example

$$M = \begin{array}{r} (A_1,B_1,C_1) = \\ (A_1{=}0,B_1{=}0) \\ (A_1{=}0,B_1{=}1) \\ (A_1{=}1,B_1{=}0) \\ (A_1{=}1,B_1{=}1) \\ (B_1{=}0,C_1{=}0) \\ (B_1{=}0,C_1{=}1) \\ (B_1{=}1,C_1{=}0) \\ (B_1{=}1,C_1{=}1) \\ (A_1{=}0,C_1{=}0) \\ (A_1{=}0,C_1{=}1) \\ (A_1{=}1,C_1{=}0) \\ (A_1{=}1,C_1{=}1) \end{array} \begin{array}{c} \begin{array}{cccccccc} (0{,}0{,}0) & (0{,}0{,}1) & (0{,}1{,}0) & (0{,}1{,}1) & (1{,}0{,}0) & (1{,}0{,}1) & (1{,}1{,}0) & (1{,}1{,}1) \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix} \end{array}$$

$$\mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

Marginal Linear Program

Marginal LP:

minimize: $\emptyset \cdot x$

subject to: $\mathcal{P}^{\mathcal{J}} \succeq 0$

$M \cdot \mathcal{P}^{\mathcal{J}} = \mathcal{P}^{\mathcal{M}}$

Dual Marginal LP:

minimize: $y \cdot \mathcal{P}^{\mathcal{M}}$

subject to: $y \cdot M \succeq 0$

- If $\mathcal{P}^{\mathcal{J}}$ exists, then:

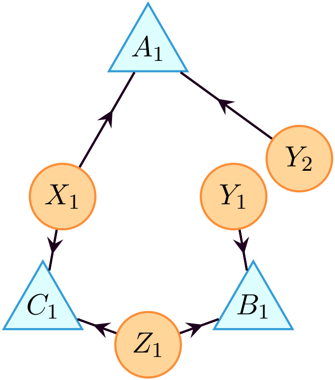
$$y \cdot M \cdot \mathcal{P}^{\mathcal{J}} = y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$$

- If not, then y is an infeasibility certificate which generates infeasibility inequality:

$$y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$$

- Most linear programing toolkits return certificates (*Mosek*, *Gurobi*, *CPLEX*, *cvxr*/*cvxopt*.)

Deflating Inequalities

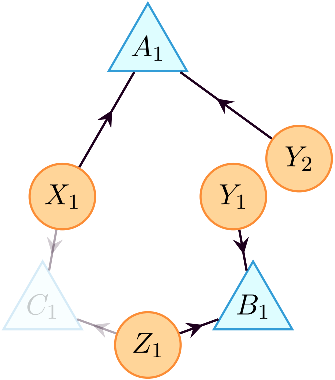


$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') = \begin{Bmatrix} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{Bmatrix}$$

$$\{A_1, B_1\} \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1B_1}(01) \leq P_{B_1C_1}(10) + P_{A_1C_1}(01)$$

Can not deflate inequality!



$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') = \begin{Bmatrix} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{Bmatrix}$$

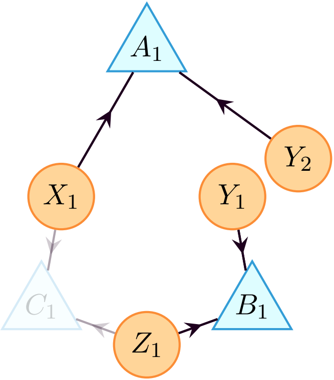
$$\{A_1, B_1\} \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1B_1}(01) \leq P_{B_1C_1}(10) + P_{A_1C_1}(01)$$

However!

$$\mathsf{AnSub}_{\mathcal{G}'}(A_1) \cap \mathsf{AnSub}_{\mathcal{G}'}(B_1) = \emptyset \iff A_1 \perp B_1$$

Deflating Inequalities



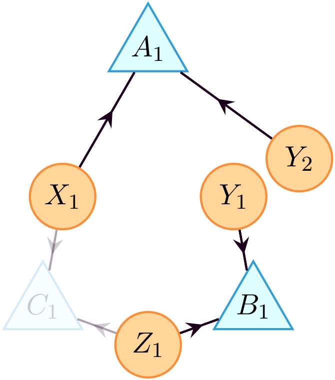
$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') = \begin{Bmatrix} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{Bmatrix}$$

$$\{A_1, B_1\} \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1B_1}(01) \leq P_{B_1C_1}(10) + P_{A_1C_1}(01)$$

$$P_{A_1}(0)P_{B_1}(1) \leq P_{B_1C_1}(10) + P_{A_1C_1}(01)$$

Deflating Inequalities



$$\mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') = \begin{Bmatrix} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{Bmatrix}$$

$$\{A_1, B_1\} \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1B_1}(01) \leq P_{B_1C_1}(10) + P_{A_1C_1}(01)$$

$$P_{A_1}(0)P_{B_1}(1) \leq P_{B_1C_1}(10) + P_{A_1C_1}(01)$$

$$P_A(0)P_B(1) \leq P_{BC}(10) + P_{AC}(01)$$

Inflation Gives Polynomial Inequalities

- Deflation only holds when inequality constrains probabilities $P_{N'}, N' \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$
- **Linear inequality** for $\mathcal{G}'$

$$
\begin{tikzcd}
P_{N'} \arrow[rr, "\substack{N'_1 \cup N'_2 = N \\ N'_1 \perp N'_2}"] \arrow[d, dash] & & P_{N'_1} P_{N'_2} \arrow[d, dash] \\
N' \notin \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \arrow[d, "\times"] & & N'_1, N'_2 \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}') \arrow[d] \\
?? & & P_{N_1} P_{N_2}
\end{tikzcd}
$$

- **Polynomial inequality** for $\mathcal{G}$!

Pre-injectable Sets

A pre-injectable set N' is:

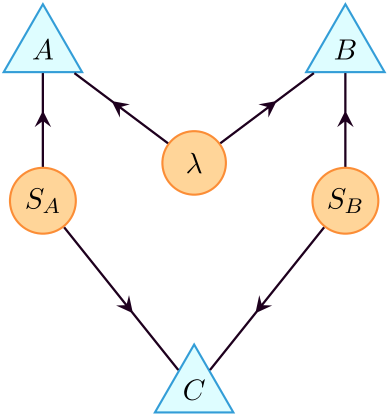
$$N' = \coprod_i N'_i \quad \forall i : N'_i \in \mathsf{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$\forall i, j : N'_i \perp N'_j \iff \mathsf{An}_{\mathcal{G}'}(N'_i) \cap \mathsf{An}_{\mathcal{G}'}\left(N'_j\right) = \emptyset$$

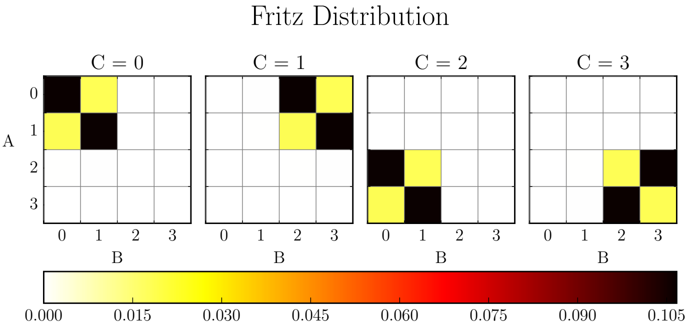
Only need to consider maximal pre-injectable sets denoted $\mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$

Appendix D: Quantum Non-classicality From Inflation

Triangle Scenario As Bell Scenario

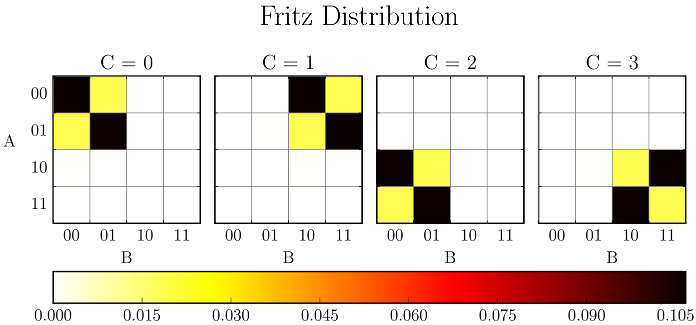


Fritz Distribution



$$\square = \frac{1}{32}\left(2 - \sqrt{2}\right) \qquad \blacksquare = \frac{1}{32}\left(2 + \sqrt{2}\right)$$

Fritz Distribution Bit-Measurements



$$= \frac{1}{32}\left(2-\sqrt{2}\right) \qquad = \frac{1}{32}\left(2+\sqrt{2}\right)$$

- States:

$$\rho_{AB} = \left|\Phi^+\right\rangle\left\langle\Phi^+\right| \quad \rho_{BC} = \rho_{CA} = \frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}$$

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- Measurements:

$$M_A = \{|0\psi_1\rangle\langle 0\psi_1|, |0\psi_5\rangle\langle 0\psi_5|, |1\psi_3\rangle\langle 1\psi_3|, |1\psi_7\rangle\langle 1\psi_7|\}$$
$$M_B = \{|\psi_6 0\rangle\langle\psi_6 0|, |\psi_2 0\rangle\langle\psi_2 0|, |\psi_0 1\rangle\langle\psi_0 1|, |\psi_4 1\rangle\langle\psi_4 1|\}$$
$$M_C = \{|00\rangle\langle 00|, |10\rangle\langle 10|, |01\rangle\langle 01|, |11\rangle\langle 11|\}$$

- Shorthand: $|\psi_n\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{in/4}|1\rangle\Big)$

Fritz Distribution Violating CHSH

- C's outcome acts as measurement "setting" for A, B; independent of ρ_{AB}
- Correlation between right bits

$$\langle A_r B_r \rangle = P_{A_r B_r}(00) + P_{A_r B_r}(11) - P_{A_r B_r}(01) - P_{A_r B_r}(10)$$

$$\langle A_r B_r | C = 0, 1, 2 \rangle = \frac{1}{\sqrt{2}} \quad \langle A_r B_r | C = 3 \rangle = -\frac{1}{\sqrt{2}}$$

- Gives CHSH violation

$$\begin{aligned} &\langle A_r B_r | C = 0 \rangle + \langle A_r B_r | C = 1 \rangle + \langle A_r B_r | C = 2 \rangle - \langle A_r B_r | C = 3 \rangle \\ &= 3\left(\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) \\ &= 2\sqrt{2} \nleq 2 \end{aligned}$$

Notes on Fritz Distribution

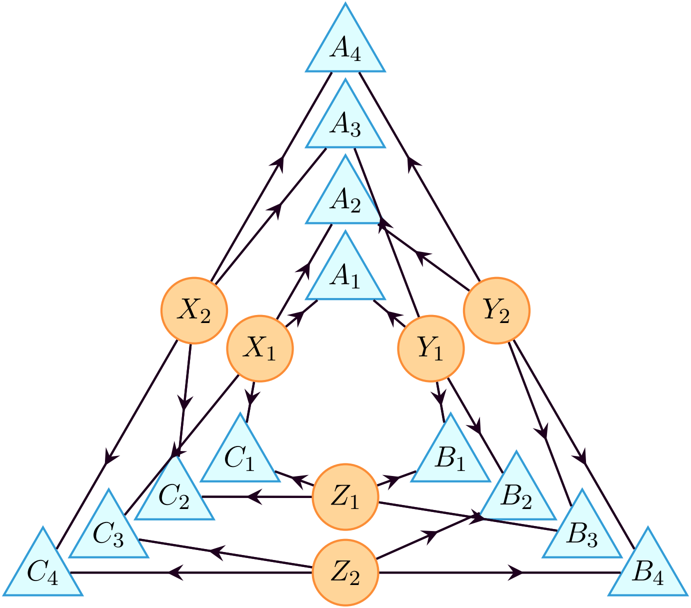
- Incompatibility proof contingent on perfect correlation between C and pseudo-settings
- Proof not robust to noise

Problem (2.17 in Fritz_2012)

Find an example of non-classical quantum correlations in TS together with a proof of its non-classicality which does not hinge on Bell's Theorem.

- "...would be helpful to have inequalities..."

Large Inflation



Large Inflation Pre-injectable Sets

Maximal Pre-injectable Sets	Ancestral Independences
$\{A_1, B_1, C_1, A_4, B_4, C_4\}$	$\{A_1, B_1, C_1\} \perp \{A_4, B_4, C_4\}$
$\{A_1, B_2, C_3, A_4, B_3, C_2\}$	$\{A_1, B_2, C_3\} \perp \{A_4, B_3, C_2\}$
$\{A_2, B_3, C_1, A_3, B_2, C_4\}$	$\{A_2, B_3, C_1\} \perp \{A_3, B_2, C_4\}$
$\{A_2, B_4, C_3, A_3, B_1, C_2\}$	$\{A_2, B_4, C_3\} \perp \{A_3, B_1, C_2\}$
$\{A_1, B_3, C_4\}$	$\{A_1\} \perp \{B_3\} \perp \{C_4\}$
$\{A_1, B_4, C_2\}$	$\{A_1\} \perp \{B_4\} \perp \{C_2\}$
$\{A_2, B_1, C_4\}$	$\{A_2\} \perp \{B_1\} \perp \{C_4\}$
$\{A_2, B_2, C_2\}$	$\{A_2\} \perp \{B_2\} \perp \{C_2\}$
$\{A_3, B_3, C_3\}$	$\{A_3\} \perp \{B_3\} \perp \{C_3\}$
$\{A_3, B_4, C_1\}$	$\{A_3\} \perp \{B_4\} \perp \{C_1\}$
$\{A_4, B_1, C_3\}$	$\{A_4\} \perp \{B_1\} \perp \{C_3\}$
$\{A_4, B_2, C_1\}$	$\{A_4\} \perp \{B_2\} \perp \{C_1\}$

Large Inflation Incidence

- Joint variables are all of the observable nodes $\mathcal{N}'_O = \mathcal{J}$

$$\mathcal{J} = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4\}$$

- Marginal scenario is composed of pre-injectable sets $\mathcal{M} = \mathsf{PreInj}_{\mathcal{G}}(\mathcal{G}')$
- Inequalities violated by Fritz distribution are inherently 4-outcome
- Incidence matrix M is **very large** $\sim 2.25\text{Gb}$
  - $\#$Columns $= 4^{12} = 16,777,216$
  - $\#$Rows $= 4 \times 4^6 + 8 \times 4^3 = 16,896$
  - $\#$Non-zero Entries $= 201,326,592$

Appendix F: Maximal Violations & Noise

Numerical Optimization

Minimize objective function $f(\lambda) \in \mathbb{R}$:

1. Real-valued parameters $\lambda = (\lambda_0, \dots, \lambda_n)$
2. Quantum states/measurements $\rho_{AB}, \rho_{BC}, \rho_{CA}, M_A, M_B, M_C$

$$P_{ABC}(abc) = \mathrm{Tr}[\Pi^\intercal \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$

3. Distribution P_{ABC}
4. Plug into inequality I in homogeneous form $I(P_{ABC}) \geq 0$
5. Output is objective value $I(P_{ABC})$

Numerical Optimization Methods

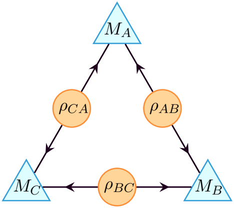
- Numerical minimization of $f(\lambda)$

$$f(\lambda_{(k+1)}) = \lambda_{(k)} - \gamma_{(k)} \nabla f(\lambda_{(k)})$$

- Non-convex, non-linear, smooth/continuous
- Gradient Descent, BFGS Method, Nelder-Mead simplex method
- Stochastic methods: simulated annealing, basin-hopping

Quantum Model On Triangle Scenario

$$P_{ABC}(abc) = \mathrm{Tr}[\Pi^{\intercal}\rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA}\Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$



- Each latent resource $\rho \in (\rho_{AB}, \rho_{BC}, \rho_{CA})$ modeled as bipartite **qubit** state acting on $\mathcal{H}^2 \otimes \mathcal{H}^2$
- Each party (A, B, C) is assigned 4-outcome PVM set (M_A, M_B, M_C)
- Parameterized using unitary transformations $U \in \mathcal{U}(4)$

- **Spengler_2010_Unitary** parameterization of $\mathcal{U}(d)$

$$\lambda = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,d} \\ \vdots & \ddots & \vdots \\ \lambda_{d,1} & \cdots & \lambda_{d,d} \end{pmatrix}$$

$$U = \left[\prod_{m=1}^{d-1} \left(\prod_{n=m+1}^{d} R_{m,n} RP_{n,m} \right) \right] \cdot \left[\prod_{l=1}^{d} GP_l \right]$$

- Global Phase Terms: $GP_l = \exp(iP_l \lambda_{l,l})$
- Relative Phase Terms: $RP_{n,m} = \exp(iP_n \lambda_{n,m})$
- Rotation Terms: $R_{m,n} = \exp(i\sigma_{m,n} \lambda_{m,n})$
- Projection Operators: $P_l = |l\rangle\langle l|$
- Anti-symmetric σ-matrices: $\sigma_{m,n} = -i|m\rangle\langle n| + i|n\rangle\langle m|$
- Parameters $\lambda_{n,m} \in [0, 2\pi]$

Parameterizing Unitary Group Cont'd

- Each parameter $\lambda_{n,m}$ has physical interpretation
- Degeneracies are easily eliminated such as global phase

$$\forall l = 1, \ldots, d : \lambda_{l,l} = 0 \implies GP_l = \mathbb{1}$$

- Parameterize $U \in \mathcal{U}(d)$ up to global phase denoted $\tilde{U} \in \mathcal{U}(d)$

$$\tilde{U} = \prod_{m=1}^{d} \left(\prod_{n=m+1}^{d} R_{m,n} RP_{n,m} \right)$$

- Computationally efficient (no matrix exponentials)

$$\begin{aligned} GP_l &= \mathbb{1} + P_l \left(e^{i\lambda_{l,l}} - 1 \right) \\ RP_{n,m} &= \mathbb{1} + P_n \left(e^{i\lambda_{n,m}} - 1 \right) \\ R_{m,n} &= \mathbb{1} + (|m\rangle\langle m| + |n\rangle\langle n|)(\cos\lambda_{n,m} - 1) \\ &\quad + (|m\rangle\langle n| - |n\rangle\langle m|) \sin\lambda_{n,m} \end{aligned}$$

Parameterizing PVMs

- Each part is assigned d element projective-valued measures (PVMs) acting on $\mathcal{H}^d$,

$$M = \{M_1, \ldots, M_d\} \qquad \sum_{i=1}^{d} M_i = \mathbb{1}$$

$$\forall i : \forall |\psi\rangle \in \mathcal{H}^d : \langle\psi|M_i|\psi\rangle \geq 0 \quad M_i = M_i^\dagger$$

$$M_i M_j = \delta_{ij} M_i \quad M_i = |m_i\rangle\langle m_i|$$

- Unitary transform maps M to computational basis

$$\{|m_1\rangle, \ldots, |m_d\rangle\} = \{U|1\rangle, \ldots, U|d\rangle\}$$

- Global phase irrelevant: $\tilde{U}$ requires $d(d-1)$ real-valued parameters
- PVMs are computationally more efficient than POVMs

$$P_{ABC}(abc) = \langle m_{A,a} m_{B,b} m_{C,c}|\Pi^\intercal \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi|m_{A,a} m_{B,b} m_{C,c}\rangle$$

Parameterizing States

- Exploit spectral decomposition

$$\rho = \sum_{i=1}^{d} p_i |\psi_i\rangle\langle\psi_i| \qquad p_i \geq 0, \sum_i p_i = 1$$

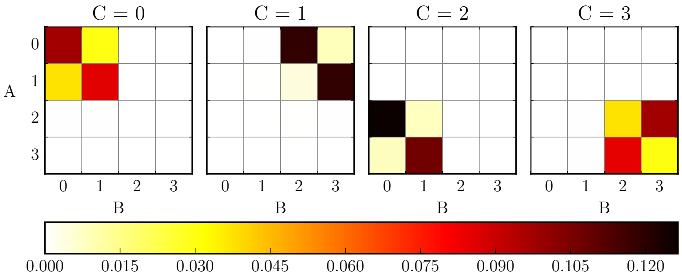
- Unitary transform maps $\{|\psi_i\rangle\}$ to computational basis

$$\{|\psi_1\rangle, \ldots, |\psi_d\rangle\} = \{U|1\rangle, \ldots, U|d\rangle\}$$

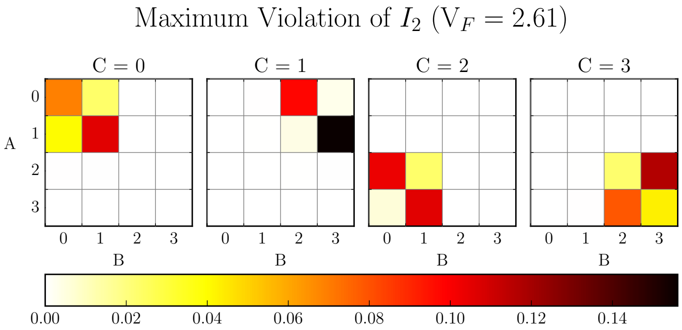
- Global phase irrelevant again $\tilde{U}$ requires $d(d-1)$ real-valued parameters
- Eigenvalues $\{p_i\}$ parameterized using hyper-spherical coordinates: $d-1$ real-valued parameters

Maximal Violation

Maximum Violation of I_0 ($V_F = 1.501$)

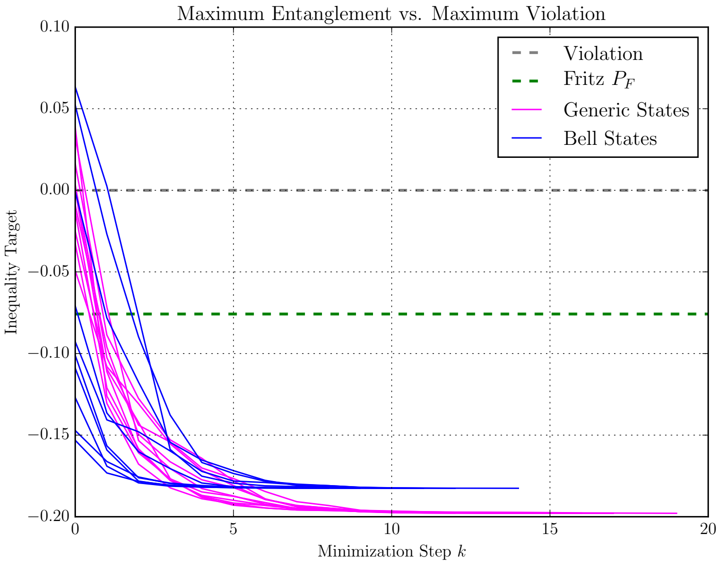


Relative Violation: $$V_F = \frac{\min_P\{I(P)\}}{I(P_F)}$$



Relative Violation: $$V_F = \frac{\min_P\{I(P)\}}{I(P_F)}$$

Max Entangled vs. Max Violating

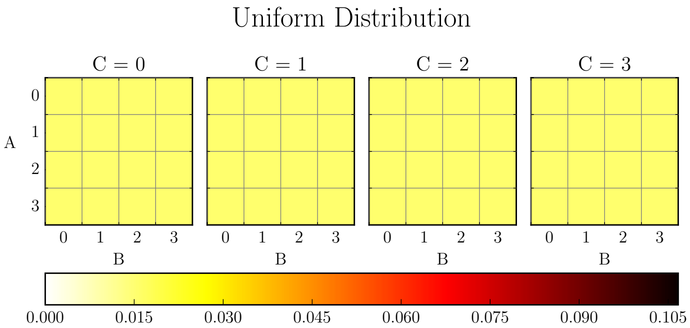


Maximally Violating Distributions

- Able to out-perform violation provided by Fritz distribution
- Maximally-violating states are not maximally-entangled; similar to detection loop-hole example of **Methot_2006**
- Both symmetric and asymmetric inequalities exhibit same qualitative features

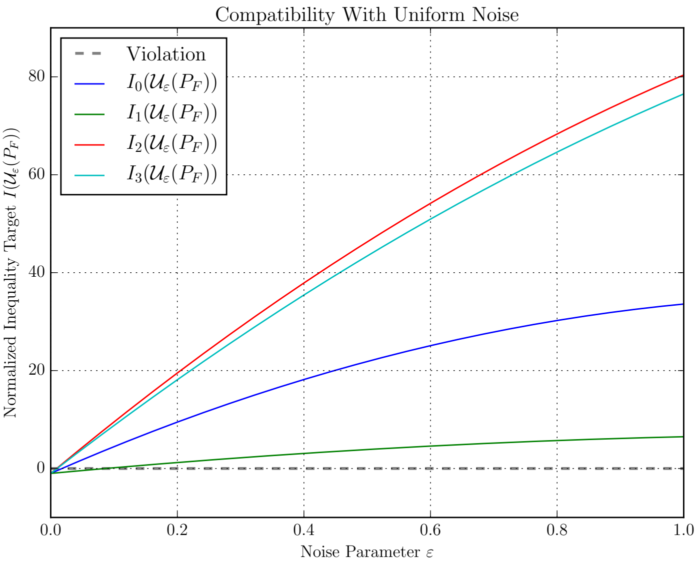
Uniform Noise

$$\mathcal{U}_\varepsilon(P) = (1-\varepsilon)P + \varepsilon\mathcal{U}$$

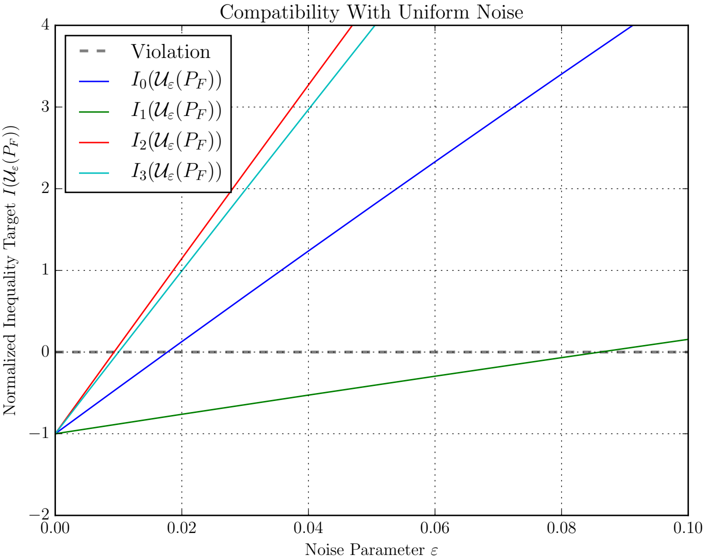


$$\square = \frac{1}{64}$$

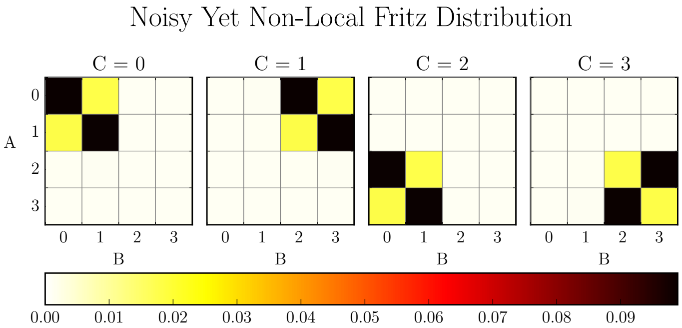
Robust to Noise



Robust to Noise Zoomed



Noisy Non-locality



$\square = 0.00133$

Conclusions

1. **Inflation technique** capable of producing polynomial inequalities with quantum/classical witnesses
2. Fritz witness-able by **party-symmetric inequalities**
3. Maximally violating distributions require **non-maximally entangled states**

Unanswered Questions

1. Do these new non-local distributions suggest **new quantum resources** in the triangle scenario?
2. Can any non-local quantum correlations in the triangle scenario **satisfy CHSH inequalities** (under Fritz type coarse graining)?
3. Which inequalities are **most robust** to noise? Facets?