

The current WagonWheel inequality we are using in the paper is the following:

$$\begin{aligned}
& I_{\text{WagonWheel}} : \\
& +P_{ABC}(000)P_C(3) - 2P_{ABC}(203)P_C(0) - 2P_{ABC}(303)P_C(0) - P_{ABC}(003)P_C(0) \\
& -P_{ABC}(013)P_C(0) - P_{ABC}(020)P_C(2) - P_{ABC}(022)P_C(2) - P_{ABC}(023)P_C(0) \\
& -P_{ABC}(023)P_C(2) - P_{ABC}(030)P_C(2) - P_{ABC}(031)P_C(2) - P_{ABC}(032)P_C(2) \\
& -P_{ABC}(033)P_C(0) - P_{ABC}(033)P_C(2) - P_{ABC}(103)P_C(0) - P_{ABC}(113)P_C(0) \\
& -P_{ABC}(123)P_C(0) - P_{ABC}(133)P_C(0) - P_{ABC}(200)P_C(0) - P_{ABC}(200)P_C(1) \\
& -P_{ABC}(200)P_C(2) - P_{ABC}(200)P_C(3) - P_{ABC}(201)P_C(0) - P_{ABC}(201)P_C(1) \\
& -P_{ABC}(201)P_C(2) - P_{ABC}(201)P_C(3) - P_{ABC}(203)P_C(1) - P_{ABC}(203)P_C(2) \\
& -P_{ABC}(203)P_C(3) - P_{ABC}(213)P_C(0) - P_{ABC}(223)P_C(0) - P_{ABC}(300)P_C(0) \\
& -P_{ABC}(300)P_C(1) - P_{ABC}(300)P_C(2) - P_{ABC}(300)P_C(3) - P_{ABC}(301)P_C(0) \\
& -P_{ABC}(301)P_C(1) - P_{ABC}(301)P_C(2) - P_{ABC}(301)P_C(3) - P_{ABC}(302)P_C(1) \\
& -P_{ABC}(303)P_C(1) - P_{ABC}(303)P_C(2) - P_{ABC}(303)P_C(3) - P_{ABC}(313)P_C(0) \leq 0
\end{aligned} \tag{1}$$

Which is currently simplified to:

$$\begin{aligned}
& I_{\text{WagonWheel}} : \\
& +P_{ABC}(000)P_C(3) + P_{ABC}(021)P_C(2) + P_{ABC}(202) + P_{ABC}(302) + P_{ABC}(233)P_C(0) + P_{AC}(33)P_C(0) \\
& -P_{ABC}(303)P_C(0) - P_{ABC}(313)P_C(0) - P_{ABC}(302)P_C(1) - P_{AB}(02)P_C(2) - \\
& -P_{AB}(03)P_C(2) - P_{AB}(20) - P_{AB}(30) - P_C(3)P_C(0) \leq 0
\end{aligned} \tag{2}$$

I went ahead and converted to the bit-notation and obtained the following:

$$\begin{aligned}
& I_{\text{WagonWheel}} : \\
& -P_{A_l B_l B_r}(100) \\
& -P_{C_l C_r}(11)P_{C_l C_r}(00) \\
& +P_{A_l B_l B_r C_l C_r}(10010) \\
& +P_{A_l A_r C_l C_r}(1111)P_{C_l C_r}(00) \\
& -P_{A_l A_r B_l B_r C_l C_r}(110011)P_{C_l C_r}(00) \\
& -P_{A_l A_r B_l B_r C_l C_r}(110010)P_{C_l C_r}(01) \\
& +P_{A_l A_r B_l B_r C_l C_r}(001001)P_{C_l C_r}(10) \\
& -P_{A_l A_r B_l}(001)P_{C_l C_r}(10) \\
& +P_{A_l A_r B_l B_r C_l C_r}(000000)P_{C_l C_r}(11) \\
& \leq 0
\end{aligned} \tag{3}$$

The following inequality is the one that Elie Wolfe found, and was proposed as a replacement for the paper due to its algebraic clarity.

$$\begin{aligned}
& I_{\text{WagonWheelAlternative}} : \\
& -P_{A_l B_l}(10) \\
& +P_{A_l B_l C_l C_r}(1010) \\
& -P_{A_l B_l}(01)P_{C_l C_r}(10) \\
& -P_{C_l C_r}(00)P_{C_l C_r}(11) \\
& +P_{C_l C_r}(00)P_{A_l A_r B_l B_r C_l C_r}(101111) \\
& +P_{C_l C_r}(00)P_{A_l A_r B_l B_r C_l C_r}(111011) \\
& -P_{C_l C_r}(01)P_{A_l A_r B_l B_r C_l C_r}(100110) \\
& -P_{C_l C_r}(01)P_{A_l A_r B_l B_r C_l C_r}(110010) \\
& +P_{C_l C_r}(10)P_{A_l A_r B_l B_r C_l C_r}(001001) \\
& +P_{C_l C_r}(10)P_{A_l A_r B_l B_r C_l C_r}(011101) \\
& +P_{C_l C_r}(11)P_{A_l A_r B_l B_r C_l C_r}(000000) \\
& +P_{C_l C_r}(11)P_{A_l A_r B_l B_r C_l C_r}(010100) \\
& \leq 0
\end{aligned} \tag{4}$$

There are some observations:

- $I_{\text{WagonWheel}}$ in bit-notation is actually shorter than $I_{\text{WagonWheelAlternative}}$
- The paper mentions the exact violation ($0.0129 \not\leq 0$) of $I_{\text{WagonWheel}}$, something I would have to calculate for $I_{\text{WagonWheelAlternative}}$

Swapping the bits C_r and B_l ,

$$\begin{aligned}
& I_{\text{WagonWheelAlternativeProperSwap}} : \\
& \quad + P_{A_l B_l}(11) \\
& \quad - P_{A_l B_l C_l C_r}(1111) \\
& \quad + P_{A_l B_l}(00) P_{C_l C_r}(11) \\
& \quad + P_{C_l C_r}(01) P_{C_l C_r}(10) \\
& \quad - P_{C_l C_r}(11) P_{A_l A_r B_l B_r C_l C_r}(000000) \\
& \quad - P_{C_l C_r}(11) P_{A_l A_r B_l B_r C_l C_r}(010100) \\
& \quad - P_{C_l C_r}(10) P_{A_l A_r B_l B_r C_l C_r}(001001) \\
& \quad - P_{C_l C_r}(10) P_{A_l A_r B_l B_r C_l C_r}(011101) \\
& \quad - P_{C_l C_r}(01) P_{A_l A_r B_l B_r C_l C_r}(100110) \\
& \quad - P_{C_l C_r}(01) P_{A_l A_r B_l B_r C_l C_r}(110010) \\
& \quad + P_{C_l C_r}(00) P_{A_l A_r B_l B_r C_l C_r}(101111) \\
& \quad + P_{C_l C_r}(00) P_{A_l A_r B_l B_r C_l C_r}(111011) \\
& \leq 0
\end{aligned} \tag{5}$$

The violation using $I_{\text{WagonWheelAlternativeProperSwap}}$,

$$\begin{aligned}
& + P_{A_l B_l}(11) = +\frac{1}{4} \\
& - P_{A_l B_l C_l C_r}(1111) = -\frac{1}{4} \\
& + P_{A_l B_l}(00) P_{C_l C_r}(11) = +\left(\frac{1}{4}\right)^2 \\
& + P_{C_l C_r}(01) P_{C_l C_r}(10) = +\left(\frac{1}{4}\right)^2 \\
& - P_{C_l C_r}(01) P_{A_l A_r B_l B_r C_l C_r}(100110) = -\frac{1}{4} \left(\frac{1}{32} (2 - \sqrt{2}) \right) \\
& - P_{C_l C_r}(01) P_{A_l A_r B_l B_r C_l C_r}(110010) = -\frac{1}{4} \left(\frac{1}{32} (2 - \sqrt{2}) \right) \\
& - P_{C_l C_r}(00) P_{A_l A_r B_l B_r C_l C_r}(101111) = -\frac{1}{4} \left(\frac{1}{32} (2 + \sqrt{2}) \right) \\
& - P_{C_l C_r}(00) P_{A_l A_r B_l B_r C_l C_r}(111011) = -\frac{1}{4} \left(\frac{1}{32} (2 + \sqrt{2}) \right) \\
& - P_{C_l C_r}(11) P_{A_l A_r B_l B_r C_l C_r}(000000) = -\frac{1}{4} \left(\frac{1}{32} (2 + \sqrt{2}) \right) \\
& - P_{C_l C_r}(11) P_{A_l A_r B_l B_r C_l C_r}(010100) = -\frac{1}{4} \left(\frac{1}{32} (2 + \sqrt{2}) \right) \\
& + P_{C_l C_r}(10) P_{A_l A_r B_l B_r C_l C_r}(001001) = +\frac{1}{4} \left(\frac{1}{32} (2 + \sqrt{2}) \right) \\
& + P_{C_l C_r}(10) P_{A_l A_r B_l B_r C_l C_r}(011101) = +\frac{1}{4} \left(\frac{1}{32} (2 + \sqrt{2}) \right) \\
& 0.0625 = \frac{1}{16} \not\leq 0
\end{aligned}$$

If we restrict the domain of $I_{\text{WagonWheelAlternativeProperSwap}}$ to only consider distributions where $C_l = A_l, C_r = B_l$ and P_C is uniform, we obtain the following constraint,

$$\begin{aligned}
 & I_{\text{WagonWheelRestricted}} : \\
 & +\frac{1}{2} - P_{A_r B_r C_l C_r}(0110) - P_{A_r B_r C_l C_r}(1010) - P_{A_r B_r C_l C_r}(0000) - P_{A_r B_r C_l C_r}(1100) \\
 & - P_{A_r B_r C_l C_r}(0001) - P_{A_r B_r C_l C_r}(1101) + P_{A_r B_r C_l C_r}(0111) + P_{A_r B_r C_l C_r}(1011) \leq 0
 \end{aligned}$$