Problem 1:
Consider the following linear system of equations:

$$x + y - z = 2$$
$$x + 2y + z = 3$$
$$x + y + (a^2 - 5)z = a$$

Find the values of a, for which the resulting system has:

i) no solutions; (pivot in the last column)

ii) a unique solution; ($A_{3\times 3}$ must have 3 pivots)

iii) infinitely many solutions.

First form the augmented matrix

$$[A \,\vdots\, b] = \begin{array}{c} \begin{array}{cccc} x & y & z & \text{RHS} \end{array} \\ \left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \end{array}$$

and start elimination:

$$\left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

pivots

candidates for the next pivot

$a^2 - 4 = 0 \iff a \in \{-2, 2\}$

$a - 2 = 0 \iff a = 2$ So we will examine these values.

(i) If $a = -2$, we have

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right] \end{array}$$

pivot in the last column of the augmented matrix

$\to 0.x + 0.y + 0.z = -4 \Rightarrow 0 = -4$ ✗ impossible.

$\Rightarrow$ inconsistent system, i.e. system has no solutions when $a = -2$.

(ii) Recall: Given $A \in M_{n\times n}(\mathbb{R})$, $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$
$\iff$ A is invertible $\iff$ $\det A \neq 0$.

$$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & a^2-5 \end{vmatrix} \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & a^2-4 \end{vmatrix} = 1.1.(a^2-4) = a^2 - 4 \neq 0 \iff a \in \mathbb{R}\setminus\{\mp 2\}$$

(upper-triangular)

(determinant does not change)

(iii) If $a = 2$, we have

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc:c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \end{array}$$

only pivots

(No pivot in the last column $\Rightarrow$ system is consistent).

No pivot in the column of z $\Rightarrow$ z is a free variable.

$\Rightarrow$ system has infinitely many solutions.

Problem 2:
Let $v_1 = (1, 2, -1)$, $v_2 = (t^2, 2, -1)$ and $v_3 = (-1, -2, t)$ be vectors in $\mathbb{R}^3$.

a) Find for which values of t, $S = \{v_1, v_2, v_3\}$ forms a basis for $\mathbb{R}^3$.

b) Let $W = SpanS = < v_1, v_2, v_3 >$. Determine $\dim(W)$ for each value of t.

a) $\underline{\dim \mathbb{R}^3 = 3}$ and $\underline{|S| = 3}$. $\Rightarrow$ S will form a basis for $\mathbb{R}^3$

$\Leftrightarrow$ S is linearly independent $\Leftrightarrow$ Span $S = \mathbb{R}^3$.

(Checking either one is enough).

S is linearly independent if $c_1v_1 + c_2v_2 + c_3v_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$

i.e. $c_1(1,2,-1) + c_2(t^2, 2, -1) + c_3(-1,-2,t) = (0,0,0) \Rightarrow c_1 = c_2 = c_3 = 0$

$$\Leftrightarrow \underbrace{\begin{bmatrix} 1 & t^2 & -1 \\ 2 & 2 & -2 \\ -1 & -1 & t \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0.$$

(i.e. $Ax = 0$ has only the trivial solution $\Leftrightarrow$ A is invertible. $\Leftrightarrow \det A \neq 0$).

expand along the 1st row.

$$\det A = \begin{vmatrix} 1 & t^2 & -1 \\ 2 & 2 & -2 \\ -1 & -1 & t \end{vmatrix} \overset{\downarrow}{=} \underbrace{1\cdot(-1)^{1+1}}_{=1} \underbrace{\begin{vmatrix} 2 & -2 \\ -1 & t \end{vmatrix}}_{2t-2} + t^2 \cdot \underbrace{(-1)^{1+2}}_{=-1} \underbrace{\begin{vmatrix} 2 & -2 \\ -1 & t \end{vmatrix}}_{2t-2} + (-1)\cdot(-1)^{1+3} \underbrace{\begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix}}_{0}$$

(proportional rows/columns).

$$\det A = 2t - 2 - 2t^3 + 2t^2 = -2(\underbrace{t^3 - t^2 - t + 1}_{t^2(t-1)-(t-1)}) = -2(t^2-1)(t+1) \neq 0$$

$\Leftrightarrow t \in \mathbb{R} \setminus \{-1, 1\}$.

Conclusion: S will form a basis of $\mathbb{R}^3 \Leftrightarrow t \in \mathbb{R} \setminus \{-1,1\}$.

b) $W = \langle v_1, v_2, v_3 \rangle$. By part(a), $W = \langle v_1, v_2, v_3 \rangle = \mathbb{R}^3$ if $t \in \mathbb{R}\setminus\{-1,1\}$.

$\Rightarrow \dim(W) = \dim(\mathbb{R}^3) = 3$ if $t \in \mathbb{R} \setminus \{-1,1\}$.

When $t = 1 \Rightarrow \underset{W}{\underset{\parallel}{\langle v_1, v_2, v_3 \rangle}} = \langle (1,2,-1), \underbrace{(1,2,-1), (-1,-2,1)}_{\text{redundant}} \rangle = \langle (1,2,-1) \rangle$

$W = \langle (1,2,-1) \rangle \Rightarrow B = \{(1,2,-1)\}$ form a basis for $W \Rightarrow \dim W = 1$ if $t = 1$.

When $t = -1 \Rightarrow \underset{W}{\underset{\parallel}{\langle v_1, v_2, v_3 \rangle}} = \langle (1,2,-1), \underbrace{(1,2,-1)}_{\text{redundant}}, (-1,-2,-1) \rangle = \langle \underbrace{(1,2,-1), (-1,-2,-1)}_{} \rangle$

linearly independent, not scalar multiple of one another.

$\Rightarrow B = \{(1,2,-1), (-1,-2,-1)\}$ forms a basis for W.

$\Rightarrow \dim W = 2$ if $t = -1$.

Say you're given $\underbrace{(2,-1)}, \underline{\underline{(0,8)}}, \underbrace{(-4,3)}$ (3 vectors in $\mathbb{R}^2 \Rightarrow$ they must be L.D.).

How to find the dependency quickly? Watch the 0: $\underset{2}{\underset{\Uparrow}{a}}(2,-1) + \underset{1}{\underset{\Uparrow}{b}}(-4,3) = (\underline{\underline{0}}, 1)$

$\Rightarrow (0,8) = 8(0,1) = 16(2,-1) + 8(-4,3)$ ✓

Problem 3:

Let $A = \begin{bmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{bmatrix} \xrightarrow[-2R_1+R_3\to R_3]{R_1+R_2\to R_2} \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_2+R_3\to R_3} \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{-3R_2+R_1\to R_1} \underbrace{\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{pivot columns of } R} = \text{r.r.e.f of } A = R$

a) Find a basis for the row space of A.

b) Find a basis for the column space of A.

c) Find a basis for the null space of A.

$Null(A^T) \leftarrow$ d) Find a basis for the left null space of A. (Exercise).

e) Determine rank$A$ and $\dim(Null(A))$ and $\dim(Null(A^T))$.

a) Nonzero rows (i.e. linearly independent rows) of R will form a basis for $Row(R) = Row(A)$.

$\Rightarrow B = [(1,-2,0,-1), (0,0,1,1)]$ forms a basis for $Row(A)$.

b) $B = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right]$ will form a basis for $Col(R)$.

the columns of A corresponding to the pivot columns of R will form a basis for $Col(A)$.

$\Rightarrow B' = \left[\begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}3\\-2\\5\end{pmatrix}\right]$ will form a basis for $Col(A)$.

Recall: $A = \begin{bmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{bmatrix}$, columns $= C_1, C_2, C_3, C_4$ with $C_2 = -2C_1$, $C_4 = -C_1 + C_3$

$R = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, columns $= C_1', C_2', C_3', C_4'$

$C_2' = -2C_1'$

$C_4' = -C_1' + C_3'$

c) $Null(A) :=$ solution space of $Ax = 0$ (Recall: e.r.o.s. does not change the nullspace)

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$\rightarrow x_1 - 2x_2 - x_4 = 0 \Rightarrow x_1 = 2x_2 + x_4 = 2s + t$

$\rightarrow x_3 + x_4 = 0 \Rightarrow x_3 = -x_4 = -t$

x_2, x_4 are free $\Rightarrow x_2 = s, x_4 = t, s,t \in \mathbb{R}$

$\Rightarrow Null(A) = \left\{ \begin{pmatrix} 2s+t \\ s \\ -t \\ t \end{pmatrix} : s,t \in \mathbb{R} \right\}$

$\begin{pmatrix} 2s+t \\ s \\ -t \\ t \end{pmatrix} = s\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}1\\0\\-1\\1\end{pmatrix} \Rightarrow Null(A) = \langle \underbrace{\begin{pmatrix}2\\1\\0\\0\end{pmatrix}}_{s_1}, \underbrace{\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}}_{s_2} \rangle \Rightarrow B = [s_1, s_2]$ forms a basis for $Null(A)$.

linearly independent

$s=1, t=0$; $s=0, t=1$ $\longrightarrow$ by this.

e) $\text{rank}(A) = \dim(Col(A)) = \dim(Row(A)) = 2$ (# of pivots). (by parts a and b)

$+ \ \dim(Null(A)) = 2$ (by part c) (# of free variables in $Ax=0$).

$\text{rank}(A) + \dim(Null(A)) = 4 \overset{\checkmark}{=}$ # of columns in A.

Problem 4:

Suppose that $A \in M_{5\times 16}(\mathbb{R})$. $\Rightarrow A^T \in M_{16\times 5}(\mathbb{R})$

a) What is the maximum possible value for $rank(A)$?

b) What is the minimum possible value for $dim(null(A))$?

c) Suppose that $\boxed{dim(col(A)) = 5.}$ What is $dim(null(A^T))$?

$\hookrightarrow rank(A) = 5$

Consider the linear transformation $T : \mathbb{R}^{16} \to \mathbb{R}^5$ defined as $T(x) = Ax$.

d) What does $dim(null(A))$ represent in terms of T?

e) What is the dimension of the image of T if the $dim(Null(A))$ is at its minimum value? Is T then surjective?

a) $Row(A)$ is a subspace of $\mathbb{R}^{16} \Rightarrow \dim(Row(A)) \le \dim(\mathbb{R}^{16}) = 16$

$Col(A)$ is a subspace of $\mathbb{R}^5 \Rightarrow \dim(Col(A)) \le \dim(\mathbb{R}^5) = 5$

$rank(A) \le \min\{5, 16\} = 5$

$\Rightarrow rank(A) \le 5 \Rightarrow$ maximum possible value of $rank(A) = 5$.

b) $\dim(Null(A)) + rank(A) = 16$ ($= \#$ of columns of A).

$\Rightarrow \dim(Null(A)) = 16 - rank(A)$. $\underset{rank(A)\le 5}{\Rightarrow}$ $\dim(Null(A)) \ge 11$

$\Rightarrow$ minimum possible value of $\dim(Null(A)) = 11$

c) $\dim(Null(A^T)) + \underbrace{rank(A^T)}_{= rank(A) = 5} = 5$ ($\#$ of columns of A^T). $\} \Rightarrow \dim(Null(A^T)) = 0$.

d) $T: \mathbb{R}^{16} \to \mathbb{R}^5$ defined as $T(x) = Ax$.

$T(x) = \underset{5\times 16}{A}\ \underset{16\times 1}{x} = \underset{5\times 1}{0}$

Recall: $\ker(T) = \{x \in \mathbb{R}^{16} : T(x) = 0\} \subseteq \mathbb{R}^{16}$

$= \{x \in \mathbb{R}^{16} : Ax = 0\} = Null(A)$.

Thus, $\dim(\ker(T)) = \dim(Null(A)) = nullity(A)$.

e) Recall: $Im(T) = \{T(x) : x \in \mathbb{R}^{16}\} \subseteq \mathbb{R}^5$

$= \{Ax : x \in \mathbb{R}^{16}\} = Col(A)$. $\Rightarrow \dim(Im(T)) = \dim(Col(A)) = rank(A)$.

l.c. of the columns of A.

(As x runs through $\mathbb{R}^{16}$, we will get all possible l.c. of the columns of A, i.e. we will get the span of the columns of A).

Thus, by parts (a) and (b) $\underset{\text{if } \dim(Null(A)) = 11}{\Rightarrow}$ $rank(A) = \dim(Col(A)) = \dim(Im(T)) = 5$

Since $Im(T)$ is a subspace of $\mathbb{R}^5$ and $\dim(Im(T)) = 5$

$\Rightarrow Im(T) = \mathbb{R}^5 \Rightarrow T$ is surjective.

I realized that I already solved previous Problem 5 on week 8. That's why I changed it.

Recall: $T: V \to W$ linear where $\dim V = n < \infty$

(1) $\dim(Ker(T)) + \dim(Im(T)) = \dim V$

(2) T is injective $\iff Ker(T) = \{0_V\}$.

(3) T is surjective $\iff Im(T) = W$.

Problem 5:

Let $\boxed{T : \mathbb{R}^3 \to \mathbb{R}^2}$ be the linear transformation defined as

$$T(a_1, a_2, a_3) = (a_1 + a_2, a_1 - a_3).$$

a) Find a basis for $Ker(T)$ and determine $dim(Ker(T)) = nullityT$.

b) Find a basis for $Im(T)$ and determine $dim(Im(T)) = rankT$.

c) Determine whether T is injective.

d) Determine whether T is surjective.

a) $Ker(T) = \{(a_1, a_2, a_3) : T(a_1, a_2, a_3) = (0,0)\}$

$T(a_1,a_2,a_3) = (a_1+a_2, a_1-a_3) = (0,0) \iff a_1 = -a_2$ and $a_1 = a_3$.

$\Rightarrow Ker(T) = \{(a_1, a_2, a_3) : a_1 = -a_2 \text{ and } a_1 = a_3\}$

$= \{\underbrace{(a_1, -a_1, a_1)}_{= a_1(1,-1,1)} : a_1 \in \mathbb{R}\} = \langle (1,-1,1) \rangle \Rightarrow Ker(T) = \langle (1,-1,1) \rangle$

$\Rightarrow B = [(1,-1,1)]$ forms a basis for $Ker(T)$ (it clearly spans $Ker(T)$ and is L.I.).

$\Rightarrow \dim(Ker(T)) = 1 = nullity(T)$

b) Recall: $T: V \to W$ linear, V finite dimensional, then $Im(T) = \langle T(v_1), T(v_2), \dots, T(v_n) \rangle$ where $B = [v_1, v_2, \dots, v_n]$ is any basis of V.

Let's take $B = [\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}]$ standard basis of $\mathbb{R}^3$.

Then $Im(T) = \langle \underset{=(1,1)}{T(1,0,0)}, \underset{=(1,0)}{T(0,1,0)}, \underset{=(0,-1)}{T(0,0,1)} \rangle = \langle (1,1), (1,0), \underbrace{(0,-1)}_{\text{redundant}} \rangle$

$(0,-1) = -((1,1) - (1,0))$

$\Rightarrow Im(T) = \langle \underbrace{(1,1),(1,0)}_{\text{linearly independent.}} \rangle$ $\Rightarrow B = [(1,1),(1,0)]$ forms a basis for $Im(T)$.

$\Rightarrow \dim(Im(T)) = rank(T) = 2$.

Let's check: $\underbrace{\dim(Ker(T))}_{=1} + \underbrace{\dim(Im(T))}_{=2} \overset{?}{=} \dim(\underset{=3}{\mathbb{R}^3})$ ✓

c) T is not injective since $Ker(T) = \langle (1,-1,1) \rangle \neq \{(0,0,0)\}$ (i.e. kernel is not the trivial subspace of $\mathbb{R}^3$).

d) we have $\dim(Im(T)) = rank(T) = 2$ and $Im(T)$ is a subspace of $\mathbb{R}^2$

$\Rightarrow$ we must have $Im(T) = \mathbb{R}^2$

$\Rightarrow T$ is surjective.

Problem 6:

Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear operator defined as $T(M) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} M$.

a) Find the matrix of T relative to the basis $B = [\underbrace{\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}}_{= v_1}, \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}}_{= v_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}}_{= v_3}, \underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}}_{= v_4}]$, i.e. find $[T]_B$.

b) Determine whether T is an isomorphism.

a) By definition, $[T]_B = \Big[[T(v_1)]_B \,\vdots\, [T(v_2)]_B \,\vdots\, [T(v_3)]_B \,\vdots\, [T(v_4)]_B \Big]$ 4×4 " $\dim(M_{2\times 2}(\mathbb{R}))$

$T(v_1) = T\left(\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$ ① $= c_1.v_1 + c_2.v_2 + c_3.v_3 + c_4.v_4$

$T(v_2) = T\left(\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ② $= 0.v_1 + 0.v_2 + 0.v_3 + 0.v_4$ (Easy)

$T(v_3) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ ③ $= d_1.v_1 + d_2.v_2 + d_3.v_3 + d_4.v_4$

$T(v_4) = T\left(\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ ④ $= 0.v_1 + 0.v_2 + 0.v_3 + (-1).v_4$ (Easy)

$c_1.v_1 + c_2.v_2 + c_3.v_3 + c_4.v_4 = \begin{pmatrix} -c_1 + 2c_2 & c_2 - c_4 \\ -c_1 + c_3 & 2c_3 + c_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \iff \left\{\begin{array}{l} -c_1 + 2c_2 = -1 \\ -c_1 + c_3 = 1 \\ c_2 = c_4 \\ -2c_3 = c_4 \end{array}\right.$ $\Rightarrow 2c_2 - c_3 = -2$ $\Rightarrow 2c_4 + \frac{c_4}{2} = -2$ $\Rightarrow \boxed{c_4 = -\frac{4}{5}}$

$c_4 = -\frac{4}{5} \Rightarrow c_2 = -4/5,\ c_3 = \frac{2}{5},\ c_1 = -\frac{3}{5}$

Similarly $d_1.v_1 + d_2.v_2 + d_3.v_3 + d_4.v_4 = \begin{pmatrix} -d_1 + 2d_2 & d_2 - d_4 \\ -d_1 + d_3 & 2d_3 + d_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$

$\Rightarrow d_1 = \frac{3}{5},\ d_2 = \frac{4}{5},\ d_3 = -\frac{2}{5},\ d_4 = \frac{-6}{5}$

Thus $[T]_B = \begin{bmatrix} -3/5 & 0 & 3/5 & 0 \\ -4/5 & 0 & 4/5 & 0 \\ 2/5 & 0 & -2/5 & 0 \\ -4/5 & 0 & -6/5 & -1 \end{bmatrix}_{4\times 4}$ since

$T(v_1) = -\frac{3}{5}v_1 - \frac{4}{5}v_2 + \frac{2}{5}v_3 - \frac{4}{5}v_4$

$T(v_2) = 0.v_1 + 0.v_2 + 0.v_3 + 0.v_4$

$T(v_3) = 3/5\, v_1 + \frac{4}{5}v_2 - \frac{2}{5}v_3 - \frac{6}{5}v_4$

$T(v_4) = 0.v_1 + 0.v_2 + 0.v_3 - 1.v_4$

b) $T: V \to V$ (where dim V = nd bo) is injective $\iff [T]_B$ is invertible.

In our example, note that $\det([T]_B) = 0$ (expand along the 2nd column)

$\Rightarrow [T]_B$ is not invertible $\Rightarrow T$ is not injective

$\Rightarrow T$ is not an isomorphism.

Problem 7:
Consider the linear operator $T : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$, given by $T(p(x)) = (x+2)p'(x)$ where $p'(x)$ the first derivative of $p(x)$.

$\hookrightarrow B = [1, x, x^2, x^3]$

i) Find the matrix $[T]_B$ relative to the basis B is the standard basis of $\mathbb{R}_3[x]$.

ii) Find the characteristic polynomial of T and determine the eigenvalues of T.

iii) Is T diagonalizable? Explain briefly.

iv) Is T an isomorphism? Explain briefly.

v) Find a basis for the eigenspaces of T corresponding to each eigenvalue.

(i) $[T]_B = \left[[T(1)]_B \,\vdots\, [T(x)]_B \,\vdots\, [T(x^2)]_B \,\vdots\, [T(x^3)]_B \right]_{4\times 4}$

$T(1) = (x+2).(1)' = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$

$T(x) = (x+2).(x)' = 2 + x = 2.1 + 1.x + 0.x^2 + 0.x^3$

$T(x^2) = (x+2).(x^2)' = 4x + 2x^2 = 0.1 + 4.x + 2.x^2 + 0.x^3$

$T(x^3) = (x+2).(x^3)' = 6x^2 + 3x^3 = 0.1 + 0.x + 6.x^2 + 3.x^3$

$$[T]_B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

upper-triangular 4×4

(ii) $p_T(\lambda) = p_{[T]_B}(\lambda) = \det(\lambda I_4 - [T]_B) = \begin{vmatrix} \lambda & -2 & 0 & 0 \\ 0 & \lambda-1 & -4 & 0 \\ 0 & 0 & \lambda-2 & -6 \\ 0 & 0 & 0 & \lambda-3 \end{vmatrix} = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$

upper triangular

$\Rightarrow$ Eigenvalues of T are the roots of $p_T(\lambda)$: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 3$

(iii) T has $4\,(= \dim \mathbb{R}_3[x])$ distinct eigenvalues thus T is diagonalizable.

(iv) Note that $\det([T]_B) = 0.1.2.3 = 0 \Rightarrow T$ is not injective $\Rightarrow T$ is not an isomorphism.

(v) $E_{[T]_B}(3) = \text{Null}(3I_4 - [T]_B)$:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 3 & -2 & 0 & 0 & 0 \\ 0 & 2 & -4 & 0 & 0 \\ 0 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$\underset{\text{free}}{x_4}$

$\Rightarrow x_1 = \frac{2x_2}{3} \Rightarrow x_1 = 8t$

$\Rightarrow 2x_2 = 4x_3 \Rightarrow x_2 = 2x_3 \Rightarrow x_2 = 12t$

$\Rightarrow x_3 = 6x_4 \Rightarrow x_3 = 6t$

$x_4 = t,\ t \in \mathbb{R}$.

$$E_{[T]_B}(3) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8t \\ 12t \\ 6t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 8 \\ 12 \\ 6 \\ 1 \end{pmatrix} \right\rangle.$$

$B' = \{(8, 12, 6, 1)\}$ forms a basis for $E_{[T]_B}(3)$.

$\quad\ \ \| \\ (v)_B$ where $B = \{1, x, x^2, x^3\}$

$\Rightarrow B'' = [\,8.1 + 12.x + 6.x^2 + 1.x^3\,]$ forms a basis for $E_T(3)$.

Let's verify: $T(8 + 12x + 6x^2 + x^3) \overset{?}{=} 3(8 + 12x + 6x^2 + x^3)$

$\| $

$(x+2).(12 + 12x + 3x^2)$

$\|$

$24 + 36x + 18x^2 + 3x^3$ ✓

As an exercise, do the same for the rest of the eigenvalues.

Problem 8:

Let $B = [e_1, e_2, e_3, e_4]$ (old) and $B' = [v_1, v_2, v_3, v_4]$ (new) where $v_1 = (1,1,0,0)$, $v_2 = (0,0,1,1)$, $v_3 = (1,0,0,4)$ and $v_4 = (0,0,0,2)$ be bases for $\mathbb{R}^4$, where S is the standard basis.

a) Find the transition matrix $[Id]_{B',B}$.

b) Find $B'' = [u_1, u_2, u_3, u_4]$, which is an other basis for $\mathbb{R}^4$, if (Exercise, we did something very similar in Week 9).

$$[Id]_{B',B''} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

a) Recall: $[Id]_{\underbrace{B'}_{\text{codomain}},\underbrace{B}_{\text{domain}}}$: transition matrix from $\underbrace{B}_{\substack{\text{old}\\ \text{(domain)}}}$ to $\underbrace{B'}_{\substack{\text{new}\\ \text{(codomain)}}}$ (to find it, take the old basis vectors and represent them in terms of the new basis vectors).

$$[Id]_{B',B} = \Big[\, [e_1]_{B'} \,\vdots\, [e_2]_{B'} \,\vdots\, [e_3]_{B'} \,\vdots\, [e_4]_{B'} \Big]_{4\times 4}$$

$4 \times 4 = \dim(\mathbb{R}^4) = 4$

$e_1 = (1,0,0,0) = (1,0,0,4) - 2(0,0,0,2) = \underline{0}.v_1 + \underline{0}.v_2 + \underline{1}.v_3 \underline{-2}v_4$

$e_2 = (0,1,0,0) = (1,1,0,0) - (1,0,0,4) + 2(0,0,0,2) = \underline{1}.v_1 + \underline{0}.v_2 + \underline{(-1)}.v_3 + \underline{2}.v_4$

$e_3 = (0,0,1,0) = (0,0,1,1) - \frac{1}{2}(0,0,0,2) = \underline{0}.v_1 + \underline{1}.v_2 + \underline{0}.v_3 + \underline{(-\frac{1}{2})}v_4$

$e_4 = (0,0,0,1) = \frac{1}{2}(0,0,0,2) = \underline{0}.v_1 + \underline{0}.v_2 + \underline{0}.v_3 + \underline{(\frac{1}{2})}v_4$

$$\Rightarrow [Id]_{B',B} = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & -1/2 & 1/2 \end{bmatrix} \\ \begin{array}{cccc} [e_1]_{B'} & [e_2]_{B'} & [e_3]_{B'} & [e_4]_{B'} \end{array} \end{array}$$

How do we use $[Id]_{B',B}$? To relate $[v]_B$ and $[v]_{B'}$. We use it as follows:

$$\boxed{[v]_{B'} = [Id]_{B',B}\,[v]_B} \quad (*)$$

OR

$\boxed{\text{(Conversely): } [v]_B = [Id]_{B,B'}\,[v]_{B'}}$

i.e. if you're given $[Id]_{B',B}$ and $[v]_B$, you can find the coordinates of v relative to the new basis B' using $(*)$.

If $(v)_B = (2,-1,0,4) \Rightarrow [v]_{B'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 2 & -1/2 & 1/2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \end{bmatrix}}_{[v]_B} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ -4 \end{bmatrix}$

$\Downarrow$

$v = 2.e_1 + (-1).e_2 + 0.e_3 + 4.e_4$

$v = (2,-1,0,4)$

Let's verify!

$(v)_{B'} = (-1,0,3,-4) \Rightarrow$

$v = (-1).\underbrace{(1,1,0,0)}_{v_1} + 0.\underbrace{(0,0,1,1)}_{v_2} + 3.\underbrace{(1,0,0,4)}_{v_3} + (-4).\underbrace{(0,0,0,2)}_{v_4}$

$= (2,-1,0,4)$

Problem 9:

Given the matrix $A = \begin{pmatrix} 7 & 0 & 9 \\ 0 & 2 & 0 \\ 9 & 0 & 7 \end{pmatrix}$, find an orthogonal matrix Q such that

$Q^TAQ = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 16 \end{pmatrix}$. The eigenvalues of A are $-2, 2, 16$.

Form $\lambda I_3 - A = \begin{bmatrix} \lambda-7 & 0 & -9 \\ 0 & \lambda-2 & 0 \\ -9 & 0 & \lambda-7 \end{bmatrix}$

First find a basis for each eigenspace:

$\mathcal{E}_A(-2) = \text{Null}(-2I_3 - A)$: $\left[\begin{array}{ccc|c} -9 & 0 & -9 & 0 \\ 0 & -4 & 0 & 0 \\ -9 & 0 & -9 & 0 \end{array}\right] \xrightarrow{-R_1+R_3 \to R_3} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} -9 & 0 & -9 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & \underset{\text{free}}{0} & 0 \end{array}\right] \end{array}$

$\to -9x_1 = 9x_3 \Rightarrow x_1 = -x_3 = -t$

$\to -4x_2 = 0 \Rightarrow x_2 = 0$

$x_3 = t, t \in \mathbb{R}$

$\mathcal{E}_A(-2) = \{(x_1,x_2,x_3) = (-t,0,t) : t \in \mathbb{R}\} = \langle(-1,0,1)\rangle \Rightarrow \boxed{B_1 = [(-1,0,1)] \text{ forms a basis of } \mathcal{E}_A(-2).}$

$\mathcal{E}_A(2) = \text{Null}(2I_3 - A)$: $\left[\begin{array}{ccc|c} -5 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 \\ -9 & 0 & -5 & 0 \end{array}\right] \xrightarrow{-9/5 R_1+R_3 \to R_3} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} -5 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \underset{\text{free}}{0} & 56/5 & 0 \end{array}\right] \end{array}$

$\to -5x_1 = 9x_3 \Rightarrow x_1 = 0$

$x_2 = t, t \in \mathbb{R}$

$\to \frac{56}{5}x_3 = 0 \Rightarrow x_3 = 0$

$\mathcal{E}_A(2) = \{(x_1,x_2,x_3) = (0,t,0) : t \in \mathbb{R}\} = \langle(0,1,0)\rangle \Rightarrow \boxed{B_2 = [(0,1,0)] \text{ forms a basis of } \mathcal{E}_A(2).}$

$\mathcal{E}_A(16) = \text{Null}(16I_3 - A)$: $\left[\begin{array}{ccc|c} 9 & 0 & -9 & 0 \\ 0 & 14 & 0 & 0 \\ -9 & 0 & 9 & 0 \end{array}\right] \xrightarrow{R_1+R_3 \to R_3} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 9 & 0 & -9 & 0 \\ 0 & 14 & 0 & 0 \\ 0 & 0 & \underset{\text{free}}{0} & 0 \end{array}\right] \end{array}$

$\to 9x_1 = 9x \Rightarrow x_1 = x_3 = t$

$\to 14x_2 = 0 \Rightarrow x_2 = 0$

$x_3 = t, t \in \mathbb{R}$

$\mathcal{E}_A(16) = \{(x_1,x_2,x_3) = (t,0,t) : t \in \mathbb{R}\} = \langle(1,0,1)\rangle \Rightarrow \boxed{B_3 = [(1,0,1)] \text{ forms a basis of } \mathcal{E}_A(16).}$

Now collect B_1, B_2 and B_3 and form $B = B_1 \cup B_2 \cup B_3$

$\boxed{B = [\underset{v_1}{(-1,0,1)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(1,0,1)}]}$

$v_1 \in \mathcal{E}_A(-2)$, $v_2 \in \mathcal{E}_A(2)$, $v_3 \in \mathcal{E}_A(16)$

Check that $v_1 \cdot v_2 = (-1,0,1)\cdot(0,1,0) = 0$

$v_1 \cdot v_3 = (-1,0,1)\cdot(1,0,1) = -1+1 = 0$

$v_2 \cdot v_3 = (0,1,0)\cdot(1,0,1) = 0$

$\Rightarrow$ B is a set of nonzero orthogonal vectors $\Rightarrow B$ is L.I $\Rightarrow B$ forms an orthogonal basis of $\mathbb{R}^3$.

Transform B into an orthonormal basis B' of $\mathbb{R}^3$ by normalizing the vectors.

$\|v_1\| = \sqrt{2}$, $\|v_2\| = 1$, $\|v_3\| = \sqrt{2}$ $\Rightarrow B' = \left[\underset{w_1}{\left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}, \underset{w_2}{(0,1,0)}, \underset{w_3}{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}\right]$ forms an orthonormal basis of $\mathbb{R}^3$ (*)

Form $Q = \left[\begin{array}{c|c|c} w_1 & w_2 & w_3 \end{array}\right] = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$ Q is an orthogonal matrix by (*).

Finally $\underset{Q^{-1}}{Q^T}AQ = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 16 \end{bmatrix} \Leftrightarrow AQ = Q\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 16 \end{bmatrix}$

$AQ = \left[\begin{array}{c|c|c} Aw_1 & Aw_2 & Aw_3 \end{array}\right] = \left[\begin{array}{c|c|c} -2w_1 & 2w_2 & 16w_3 \end{array}\right]$ ✓

$Aw_1 = -2w_1$, $Aw_2 = 2w_2$, $Aw_3 = 16w_3$

Since $w_1 \in \mathcal{E}_A(-2)$

$w_2 \in \mathcal{E}_A(2)$

$w_3 \in \mathcal{E}_A(16)$

Problem 10:

Decide if the statements below are true or false. (For true or false problems you either justify(prove) that the statement is true or you show by a counter-example or by logical reasoning that it is false.)

1) If A and B are $n \times n$ similar matrices, then $\det A = \det B$.

True. If A and B are $n\times n$ similar matrices, then there exists an invertible $P \in M_{n\times n}(\mathbb{R})$ invertible such that $B = P^{-1}AP$. Take the det. of both sides:

$\Rightarrow \det(B) = \det(P^{-1}AP) = \det(P^{-1}).\det(A).\det(P)$

$= \frac{1}{\cancel{\det} P}.\det(A).\cancel{\det}(P) = \det(A).$

2) Given 3 linearly independent vectors in $M_{2\times 2}(\mathbb{R})$, those vectors will form a basis for $M_{2\times 2}(\mathbb{R})$.

False: Recall: $\dim(M_{2\times 2}(\mathbb{R})) = 4$. That is, every basis of $M_{2\times 2}(\mathbb{R})$ must contain exactly 4 elements. So, a "3 element set" can not form a basis for $M_{2\times 2}(\mathbb{R})$. (Simply it will not span $M_{2\times 2}(\mathbb{R})$).

3) If 0 is an eigenvalue of a given matrix A, then $dim(Null(A)) > 0$.

True. 0 is an eigenvalue of $A \in M_{n\times n}(\mathbb{R})$ $\Rightarrow A.x = 0.x$ for some nonzero $x \in \mathbb{R}^n$

$\Rightarrow A.x = 0$ " " ".

$\Rightarrow A$ is not invertible.

$\Rightarrow \text{rank } A < n \Rightarrow \dim(\text{Null}(A)) = n - \text{rank } A > 0.$

4) If the matrix A has the characteristic polynomial $\lambda^3 - \lambda$, then A is diagonalizable.

True. Note that $P_A(\lambda) = \lambda^3 - \lambda \Rightarrow A \in M_{3\times 3}(\mathbb{R})$

$P_A(\lambda) = \lambda(\lambda^2 - 1) = \lambda(\lambda-1)(\lambda+1) \Rightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1$ are the eigenvalues of A.

$A \in M_{3\times 3}(\mathbb{R})$ has $\underline{\underline{3}}$ $\underline{\underline{\text{distinct}}}$ eigenvalues $\Rightarrow A$ is diagonalizable.

5) If $A \in M_{3\times 3}(\mathbb{R})$ is diagonalizable, then A has three distinct eigenvalues.

False. Counter-example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ which is clearly diagonalizable. However, A has a repeated eigenvalues ($AM(2) = 3$).

6) If v is orthogonal to every vector of a subspace W, then $v = 0$.

False. Counter-example: Let $W = \langle(1,0)\rangle = \{(a,0) : a \in \mathbb{R}\}$ – x-axis.

Let $v = (0,2)$. Note that $(0,2)\cdot(1,0) = 0 \xRightarrow{\text{Theorem 4 of Week 13}} (0,2)\cdot(a,0) = 0$ for any $a \in \mathbb{R}$.

$\Rightarrow v = (0,2)$ is orthogonal to every vector in W.

Clearly, $v = (0,2) \neq (0,0)$.

False 7) If A is a 5×4 matrix with rank 3, then $Ax = 0$ has only the trivial solution.

Given $A \in M_{5\times 4}(\mathbb{R}) \Rightarrow \text{rank}(A) + \dim(\text{Null}(A)) = 4$ (= # of columns in A)

If $\text{rank}(A) = 3 \Rightarrow \dim(\text{Null}(A)) = 1$

$\Rightarrow$ Null(A) has infinitely many elements.

$\Rightarrow Ax = 0$ has infinitely many solutions.

False 8) The column vectors of a 3×4 matrix can be linearly independent.

$\boxed{A_{3\times 4}} \Rightarrow$ Let C_1, C_2, C_3, C_4 denote the columns of A. Note that each $C_i \in \mathbb{R}^3$. Recall: $\dim(\mathbb{R}^3) = 3$.

$S = \{C_1, C_2, C_3, C_4\}$ cannot be linearly independent since $\underbrace{|S|}_{\text{\# of elements in } S} = 4 > 3 = \dim(\mathbb{R}^3)$

9) If A is a square matrix, then AA^T and A^TA are orthogonally diagonalizable.

True. $(AA^T)^T = (A^T)^T A^T = AA^T \Rightarrow AA^T$ is symmetric

$(A^TA)^T = A^T(A^T)^T = A^TA \Rightarrow A^TA$ is symmetric.

By Theorem 10 of Week 13, AA^T and A^TA are orthogonally diagonalizable.

10) The vector space $Span((1,1,1),(1,0,1),(0,1,0))$ is isomorphic to the vector space $\mathbb{R}_2[x]$. Fact: Isomorphic v.s.'s must have the same dimension.

False. We know that $\dim(\mathbb{R}_2[x]) = 3$. Let's find the dimension of W.

$W = \langle (1,1,1), (1,0,1), \underbrace{(0,1,0)}_{\text{redundant}} \rangle$ since $(1,1,1) - (1,0,1) = (0,1,0)$

$\Rightarrow W = \langle \underbrace{(1,1,1), (1,0,1)}_{\text{L.I.}} \rangle \Rightarrow B = [(1,1,1),(1,0,1)]$ forms a basis for $W \Rightarrow \dim W = 2$.

Since $\dim W \neq \dim \mathbb{R}_2[x] \Rightarrow$ they can NOT be isomorphic.

11) If A and B are row-equivalent matrices, then their determinants are equal.

False. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. A and B are row-equivalent.

However, $\det A = 1$ and $\det B = -1$. $\Rightarrow \det A \neq \det B$.