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Q: $T: \mathbb{R}_3[x] \longrightarrow \mathbb{R}_3[x]$, given by $T(p(x)) = (x+1).p'(x) - p(x)$

Solution:

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a) standard basis of $\mathbb{R}_3[x] \longrightarrow B = [1, x, x^2, x^3]$

$T(1) = (x+1).0 - 1 = -1 = -1.1 + 0.x + 0.x^2 + 0.x^3$

$T(x) = (x+1).1 - x = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$

$T(x^2) = (x+1).2x - x^2 = x^2 + 2x = 0.1 + 2.x + 1.x^2 + 0.x^3$

$T(x^3) = (x+1)3x^2 - x^3 = 2x^3 + 3x^2 = 0.1 + 0.x + 3.x^2 + 2.x^3$

$$[T]_B = \left[[T(1)]_B \,\vdots\, [T(x)]_B \,\vdots\, [T(x^2)]_B \,\vdots\, [T(x^3)]_B \right] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}_{4\times 4}$$ (upper triangular)

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b) isomorphism requires $\rightarrow$ linear ✓ (given in the question)
$\rightarrow$ injective ✗
$\rightarrow$ surjective

T is injective $\leftrightarrow [T]_B$ is invertible
$\leftrightarrow \det([T]_B) \neq 0$

$\Big\}$ $\det([T]_B) = (-1).0.1.3 = 0 \rightarrow$ Not injective

$\rightarrow$ T is not an isomorphism.

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c) $P_T(\lambda) = P_{[T]_B}(\lambda)$ where B is any basis of $\mathbb{R}_3[x]$.

upper triangular

$$P_T(\lambda) = P_{[T]_B}(\lambda) = \det(\lambda I_4 - [T]_B) = \begin{vmatrix} \lambda+1 & -1 & 0 & 0 \\ 0 & \lambda & -2 & 0 \\ 0 & 0 & \lambda-1 & -3 \\ 0 & 0 & 0 & \lambda-2 \end{vmatrix} = (\lambda+1)\lambda(\lambda-1)(\lambda-2)$$

The roots of $P_T(\lambda)$ are the eigenvalues of T:

$\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 1$, $\lambda_4 = 2$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

$AM(-1) = 1$ $AM(0) = 1$ $AM(1) = 1$ $AM(2) = 1$

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d) According to Theorem 4: ($\dim V = n < \infty$)

T is diagonalizable $\iff$ i) $P_T(\lambda)$ has n roots

ii) $AM(\lambda) = GM(\lambda)$ for each eigenvalue of T.

$\rightarrow \dim(\mathbb{R}_3[x]) = 4$ and $P_T(\lambda)$ has 4 roots ✓ (i)

$\rightarrow$ $\left.\begin{array}{l} AM(-1) = GM(-1) = 1 \\ AM(0) = GM(0) = 1 \\ AM(1) = GM(1) = 1 \\ AM(2) = GM(2) = 1 \end{array}\right\}$ ✓ (ii) By (i) and (ii), T is diagonalizable.

(Geometric multiplicities are all 1. Because of this rule: $1 \leq GM \leq AM$)

e) $\lambda_4 = 2$:

v is an eigenvector of T corresponding to $\lambda = 2$ $\iff$ $[v]_B$ is an eigenvector of $[T]_B$ corresponding to $\lambda = 2$.

$$E_{[T]_B}(2) = \text{Null}(2.I_4 - [T]_B) = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 3 & -1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} & & & \underset{\text{free}}{\downarrow} \end{array} \end{array} \rightarrow \begin{array}{l} 3x_1 - x_2 = 0 \rightarrow x_1 = t \\ 2x_2 - 2x_3 = 0 \rightarrow x_2 = 3t \\ x_3 - 3x_4 = 0 \rightarrow x_3 = 3t \\ x_4 = t,\ t \in \mathbb{R} \end{array}$$

$$E_{[T]_B}(2) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \begin{pmatrix} t \\ 3t \\ 3t \\ t \end{pmatrix},\ t \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix} \right\rangle$$

An eigenvector of $[T]_B$ corresponding to $\lambda_4 = 2$ is $\underset{[v]_B}{\begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}}$

An eigenvector of T corresponding to $\lambda_4 = 2$ is $v = 1.1 + 3.x + 3.x^2 + 1.x^3$
$= 1 + 3x + 3x^2 + x^3$

$E_T(2) = \langle 1 + 3x + 3x^2 + x^3 \rangle \Rightarrow B := [1 + 3x + 3x^2 + x^3]$ forms a basis for $E_T(2)$.

$\Rightarrow GM(2) = \dim(E_T(2)) = 1$.