Name: Elif Cemre Durgut   Recitation Section: A18

Student ID: 26493   Signature: Elif

math201 - Solutions of worksheet weeks 7-8

Problem 1:

$$A=\begin{bmatrix}1&3&-2&-5&2&1\\3&9&-5&-13&6&3\\-2&-6&8&18&-4&-1\end{bmatrix}\xrightarrow[2R_1+R_3\to R_3]{-3R_1+R_2\to R_2}\begin{bmatrix}1&3&-2&-5&2&1\\0&0&1&2&0&0\\0&0&4&8&0&1\end{bmatrix}\xrightarrow{-4R_2+R_3\to R_3}\begin{bmatrix}1&3&-2&-5&2&1\\0&0&1&2&0&0\\0&0&0&0&0&1\end{bmatrix}$$

$$\xrightarrow{-R_3+R_1\to R_1}\begin{bmatrix}1&3&-2&-5&2&0\\0&0&1&2&0&0\\0&0&0&0&0&1\end{bmatrix}\xrightarrow{2R_2+R_1\to R_1}\begin{bmatrix}1&3&0&-1&2&0\\0&0&1&2&0&0\\0&0&0&0&0&1\end{bmatrix}\Rightarrow \text{rref}(A)=R$$

a) $B=[(1,3,0,-1,2,0),(0,0,1,2,0,0),(0,0,0,0,0,1)]$ will form a basis for $Row(R)=Row(A)$.

$\Rightarrow \dim(Row(A))=3=\#$ of pivots

Conclusion: $Row(A)$ is a subspace of $\mathbb{R}^6$ with dimension 3.

b) $B'=\left[\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$ will form a basis for $Col(R)$

$B=\left[\begin{pmatrix}1\\3\\-2\end{pmatrix},\begin{pmatrix}-2\\-5\\8\end{pmatrix},\begin{pmatrix}1\\3\\-1\end{pmatrix}\right]$ will form a basis for $Col(A)$.

$\Rightarrow \dim(Col(A))=3=\#$ of pivots

Conclusion: $Col(A)$ is a subspace of $\mathbb{R}^3$ with dimension 3.

c) $Null(A)=\{x\in\mathbb{R}^6 : Ax=0\}$ $[A\,|\,0]\dashrightarrow[R\,|\,0]$

$$\begin{array}{cccccc}x_1&x_2&x_3&x_4&x_5&x_6\end{array}$$
$$\left[\begin{array}{cccccc|c}1&3&0&-1&2&0&0\\0&0&1&2&0&0&0\\0&0&0&0&0&1&0\end{array}\right]\rightarrow \begin{aligned}&x_1+3x_2-x_4+2x_5=0\\&x_3+2x_4=0\\&x_6=0\end{aligned}\Rightarrow\begin{aligned}&x_1=x_4-3x_2-2x_5\\&x_3=-2x_4\\&x_6=0\end{aligned}$$

x_2,x_4,x_5 (free) $\in\mathbb{R}$.

$$Null(A)=\left\{\begin{pmatrix}x_4-3x_2-2x_5\\x_2\\-2x_4\\x_4\\x_5\\0\end{pmatrix}\in\mathbb{R}^6 : x_2,x_4,x_5\in\mathbb{R}\right\}=\left\{x_2\begin{pmatrix}-3\\1\\0\\0\\0\\0\end{pmatrix}+x_4\begin{pmatrix}1\\0\\-2\\1\\0\\0\end{pmatrix}+x_5\begin{pmatrix}-2\\0\\0\\0\\1\\0\end{pmatrix} : x_2,x_4,x_5\in\mathbb{R}\right\}$$

$$=\left\langle \underbrace{\begin{pmatrix}-3\\1\\0\\0\\0\\0\end{pmatrix}}_{S_1},\underbrace{\begin{pmatrix}1\\0\\-2\\1\\0\\0\end{pmatrix}}_{S_2},\underbrace{\begin{pmatrix}-2\\0\\0\\0\\1\\0\end{pmatrix}}_{S_3}\right\rangle$$

$[S_1,S_2,S_3]$ will form a basis for $Null(A)$.
(it spans $Null(A)$ and it is L.I.)

$\dim(Null(A))=3$

Conclusion: $Null(A)$ is a subspace of $\mathbb{R}^6$ with dimension 3.

d) rank $A=\dim(Row(A))=\dim(Col(A))=3$
$=\#$ of pivots

$\dim(Null(A))=\#$ of columns $-\#$ pivots
$=6-3$
$=3$

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Problem 2:

$T(A) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} A$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R})$

$T(A) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -c & -d \end{bmatrix} \quad (\bigstar)$

a) $Ker(T) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{R}) : T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \xrightarrow{\text{This means}} \begin{array}{l} a+2c=0 \\ b+2d=0 \\ -c=0 \\ -d=0 \end{array} \Rightarrow a=b=c=d=0$

$Ker(T) = \underbrace{\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}}_{\text{trivial}} = \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle \qquad \dim(Ker(T)) = 0$

b) By dimension theorem:

$\underbrace{\dim(Ker(T))}_{0} + \dim(Im(T)) = \underbrace{\dim(M_{2\times2}(\mathbb{R}))}_{4}$

$\Rightarrow \dim(Im(T)) = 4 \Rightarrow \boxed{Im(T) = \mathbb{R}^4}$ //

c) $B = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$ → standard basis of $M_{2\times2}(\mathbb{R})$

$[T_B] = \left[\left[T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_B \middle| \left[T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]_B \middle| \left[T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]_B \middle| \left[T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]_B \right]_{4\times4}$

using $(\bigstar)$:

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}$

$[T_B] = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{4\times4}$ //