

Causal Compatibility Inequalities Admitting of Quantum Violations in the Triangle Scenario

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Phys 437 Oral Report, Jan. 2017

Quantum Mechanics can do stuff that Classical Mechanics can't!

- Quantum Information/Communication
 - Quantum Key Distribution (QKD)
- Quantum Algorithms/Computation
 - Shor's Algorithm, Quantum Simulated Annealing, String Matching, ...
 - Polynomial or even exponential advantage
 - <http://math.nist.gov/quantum/zoo/>
- Quantum Foundations
 - Derivation of Bell inequalities
 - Abandoning local hidden variable theories (LHVs)

Important to determine *precisely* where quantum/classical differ, but how?

Given a set of observations, are they *compatible* with a given classical model?

Causal Structure \mathcal{G}

(classical model)

$$\mathcal{G} = (\mathcal{N}, \mathcal{E})$$

\mathcal{E} : causal influence

\mathcal{N} : random variables

\mathcal{N}_O : observable variables

\mathcal{N}_L : latent variables

$$\mathcal{N} = \mathcal{N}_O \cup \mathcal{N}_L$$

Marginal Model $P^{\mathcal{M}}$

(set of observations)

\mathcal{M} : marginal scenario

$$\mathcal{M} = \{V \mid V \subseteq \mathcal{N}_O\}$$

$$P^{\mathcal{M}} = \{P_V \mid V \in \mathcal{M}\}$$

Question: Can your model \mathcal{G} explain the observation $P^{\mathcal{M}}$?

Question: Is $P^{\mathcal{M}}$ **compatible** with \mathcal{G} ?

Answer: If $P^{\mathcal{M}}$ is to be compatible with \mathcal{G} then:

- 1 $P^{\mathcal{M}}$ must admit a **joint distribution** $P_{\mathcal{N}}$:

$$P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$$

- 2 $P_{\mathcal{N}}$ must obey **Markov conditions** (independence given parents):

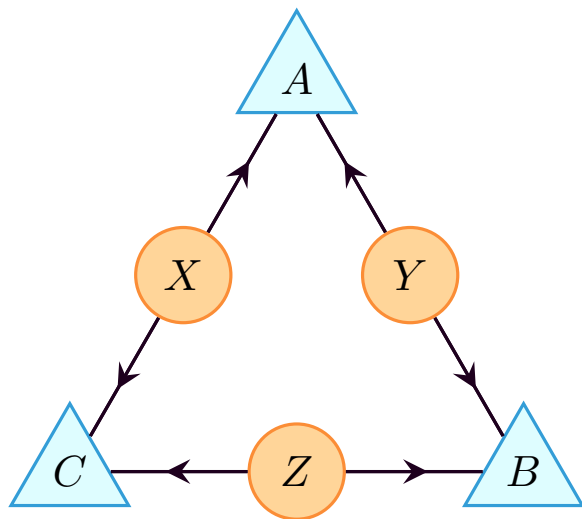
$$P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\text{Pa}_{\mathcal{G}}(n)}$$

Task: Determine if $P^{\mathcal{M}}$ is **compatible** or **incompatible** with \mathcal{G}

Tool: A **casual compatibility inequality** I is an inequality satisfied by all compatible $P^{\mathcal{M}}$

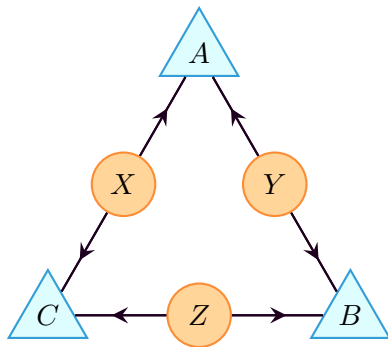
- Inequalities I are **extremely useful**:
- If $P^{\mathcal{M}}$ satisfies *all* such constraints I , then it is *compatible* with \mathcal{G}
- If $P^{\mathcal{M}}$ violates *any* constraint I , then it is *incompatible* with \mathcal{G}
- Inequalities act as *measures* of non-classicality
- I identifies *fundamentally* new resources or quantum advantages!

Case Study: The Triangle Scenario



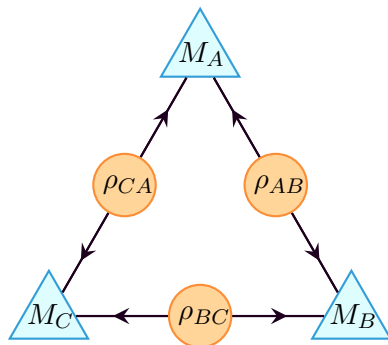
Classical (Compatible) Distributions for Triangle Scenario

$$P_{ABC} = \sum_{X,Y,Z} P_{A|X,Y} P_{B|Y,Z} P_{C|Z,X} P_X P_Y P_Z$$



Quantum Distributions for Triangle Scenario

$$P_{ABC} = \text{Tr}[\Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_A \otimes M_B \otimes M_C]$$



Why Triangle Scenario?

- The TS could be home to **brand new** entanglement resources which could lead to new technologies and fundamental physics, albeit very difficult to find
- In **Branciard_2012** it was noted that characterizing quantum non-classicality in TS remained an **open problem** and that identifying compatibility inequalities **seemed challenging**
- In **Fritz_2012** Fritz demonstrated that TS is the **smallest** correlation scenario in which quantum non-classicality is manifest (proof without inequalities)
- In **Henson_2014** TS was classified as an **interesting** causal structure: conditional independence relations are not a sufficient characterization of compatibility (there are none)
- Several other authors (see **Steudel_2010 Chaves_2014 Inflation ...**) have been unable to find inequalities with quantum violations

Deriving Inequalities

- Two necessary components to compatibility: joint distribution, Markov separability
- It is possible to cast Markov separability problem into a joint distribution problem (at least partially)!
- **Inflation Technique??** developed by E. Wolfe, R. Spekkens, T. Fritz provides tools for solving compatibility problem
- **Definite Extension Procedure** developed by myself, E. Wolfe and others allows one to derive inequalities for **very large** causal structures

$$P_{ABC}(000)P_C(3)$$

$$\leq$$

$$\begin{aligned} & 2P_{ABC}(203)P_C(0) + 2P_{ABC}(303)P_C(0) + P_{ABC}(003)P_C(0) + P_{ABC}(013)P_C(0) + \\ & P_{ABC}(020)P_C(2) + P_{ABC}(022)P_C(2) + P_{ABC}(023)P_C(0) + P_{ABC}(023)P_C(2) + \\ & P_{ABC}(030)P_C(2) + P_{ABC}(031)P_C(2) + P_{ABC}(032)P_C(2) + P_{ABC}(033)P_C(0) + \\ & P_{ABC}(033)P_C(2) + P_{ABC}(103)P_C(0) + P_{ABC}(113)P_C(0) + P_{ABC}(123)P_C(0) + \\ & P_{ABC}(133)P_C(0) + P_{ABC}(200)P_C(0) + P_{ABC}(200)P_C(1) + P_{ABC}(200)P_C(2) + \\ & P_{ABC}(200)P_C(3) + P_{ABC}(201)P_C(0) + P_{ABC}(201)P_C(1) + P_{ABC}(201)P_C(2) + \\ & P_{ABC}(201)P_C(3) + P_{ABC}(203)P_C(1) + P_{ABC}(203)P_C(2) + P_{ABC}(203)P_C(3) + \\ & P_{ABC}(213)P_C(0) + P_{ABC}(223)P_C(0) + P_{ABC}(300)P_C(0) + P_{ABC}(300)P_C(1) + \\ & P_{ABC}(300)P_C(2) + P_{ABC}(300)P_C(3) + P_{ABC}(301)P_C(0) + P_{ABC}(301)P_C(1) + \\ & P_{ABC}(301)P_C(2) + P_{ABC}(301)P_C(3) + P_{ABC}(302)P_C(1) + P_{ABC}(303)P_C(1) + \\ & P_{ABC}(303)P_C(2) + P_{ABC}(303)P_C(3) + P_{ABC}(313)P_C(0) \end{aligned}$$

Causal Compatibility Inequalities (Simplified)

$$\begin{aligned} &P_{ABC}(000)P_C(3) + P_{ABC}(021)P_C(2) + P_{ABC}(202) + P_{ABC}(302) + \\ &\quad P_{ABC}(233)P_C(0) + P_{AC}(33)P_C(0) \\ &\quad \leq \\ &P_{ABC}(303)P_C(0) + P_{ABC}(313)P_C(0) + P_{ABC}(302)P_C(1) + \\ &P_{AB}(02)P_C(2) + P_{AB}(03)P_C(2) + P_{AB}(20) + P_{AB}(30) + P_C(3)P_C(0) \end{aligned}$$

$$\begin{aligned} &P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333) \leq \\ &2P(020)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213) + \\ &2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210) + \\ &2P(123)P(310) + 2P(130)P(213) + 2P(132)P(311) + 2P(133)P(210) + \\ &\quad + \cdots \quad 324 \text{ more terms } \cdots + \\ &P(320)P(323) + P(320)P(333) + P(323)P(330) + P(330)P(333) \end{aligned}$$

Note: $P(abc)$ shorthand for $P_{ABC}(abc)$

Certificate for Fritz Distribution (Full)

$$P(110)P(223) + P(110)P(233) + P(110)P(323) + P(110)P(333)$$

 λ

$2P(202)P(213) + 2P(023)P(210) + 2P(023)P(310) + 2P(030)P(213) + 2P(033)P(210) + 2P(033)P(310) + 2P(120)P(213) + 2P(123)P(210) + 2P(123)P(310) + 2P(130)P(213) + 2P(132)P(311) + 2P(133)P(210) + 2P(133)P(310) + P(000)P(023) + P(000)P(013) + P(000)P(023) + P(000)P(033) + P(000)P(103) + P(000)P(113) + P(000)P(123) + P(000)P(133) + P(000)P(203) + P(000)P(210) + P(000)P(213) + P(000)P(223) + P(000)P(010) + P(000)P(020) + P(003)P(020) + P(003)P(030) + P(003)P(100) + P(003)P(110) + P(003)P(120) + P(003)P(130) + P(003)P(200) + P(003)P(210) + P(003)P(220) + P(003)P(230) + P(003)P(300) + P(003)P(310) + P(003)P(320) + P(003)P(330) + P(010)P(013) + P(010)P(023) + P(010)P(033) + P(010)P(103) + P(010)P(113) + P(010)P(123) + P(010)P(133) + P(010)P(203) + P(010)P(213) + P(010)P(223) + P(010)P(230) + P(010)P(300) + P(010)P(310) + P(010)P(320) + P(010)P(330) + P(013)P(030) + P(013)P(100) + P(013)P(110) + P(013)P(120) + P(013)P(130) + P(013)P(200) + P(013)P(210) + P(013)P(220) + P(013)P(230) + P(013)P(300) + P(013)P(310) + P(013)P(320) + P(013)P(330) + P(020)P(023) + P(020)P(033) + P(020)P(103) + P(020)P(113) + P(020)P(123) + P(020)P(133) + P(020)P(203) + P(020)P(210) + P(020)P(211) + P(020)P(212) + P(020)P(223) + P(020)P(233) + P(020)P(310) + P(020)P(311) + P(020)P(312) + P(020)P(313) + P(021)P(210) + P(021)P(211) + P(021)P(212) + P(021)P(213) + P(021)P(310) + P(021)P(311) + P(021)P(312) + P(021)P(313) + P(022)P(210) + P(022)P(211) + P(022)P(212) + P(022)P(213) + P(022)P(310) + P(022)P(311) + P(022)P(312) + P(022)P(313) + P(023)P(030) + P(023)P(100) + P(023)P(110) + P(023)P(120) + P(023)P(130) + P(023)P(200) + P(023)P(210) + P(023)P(212) + P(023)P(213) + P(023)P(220) + P(023)P(230) + P(023)P(300) + P(023)P(311) + P(023)P(312) + P(023)P(313) + P(023)P(320) + P(023)P(330) + P(030)P(033) + P(030)P(103) + P(030)P(113) + P(030)P(123) + P(030)P(133) + P(030)P(203) + P(030)P(210) + P(030)P(211) + P(030)P(212) + P(030)P(223) + P(030)P(233) + P(030)P(310) + P(030)P(311) + P(030)P(312) + P(030)P(313) + P(031)P(210) + P(031)P(211) + P(031)P(212) + P(031)P(213) + P(031)P(310) + P(031)P(311) + P(031)P(312) + P(031)P(313) + P(032)P(210) + P(032)P(211) + P(032)P(212) + P(032)P(213) + P(032)P(310) + P(032)P(311) + P(032)P(312) + P(032)P(313) + P(033)P(100) + P(033)P(110) + P(033)P(120) + P(033)P(130) + P(033)P(200) + P(033)P(210) + P(033)P(212) + P(033)P(213) + P(033)P(220) + P(033)P(230) + P(033)P(300) + P(033)P(310) + P(033)P(311) + P(033)P(312) + P(033)P(313) + P(100)P(103) + P(100)P(113) + P(100)P(123) + P(100)P(133) + P(100)P(203) + P(100)P(213) + P(100)P(223) + P(103)P(110) + P(103)P(120) + P(103)P(130) + P(103)P(200) + P(103)P(210) + P(103)P(220) + P(103)P(230) + P(103)P(300) + P(103)P(310) + P(103)P(320) + P(103)P(330) + P(110)P(113) + P(110)P(123) + P(110)P(133) + P(110)P(203) + P(110)P(213) + P(110)P(223) + P(113)P(120) + P(113)P(130) + P(113)P(200) + P(113)P(210) + P(113)P(220) + P(113)P(230) + P(113)P(300) + P(113)P(310) + P(113)P(320) + P(113)P(330) + P(120)P(123) + P(120)P(133) + P(120)P(203) + P(120)P(210) + P(120)P(211) + P(120)P(212) + P(120)P(223) + P(120)P(233) + P(120)P(310) + P(120)P(311) + P(120)P(312) + P(120)P(313) + P(121)P(210) + P(121)P(211) + P(121)P(212) + P(121)P(213) + P(121)P(310) + P(121)P(311) + P(121)P(312) + P(121)P(313) + P(122)P(210) + P(122)P(212) + P(122)P(213) + P(122)P(310) + P(122)P(311) + P(122)P(312) + P(122)P(313) + P(123)P(130) + P(123)P(200) + P(123)P(211) + P(123)P(212) + P(123)P(213) + P(123)P(220) + P(123)P(230) + P(123)P(300) + P(123)P(310) + P(123)P(312) + P(123)P(313) + P(123)P(320) + P(123)P(330) + P(130)P(133) + P(130)P(203) + P(130)P(210) + P(130)P(211) + P(130)P(212) + P(130)P(223) + P(130)P(233) + P(130)P(310) + P(130)P(311) + P(130)P(312) + P(130)P(313) + P(131)P(210) + P(131)P(211) + P(131)P(212) + P(131)P(213) + P(131)P(310) + P(131)P(311) + P(131)P(312) + P(131)P(313) + P(132)P(201) + P(132)P(210) + P(132)P(211) + P(132)P(212) + P(132)P(213) + P(132)P(301) + P(132)P(310) + P(132)P(312) + P(132)P(313) + P(133)P(200) + P(133)P(211) + P(133)P(212) + P(133)P(213) + P(133)P(220) + P(133)P(230) + P(133)P(300) + P(133)P(310) + P(133)P(312) + P(133)P(313) + P(133)P(320) + P(133)P(330) + P(200)P(203) + P(200)P(213) + P(200)P(223) + P(200)P(233) + P(200)P(303) + P(200)P(313) + P(200)P(323) + P(200)P(333) + P(203)P(210) + P(203)P(220) + P(203)P(230) + P(203)P(300) + P(203)P(310) + P(203)P(320) + P(203)P(330) + P(210)P(213) + P(210)P(223) + P(210)P(233) + P(210)P(303) + P(210)P(313) + P(210)P(323) + P(210)P(333) + P(213)P(220) + P(213)P(230) + P(213)P(300) + P(213)P(310) + P(213)P(320) + P(213)P(330) + P(220)P(223) + P(220)P(233) + P(220)P(303) + P(220)P(313) + P(220)P(323) + P(220)P(333) + P(223)P(2$

Party Symmetric Inequality

$$2[P(001)P(333)]_3 + 2[P(010)P(323)]_3 + 6[P(000)P(323)]_3 + 6[P(000)P(333)]_1$$

$$\leq$$

$$12[P(031)P(302)]_6 + 12[P(033)P(303)]_6 + 12[P(103)P(130)]_6 + 12[P(203)P(230)]_6 +$$

$$+ \cdots \text{ 126 more terms } \cdots +$$

$$6[P(101)P(130)]_6 + 6[P(103)P(310)]_6 + 6[P(113)P(130)]_6 + 6[P(113)P(230)]_6 +$$

$$6[P(113)P(330)]_3 + 6[P(122)P(330)]_6 + 6[P(130)P(313)]_6 + 6[P(132)P(303)]_6 +$$

$$6[P(133)P(303)]_6 + 6[P(133)P(320)]_6 + 6[P(200)P(203)]_6 + 6[P(201)P(230)]_6 +$$

$$6[P(203)P(231)]_6 + 6[P(223)P(300)]_6 + 8[P(003)P(320)]_6 + 8[P(032)P(300)]_6$$

Note: $[P(113)P(330)]_3$ shorthand sum over permutations:

$$P(113)P(330) + P(131)P(303) + P(311)P(033)$$

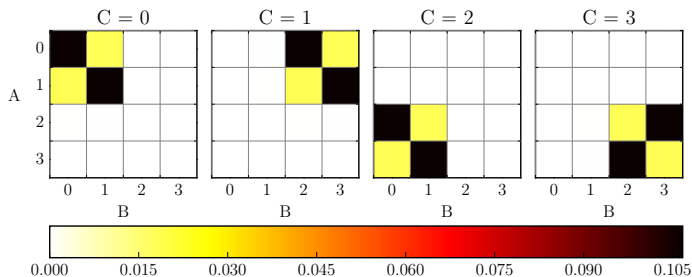
Party Symmetric Inequality (Full)

$$2[P(001)P(333)]_3 + 2[P(010)P(323)]_3 + 6[P(000)P(323)]_3 + 6[P(000)P(333)]_1$$

\leq

$$\begin{aligned} &12[P(031)P(302)]_6 + 12[P(033)P(303)]_6 + 12[P(103)P(130)]_6 + 12[P(203)P(230)]_6 + 12[P(203)P(330)]_6 + 2[P(001)P(320)]_6 + 2[P(002)P(221)]_3 + 2[P(003)P(211)]_6 + \\ &2[P(003)P(331)]_3 + 2[P(011)P(211)]_3 + 2[P(012)P(322)]_6 + 2[P(013)P(313)]_6 + 2[P(013)P(332)]_6 + 2[P(020)P(111)]_3 + 2[P(020)P(211)]_6 + 2[P(021)P(212)]_6 + \\ &2[P(022)P(211)]_3 + 2[P(022)P(212)]_6 + 2[P(022)P(322)]_3 + 2[P(023)P(232)]_6 + 2[P(030)P(212)]_3 + 2[P(031)P(231)]_6 + 2[P(032)P(331)]_6 + 2[P(033)P(333)]_3 + \\ &2[P(101)P(131)]_3 + 2[P(101)P(132)]_6 + 2[P(102)P(131)]_6 + 2[P(102)P(132)]_6 + 2[P(102)P(133)]_6 + 2[P(110)P(133)]_6 + 2[P(110)P(212)]_6 + 2[P(110)P(222)]_3 + \\ &2[P(110)P(223)]_3 + 2[P(112)P(331)]_3 + 2[P(120)P(122)]_6 + 2[P(121)P(201)]_6 + 2[P(122)P(200)]_3 + 2[P(122)P(202)]_6 + 2[P(122)P(210)]_6 + 2[P(122)P(300)]_3 + \\ &2[P(130)P(232)]_6 + 2[P(130)P(233)]_6 + 2[P(131)P(201)]_6 + 2[P(131)P(202)]_3 + 2[P(131)P(313)]_3 + 2[P(133)P(200)]_3 + 2[P(133)P(201)]_6 + 2[P(133)P(211)]_3 + \\ &2[P(133)P(212)]_6 + 2[P(133)P(300)]_3 + 2[P(202)P(231)]_6 + 2[P(210)P(222)]_6 + 2[P(220)P(222)]_3 + 2[P(220)P(313)]_6 + 2[P(221)P(313)]_6 + 2[P(222)P(331)]_3 + \\ &2[P(223)P(331)]_3 + 2[P(230)P(312)]_6 + 2[P(231)P(313)]_6 + 2[P(232)P(320)]_6 + 2[P(302)P(322)]_6 + 2[P(320)P(323)]_6 + 2[P(330)P(332)]_3 + 3[P(000)P(003)]_3 + \\ &3[P(010)P(301)]_6 + 4[P(001)P(131)]_6 + 4[P(002)P(020)]_6 + 4[P(002)P(133)]_6 + 4[P(002)P(323)]_6 + 4[P(010)P(123)]_6 + 4[P(013)P(212)]_6 + 4[P(013)P(312)]_6 + \\ &4[P(023)P(221)]_6 + 4[P(023)P(222)]_6 + 4[P(023)P(322)]_6 + 4[P(031)P(211)]_6 + 4[P(032)P(321)]_6 + 4[P(100)P(123)]_6 + 4[P(100)P(232)]_6 + 4[P(100)P(313)]_6 + \\ &4[P(112)P(310)]_6 + 4[P(122)P(203)]_6 + 4[P(122)P(302)]_6 + 4[P(130)P(222)]_6 + 4[P(130)P(223)]_6 + 4[P(222)P(310)]_6 + 4[P(223)P(320)]_6 + 4[P(231)P(301)]_6 + \\ &4[P(312)P(330)]_6 + 6[P(001)P(031)]_6 + 6[P(001)P(033)]_6 + 6[P(002)P(300)]_6 + 6[P(002)P(330)]_3 + 6[P(003)P(032)]_6 + 6[P(003)P(131)]_6 + 6[P(003)P(132)]_6 + \\ &6[P(011)P(300)]_3 + 6[P(011)P(320)]_6 + 6[P(012)P(200)]_6 + 6[P(012)P(301)]_6 + 6[P(013)P(030)]_6 + 6[P(013)P(110)]_6 + 6[P(013)P(120)]_6 + 6[P(013)P(303)]_6 + \\ &6[P(020)P(102)]_6 + 6[P(020)P(103)]_6 + 6[P(020)P(123)]_6 + 6[P(020)P(202)]_6 + 6[P(020)P(203)]_6 + 6[P(020)P(311)]_6 + 6[P(020)P(322)]_6 + 6[P(020)P(330)]_6 + \\ &6[P(022)P(303)]_6 + 6[P(030)P(033)]_6 + 6[P(030)P(101)]_3 + 6[P(030)P(133)]_6 + 6[P(030)P(202)]_3 + 6[P(030)P(303)]_3 + 6[P(030)P(332)]_6 + 6[P(031)P(203)]_6 + \\ &6[P(032)P(310)]_6 + 6[P(033)P(101)]_6 + 6[P(033)P(130)]_6 + 6[P(033)P(200)]_3 + 6[P(033)P(212)]_6 + 6[P(033)P(220)]_6 + 6[P(033)P(222)]_3 + 6[P(033)P(230)]_6 + \\ &6[P(033)P(322)]_3 + 6[P(100)P(203)]_6 + 6[P(101)P(130)]_6 + 6[P(103)P(310)]_6 + 6[P(103)P(130)]_6 + 6[P(113)P(230)]_6 + 6[P(113)P(330)]_3 + 6[P(122)P(330)]_6 + \\ &6[P(130)P(313)]_6 + 6[P(132)P(303)]_6 + 6[P(133)P(303)]_6 + 6[P(133)P(320)]_6 + 6[P(200)P(203)]_6 + 6[P(201)P(230)]_6 + 6[P(203)P(231)]_6 + 6[P(223)P(300)]_6 + \\ &8[P(003)P(320)]_6 + 8[P(032)P(300)]_6 \end{aligned}$$

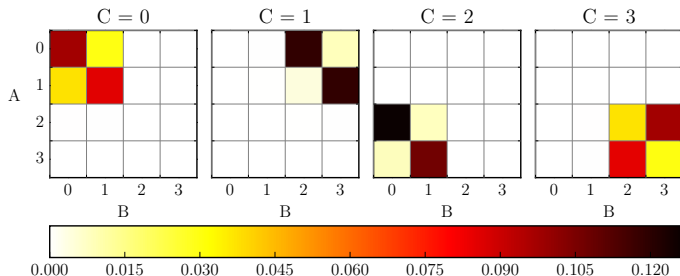
Fritz Distribution



$$\text{Yellow} = \frac{1}{32} (2 - \sqrt{2}) \quad \text{Black} = \frac{1}{32} (2 + \sqrt{2})$$

Maximal Violation

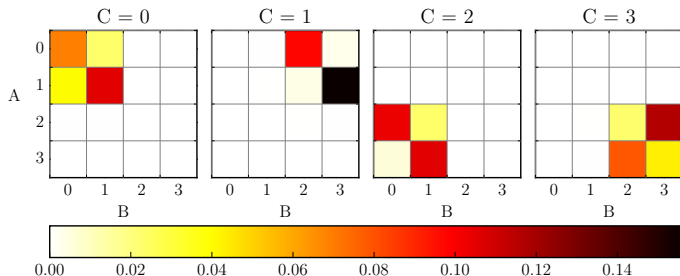
Maximum Violation of I_0 ($V_F = 1.501$)



Relative Violation:
$$V_F = \frac{\min_P \{I(P)\}}{I(P_F)}$$

Maximal Violation Symmetric

Maximum Violation of I_2 ($V_F = 2.61$)

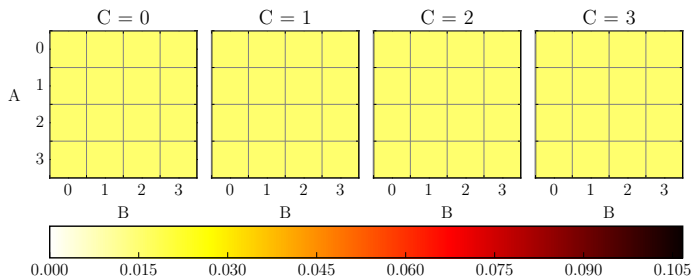


Relative Violation:
$$V_F = \frac{\min_P \{I(P)\}}{I(P_F)}$$

Uniform Noise

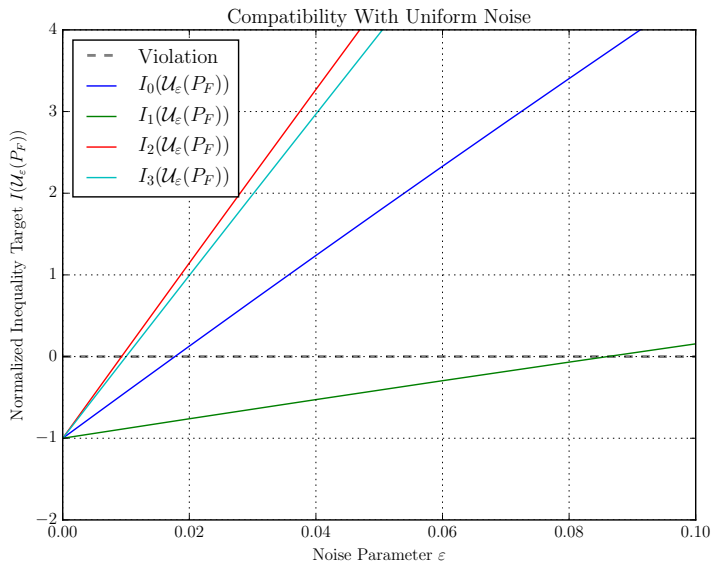
$$\mathcal{U}_\varepsilon(P) = (1 - \varepsilon)P + \varepsilon\mathcal{U}$$

Uniform Distribution



$$\text{Yellow square} = \frac{1}{64}$$

Robust to Noise Zoomed



- Quantum/Classical difference can be observed in the triangle scenario
- Successful demonstration of the inflation technique
- Developed new algorithmic tools to find such inequalities
- New measures of non-classicality which lead to new quantum mechanical resources

Appendix A: Optional Slides

Question: Which marginal models $P^{\mathcal{M}}$ are **compatible** with a causal structure \mathcal{G} ?

- **Marginal model** $P^{\mathcal{M}}$ is collection of probability distributions

$$P^{\mathcal{M}} = \{P_{V_1}, \dots, P_{V_k}\}$$

- **Marginal scenario** $\mathcal{M} = \{V_1, \dots, V_k\}$

$$V \in \mathcal{M}, V' \subseteq V \implies V' \in \mathcal{M}$$

- **Joint random variables** $\mathcal{J} = \bigcup_i V_i = \{v_1, \dots, v_n\}$
- **Causal Structure** $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is a directed acyclic graph (DAG)
- Nodes classified into **latent nodes** \mathcal{N}_L and **observed nodes** \mathcal{N}_O

Let $n, m \in \mathcal{N}$ be nodes of the graph \mathcal{G} .

- **parents of n :** $\text{Pa}_{\mathcal{G}}(n) \equiv \{m \mid m \rightarrow n\}$
- **children of n :** $\text{Ch}_{\mathcal{G}}(n) \equiv \{m \mid n \rightarrow m\}$
- **ancestry of n :** $\text{An}_{\mathcal{G}}(n) \equiv \bigcup_{i \in \mathbb{W}} \text{Pa}_{\mathcal{G}}^i(n)$

$$\text{Pa}_{\mathcal{G}}^0(n) = n \quad \text{Pa}_{\mathcal{G}}^i(n) \equiv \text{Pa}_{\mathcal{G}}(\text{Pa}_{\mathcal{G}}^{i-1}(n))$$

Notation extends to sets of nodes $N \subseteq \mathcal{N}$,

- **parents of N :** $\text{Pa}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \text{Pa}_{\mathcal{G}}(n)$
- **children of N :** $\text{Ch}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \text{Ch}_{\mathcal{G}}(n)$
- **ancestry of N :** $\text{An}_{\mathcal{G}}(N) \equiv \bigcup_{n \in N} \text{An}_{\mathcal{G}}(n)$

An **induced subgraph** of $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ due to $N \subseteq \mathcal{N}$

$$\text{Sub}_{\mathcal{G}}(N) = (N, \{e \in \mathcal{E} \mid e \subseteq N\})$$

Question: Which marginal models $P^{\mathcal{M}}$ are **compatible** with a causal structure \mathcal{G} ?

Answer: $P^{\mathcal{M}}$ is compatible with \mathcal{G} if there exists a set of **casual parameters**

$$\left\{ P_{n|\text{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N} \right\}$$

Such that for each $V \in \mathcal{M}$, P_V can be recovered:

1 $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\text{Pa}_{\mathcal{G}}(n)}$

2 $P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$

Inequality: A **casual compatibility inequality** I is an inequality over $P^{\mathcal{M}}$ that is satisfied by all compatible $P^{\mathcal{M}}$

Two necessary components to compatibility:

- 1 **Marginal problem:** $\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{N} \setminus V} P_{\mathcal{N}}$
 - Is the marginal model contextual or non-contextual?
 - 3 distinct ways to tackle this problem
 - 1 Convex hull, Polytope projection, Fourier-Motzkin
 - 2 Possibilistic Hardy Inequalities (Hypergraph transversals)
 - 3 Linear Program Feasibility/Infeasibility
- 2 **Markov Separation:** $P_{\mathcal{N}} = \prod_{n \in \mathcal{N}} P_{n|\text{Pa}_{\mathcal{G}}(n)}$
 - Much harder to determine since latent nodes \mathcal{N}_O have unspecified behaviour
 - It is possible to turn Markov Separation problem into a Marginal problem (at least partially)

Developed by Wolfe, Spekkens, and Fritz **Inflation**

Definition

An **inflation** of a causal structure \mathcal{G} is another causal structure \mathcal{G}' such that:

$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \text{AnSub}_{\mathcal{G}'}(n') \sim \text{AnSub}_{\mathcal{G}}(n)$$

Where $\text{AnSub}_{\mathcal{G}}(n)$ denotes the ancestral sub-graph of n in \mathcal{G}

$$\text{AnSub}_{\mathcal{G}}(n) = \text{Sub}_{\mathcal{G}}(\text{An}_{\mathcal{G}}(n))$$

And ' \sim ' is a **copy-index** equivalence relation

$$A_1 \sim A_2 \sim A \not\sim B_1 \sim B_2 \sim B$$

Inflation Lemma

If one has obtained \mathcal{G} , inflation \mathcal{G}' and *compatible* marginal distribution P_N where $N \subseteq \mathcal{N}$, then:

- 1 There exists causal parameters $\{P_{n|\text{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\}$ such that

$$P_N = \prod_{n \in N} P_{n|\text{Pa}_{\mathcal{G}}(n)}$$

- 2 $\text{AnSub}_{\mathcal{G}'}(n') \sim \text{AnSub}_{\mathcal{G}}(n) \implies \text{Pa}_{\mathcal{G}'}(n') \sim \text{Pa}_{\mathcal{G}}(n)$

- 3 Construct **inflated causal parameters**

$$\forall n' \in \mathcal{N}' : P_{n'|\text{Pa}_{\mathcal{G}'}(n')} \equiv P_{n|\text{Pa}_{\mathcal{G}}(n)}$$

- 4 Obtain *compatible* marginal distributions over any $N' \subseteq \mathcal{N}'$

$$P_{N'} = \prod_{n' \in N'} P_{n'|\text{Pa}_{\mathcal{G}'}(n')}$$

- Inflation procedure holds for any $N \in \mathcal{N}, N' \in \mathcal{N}'$ where $\text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')$
- Define **injectable sets of \mathcal{G}'** and **images of the injectable of \mathcal{G}**

$$\text{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : \text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')\}$$

$$\text{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : \text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')\}$$

- For $N' \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')$ there is a *unique* $N \subseteq \mathcal{N}$ such that $\text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')$
- For $N \in \text{ImInj}_{\mathcal{G}}(\mathcal{G}')$ there can *exist many* $N' \subseteq \mathcal{N}'$ such that $\text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')$

Lemma

***The Inflation Lemma: Inflation** Given a particular inflation \mathcal{G}' of \mathcal{G} , if a marginal model $\{P_N \mid N \in \text{ImInj}_{\mathcal{G}}(\mathcal{G}')\}$ is compatible with \mathcal{G} then all marginal models $\{P_{N'} \mid N' \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')\}$ are compatible with \mathcal{G}' provided that $P_N = P_{N'}$ for all instances where $N \sim N'$.*

Corollary

*Any causal compatibility inequality I' constraining the injectable sets $\text{Inj}_{\mathcal{G}}(\mathcal{G}')$ can be **deflated** into a causal compatibility inequality I constraining the images of the injectable sets $\text{ImInj}_{\mathcal{G}}(\mathcal{G}')$.*

- d -separation relations + inflation = polynomial inequalities over \mathcal{G}
- Restrict focus to sets N' that are partitioned into N'_1, N'_2 d -separated by empty set \emptyset
- A **pre-injectable set** N' :

$$N' = \coprod_i N'_i \quad \forall i : N'_i \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')$$

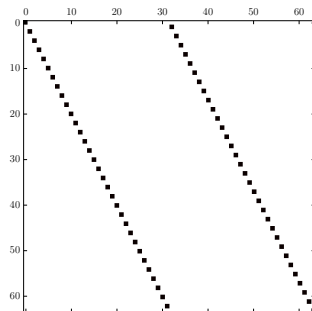
$$\forall i, j : N'_i \perp N'_j \iff \text{An}_{\mathcal{G}'}(N'_i) \cap \text{An}_{\mathcal{G}'}(N'_j) = \emptyset$$

- Only need to consider **maximal pre-injectable sets** $\text{PreInj}_{\mathcal{G}}(\mathcal{G}')$

Network Permutation Matrix

- States and measurements in the Triangle Scenario are not aligned
- Without Π , P_{ABC} would be separable
- Required to align B 's measurement over $\text{Tr}_{A,C}(\rho_{AB} \otimes \rho_{BC})$
- Π is a $2^6 \times 2^6$ matrix
- Shifts one qubit to the left

$$\Pi \equiv \sum_{|q_i\rangle \in \{|0\rangle, |1\rangle\}} |q_2 q_3 q_4 q_5 q_6 q_1\rangle \langle q_1 q_2 q_3 q_4 q_5 q_6|$$



Outcomes and Events

Definition

Each variable v has finite set of **outcomes** O_v .

Each set of variables V has finite set of **events** $\mathcal{E}(V)$:

$$\mathcal{E}(V) \equiv \{s : V \rightarrow O_V \mid \forall v \in V, s(v) \in O_v\}$$

Definition

The set of events over the joint variables $\mathcal{E}(\mathcal{J})$ are termed the **joint events**.

Definition

The set of events over the marginal contexts are the **marginal events**

$$\mathcal{E}(\mathcal{M}) \equiv \coprod_{V \in \mathcal{M}} \mathcal{E}(V)$$

Definition

The **joint distribution vector** $\mathcal{P}^{\mathcal{J}}$

$$\mathcal{P}_j^{\mathcal{J}} = P_{\mathcal{J}}(j) \quad \forall j \in \mathcal{E}(\mathcal{J})$$

Definition

The **marginal distribution vector** $\mathcal{P}^{\mathcal{M}}$

$$\mathcal{P}_m^{\mathcal{M}} = P_{\mathcal{D}(m)}(m) \quad \forall m \in \mathcal{E}(\mathcal{M}), \mathcal{D}(m) \in \mathcal{M}$$

Can now write complete marginal problem as matrix multiplication:

$$\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J} \setminus V} P_{\mathcal{J}} \iff \mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

Incidence Matrix

- **Incidence matrix** M is a bit-wise matrix
- Row-indexed by marginal events $m \in \mathcal{E}(\mathcal{M})$
- Column-indexed by joint events $j \in \mathcal{E}(\mathcal{J})$

$$M_{m,j} = \begin{cases} 1 & m = j|_{\mathcal{D}(m)} \\ 0 & \text{otherwise} \end{cases}$$

$$\# \text{Columns} = |\mathcal{E}(\mathcal{J})| = \prod_{v \in \mathcal{J}} |O_v|$$

$$\# \text{Rows} = |\mathcal{E}(\mathcal{M})| = \sum_{V \in \mathcal{M}} \prod_{v \in V} |O_v|$$

- Desirable to find compatibility inequality I such that

$$\forall \varphi \in \text{Perm}(A, B, C) : \varphi[I] = I$$

- Compatibility is independent of variable labels
 $I, \mathcal{G} \rightarrow \varphi[I], \varphi[\mathcal{G}]$
- Need $\varphi[\mathcal{G}] = \mathcal{G}$ to find new $\varphi[I]$

Definition

The **causal symmetry group** of causal structure \mathcal{G} :

$$\text{Aut}(\mathcal{G}) = \{\varphi \in \text{Perm}(\mathcal{N}) \mid \varphi[\mathcal{G}] = \mathcal{G}\}$$

Strictly speaking, one needs to preserve observable nodes:

$$\text{Aut}_{\mathcal{N}_O}(\mathcal{G}) = \{\varphi \in \text{Aut}(\mathcal{G}) \mid \varphi[\mathcal{N}_O] = \mathcal{N}_O\}$$

- Causal symmetry group for \mathcal{G}' is no good!
- Not possible to deflate inequality if it's not in terms of injectable sets

Definition

The **restricted causal symmetry group** Φ of \mathcal{G}' :

$$\Phi = \text{Aut}_{\text{PreInj}_{\mathcal{G}}}(\mathcal{G}')$$

Restricted Causal Symmetry of Large Inflation

- Φ for the large inflation is an order 48 group with 4 generators

φ_1	φ_2	φ_3	φ_4
$A_1 \rightarrow A_4$	$A_1 \rightarrow A_1$	$A_1 \rightarrow C_1$	$A_1 \rightarrow A_1$
$A_2 \rightarrow A_3$	$A_2 \rightarrow A_3$	$A_2 \rightarrow C_2$	$A_2 \rightarrow A_2$
$A_3 \rightarrow A_2$	$A_3 \rightarrow A_2$	$A_3 \rightarrow C_3$	$A_3 \rightarrow A_3$
$A_4 \rightarrow A_1$	$A_4 \rightarrow A_4$	$A_4 \rightarrow C_4$	$A_4 \rightarrow A_4$
$B_1 \rightarrow B_4$	$B_1 \rightarrow C_1$	$B_1 \rightarrow A_1$	$B_1 \rightarrow B_2$
$B_2 \rightarrow B_3$	$B_2 \rightarrow C_3$	$B_2 \rightarrow A_2$	$B_2 \rightarrow B_1$
$B_3 \rightarrow B_2$	$B_3 \rightarrow C_2$	$B_3 \rightarrow A_3$	$B_3 \rightarrow B_4$
$B_4 \rightarrow B_1$	$B_4 \rightarrow C_4$	$B_4 \rightarrow A_4$	$B_4 \rightarrow B_3$
$C_1 \rightarrow C_4$	$C_1 \rightarrow B_1$	$C_1 \rightarrow B_1$	$C_1 \rightarrow C_3$
$C_2 \rightarrow C_3$	$C_2 \rightarrow B_3$	$C_2 \rightarrow B_2$	$C_2 \rightarrow C_4$
$C_3 \rightarrow C_2$	$C_3 \rightarrow B_2$	$C_3 \rightarrow B_3$	$C_3 \rightarrow C_1$
$C_4 \rightarrow C_1$	$C_4 \rightarrow B_4$	$C_4 \rightarrow B_4$	$C_4 \rightarrow C_2$

Symmetric Incidence

- Group orbits through repeated action of $\varphi \in \Phi$ on $m \in \mathcal{E}(\mathcal{M})$ and $j \in \mathcal{E}(\mathcal{J})$

$$\Phi[m] \equiv \{\varphi[m] \mid \varphi \in \Phi\}$$

$$\Phi[j] \equiv \{\varphi[j] \mid \varphi \in \Phi\}$$

- Construct **symmetric incidence matrix** $\Phi[M]$

$$\Phi[M]_{\Phi[m], \Phi[j]} = \sum_{m' \in \Phi[m]} \sum_{j' \in \Phi[j]} M_{m', j'}$$

$$\Phi[M] = \Lambda_{\Phi[\mathcal{E}(\mathcal{M})]} \cdot M \cdot \Lambda_{\Phi[\mathcal{E}(\mathcal{J})]}$$

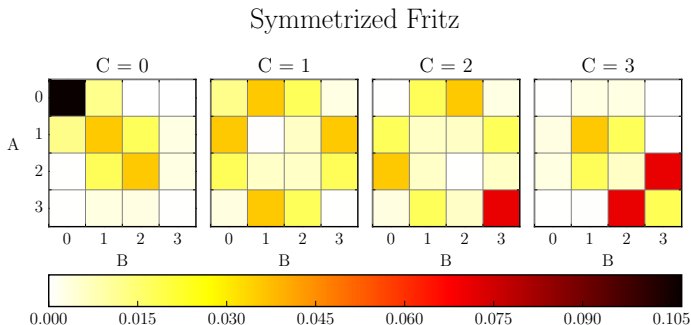
- $\Phi[M]$ not a bit-wise matrix like M
- For large inflation M is $16,896 \times 16,777,216$
- For large inflation $\Phi[M]$ is $450 \times 358,120$

Symmetric Inequalities

- Certificate inequalities are curated/constructed to witness violation of particular distribution
- Above inequality not the best candidate to search for non-locality different than Fritz distribution
- Perhaps one could obtain inequality that does not assign special preference to C
- Desirable to find compatibility inequality I such that

$$\forall \varphi \in \text{Perm}(A, B, C) : \varphi[I] = I$$

Symmetrized Fritz



- More non-local yet not quantum-accessible
- Quantum-accessible distributions form non-context set

- Each party (A, B, C) is assigned a **projective-operator valued measure (POVM)** (M_A, M_B, M_C)

$$\forall |\psi\rangle \in \mathcal{H}^d : \langle \psi | M_\chi | \psi \rangle \geq 0 \quad M_\chi = M_\chi^\dagger$$

- n -outcome measurement

$$M_\chi = \{M_{\chi,1}, \dots, M_{\chi,n}\} \quad \sum_{i=1}^n M_{\chi,i} = \mathbb{1}$$

- For $n = 2$ outcomes, a parameterization exists by constraining the eigenvalues of $M_{\chi,i}$; for $n > 2$ not aware of anything
- Warrants consideration of **projective-valued measures (PVMs)**

Cholesky Parameterization of States

- Each latent resource $\rho \in (\rho_{AB}, \rho_{BC}, \rho_{CA})$ modeled as bipartite qubit state acting on $\mathcal{H}^{d/2} \otimes \mathcal{H}^{d/2}$
- $d \times d$ positive semi-definite (PSD) hermitian matrices with unitary trace
- **Cholesky Parametrization** allows one to write any hermitian PSD as $\rho = T^\dagger T$
- For $d = 4$:

$$T = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_2 + i\lambda_3 & \lambda_4 & 0 & 0 \\ \lambda_5 + i\lambda_6 & \lambda_7 + i\lambda_8 & \lambda_9 & 0 \\ \lambda_{10} + i\lambda_{11} & \lambda_{12} + i\lambda_{13} & \lambda_{14} + i\lambda_{15} & \lambda_{16} \end{bmatrix}$$

- d^2 real-valued parameters
- Normalized $\rho = T^\dagger T / \text{Tr}(T^\dagger T)$ adds degeneracy

Appendix B: Local Minima Concerns

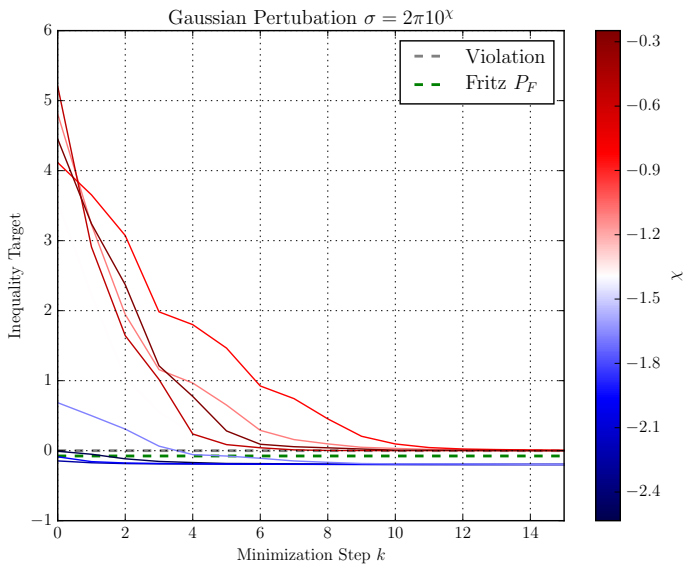
- Finding global minimum is tricky
- Difficult to converge to violation
- Noisy seed (Gaussian noise):

$$\lambda_{(0)} = \lambda_{(F)} + \delta\lambda \quad \delta\lambda_i \sim \mathcal{N}(\mu = 0, \sigma^2 = (2\pi 10^x)^2)$$

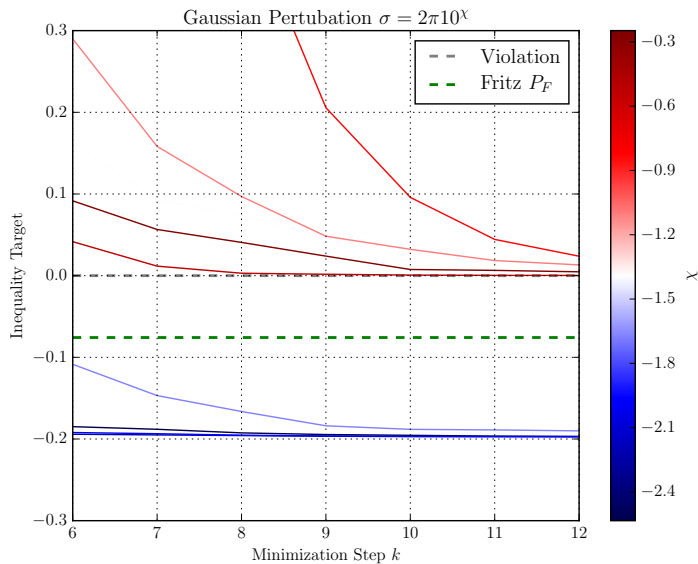
- Uniform seed:

$$\lambda_{(0),i} \sim \mathcal{U}([0, 2\pi])$$

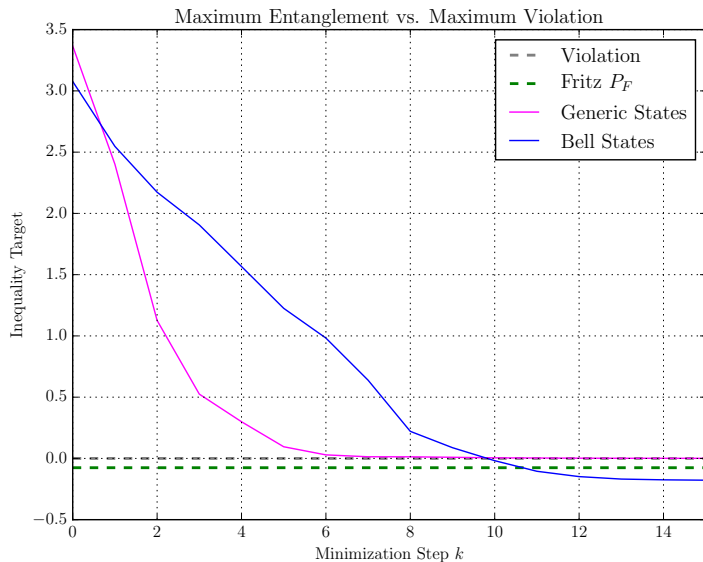
Fritz Local Minima



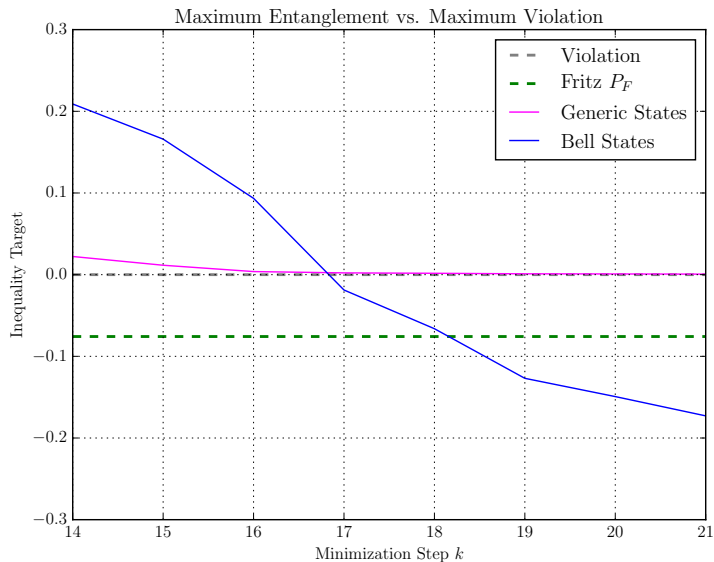
Fritz Local Minima Zoomed



Max Entangled vs. Max Violating (???)

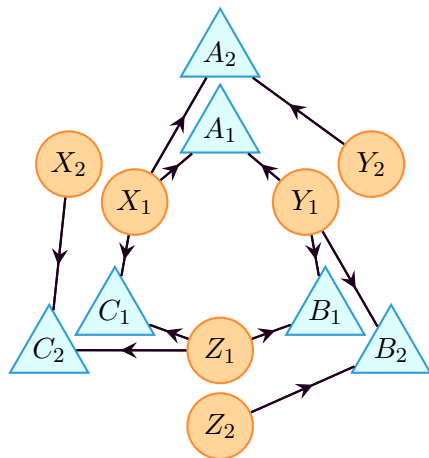
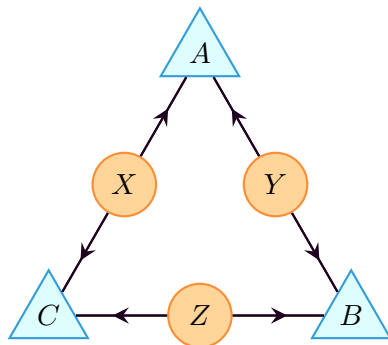


Max Entangled vs. Max Violating (???)



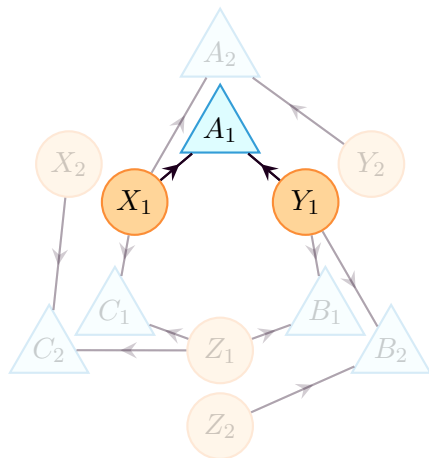
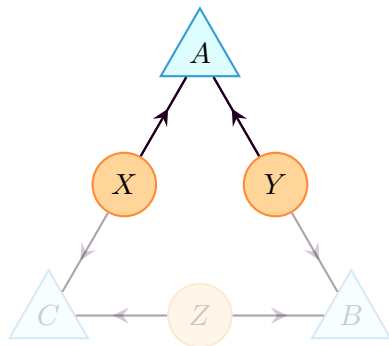
Section C: Inflation Technique

Demonstrating Inflation Technique



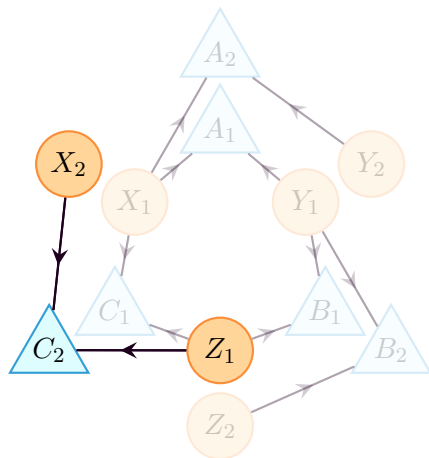
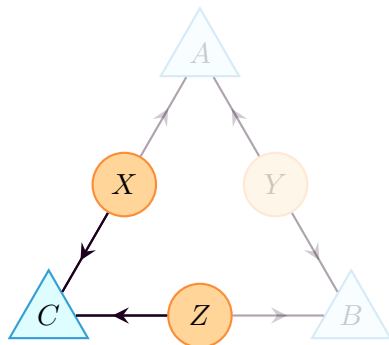
$$\forall n' \in \mathcal{N}', n' \sim n \in \mathcal{N} : \text{AnSub}_{\mathcal{G}'}(n') \sim \text{AnSub}_{\mathcal{G}}(n)$$

Demonstrating Inflation Technique



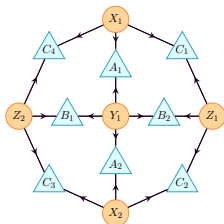
$$\text{AnSub}_G(A) \sim \text{AnSub}_{G'}(A_1)$$

Demonstrating Inflation Technique

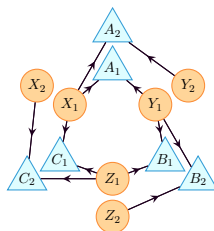


$$\text{AnSub}_G(C) \sim \text{AnSub}_{G'}(C_2)$$

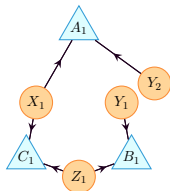
Some Inflations of the Triangle Scenario



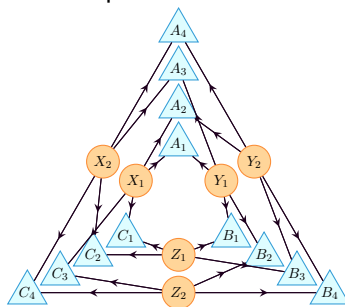
Wagon Wheel Inflation



Spiral Inflation

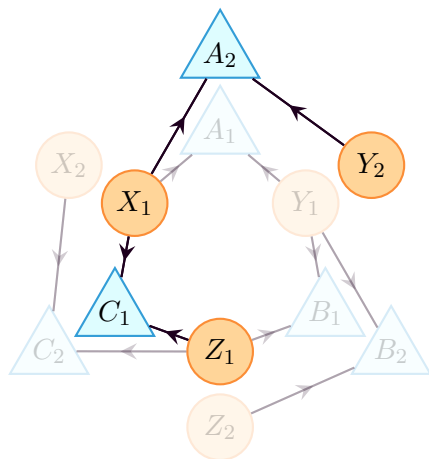
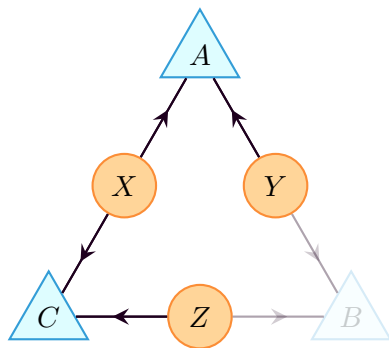


Cut Inflation



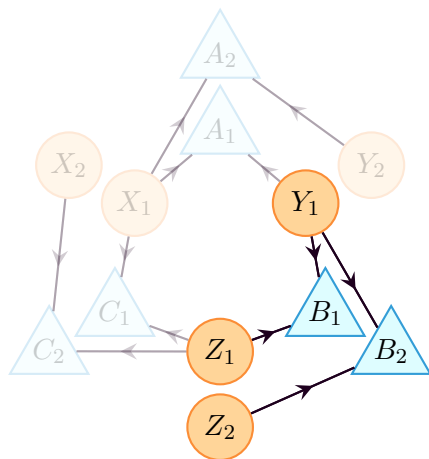
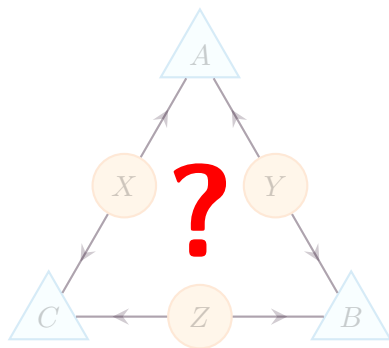
Web Inflation

What are Injectable Sets?



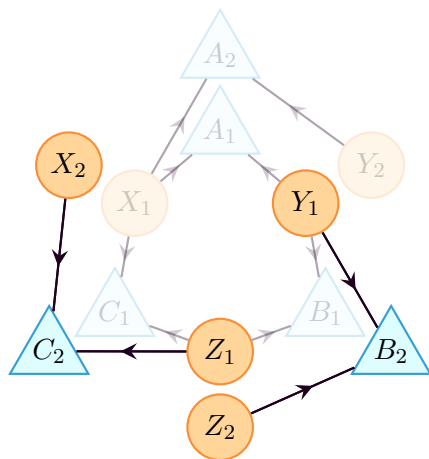
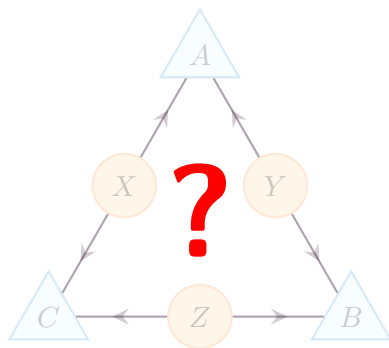
$$\text{AnSub}_{\mathcal{G}}(A, C) \sim \text{AnSub}_{\mathcal{G}'}(A_2, C_1)$$

What are Injectable Sets?



?? $\not\sim \text{AnSub}_{G'}(B_1, B_2)$

What are Injectable Sets?



?? $\not\sim \text{AnSub}_{G'}(B_2, C_2)$

The **injectable sets** in \mathcal{G}' :

$$\text{Inj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N' \subseteq \mathcal{N}' \mid \exists N \subseteq \mathcal{N} : \text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')\}$$

The **images of the injectable sets** in \mathcal{G} :

$$\text{ImInj}_{\mathcal{G}}(\mathcal{G}') \equiv \{N \subseteq \mathcal{N} \mid \exists N' \subseteq \mathcal{N}' : \text{AnSub}_{\mathcal{G}}(N) \sim \text{AnSub}_{\mathcal{G}'}(N')\}$$

What makes injectable sets useful?

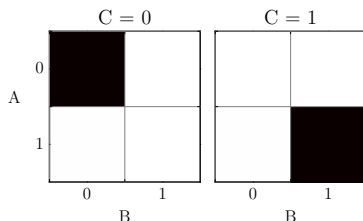
Lemma (Inflation Lemma)

Given $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ and inflation $\mathcal{G}' = (\mathcal{N}', \mathcal{E}')$:

$$\begin{array}{ccc} \underbrace{\{P_N \mid N \in \text{ImInj}_{\mathcal{G}}(\mathcal{G}')\}}_{\text{compatible with } \mathcal{G}} & \longrightarrow & \{P_{n|\text{Pa}_{\mathcal{G}}(n)} \mid n \in \mathcal{N}\} \\ & & \downarrow \text{define} \\ \underbrace{\{P_{N'} \mid N' \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')\}}_{\text{compatible with } \mathcal{G}'} & \longleftarrow & \{P_{n'|\text{Pa}_{\mathcal{G}'}(n')} \mid n' \in \mathcal{N}'\} \end{array}$$

Perfect Correlation Is Incompatible

Perfect Correlation



$$\blacksquare = \frac{1}{2}$$

$$P_{ABC}(abc) = \frac{[000] + [111]}{2}$$

$$P_{ABC}(abc) = \begin{cases} \frac{1}{2} & a = b = c \\ 0 & \text{otherwise} \end{cases}$$

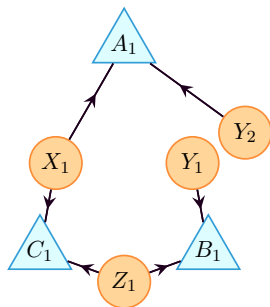
Compatibility Inequality

$$P_A(0)P_B(1) \leq P_{BC}(10) + P_{AC}(01)$$

Witnesses Perfect Correlation

$$\left(\frac{1}{2}\right)^2 \not\leq 0 + 0$$

Deriving Compatibility Inequalities



$$\mathcal{M} = \{\{A_1, B_1\}, \{B_1, C_1\}, \{A_1, C_1\}\}$$

$$P^{\mathcal{M}} = \{P_{A_1 B_1}, P_{B_1 C_1}, P_{A_1 C_1}\}$$

Compatibility requires: $\exists P_{\mathcal{J}} = P_{A_1 B_1 C_1}$

$$P_{A_1 B_1} = \sum_{C_1} P_{\mathcal{J}} \quad P_{B_1 C_1} = \sum_{A_1} P_{\mathcal{J}} \quad P_{A_1 C_1} = \sum_{B_1} P_{\mathcal{J}}$$

Deriving Compatibility Inequalities Cont'd

$$\underbrace{P_{A_1 B_1} = \sum_{C_1} P_{\mathcal{J}} \quad P_{B_1 C_1} = \sum_{A_1} P_{\mathcal{J}} \quad P_{A_1 C_1} = \sum_{B_1} P_{\mathcal{J}}}$$

$$\underbrace{\forall V \in \mathcal{M} : P_V = \sum_{\mathcal{J} \setminus V} P_{\mathcal{J}}}$$

$$\mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

$$\mathcal{P}^{\mathcal{M}} = \begin{pmatrix} P_{A_1 B_1}(00) \\ P_{A_1 B_1}(01) \\ P_{A_1 B_1}(10) \\ P_{A_1 B_1}(11) \\ \hline P_{B_1 C_1}(00) \\ P_{B_1 C_1}(01) \\ P_{B_1 C_1}(10) \\ P_{B_1 C_1}(11) \\ \hline P_{A_1 C_1}(00) \\ P_{A_1 C_1}(01) \\ P_{A_1 C_1}(10) \\ P_{A_1 C_1}(11) \end{pmatrix} \quad \mathcal{P}^{\mathcal{J}} = \begin{pmatrix} P_{A_1 B_1 C_1}(000) \\ P_{A_1 B_1 C_1}(001) \\ P_{A_1 B_1 C_1}(010) \\ P_{A_1 B_1 C_1}(011) \\ P_{A_1 B_1 C_1}(100) \\ P_{A_1 B_1 C_1}(101) \\ P_{A_1 B_1 C_1}(110) \\ P_{A_1 B_1 C_1}(111) \end{pmatrix}$$

Incidence Example

$$M = \begin{matrix} (A_1, B_1, C_1) = & (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \\ \begin{matrix} (A_1=0, B_1=0) \\ (A_1=0, B_1=1) \\ (A_1=1, B_1=0) \\ (A_1=1, B_1=1) \\ (B_1=0, C_1=0) \\ (B_1=0, C_1=1) \\ (B_1=1, C_1=0) \\ (B_1=1, C_1=1) \\ (A_1=0, C_1=0) \\ (A_1=0, C_1=1) \\ (A_1=1, C_1=0) \\ (A_1=1, C_1=1) \end{matrix} & \left(\begin{array}{cccccccc} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \end{array} \right) \end{matrix}$$

$$\mathcal{P}^{\mathcal{M}} = M \cdot \mathcal{P}^{\mathcal{J}}$$

Marginal Linear Program

Marginal LP:

minimize: $\emptyset \cdot x$

subject to: $\mathcal{P}^{\mathcal{J}} \succeq 0$

$$M \cdot \mathcal{P}^{\mathcal{J}} = \mathcal{P}^{\mathcal{M}}$$

Dual Marginal LP:

minimize: $y \cdot \mathcal{P}^{\mathcal{M}}$

subject to: $y \cdot M \succeq 0$

- If $\mathcal{P}^{\mathcal{J}}$ exists, then:

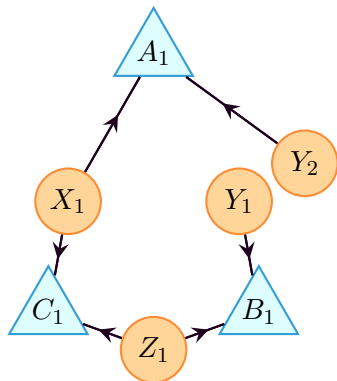
$$y \cdot M \cdot \mathcal{P}^{\mathcal{J}} = y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$$

- If not, then y is an **infeasibility certificate** which generates **infeasibility inequality**:

$$y \cdot \mathcal{P}^{\mathcal{M}} \geq 0$$

- Most linear programming toolkits return certificates (*Mosek*, *Gurobi*, *CPLEX*, *cvxr/cvxopt*.)

Deflating Inequalities



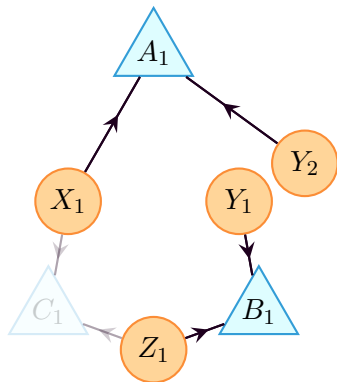
$$\text{Inj}_{\mathcal{G}}(\mathcal{G}') = \left\{ \begin{array}{l} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{array} \right\}$$

$$\{A_1, B_1\} \notin \text{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

Can not deflate inequality!

Deflating Inequalities



$$\text{Inj}_{\mathcal{G}'} = \left\{ \begin{array}{l} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{array} \right\}$$

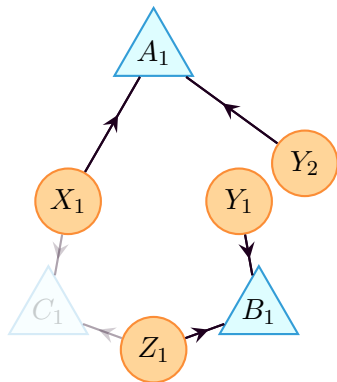
$$\{A_1, B_1\} \notin \text{Inj}_{\mathcal{G}'}(\mathcal{G}')$$

$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

However!

$$\text{AnSub}_{\mathcal{G}'}(A_1) \cap \text{AnSub}_{\mathcal{G}'}(B_1) = \emptyset \iff A_1 \perp B_1$$

Deflating Inequalities



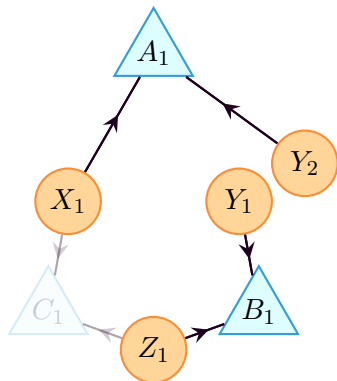
$$\text{Inj}_{\mathcal{G}}(\mathcal{G}') = \left\{ \begin{array}{l} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{array} \right\}$$

$$\{A_1, B_1\} \notin \text{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

$$P_{A_1}(0)P_{B_1}(1) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

Deflating Inequalities



$$\text{Inj}_{\mathcal{G}'} = \left\{ \begin{array}{l} \{A_1, C_1\} \\ \{B_1, C_1\} \\ \{A_1\} \\ \{B_1\} \\ \{C_1\} \end{array} \right\}$$

$$\{A_1, B_1\} \notin \text{Inj}_{\mathcal{G}'}(\mathcal{G}')$$

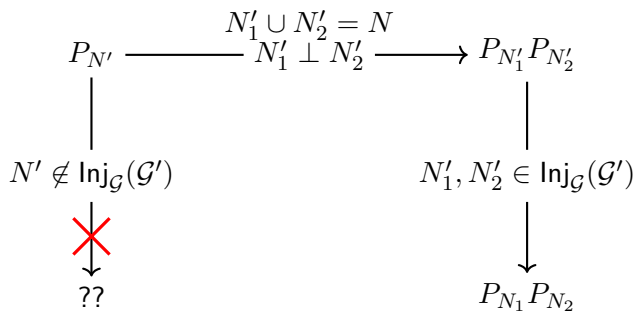
$$P_{A_1 B_1}(01) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

$$P_{A_1}(0)P_{B_1}(1) \leq P_{B_1 C_1}(10) + P_{A_1 C_1}(01)$$

$$P_A(0)P_B(1) \leq P_{BC}(10) + P_{AC}(01)$$

Inflation Gives Polynomial Inequalities

- Deflation only holds when inequality constrains probabilities
 $P_{N'}, N' \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')$
- **Linear inequality** for \mathcal{G}'



- **Polynomial inequality** for \mathcal{G} !

A **pre-injectable set** N' is:

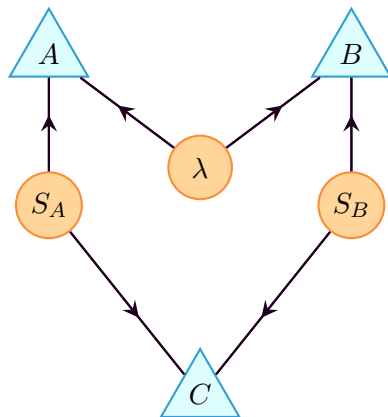
$$N' = \coprod_i N'_i \quad \forall i : N'_i \in \text{Inj}_{\mathcal{G}}(\mathcal{G}')$$

$$\forall i, j : N'_i \perp N'_j \iff \text{An}_{\mathcal{G}'}(N'_i) \cap \text{An}_{\mathcal{G}'}(N'_j) = \emptyset$$

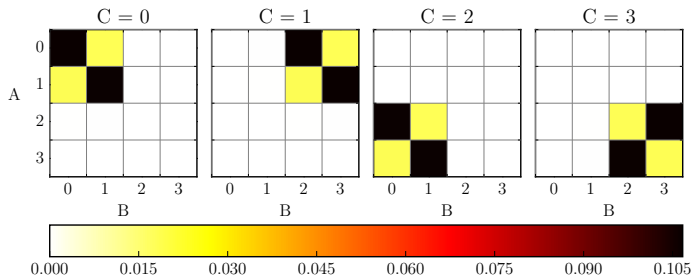
Only need to consider **maximal pre-injectable sets** denoted $\text{PreInj}_{\mathcal{G}}(\mathcal{G}')$

Appendix D: Quantum Non-classicality From Inflation

Triangle Scenario As Bell Scenario

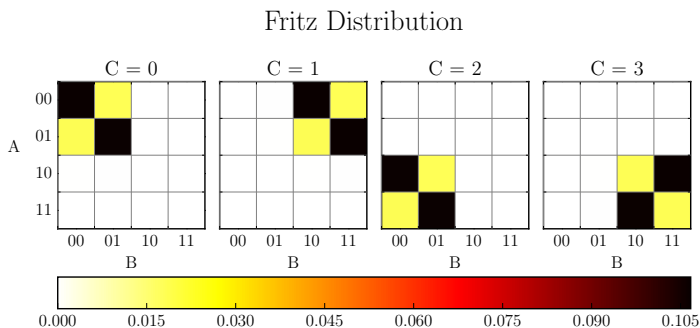


Fritz Distribution



$$\text{Yellow} = \frac{1}{32} (2 - \sqrt{2}) \quad \text{Black} = \frac{1}{32} (2 + \sqrt{2})$$

Fritz Distribution Bit-Measurements



$$\text{Yellow} = \frac{1}{32} (2 - \sqrt{2}) \quad \text{Black} = \frac{1}{32} (2 + \sqrt{2})$$

Quantum Implementation of Fritz Distribution

■ States:

$$\rho_{AB} = |\Phi^+\rangle\langle\Phi^+| \quad \rho_{BC} = \rho_{CA} = \frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

■ Measurements:

$$M_A = \{|0\psi_1\rangle\langle 0\psi_1|, |0\psi_5\rangle\langle 0\psi_5|, |1\psi_3\rangle\langle 1\psi_3|, |1\psi_7\rangle\langle 1\psi_7|\}$$

$$M_B = \{|\psi_6 0\rangle\langle \psi_6 0|, |\psi_2 0\rangle\langle \psi_2 0|, |\psi_0 1\rangle\langle \psi_0 1|, |\psi_4 1\rangle\langle \psi_4 1|\}$$

$$M_C = \{|00\rangle\langle 00|, |10\rangle\langle 10|, |01\rangle\langle 01|, |11\rangle\langle 11|\}$$

■ Shorthand: $|\psi_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{in/4}|1\rangle)$

Fritz Distribution Violating CHSH

- C 's outcome acts as measurement “setting” for A , B ;
independent of ρ_{AB}
- Correlation between right bits

$$\langle A_r B_r \rangle = P_{A_r B_r}(00) + P_{A_r B_r}(11) - P_{A_r B_r}(01) - P_{A_r B_r}(10)$$

$$\langle A_r B_r | C = 0, 1, 2 \rangle = \frac{1}{\sqrt{2}} \quad \langle A_r B_r | C = 3 \rangle = -\frac{1}{\sqrt{2}}$$

- Gives CHSH violation

$$\begin{aligned} & \langle A_r B_r | C = 0 \rangle + \langle A_r B_r | C = 1 \rangle + \langle A_r B_r | C = 2 \rangle - \langle A_r B_r | C = 3 \rangle \\ &= 3 \left(\frac{1}{\sqrt{2}} \right) - \left(-\frac{1}{\sqrt{2}} \right) \\ &= 2\sqrt{2} \not\leq 2 \end{aligned}$$

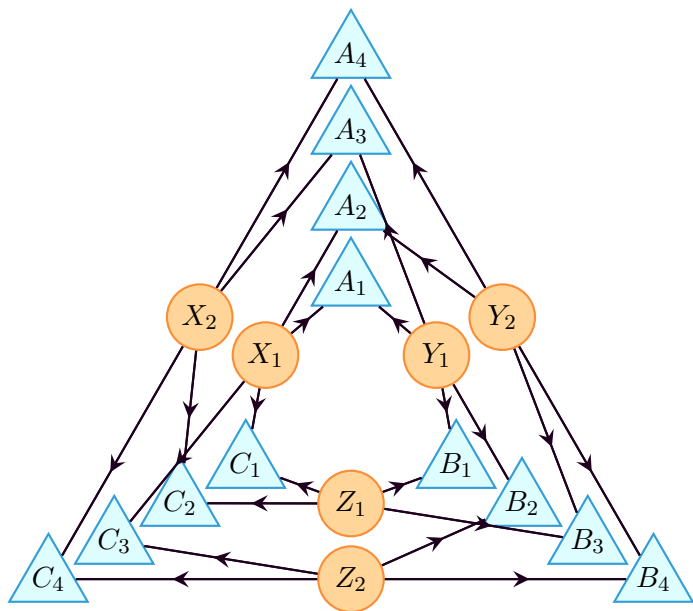
- Incompatibility proof contingent on perfect correlation between C and pseudo-settings
- Proof not robust to noise

Problem (2.17 in **Fritz_2012**)

Find an example of non-classical quantum correlations in TS together with a proof of its non-classicality which does not hinge on Bell's Theorem.

- “...would be helpful to have inequalities...”

Large Inflation



Large Inflation Pre-injectable Sets

Maximal Pre-injectable Sets

$\{A_1, B_1, C_1, A_4, B_4, C_4\}$

$\{A_1, B_2, C_3, A_4, B_3, C_2\}$

$\{A_2, B_3, C_1, A_3, B_2, C_4\}$

$\{A_2, B_4, C_3, A_3, B_1, C_2\}$

$\{A_1, B_3, C_4\}$

$\{A_1, B_4, C_2\}$

$\{A_2, B_1, C_4\}$

$\{A_2, B_2, C_2\}$

$\{A_3, B_3, C_3\}$

$\{A_3, B_4, C_1\}$

$\{A_4, B_1, C_3\}$

$\{A_4, B_2, C_1\}$

Ancestral Independences

$\{A_1, B_1, C_1\} \perp \{A_4, B_4, C_4\}$

$\{A_1, B_2, C_3\} \perp \{A_4, B_3, C_2\}$

$\{A_2, B_3, C_1\} \perp \{A_3, B_2, C_4\}$

$\{A_2, B_4, C_3\} \perp \{A_3, B_1, C_2\}$

$\{A_1\} \perp \{B_3\} \perp \{C_4\}$

$\{A_1\} \perp \{B_4\} \perp \{C_2\}$

$\{A_2\} \perp \{B_1\} \perp \{C_4\}$

$\{A_2\} \perp \{B_2\} \perp \{C_2\}$

$\{A_3\} \perp \{B_3\} \perp \{C_3\}$

$\{A_3\} \perp \{B_4\} \perp \{C_1\}$

$\{A_4\} \perp \{B_1\} \perp \{C_3\}$

$\{A_4\} \perp \{B_2\} \perp \{C_1\}$

- Joint variables are all of the observable nodes $\mathcal{N}'_O = \mathcal{J}$

$$\mathcal{J} = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4\}$$

- Marginal scenario is composed of pre-injectable sets
 $\mathcal{M} = \text{PreInj}_{\mathcal{G}}(\mathcal{G}')$
- Inequalities violated by Fritz distribution are inherently 4-outcome
- Incidence matrix M is **very large** $\sim 2.25\text{Gb}$
 - #Columns = $4^{12} = 16,777,216$
 - #Rows = $4 \times 4^6 + 8 \times 4^3 = 16,896$
 - #Non-zero Entries = 201,326,592

Appendix F: Maximal Violations & Noise

Minimize objective function $f(\lambda) \in \mathbb{R}$:

- 1 Real-valued parameters $\lambda = (\lambda_0, \dots, \lambda_n)$
- 2 Quantum states/measurements $\rho_{AB}, \rho_{BC}, \rho_{CA}, M_A, M_B, M_C$

$$P_{ABC}(abc) = \text{Tr}[\Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$

- 3 Distribution P_{ABC}
- 4 Plug into inequality I in homogeneous form $I(P_{ABC}) \geq 0$
- 5 Output is objective value $I(P_{ABC})$

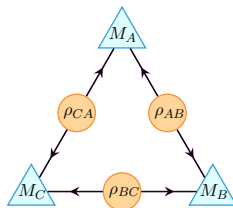
- Numerical minimization of $f(\lambda)$

$$f(\lambda_{(k+1)}) = \lambda_{(k)} - \gamma_{(k)} \nabla f(\lambda_{(k)})$$

- Non-convex, non-linear, smooth/continuous
- Gradient Descent, BFGS Method, Nelder-Mead simplex method
- Stochastic methods: simulated annealing, basin-hopping

Quantum Model On Triangle Scenario

$$P_{ABC}(abc) = \text{Tr}[\Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi M_{A,a} \otimes M_{B,b} \otimes M_{C,c}]$$



- Each latent resource $\rho \in (\rho_{AB}, \rho_{BC}, \rho_{CA})$ modeled as bipartite **qubit** state acting on $\mathcal{H}^2 \otimes \mathcal{H}^2$
- Each party (A, B, C) is assigned 4-outcome PVM set (M_A, M_B, M_C)
- Parameterized using unitary transformations $U \in \mathcal{U}(4)$

Parameterizing Unitary Group

- **Spengler_2010_Unitary** parameterization of $\mathcal{U}(d)$

$$\lambda = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,d} \\ \vdots & \ddots & \vdots \\ \lambda_{d,1} & \cdots & \lambda_{d,d} \end{pmatrix}$$

$$U = \left[\prod_{m=1}^{d-1} \left(\prod_{n=m+1}^d R_{m,n} R P_{n,m} \right) \right] \cdot \left[\prod_{l=1}^d G P_l \right]$$

- Global Phase Terms: $G P_l = \exp(i P_l \lambda_{l,l})$
- Relative Phase Terms: $R P_{n,m} = \exp(i P_n \lambda_{n,m})$
- Rotation Terms: $R_{m,n} = \exp(i \sigma_{m,n} \lambda_{m,n})$
- Projection Operators: $P_l = |l\rangle\langle l|$
- Anti-symmetric σ -matrices: $\sigma_{m,n} = -i|m\rangle\langle n| + i|n\rangle\langle m|$
- Parameters $\lambda_{n,m} \in [0, 2\pi]$

Parameterizing Unitary Group Cont'd

- Each parameter $\lambda_{n,m}$ has physical interpretation
- Degeneracies are easily eliminated such as global phase

$$\forall l = 1, \dots, d : \lambda_{l,l} = 0 \implies GP_l = \mathbb{1}$$

- Parameterize $U \in \mathcal{U}(d)$ up to global phase denoted $\tilde{U} \in \mathcal{U}(d)$

$$\tilde{U} = \prod_{m=1}^d \left(\prod_{n=m+1}^d R_{m,n} R P_{n,m} \right)$$

- Computationally efficient (no matrix exponentials)

$$GP_l = \mathbb{1} + P_l \left(e^{i\lambda_{l,l}} - 1 \right)$$

$$RP_{n,m} = \mathbb{1} + P_n \left(e^{i\lambda_{n,m}} - 1 \right)$$

$$\begin{aligned} R_{m,n} = \mathbb{1} &+ (|m\rangle\langle m| + |n\rangle\langle n|)(\cos \lambda_{n,m} - 1) \\ &+ (|m\rangle\langle n| - |n\rangle\langle m|) \sin \lambda_{n,m} \end{aligned}$$

Parameterizing PVMs

- Each part is assigned d element **projective-valued measures (PVMs)** acting on \mathcal{H}^d ,

$$M = \{M_1, \dots, M_d\} \quad \sum_{i=1}^d M_i = \mathbb{1}$$

$$\forall i : \forall |\psi\rangle \in \mathcal{H}^d : \langle \psi | M_i | \psi \rangle \geq 0 \quad M_i = M_i^\dagger$$

$$M_i M_j = \delta_{ij} M_i \quad M_i = |m_i\rangle\langle m_i|$$

- Unitary transform maps M to computational basis

$$\{|m_1\rangle, \dots, |m_d\rangle\} = \{U|1\rangle, \dots, U|d\rangle\}$$

- Global phase irrelevant: \tilde{U} requires $d(d-1)$ real-valued parameters
- PVMs are computationally more efficient than POVMs

$$P_{ABC}(abc) = \langle m_{A,a} m_{B,b} m_{C,c} | \Pi^\top \rho_{AB} \otimes \rho_{BC} \otimes \rho_{CA} \Pi | m_{A,a} m_{B,b} m_{C,c} \rangle$$

Parameterizing States

- Exploit spectral decomposition

$$\rho = \sum_{i=1}^d p_i |\psi_i\rangle\langle\psi_i| \quad p_i \geq 0, \sum_i p_i = 1$$

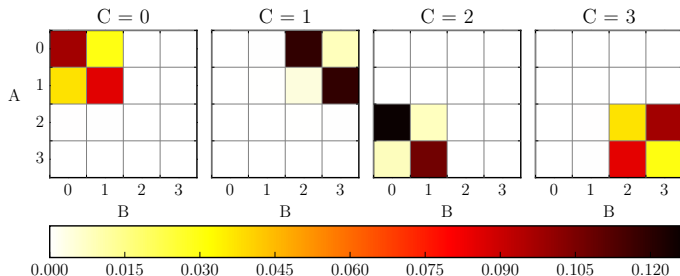
- Unitary transform maps $\{|\psi_i\rangle\}$ to computational basis

$$\{|\psi_1\rangle, \dots, |\psi_d\rangle\} = \{U|1\rangle, \dots, U|d\rangle\}$$

- Global phase irrelevant again \tilde{U} requires $d(d-1)$ real-valued parameters
- Eigenvalues $\{p_i\}$ parameterized using hyper-spherical coordinates: $d-1$ real-valued parameters

Maximal Violation

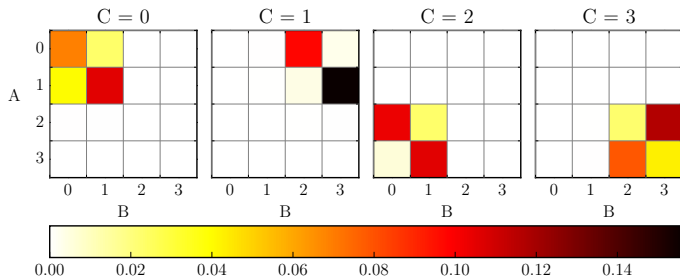
Maximum Violation of I_0 ($V_F = 1.501$)



Relative Violation:
$$V_F = \frac{\min_P \{I(P)\}}{I(P_F)}$$

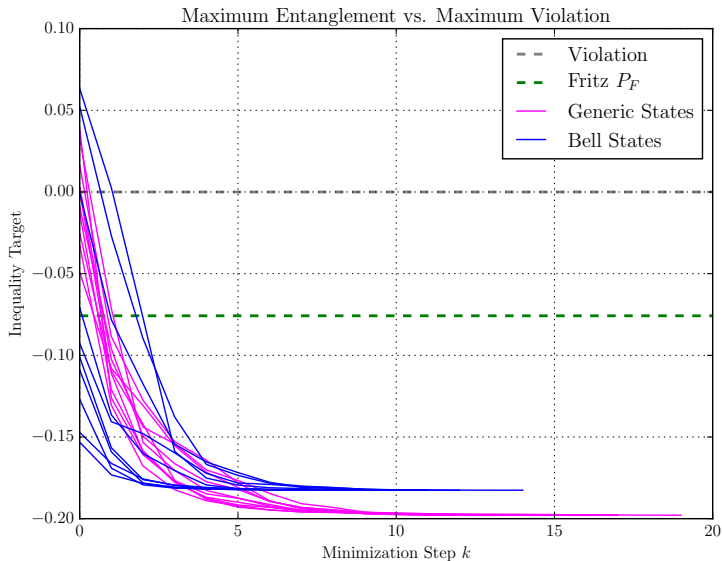
Maximal Violation Symmetric

Maximum Violation of I_2 ($V_F = 2.61$)



Relative Violation:
$$V_F = \frac{\min_P \{I(P)\}}{I(P_F)}$$

Max Entangled vs. Max Violating



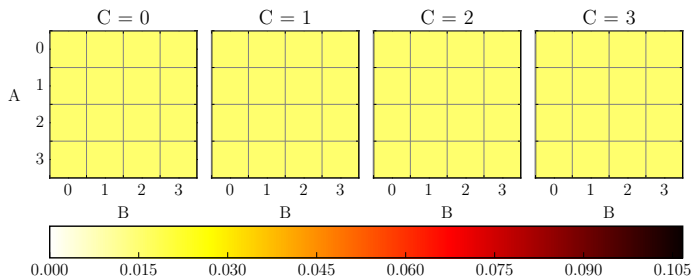
Maximally Violating Distributions

- Able to out-perform violation provided by Fritz distribution
- Maximally-violating states are not maximally-entangled; similar to detection loop-hole example of **Methot_2006**
- Both symmetric and asymmetric inequalities exhibit same qualitative features

Uniform Noise

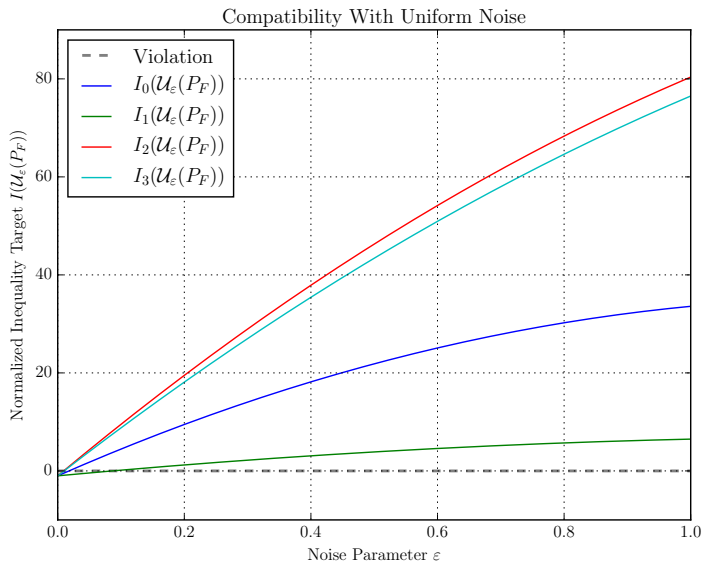
$$\mathcal{U}_\varepsilon(P) = (1 - \varepsilon)P + \varepsilon\mathcal{U}$$

Uniform Distribution

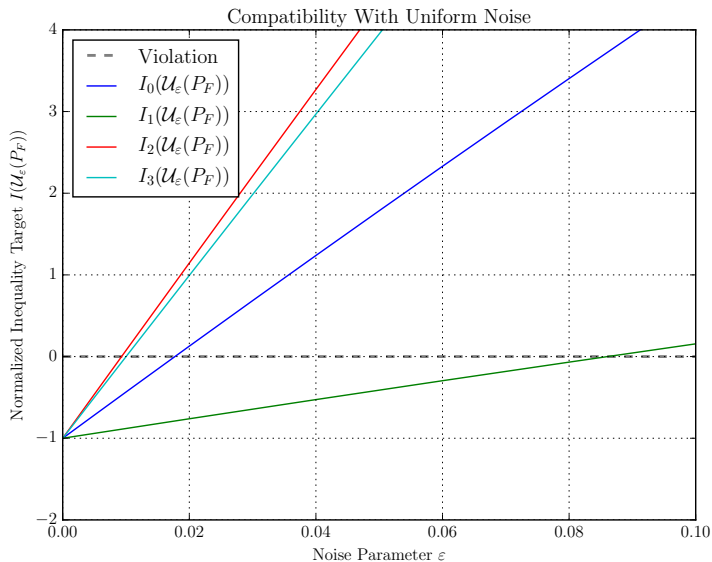


$$\text{Yellow square} = \frac{1}{64}$$

Robust to Noise

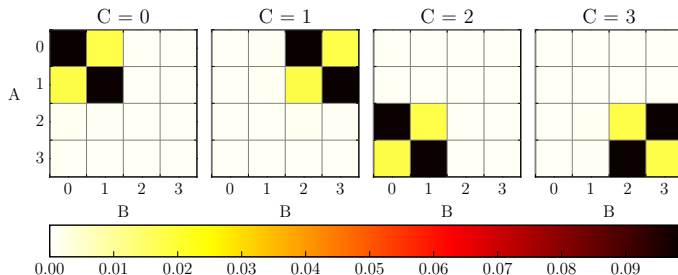


Robust to Noise Zoomed



Noisy Non-locality

Noisy Yet Non-Local Fritz Distribution



$$\square = 0.00133$$

- 1 **Inflation technique** capable of producing polynomial inequalities with quantum/classical witnesses
- 2 Fritz witness-able by **party-symmetric inequalities**
- 3 Maximally violating distributions require **non-maximally entangled states**

- 1 Do these new non-local distributions suggest **new quantum resources** in the triangle scenario?
- 2 Can any non-local quantum correlations in the triangle scenario **satisfy CHSH inequalities** (under Fritz type coarse graining)?
- 3 Which inequalities are **most robust** to noise? Facets?