Solution Key

NAME, LAST NAME: ____________________

STUDENT NUMBER: ______________ SECTION (A OR B): ____

Math 204 – Midterm 1 – March 7, 2020

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Time allowed is 120 minutes. There are ?? problems worth a total of ?? points.
You MUST show your work and give explanations.

Problem 1: (15 points)
Determine if the following statements are true (**T**) or false(**F**). (**No explanation required**)

1. $|\{\emptyset\}| = 1$ **[T]** [F]
2. $\mathcal{P}(\{\emptyset, \mathcal{P}(\emptyset)\})$ has 2 element. [T] **[F]**
3. $\forall x(P(x) \vee Q(x))$ is logically equivalent to $\forall x P(x) \vee \forall x Q(x)$. [T] **[F]**
4. $\forall x(P(x) \wedge Q(x))$ is logically equivalent to $\forall x P(x) \wedge \forall x Q(x)$. **[T]** [F]
5. $A \times B \times C = A \times (B \times C)$. [T] **[F]**
6. Let S and T be two finite sets such that $f : S \to T$ is surjective. Then $|S| \geq |T|$. **[T]** [F]
7. $p \to q$ is logically equivalent to $\neg q \to \neg p$. **[T]** [F]
8. If $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, then $\lceil x - n \rceil = \lceil x \rceil - n$. **[T]** [F]
9. Let $a_1, a_2, \ldots$ be a sequence. Then $\sum_{i=1}^{n}(a_i - a_{i+1}) = a_1 - a_{n+1}$. **[T]** [F]
10. If A and B are countable then $A \cup B$ is not necessarily countable. [T] **[F]**
11. For any set S, we have $|S| < |P(S)|$. **[T]** [F]
12. If A and B are uncountable sets then $A \cap B$ must be uncountable. [T] **[F]**
13. The function $f(x) = \frac{x+1}{x+2}$ defines a bijection from $\mathbb{R}$ to $\mathbb{R}$. [T] **[F]**
14. Let f and g be two bijective functions from A to B. Then $f + g$ is also a bijective function from A to B. [T] **[F]**
15. A map $f : A \to B$ is injective if $x_1 = x_2$ then $f(x_1) = f(x_2)$, for every $x_1, x_2 \in A$. [T] **[F]**

Problem 2: (12 points)

1. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.

   (a) The absolute value of the product of two integers is the product of their absolute values.

$\forall x \forall y\ (x, y \in \mathbb{Z} \rightarrow |xy| = |x||y|)$ OR $\forall x \forall y\ (|xy| = |x||y|)$

   (b) For every non-zero real number, there exists a real number such that their product is 1.

$\forall x \exists y\ (x \neq 0 \rightarrow xy = 1)$

2. Express the negations of each of the following statements so that all negation symbols immediately precede predicates.

   (a) $\exists y\ (\exists x R(x,y) \vee \forall x S(x,y))$

$\forall y\ (\forall x \neg R(x,y) \wedge \exists x \neg S(x,y))$

   (b) $\forall x \forall y (x^2 = y^2 \rightarrow x = y)$

$\exists x \exists y\ (x^2 \neq y^2 \wedge x = y)$

3. Determine the truth value of each of the following statements if the domain for each variable is set of real numbers. Explain your answer briefly.

   (a) $\exists x \forall y (y \neq 0 \rightarrow x = \frac{1}{y})$

FALSE, because

if $y_1 \neq y_2$, $y_1 \neq 0$, $y_2 \neq 0$, then $\frac{1}{y_1} \neq \frac{1}{y_2}$.

   (b) $\forall x \exists y (y^2 < x)$

FALSE, because

if $x < 0$, then no such y exists.

Problem 3: (3 points)
Let P be the statement that you have a flu, Q be the statement that you miss the final exam and R be the statement that you pass the course. Translate the following proposition into English sentence.

$$(P \rightarrow \neg R) \vee (Q \rightarrow \neg R)$$

If you have a flue, then you will not pass the course; <u>or</u> if you miss the final exam, then you will not pass the course

$\equiv$ You do not have flue <u>or</u> you do not miss the the final exam <u>or</u> you do not pass the course

WRONG: $(P \vee Q) \rightarrow \neg R$

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Problem 4: (7 points)
Determine whether $(p \iff q) \oplus (p \iff \neg q)$ is a tautology by construction truth table. Explain your answer briefly.

p	q	$p \leftrightarrow q$	$\neg q$	$p \leftrightarrow \neg q$	$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	F	T	T
F	F	T	T	F	T

Answer:
Since the argument $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$ is always true (T), it is a tautology.

Problem 5: (10 points)
Let A, B and C be sets. **Using set identities**, show that $(B \cup (A - C)) \cap (B \cup (A \cap C)) = B$.
Note: Proof by using Venn diagram is NOT allowed.

$(B \cup (A-C)) \cap (B \cup (A \cap C))$

$= B \cup ((A-C) \cap (A \cap C))$ by distributiy law

$= B \cup ((A \cap \overline{C}) \cap (A \cap C))$ since $A - C = A \cap \overline{C}$

$= B \cup (A \cap \overline{C} \cap C)$

$= B \cup (A \cap \emptyset)$ since $\overline{C} \cap C = \emptyset$

$= B \cup \emptyset$ since $A \cap \emptyset = \emptyset$

$= B$.

Problem 6: (10 points)
Let $A_1, A_2, \ldots, A_n$ and B be sets. Use mathematical induction to prove that

$$(A_1 \cup A_2 \cup \ldots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B)$$

Base case: $n=1$.

Left hand side = Right hand side = $A_1 \cap B$.
Hence the argument is trivially holds for $n=1$.
OR
$n=2$, we have $(A_1 \cup A_2) \cap B = (A_1 \cap B) \cap (A_2 \cap B)$
by distributive law. Hence, the argument is true for $n=2$.

Inductive step:

Assume that the argument is true for an integer $k \geq 1$.
That is, we have
$(A_1 \cup A_2 \cup \cdots \cup A_k) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B)$.
We show that the argument is also true for $k+1$.

$(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}) \cap B$
$= ((A_1 \cup A_2 \cup \cdots \cup A_k) \cap B) \cup (A_{k+1} \cap B)$ by distributive law.
$= ((A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B)) \cup (A_{k+1} \cap B)$
by induction hypothesis
$= (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B) \cup (A_{k+1} \cap B)$

This gives the desired conclusion.

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Problem 7: (15 points)

1. Let $x \in \mathbb{R} \setminus \mathbb{Z}$. Decide what is the value of $\lceil x \rceil - \lfloor x \rfloor$.

Let $x \in \mathbb{R} \setminus \mathbb{Z}$. Then there exists $n \in \mathbb{Z}$ such that $n < x < n+1$ as $x \notin \mathbb{Z}$. Hence, $\lfloor x \rfloor = n$ and $\lceil x \rceil = n+1$

Then $\lceil x \rceil - \lfloor x \rfloor = (n+1) - n = 1$

2. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if $A_i = \{-i, \ldots, i\}$, for every postive integer i.

$A_1 = \{-1, 0, 1\}$, $A_2 = \{-2, -1, 0, 1, 2\}$, $A_3 = \{-3, -2, -1, 0, 1, 2, 3\}, \ldots$

Note that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$

Hence $\bigcap_{i=1}^{\infty} A_i = A_1 = \{-1, 0, 1\}$, and

$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$

3. Give an example of two uncountable sets such that their difference is countably infinite?

Let $A = \mathbb{R}$ and $B = \mathbb{R} \setminus \mathbb{Z}$. Then we have

$A \setminus B = \mathbb{R} \setminus (\mathbb{R} \setminus \mathbb{Z}) = \mathbb{Z}$, which is countably infinite

Problem 8: (8 points)
Let $n \in \mathbb{Z}$. Prove that $3n + 2$ is odd if and only if $9n + 5$ is even .

$(\Rightarrow)$ Suppose that $3n+2$ is odd. Then $3n$ is odd. This shows that n must be odd. Hence $9n$ is odd. Since $9n$ and 5 are odd, $9n+5$ is even

$(\Leftarrow)$ Suppose that $9n+5$ is even. Then $9n$ is odd. This shows that n must be odd, Hence $3n$ is odd. Since $3n$ is odd and 2 is even, $3n+2$ is odd.

Problem 9: (10 points)
Prove that $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{-(n+1)}{2} & \text{if } n \text{ is odd} \end{cases}$$

is bijective.

Note that we have the following correspondence:

$$\begin{array}{ll} \mathbb{N}: & \{0, 1, 2, 3, 4, 5, 6, \ldots\} \\ & \;\,\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow\;\;\downarrow \\ \mathbb{Z}: & \{0, -1, 1, -2, 2, -3, 3, \ldots\} \end{array}$$

We need to show f is surjective and injective!

Surjective: Let $k \in \mathbb{Z}$. We will show that there exist $n \in \mathbb{N}$ such that $f(n)=k$.

case (i): $k=0$

Then $n=0$ since $f(0)=0$.

case (ii): $k<0$.

Then $n=-2k-1>0$ since $f(-2k-1) = -\frac{(-2k-1+1)}{2} = \frac{2k}{2} = k$

case (iii): $k>0$

Then $n=2k$ since $f(2k) = \frac{2k}{k} = k$.

Injective: Let $f(n_1)=f(n_2)$ for some $n_1, n_2 \in \mathbb{N}$

case (i): Both n_1 and n_2 are odd

$f(n_1)=f(n_2) \Leftrightarrow -\frac{(n_1+1)}{2} = -\frac{(n_2+1)}{2} \Leftrightarrow n_1=n_2$.

case (ii) Both n_1 and n_2 are even

$f(n_1)=f(n_2) \Leftrightarrow \frac{n_1}{2} = \frac{n_2}{2} \Leftrightarrow n_1=n_2$

case (iii): One of them is odd and one of them even.

Say, n_1 is odd and n_2 is even.

$f(n_1)=f(n_2) \Leftrightarrow -\frac{(n_1+1)}{2} = \frac{n_2}{2} \Leftrightarrow -n_1-1=n_2 \Leftrightarrow n_1+n_2=-1$, which is not possible since $n_1, n_2 \in \mathbb{N}$. Contradiction.

Hence f is 1-1 since we have $f(n_1)=f(n_2) \Rightarrow n_1=n_2$.