MATH 201 FINAL EXAM

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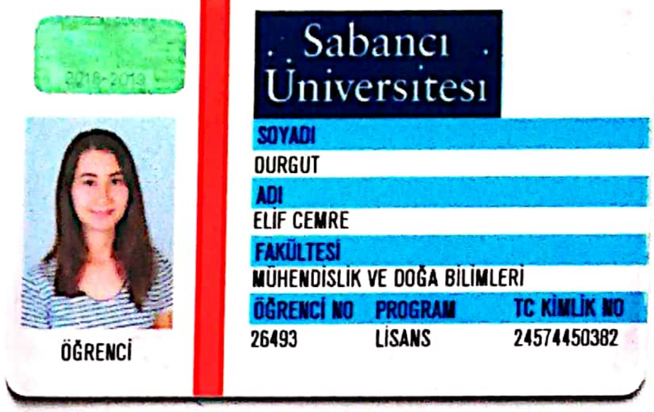

Problem 1

a) $\dim \mathbb{R}^3 = 3$ and $|S| = 3$

$\Rightarrow$ S will form a basis for $\mathbb{R}^3 \Leftrightarrow$ S is linearly independent.

$\Leftrightarrow \boxed{c_1v_1 + c_2v_2 + c_3v_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0}$

$\Rightarrow c_1(t,-3,2) + c_2(0,1,1) + c_3(4,1,t) = 0$

$$\begin{bmatrix} t & 0 & 4 \\ -3 & 1 & 1 \\ 2 & 1 & t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It has only the trivial solution if and only if A is invertible ($Ax=0$) ($\det A \neq 0$)

$\det A = \begin{vmatrix} t & 0 & 4 \\ -3 & 1 & 1 \\ 2 & 1 & t \end{vmatrix}$ expansion along the 2^{nd} column

$$= \underbrace{\cancel{a_{12}C_{12}}}_{0} + \underbrace{a_{22}}_{1}C_{22} + \underbrace{a_{32}}_{1}C_{32} = (-1)^{2+2}\begin{vmatrix} t & 4 \\ 2 & t \end{vmatrix} + (-1)^{3+2}\begin{vmatrix} t & 4 \\ -3 & 1 \end{vmatrix}$$

$$= t^2 - 8 + (-1)(t+12) = t^2 - t - 20 \neq 0$$

$(t-5)(t+4) \neq 0$

$t \neq 5 \quad t \neq -4$

Conclusion: S will form a basis of $\mathbb{R}^3$ if and only if $t \in \mathbb{R} - \{-4,5\}$ //

b) By part a), $\dim(W) = \dim(\mathbb{R}^3) = 3$ if $t \in \mathbb{R} - \{-4,5\}$

If $t = -4 \Rightarrow W = \langle v_1, v_2, v_3 \rangle = \langle (-4,-3,2), (0,1,1), (4,1,-4) \rangle = \langle \overbrace{(-4,-3,2), (4,1,-4)}^{L.I.} \rangle$

$(-4,-3,2) + (4,1,-4) = -2(0,1,1) \Rightarrow (0,1,1)$ redundant

$\Rightarrow W = \langle (-4,-3,2), (4,1,-4) \rangle$ forms a basis for $W \Rightarrow \dim W = 2$ if $t = -4$

If $t = 5 \Rightarrow W = \langle v_1, v_2, v_3 \rangle = \langle (5,-3,2), (0,1,1), (4,1,5) \rangle = \langle (5,-3,2), (4,1,5) \rangle$

$-4(5,-3,2) + 5(4,1,5) = 17(0,1,1) \Rightarrow (0,1,1)$ is redundant

$\Rightarrow W = \langle (5,-3,2), (4,1,5) \rangle$ forms a basis for $W \Rightarrow \dim W = 2$ if $t = 5$

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Problem 2

a)

$$\text{Standard basis } B \text{ of } M_{2\times 2}(\mathbb{R}) = \left[\underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right]$$

$$[T]_B = \left[[T(v_1)]_B \,\middle|\, [T(v_2)]_B \,\middle|\, [T(v_3)]_B \,\middle|\, [T(v_4)] \right] = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 4 & -7 & 0 \\ 0 & -7 & 4 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}_{4\times 4}$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} = -3v_1 + 0\,v_2 + 0\,v_3 + 0\,v_4$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -7 & 0 \end{bmatrix} = 0.v_1 + 4.v_2 - 7.v_3 + 0.v_4$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 4 & 0 \end{bmatrix} = 0.v_1 - 7.v_2 + 4.v_3 + 0.v_4$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} = 0.v_1 + 0.v_2 + 0.v_3 - 3.v_4$$

b)

$$P_T(\lambda) = P_{[T]_B}(\lambda) = \det(\lambda I_4 - [T]_B) = \begin{vmatrix} \lambda+3 & 0 & 0 & 0 \\ 0 & \lambda-4 & 7 & 0 \\ 0 & 7 & \lambda-4 & 0 \\ 0 & 0 & 0 & \lambda+3 \end{vmatrix}$$

expansion along the 1st row

$$= a_{11}\cdot C_{11} + a_{12}\cancel{C_{12}} + a_{13}\cancel{C_{13}} + a_{14}\cancel{C_{14}} = (\lambda+3)(-1)^{1+1}\cdot \underbrace{\begin{vmatrix} \lambda-4 & 7 & 0 \\ 7 & \lambda-4 & 0 \\ 0 & 0 & \lambda+3 \end{vmatrix}}$$

expansion along the 3rd row

$$= (\lambda+3)(-1)^{3+3}\cdot \begin{vmatrix} \lambda-4 & 7 \\ 7 & \lambda-4 \end{vmatrix}$$

$$= (\lambda+3)(\lambda^2 - 8\lambda - 33)$$

$$P_T(\lambda) = (\lambda+3)^2 \overset{\det<0}{(\lambda^2 - 8\lambda - 33)} = 0 \quad \Leftarrow$$

$\hookrightarrow \lambda = -3 \rightarrow AM(-3) = 2$

c) $P_T(\lambda) = P_{[T]_B}(\lambda)$ where B is any basis of V. Because characteristic polynomial is independent of the choice of basis B of V.

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$$\lambda I_3 - A = \begin{bmatrix} \lambda-5 & 0 & -2 \\ 0 & \lambda-4 & 0 \\ -2 & 0 & \lambda-5 \end{bmatrix}$$

Problem 3

a) For $\lambda_1 = 3$, we solve $(3I_3 - A)x = 0$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \end{array} \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \xrightarrow{2R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$$

free

$x_1 + x_3 = 0 \to x_1 = -x_3$

$-x_2 = 0$

$x_2 = 0$

$x_3 = t, \; t \in \mathbb{R}$

$$E_A(3) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}, t \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$B_1 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right]$ forms a basis for $E_A(3)$

For $\lambda_2 = 4$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_3 \to R_3} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \end{array} \to$$

free

$-x_1 - 2x_3 = 0 \to x_1 = 0$

$x_2 = t, \; t \in \mathbb{R}$

$3x_3 = 0 \to x_3 = 0$

$$E_A(4) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$B_2 = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$ forms a basis for $E_A(4)$

For $\lambda_3 = 7$

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 3 & 0 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$$

free

$2x_1 - 2x_3 = 0 \to x_1 = t$

$3x_2 = 0$

$x_2 = 0 \quad x_3 = t, \; t \in \mathbb{R}$

$$E_A(7) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}, t \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$B_3 = \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$ forms a basis for $E_A(7)$.

b) $B' = B_1 \cup B_2 \cup B_3 = [(-1,0,1), (0,1,0), (1,0,1)] \to$ Linearly independent

$\hookrightarrow B'$ forms a basis of $\mathbb{R}^3$

pairwise $v_i \cdot v_j$ should be zero to be orthonormal:

$v_1 \cdot v_2 = (-1,0,1) \cdot (0,1,0) = 0$ ✓

$v_1 \cdot v_3 = (-1,0,1) \cdot (1,0,1) = 0$ ✓ $\to$ If we normalize them $\|v_i\| = 1$ $\to$ Then it is orthonormal ✓

$v_2 \cdot v_3 = (0,1,0) \cdot (1,0,1) = 0$ ✓

c) $$[T]_{B'} = \Big[\underbrace{[T(v_1)]_{B'}}_{3v_1} \,\Big|\, \underbrace{[T(v_2)]_{B'}}_{4v_2} \,\Big|\, \underbrace{[T(v_3)]_{B'}}_{7v_3} \Big] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix} //$$

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Problem 4

"$T_2 \circ T_1$ is injective $\Leftrightarrow$ $[T_2 \circ T_1]_B$ is invertible

$$[T_2 \circ T_1]_B = [T_2]_{B', B} \cdot [T_1]_{B, B'}$$