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Recitation : A18

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<u>Question:</u> $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x) = Ax$.

<u>Solution:</u>

upper triangular matrix ☆

a) $P_A(\lambda) = \det(\lambda I_3 - A) = \begin{vmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda-2 & -2 \\ 0 & 0 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3)$

$\boxed{P_A(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3)}$

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b) Eigenvalues are the roots of characteristic polynomial of A.

$P_A(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3) = 0 \longrightarrow \lambda_1 = 1 \longrightarrow AM(1) = 1$

$\searrow \lambda_2 = 2 \longrightarrow AM(2) = 1$

$\searrow \lambda_3 = 3 \longrightarrow AM(3) = 1$

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c) i) For $\lambda_1 = 1$, we solve $(I_3 - A)x = 0$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \end{array} \xrightarrow{-R_1+R_2 \to R_2} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \end{array} \to \begin{array}{l} -x_2 = 0 \longrightarrow x_2 = 0 \\ -2x_3 = 0 \longrightarrow x_3 = 0 \\ x_1 = t,\ t \in \mathbb{R} \end{array}$$

x_1: free

$E_A(1) = \mathrm{Null}(I_3 - A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ $\quad B_1 = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$ forms a basis for $E_A(1)$.

$\hookrightarrow \dim(E_A(1)) = GM(1) = 1$ //

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ii) For $\lambda_2 = 2$, we solve $(2I_3 - A)x = 0$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \end{array} \xrightarrow{-\frac{1}{2}R_2+R_3 \to R_3} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \to \begin{array}{l} x_1 = x_2 = t \\ -2x_3 = 0 \\ x_3 = 0 \\ x_2 = t,\ t \in \mathbb{R} \end{array}$$

x_2: free

$E_A(2) = \mathrm{Null}(2I_3 - A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$ $\quad B_2 = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right]$ forms a basis for $E_A(2)$

$\to \dim(E_A(2)) = GM(2) = 1$ //

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iii) For $\lambda_3 = 3$, we solve $(3I_3 - A)x = 0$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} 2x_1 = x_2 \longrightarrow x_1 = t \\ x_2 = 2x_3 \longrightarrow x_2 = 2t \\ x_3 = t,\ t \in \mathbb{R} \end{array}$$

x_3: free

$B_3 = \left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right]$ forms a basis for $E_A(3)$

$\hookrightarrow \dim(E_A(3)) = GM(3) = 1$ //

$E_A(3) = \mathrm{Null}(3I_3 - A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

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d) $B' = B_1 \cup B_2 \cup B_3$

$B' = [(1,0,0), (1,1,0), (1,2,1)]$

$\dim \mathbb{R}^3 = 3 = |B'| \longrightarrow$ So, B' forms a basis of $\mathbb{R}^3$ if B' is linearly independent.

$c_1(1,0,0) + c_2(1,1,0) + c_3(1,2,1) = (0,0,0)$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]}_{D}$$

B' is linearly independent $\iff \det D \neq 0$

expansion along 3rd row

$$\det D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \overset{\downarrow}{=} 1 \cdot (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow B \text{ is L.I.}$$

$\Rightarrow [(1,0,0), (1,1,0), (1,2,1)]$ forms a basis of $\mathbb{R}^3$. //

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e) $B' = [\underbrace{(1,0,0)}_{u_1}, \underbrace{(1,1,0)}_{u_2}, \underbrace{(1,2,1)}_{u_3}]$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = A\begin{pmatrix}1\\0\\0\end{pmatrix} = 1 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow T(u_1) = Au_1 = 1u_1 = 1u_1 + 0u_2 + 0u_3 \Rightarrow [T(u_1)]_{B'} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = A\begin{pmatrix}1\\1\\0\end{pmatrix} = 2\begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow T(u_2) = Au_2 = 2u_2 = 0u_1 + 2u_2 + 0u_3 \Rightarrow [T(u_2)]_{B'} = \begin{bmatrix}0\\2\\0\end{bmatrix}$$

$$T\begin{pmatrix}1\\2\\1\end{pmatrix} = A\begin{pmatrix}1\\2\\1\end{pmatrix} = 3\begin{pmatrix}1\\2\\1\end{pmatrix} \Rightarrow T(u_3) = Au_3 = 3u_3 = 0u_1 + 0u_2 + 3u_3 \Rightarrow [T(u_3)]_{B'} = \begin{bmatrix}0\\0\\3\end{bmatrix}$$

$$[T]_{B'} = \left[\, [T(u_1)]_{B'} \;\middle|\; [T(u_2)]_{B'} \;\middle|\; [T(u_3)]_{B'} \,\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} //$$