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Question 1

a) $f_Y(y) = \int_x f(x,y)\,dx = \int_{x=0}^{y} 6x\,dx = (3x^2)\Big|_{x=0}^{y} = 3y^2 \quad , \quad 0<y<1$

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b) To be independent:

$\boxed{f(x,y) = f_X(x) \cdot f_Y(y)}$ → this should satisfy.

we need to find $f_X(x)$:

$f_X(x) = \int_y f(x,y)\,dy = \int_{y=0}^{1} 6x\,dy = (6xy)\Big|_{y=0}^{1} = 6x - 0 = 6x \quad , \quad 0<x<y$

$f(x,y) \overset{?}{=} f_X(x) \cdot f_Y(y)$

$6x \overset{?}{=} 6x \cdot 3y^2$ → Not equal ⇒ X and Y are dependent.

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c) Conditional density: $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

$f_{X|Y}(x|\tfrac{1}{2}) = \dfrac{f(x,1/2)}{f_Y(1/2)} = \dfrac{6x}{3 \cdot \frac{1}{4}} = \dfrac{2x}{1/4} = 8x \quad , \quad 0<x<\frac{1}{2}$

Because y was given as $\frac{1}{2}$.

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Question 2

a) $\text{cov}(X,Y) = E[XY] - E[X].E[Y]$

First, we need to find $f_X(x)$ and $f_Y(y)$ to find $E[X]$ and $E[Y]$

$$f_X(x) = \sum_y f(x,y) = \sum_{y=0}^{2} f(x,y) = f(x,0) + f(x,1) + f(x,2)$$

$$= \frac{1}{12}(x^2+0) + \frac{1}{12}(x^2+1) + \frac{1}{12}(x^2+4)$$

$$= \frac{x^2}{4} + \frac{5}{12}, \quad x = -1, 1$$

-1	0
-1	1
-1	2
1	0
1	1
1	2

$$f_Y(y) = \sum_x f(x,y) = f(-1,y) + f(1,y)$$

$$= \frac{1}{12}(1+y) + \frac{1}{12}(1+y) = \frac{1}{6} + \frac{y}{6}, \quad y = 0,1,2$$

$$E[X] = \sum_x x.f_X(x) = (-1).f_X(-1) + 1.f_X(1) = -1\left(\frac{1}{4}+\frac{5}{12}\right) + 1\left(\frac{1}{4}+\frac{5}{12}\right) = 0$$

$$E[Y] = \sum_y y.f_Y(y) = 0.f_Y(0) + 1.f_Y(1) + 2.f_Y(2) = 1.\left(\frac{1}{6}+\frac{1}{6}\right) + 2.\left(\frac{1}{6}+\frac{2}{6}\right) = \frac{4}{3}$$

$$E[XY] = \sum_x \sum_y x.y\,f(x,y) = \cancel{(-1).0.f(-1,0)} + (-1).1.f(-1,1) + (-1).1.f(-1,2)$$

$$+ \cancel{1.0.f(1,0)} + 1.1.f(1,1) + 1.2.f(1,2)$$

$$= -2/12 - 3/12 + 2/12 + 3/12 = 0$$

$$\text{cov}(X,Y) = E[XY] - E[X].E[Y] = 0 - 0.\,4/3 = 0 \,//$$

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b) Independency: $f(x,y) = \underbrace{f_X(x).f_Y(y)}$

$$\left(\frac{x^2}{4}+\frac{5}{12}\right)\left(\frac{1}{6}+\frac{y}{6}\right) = \frac{x^2}{24} + \frac{x^2y}{24} + \frac{5}{72} + \frac{5y}{72} = \frac{1}{12}\left(\frac{x^2}{2} + \frac{x^2y}{2} + \frac{5}{6} + \frac{5y}{6}\right)$$

$$\neq \frac{1}{12}(x^2+y)$$

Equality does not hold.

↳ They are dependent //

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Question 3

a) X : number of miles you can drive

Y : source of problem

↳ $\{Y=1\}$: problem is brake

↳ $\{Y=2\}$: problem is engine

Given in the question

- $P(Y=1) = 0.6$
- $P(Y=2) = 0.4$
- $f_{X|Y}(x|1) = \frac{1}{500} \cdot e^{\frac{-x}{500}}$
- $f_{X|Y}(x|2) = \frac{1}{100} \cdot e^{\frac{-x}{100}}$

Exp. Distr: $f(x;\theta) = \frac{1}{\theta} e^{\frac{-x}{\theta}}$ $\mu = \theta$

$P(X > 200) = ?$

$$\int_{x=200}^{\infty} f(x,y)dx = \int_{x=200}^{\infty} f(x,1) + f(x,2)dx = \int_{x=200}^{\infty} f_{X|Y}(x|1) \cdot \underbrace{f_Y(1)}_{0.6} + f_{X|Y}(x|2) \cdot \underbrace{f_Y(2)}_{0.4} \, dx$$

$$= 0.6 \int_{200}^{\infty} \frac{1}{500} \cdot e^{\frac{-x}{500}} dx + 0.4 \int_{200}^{\infty} \frac{1}{100} \cdot e^{\frac{-x}{100}} dx = 0.6 \cdot e^{\frac{-2}{5}} + 0.4 \cdot e^{2} \text{ //}$$

b) $E[X] = ?$

$$\int_{x=0}^{\infty} x \sum_{y=1}^{2} f(x,y)\, dx = \int_{0}^{\infty} x \cdot f(x,1) + x\, f(x,2)dx = \int_{0}^{\infty} x \cdot 0.6 \cdot \frac{1}{500} \cdot e^{\frac{-x}{500}} + x \cdot 0.4 \cdot \frac{1}{100} \cdot e^{\frac{-x}{100}} dx$$

$= 0.6 \times 500 + 0.4 \times 100 = 300 + 40 = 340$ miles //

$x \cdot f(x)$

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Question 4:

a) Uniform Distribution $\Rightarrow f(x;\alpha,\beta)=\frac{1}{\beta-\alpha}$, $\mu=\frac{\beta+\alpha}{2}$, $\sigma^2=\frac{(\beta-\alpha)^2}{12}$

$f(x)=\frac{1}{1-0}=1$

$\bar{x}$ · sample mean

$E[x]=E[\bar{x}]=\frac{\alpha+\beta}{2}=\frac{1+0}{2}=\frac{1}{2} \Rightarrow$ mean of sample mean: $1/2$ //

$Var[x]=\frac{(\beta-\alpha)^2}{12}=\frac{(1-0)^2}{12}=\frac{1}{12} \rightarrow$ standard deviation $\sigma=\frac{1}{\sqrt{12}}$

standard deviation of sample mean: $\frac{\sigma}{\sqrt{n}}=\frac{1/\sqrt{12}}{\sqrt{48}}=\frac{1}{2\sqrt{3}\cdot4\sqrt{3}}=\frac{1}{24}$ //

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b) X_i : stress score of i^{th} person

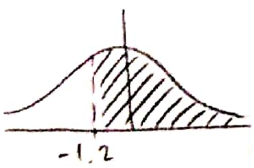
$$P\left(\frac{X_1+X_2+X_3+\cdots+X_{48}}{48}>0.45\right)=P(\bar{X}>0.45)$$

$$P(\bar{X}>0.45)=P\left(\frac{\bar{X}-1/2}{1/24}>\frac{0.45-1/2}{1/24}\right)=P(Z>-1.2)$$

$= 0.5 + \text{table}(1.2)$

$= 0.5 + 0.3849$

$= 0.8849$



$\downarrow$

%88 , the average will be larger than 0.45.

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Question 5

Question gives us:

a) X: midterm grade

Y: final grade

G: overall grade

$G = \frac{X}{2} + \frac{Y}{2}$ $\mu_X = 60$ $\sigma_X = 18$

$\mu_Y = 40$ $\sigma_Y = 24$

$P(G > 80) = ?$

$Z = \frac{X-\mu}{\sigma}$

$P(G>80) = P\left(\frac{X}{2} + \frac{Y}{2} > 80\right) = P(X+Y > 160)$

table2

$= P\left(Z > \frac{160-100}{30}\right) = P(Z > 2) = 0.5 - 0.4772$

$= 0.03 \ldots$ //

$E[X+Y] = E[X] + E[Y]$

$= 60 + 40$

$= 100$

$Var[X+Y] = Var[X] + Var[Y]$ since independent

$= 18 + 24$

$= 30^2$

324
576

24×24: 96, 48, 576

2

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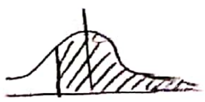
b) M: passing grade

$P(G > M) = \frac{84.13}{100} \rightarrow P\left(\frac{X+Y}{2} > M\right) = P(X+Y > 2M) = 0.8413$

$= 0.5 + 0.3413$

$= 0.5 + \text{table}(1)$

$= P(Z > -1)$



-1

$\frac{2M-\mu}{\sigma} = -1$

$\frac{2M-100}{\sqrt{42}} = -1$

$m = \frac{100-\sqrt{42}}{2}$ //

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Question 6

if minimum of x_1 and x_2 is greater than y, both of them are greater than y

$\Uparrow$

a) $1-F_Y(y) = 1-P(Y \le y) = P(Y>y) = P(\min(X_1, X_2) > y) = P(X_1 > y, X_2 > y)$

Since they are independent $\rightarrow P(X_1 > y, X_2 > y) = P(X_1>y) \cdot P(X_2>y)$

$$= P(X_1>y)\cdot P(X_2>y) = \int_{x_1=y}^{2} \frac{x_1}{2}\,dx_1 \cdot \int_{x_2=y}^{2} \frac{x_2}{2}\,dx_2 = \left(\frac{x_1^2}{4}\right)\Big|_y^2 \cdot \left(\frac{x_2^2}{4}\right)\Big|_y^2$$

$$= \left(1-\frac{y^2}{4}\right)\left(1-\frac{y^2}{4}\right) = 1-\frac{y^2}{2}+\frac{y^4}{4}$$

$$1-F_Y(y) = 1-\frac{y^2}{2}+\frac{y^4}{4} \Rightarrow F_Y(y) = \frac{y^2}{2}-\frac{y^4}{4}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d\left(\frac{y^2}{2}-\frac{y^4}{4}\right)}{dy} = y-y^3 \quad , \quad 0<y<2 \quad //$$

b) $$P(Y>1) = \int_{y=1}^{2} y-y^3\,dy = \left(\frac{y^2}{2}-\frac{y^4}{4}\right)\Big|_{y=1}^{2} = (2-4)-\left(\frac{1}{2}-\frac{1}{4}\right) = -2-\frac{1}{2} = \frac{-5}{2} //$$

?

It should not be negative

c) $$E[Y] = \int_y y\cdot f_Y(y)\,dy = \int_0^2 y^2-y^4\,dy = \frac{y^3}{3}-\frac{y^5}{5}\Big|_0^2 = \frac{8}{3}-\frac{32}{5} = \frac{-56}{15} //$$