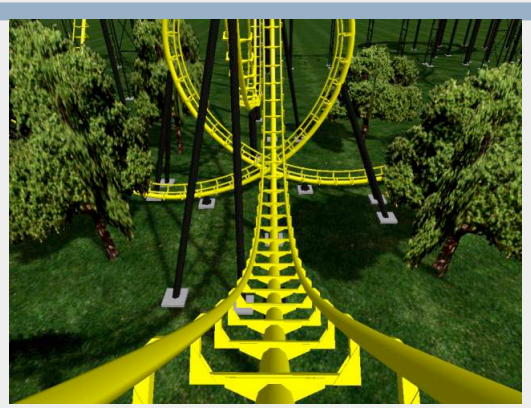
#15: Final Project: Roller Coaster!

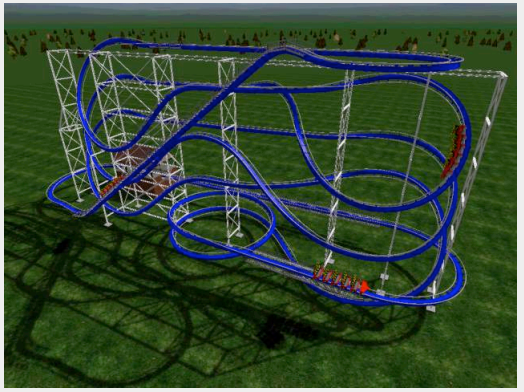
CSE167: Computer Graphics

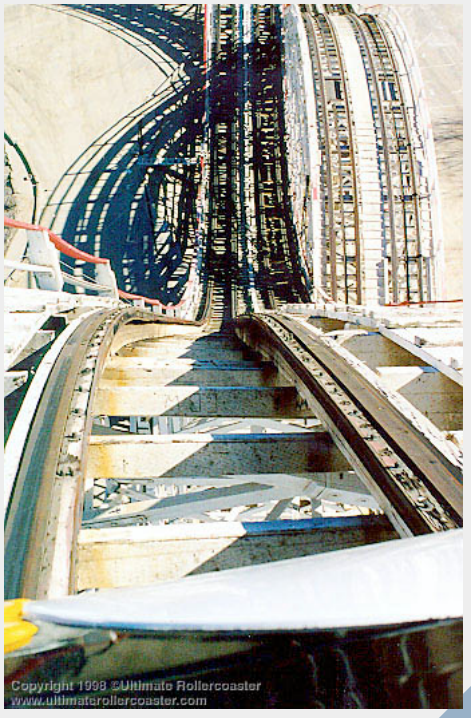
Instructor: Ronen Barzel

UCSD, Winter 2006

Roller coasters…







Final project

- Build a roller coaster, with a car riding on it
- More open-ended than previous projects
- Be creative!
  - We do have some specific capabilities/features we'll look for
  - But we'll be impressed by "style" too!

Final project

- More on your own than previous projects
- No “base code”
  - though you may use base code from previous assignments
- Today I’ll go over the key techniques

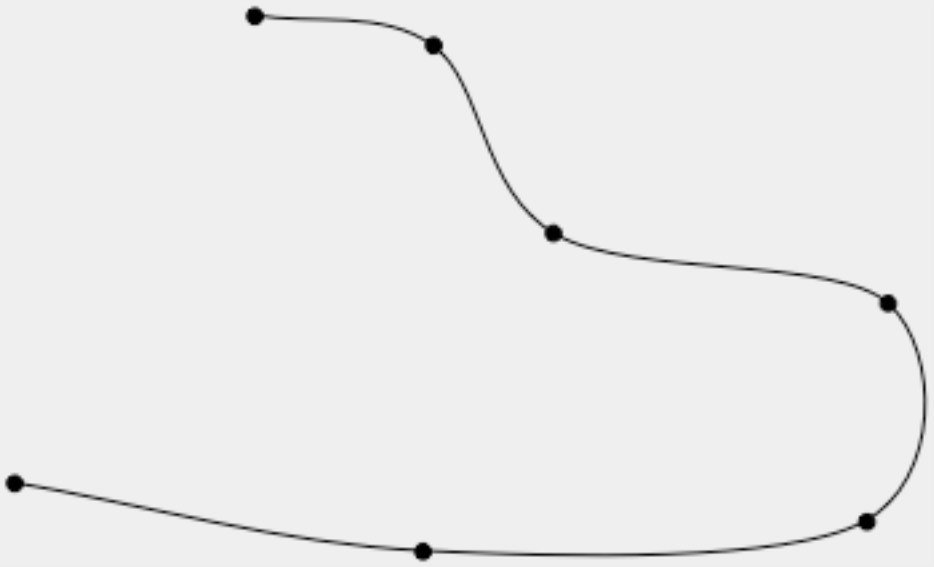
Roller coaster steps

(see web page for details)

1. Make a piecewise-cubic curve
2. Create coordinate frames along a piecewise-cubic curve
   - will be used to orient along the path
   - include a roll angle at each curve point, to adjust tilt
3. Create a swept surface
   - the actual track along the of the roller coaster
4. Design and build the roller coaster track
5. Run a car on the track

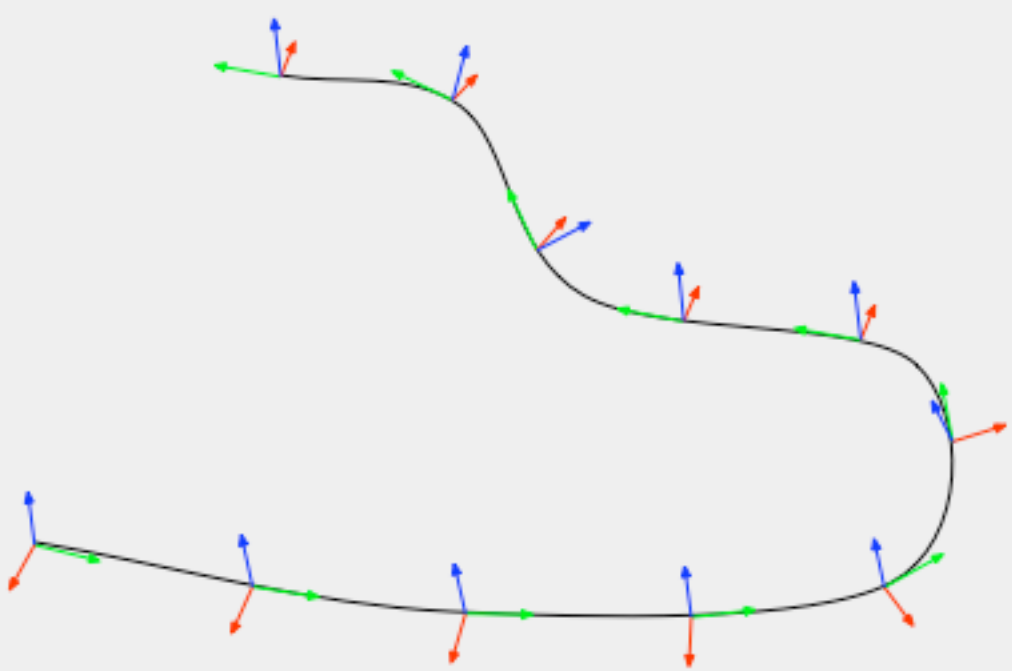
Step 1. Piecewise-Cubic Curve

- Specify as a series of points
- Will be used for the path of the roller coaster



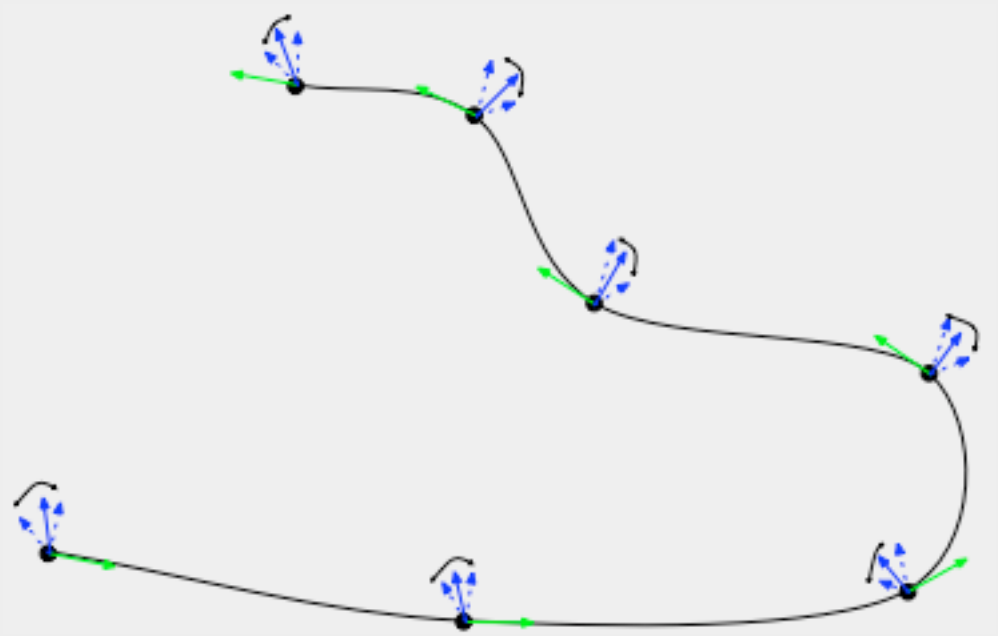
Step 2. Coordinate Frames on curve

- Describes orientation along the path



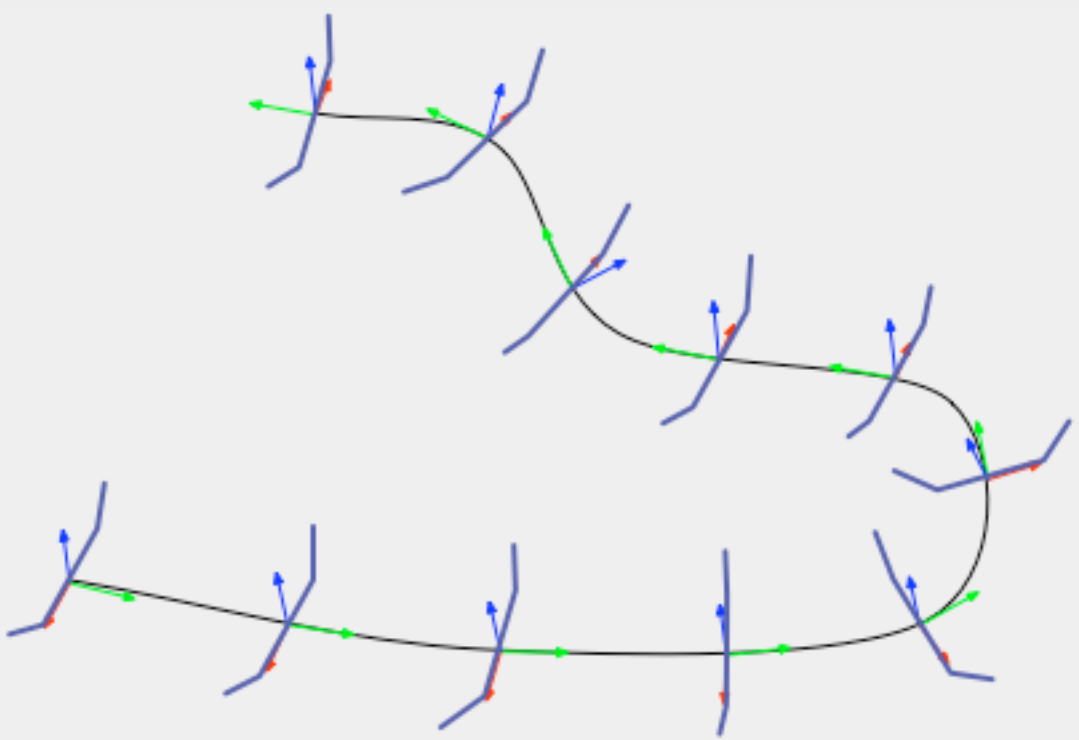
Step 2b. Tweak coordinate frames

- Control lean of coordinate frames
- Specify “roll” angle offset at each control point



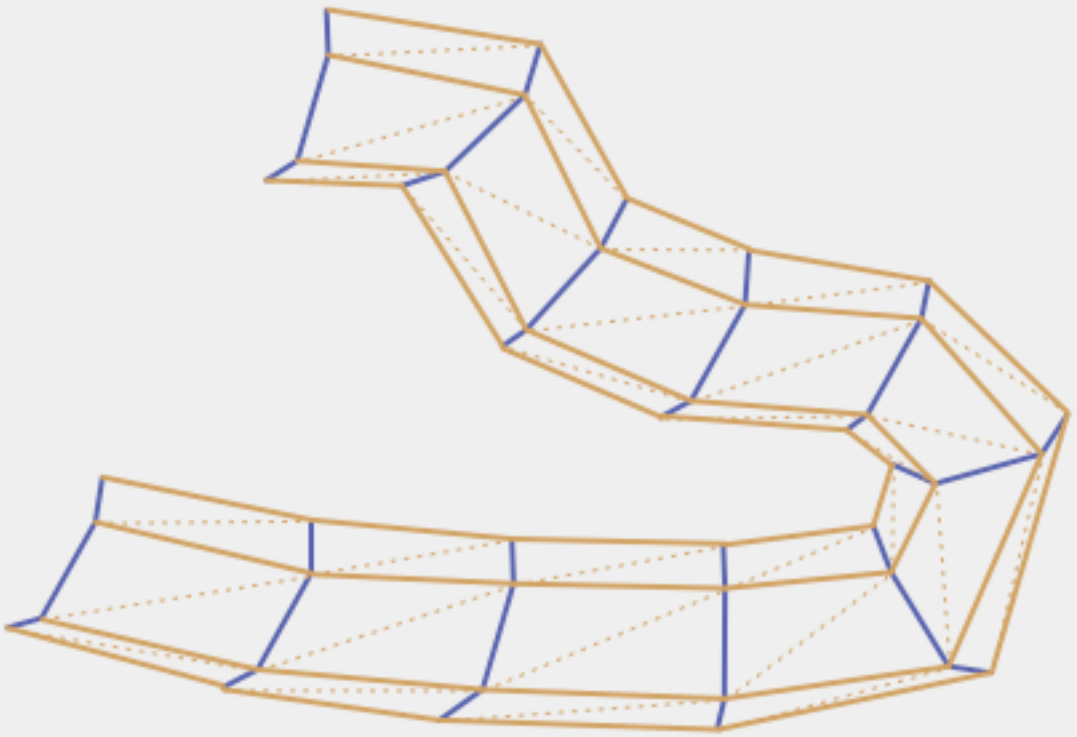
Step 3a. Sweep a cross section

- Define a cross section curve (piecewise-linear is OK)
- Sweep it along the path, using the frames

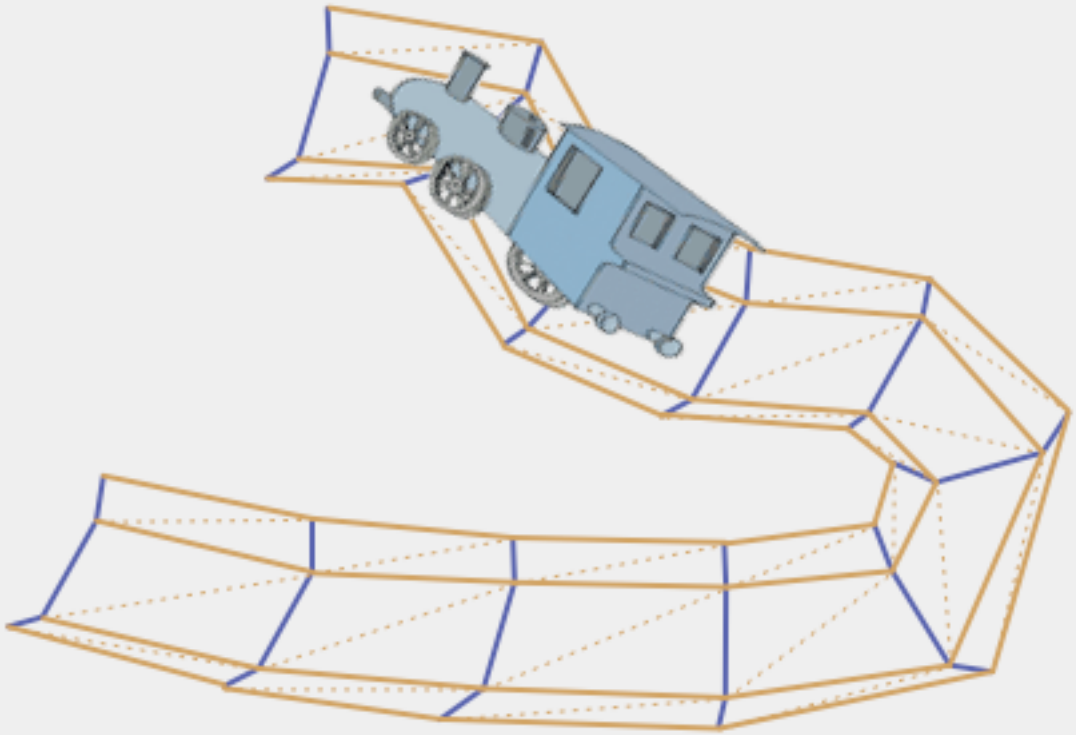


3b. Tessellate

- Construct triangles using the swept points
- (sweep more finely than this drawing, so it'll be smoother)

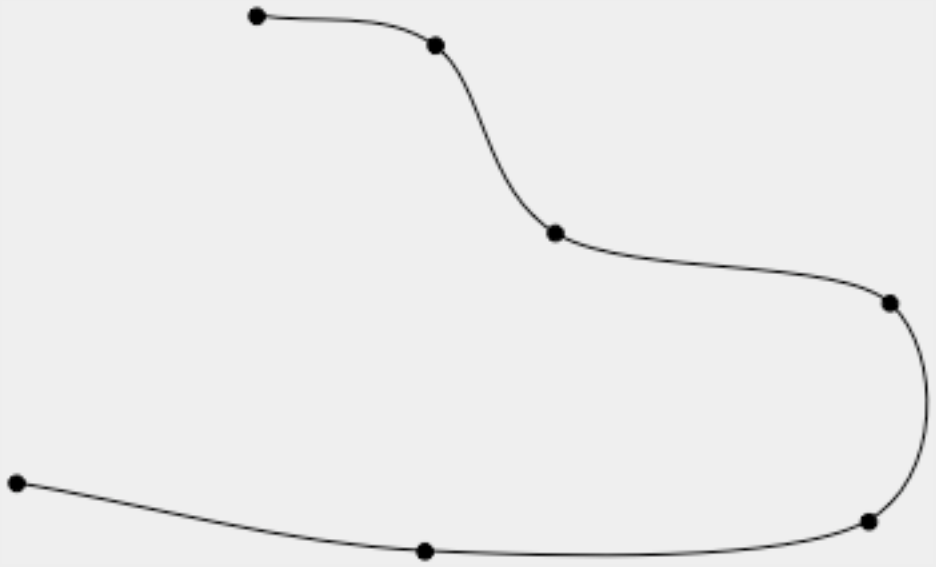


4. Run a car on the track



Step 1. Piecewise-Cubic Curve

- Specify as a series of points
- Will be used for the path of the roller coaster



Piecewise-Cubic Curve

- Requirements:
  - Given an array of N points
  - Be able to evaluate curve for any value of t (in 0…1)
  - Curve should be C1-continuous

- Best approach: define a class, something like

```
class Curve {
  Curve(int Npoints, Point3 points[]);
  Point3 Eval(float t);
}
```

Piecwise-Cubic Curves

- Three recommended best choices:
  - Bézier segments
    - Advantages: You did a Bézier segment in Project 5
    - Disadvantages: Some work to get C1 continuity
      - Will discuss technique to do this
  - B-Spline
    - Advantages: simple, always C1 continuous
    - Disadvantages: New thing to implement, doesn't interpolate
  - Catmull-Rom spline
    - Advantages: interpolates points
    - Disadvantages: Less predictable than B-Spline. New thing to implement

Piecewise Bézier curve review

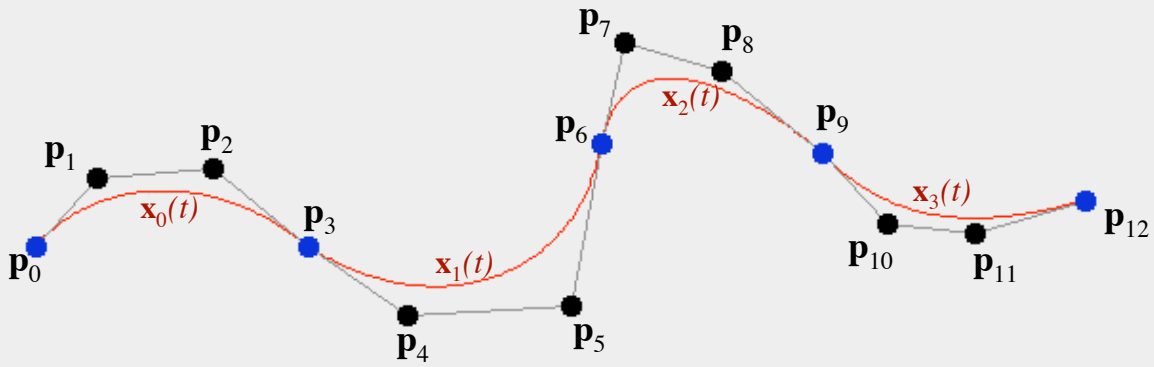
- We have a Bézier segment function $Bez(t,\mathbf{p}_0,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$
  - Given four points
  - Evaluates a Bézier segment for t in $0\ldots1$
  - Implemented in Project 5

- Now define a piecewise Bézier curve $x(u)$
  - Given $3N+1$ points $\mathbf{p}_0,\ldots\ \mathbf{p}_{3N}$
  - Evaluates entire curve for u in $0\ldots1$
  - Note: In class 8, defined curve for u in $0\ldots N$
    - Today will define it for u in $0\ldots1$

Piecewise Bézier curve: segments

- Given $3N+1$ points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{3N}$
- Define N Bézier segments:

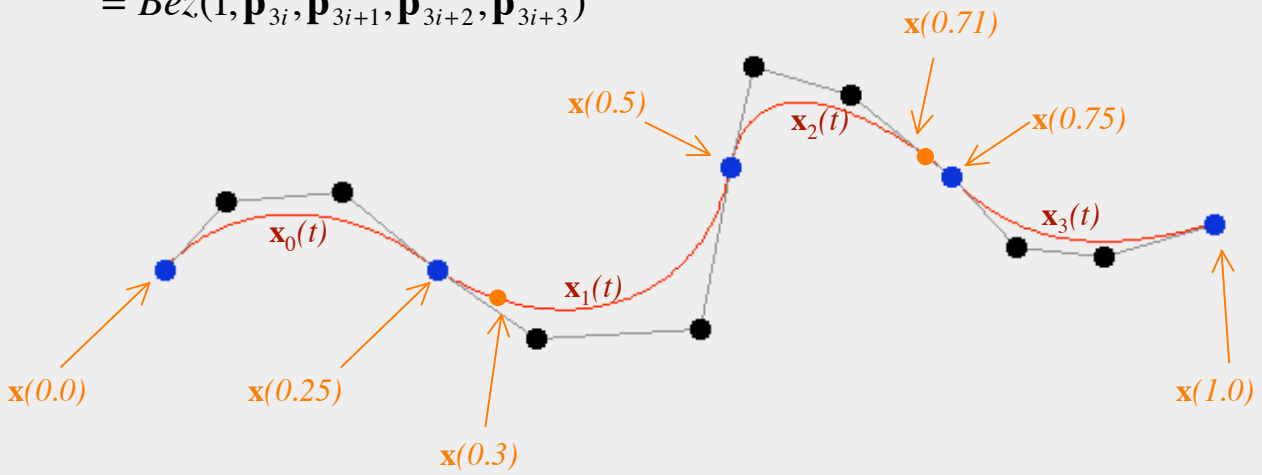
$$\mathbf{x}_0(t) = Bez(t, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$$
$$\mathbf{x}_1(t) = Bez(t, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6)$$
$$\vdots$$
$$\mathbf{x}_{N-1}(t) = Bez(t, \mathbf{p}_{3N-3}, \mathbf{p}_{3N-2}, \mathbf{p}_{3N-1}, \mathbf{p}_{3N})$$



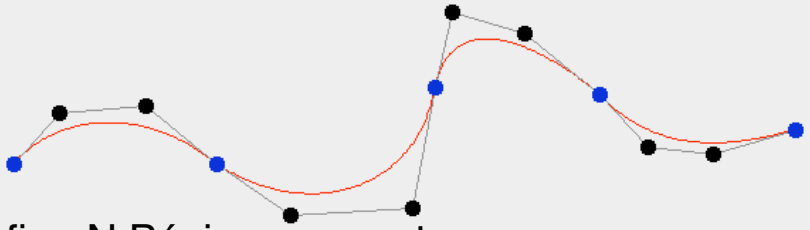
Piecewise Bézier curve

$$\mathbf{x}(u) = \begin{cases} \mathbf{x}_0(Nu), & 0 \le u \le \frac{1}{N} \\ \mathbf{x}_1(Nu-1), & \frac{1}{N} \le u \le \frac{2}{N} \\ \vdots & \vdots \\ \mathbf{x}_{N-1}(Nu-(N-1)), & \frac{N-1}{N} \le u \le 1 \end{cases}$$

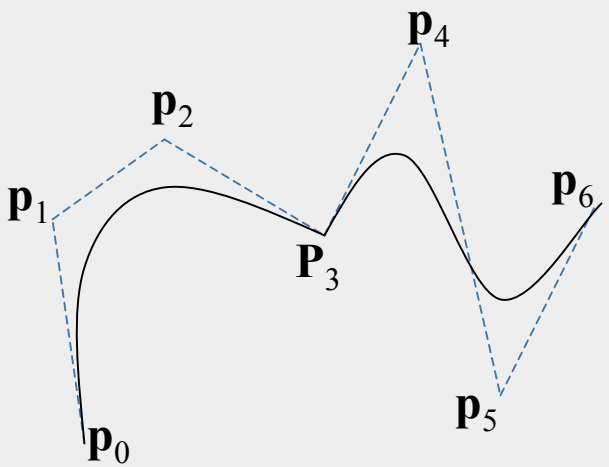
$$\begin{aligned} \mathbf{x}(u) &= \mathbf{x}_i\left(Nu-i\right), \text{ where } i = \lfloor Nu \rfloor, u < 1; \\ &= Bez(Nu-i, \mathbf{p}_{3i}, \mathbf{p}_{3i+1}, \mathbf{p}_{3i+2}, \mathbf{p}_{3i+3}) \\ \mathbf{x}(1) &= \mathbf{p}_{3N} \\ &= Bez(1, \mathbf{p}_{3i}, \mathbf{p}_{3i+1}, \mathbf{p}_{3i+2}, \mathbf{p}_{3i+3}) \end{aligned}$$



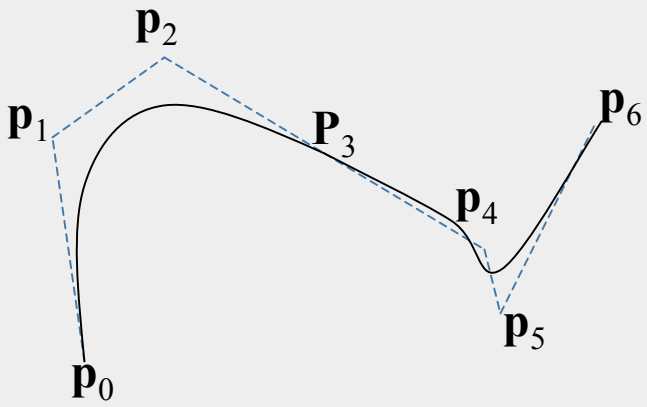
Piecewise Bézier curve



- 3N+1 points define N Bézier segments
- $\mathbf{x}(i/N)=\mathbf{p}_{3i}$
- C^0 continuous by construction
- G^1 continuous at $\mathbf{p}_{3i}$ when $\mathbf{p}_{3i-1}$, $\mathbf{p}_{3i}$, $\mathbf{p}_{3i+1}$ are collinear
- C^1 continuous at $\mathbf{p}_{3i}$ when $\mathbf{p}_{3i}-\mathbf{p}_{3i-1} = \mathbf{p}_{3i+1}-\mathbf{p}_{3i}$



C^1 discontinuous



C^1 continuous

Tangent to piecewise Bézier curve

- Tangent to the piecewise curve
  - same as the tangent to each segment (from Project 5)
    - mathematically speaking, needs to be scaled
    - in practice we often normalize anyway
  - At the shared points, the tangents on the two sides might not agree
    - Unless we arrange for C1 continuity

$$\left.\begin{aligned}\mathbf{x}(u) &= Bez(t, \mathbf{p}_{3i}, \mathbf{p}_{3i+1}, \mathbf{p}_{3i+2}, \mathbf{p}_{3i+3}) \\ \vec{\mathbf{x}}'(u) &= N \;\; Bez'(t, \mathbf{p}_{3i}, \mathbf{p}_{3i+1}, \mathbf{p}_{3i+2}, \mathbf{p}_{3i+3})\end{aligned}\right\}$$

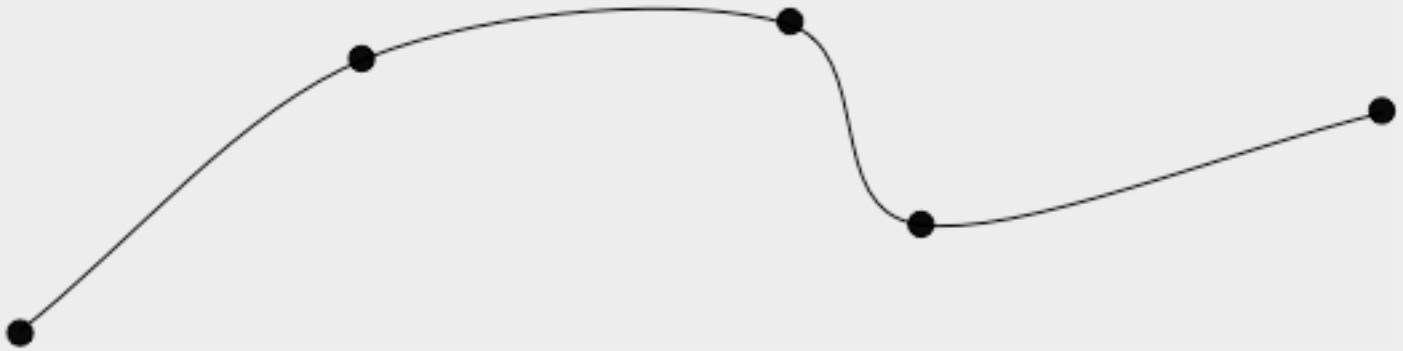
$$\text{where } \begin{cases} u < 1: & i = \lfloor Nu \rfloor \text{ and } t = Nu - i \\ u = 1: & i = N - 1 \text{ and } t = 1 \end{cases}$$

Piecewise Bézier curves

- Inconveniences:
  - Must have 4 or 7 or 10 or 13 or … (1 plus a multiple of 3) points
  - Not all points are the same
  - Have to fiddle to keep curve C1-continuous

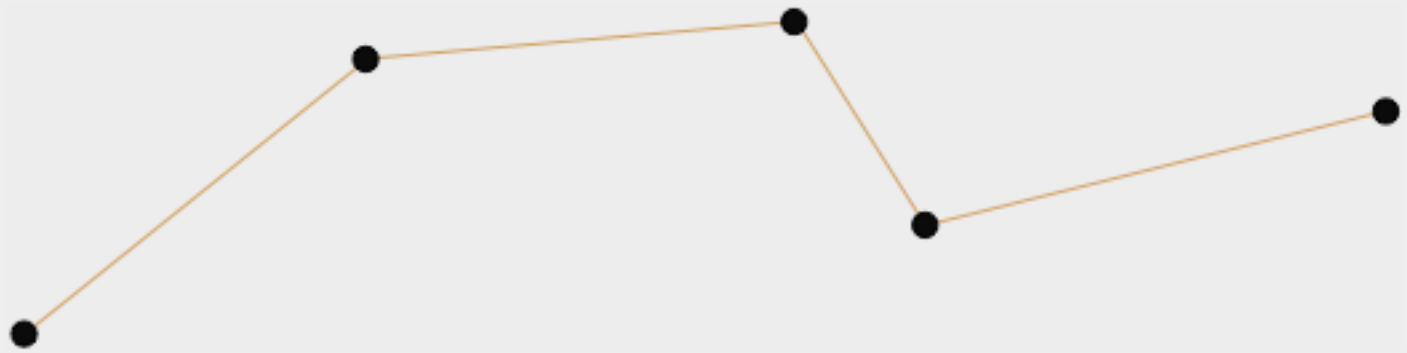
Making a C1 Bézier curve

- A hack to construct a C1-continuous Bézier curve
  - Actually, this is essentially implementing Catmull-Rom splines
  - Instead, could just implement Catmull-Rom splines directly
    - Same amount of work as implementing B-Spline
- Given M points:
  - want a piecewise-Bézier curve that interpolates them
  - Bézier segments interpolate their endpoints
  - Need to construct the intermediate points for the segments



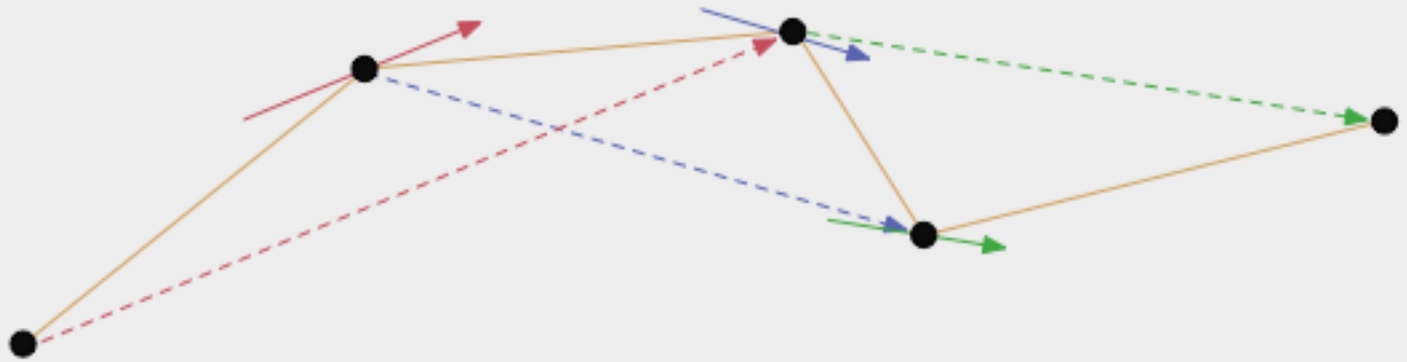
Auto-creation of intermediate points

- Given M original points
  - Will have M-1 segments
  - Will have N=3*(M-1)+1 final points



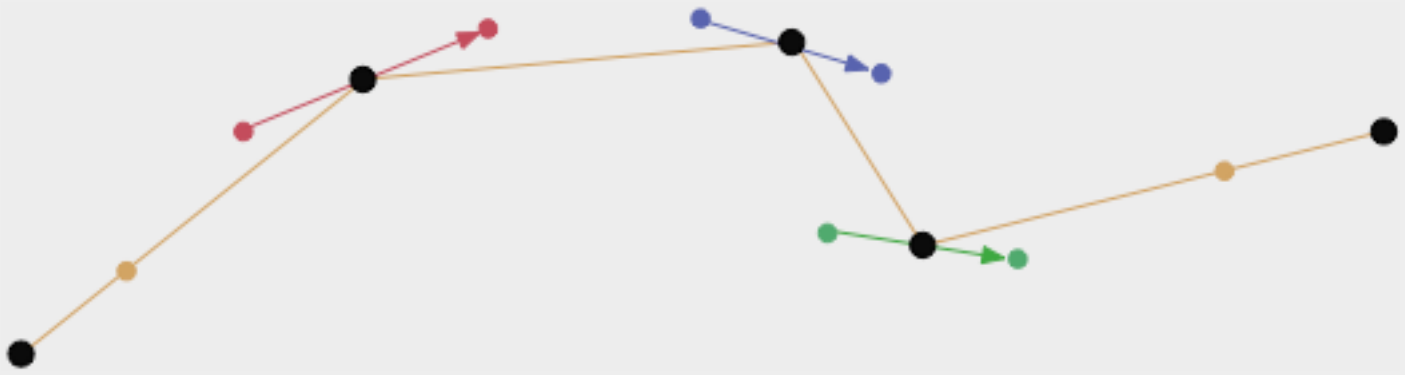
Auto-creation of intermediate points

- Need tangent directions at each original point
  - Use direction from previous point to next point



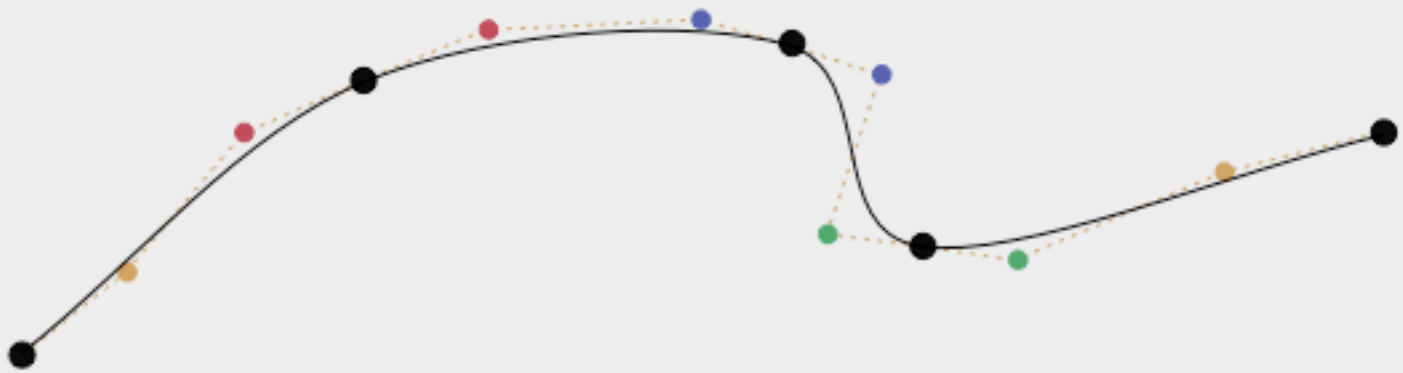
Auto-creation of intermediate points

- Introduce intermediate points
  - Move off each original point in the direction of the tangent
  - “Handle” length = 1/3 distance between previous and next points
  - special-case the end segments.
    - or for closed loop curve, wrap around



Auto-creation of intermediate points

- Resulting curve:



Auto-creation of intermediate points

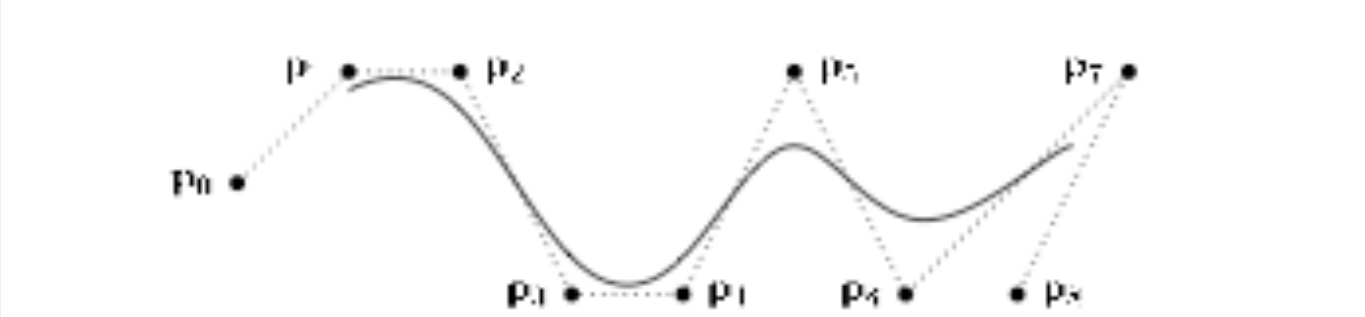
- Summary of algorithm:

```
Given M originalPoints
Allocate newPoints array, N = 3*(M-1)+1 new points
for each original point i
   p = originalPoints[i]
   v = originalPoints[i+1]-originalPoints[i-1]
   newPoints[3*i-1] = p-v/6
   newPoints[3*i] = p
   newPoints[3*i-1] = p+v/6
```

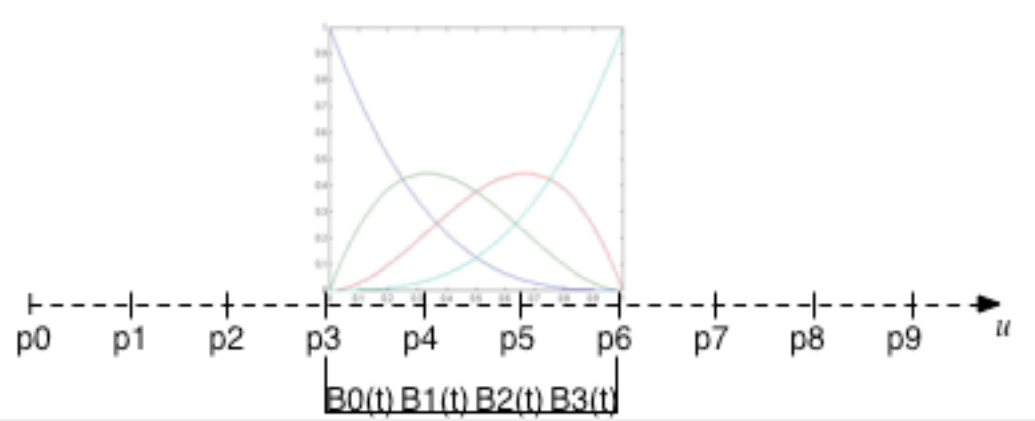
- Of course, must be careful about start and end of loop
- Can allow the curve to be a closed loop: wrap around the ends

Other option… B-spline

- Need at least 4 points, i.e. N+3 points
- B-spline curve uses given N+3 points as is
  - Always C2 continuous
    - no intermediate points needed
  - Approximates rather than interpolates
    - doesn't reach the endpoints
    - not a problem for closed loops: wrap around.
  - simple algorithm to compute
    - (may need to extend matrix library to support Vector4)



Blending Functions Review

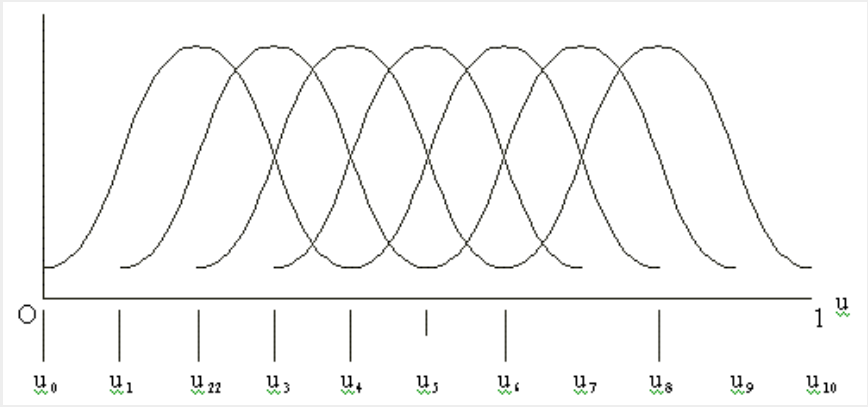


- Evaluate Bézier using "Sliding window"
  - Window moves forward by 3 units when *u* passes to next Bézier segment
  - With 3N+1 points, N positions for window, ie. N segments
  - Evaluate matrix in window:

$$\mathbf{x}(u) = \underbrace{\begin{bmatrix}\mathbf{p}_{3i} & \mathbf{p}_{3i+1} & \mathbf{p}_{3i+2} & \mathbf{p}_{3i+3}\end{bmatrix}}_{\mathbf{G}_{Bez}} \underbrace{\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}}_{\mathbf{B}_{Bez}} \underbrace{\begin{bmatrix}t^3\\ t^2\\ t\\ 1\end{bmatrix}}_{\mathbf{T}}$$

where $\begin{cases} u<1: & i=\lfloor Nu \rfloor \text{ and } t = Nu - i \\ u=1: & i = N-1 \text{ and } t = 1 \end{cases}$

B-spline blending functions



- Still a sliding “window”
  - Shift “window” by 1, not by 3
  - With N+3 points, N positions for window, i.e. N segments

$$\mathbf{x}(u) = \underbrace{\begin{bmatrix}\mathbf{p}_i & \mathbf{p}_{i+1} & \mathbf{p}_{i+2} & \mathbf{p}_{i+3}\end{bmatrix}}_{\mathbf{G}} \underbrace{\frac{1}{6}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 0 & 4\\ -3 & 3 & 3 & 1\\ 1 & 0 & 0 & 0\end{bmatrix}}_{\mathbf{B}_{B-spline}} \underbrace{\begin{bmatrix}t^3\\ t^2\\ t\\ 1\end{bmatrix}}_{\mathbf{T}} \text{ where } \begin{cases} u<1: & i=\lfloor Nu \rfloor \text{ and } t = Nu - i \\ u=1: & i = N-1 \text{ and } t = 1 \end{cases}$$

Tangent to B-spline

- Same formulation, but use T' vector (and scale by N)

$$\mathbf{x}(u) = \underbrace{\begin{bmatrix}\mathbf{p}_i & \mathbf{p}_{i+1} & \mathbf{p}_{i+2} & \mathbf{p}_{i+3}\end{bmatrix}}_{\mathbf{G}} \underbrace{\frac{1}{6}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 0 & 4\\ -3 & 3 & 3 & 1\\ 1 & 0 & 0 & 0\end{bmatrix}}_{\mathbf{B}_{B-spline}} \underbrace{\begin{bmatrix}t^3\\ t^2\\ t\\ 1\end{bmatrix}}_{\mathbf{T}} \text{ where } \begin{cases} u<1: & i=\lfloor Nu \rfloor \text{ and } t = Nu - i \\ u=1: & i = N-1 \text{ and } t = 1 \end{cases}$$

$$\vec{\mathbf{x}}'(u) = \underbrace{\begin{bmatrix}\mathbf{p}_i & \mathbf{p}_{i+1} & \mathbf{p}_{i+2} & \mathbf{p}_{i+3}\end{bmatrix}}_{\mathbf{G}} \underbrace{\frac{1}{6}\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 0 & 4\\ -3 & 3 & 3 & 1\\ 1 & 0 & 0 & 0\end{bmatrix}}_{\mathbf{B}_{B-spline}} N \underbrace{\begin{bmatrix}3t^2\\ 2t\\ 1\\ 0\end{bmatrix}}_{\mathbf{T}'} \text{ where } \begin{cases} u<1: & i=\lfloor Nu \rfloor \text{ and } t = Nu - i \\ u=1: & i = N-1 \text{ and } t = 1 \end{cases}$$

Catmull-Rom

- Use same formulation as B-spline
  - But a different basis matrix
- Interpolates all its control points
  - Except it doesn't reach the endpoints
  - Add extra points on the ends, or wrap around for closed loop

$$\mathbf{x}(u) = \underbrace{\begin{bmatrix}\mathbf{p}_i & \mathbf{p}_{i+1} & \mathbf{p}_{i+2} & \mathbf{p}_{i+3}\end{bmatrix}}_{\mathbf{G}} \underbrace{\frac{1}{2}\begin{bmatrix}-1 & 2 & -1 & 0\\ 3 & -5 & 0 & 2\\ -3 & 4 & 1 & 0\\ 1 & -1 & 0 & 0\end{bmatrix}}_{\mathbf{B}_{Catmull-Rom}} \underbrace{\begin{bmatrix}t^3\\ t^2\\ t\\ 1\end{bmatrix}}_{\mathbf{T}} \text{ where } \begin{cases} u<1: & i=\lfloor Nu \rfloor \text{ and } t = Nu - i \\ u=1: & i = N-1 \text{ and } t = 1 \end{cases}$$

$$\vec{\mathbf{x}}'(u) = \underbrace{\begin{bmatrix}\mathbf{p}_i & \mathbf{p}_{i+1} & \mathbf{p}_{i+2} & \mathbf{p}_{i+3}\end{bmatrix}}_{\mathbf{G}} \underbrace{\frac{1}{2}\begin{bmatrix}-1 & 2 & -1 & 0\\ 3 & -5 & 0 & 2\\ -3 & 4 & 1 & 0\\ 1 & -1 & 0 & 0\end{bmatrix}}_{\mathbf{B}_{Catmull-Rom}} N \underbrace{\begin{bmatrix}3t^2\\ 2t\\ 1\\ 0\end{bmatrix}}_{\mathbf{T}'} \text{ where } \begin{cases} u<1: & i=\lfloor Nu \rfloor \text{ and } t = Nu - i \\ u=1: & i = N-1 \text{ and } t = 1 \end{cases}$$

Step 1, Summary

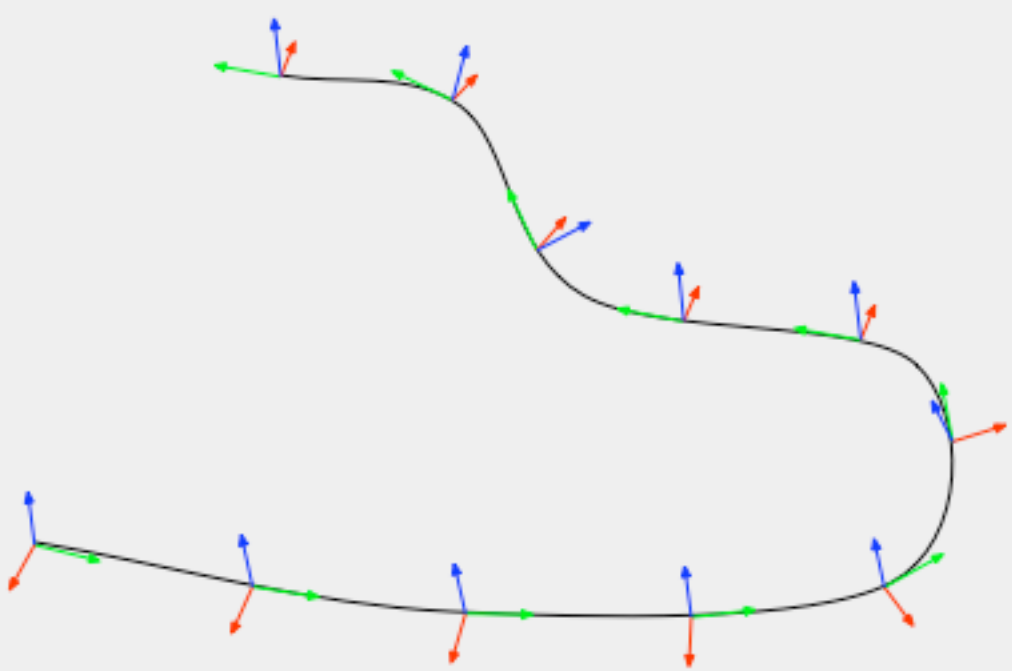
- We can define a curve:
  - Given N points
  - We know how to evaluate the curve for any u in 0..1
  - We know how to evaluate the tangent for any u in 0..1
  - Might be implemented via Bézier, B-Spline, Catrom

- Bundle in a class

```
class Curve {
  Curve(int Npoints, Point3 points[]);
  Point3 Eval(float u);
  Vector3 Tangent(float u);
}
```

Step 2. Coordinate Frames on curve

- Describes orientation along the path



Finding coordinate frames

- To be able to build the track, we need to define coordinate frames along the path.
  - We know the tangent, this gives us one axis, the “forward” direction
  - We need to come up with vectors for the “right” and “up” directions.
- There’s no automatic solution
- Most straightforward :
  - Use the same approach as camera lookat
  - Pick an “up” vector.

Constructing a frame

- Same as constructing an object-to-world transfrom

  Given that we can compute position $\mathbf{x}(t)$ and tangent $\vec{\mathbf{x}}'(t)$

  Given a global "up" vector $\vec{\mathbf{u}}$

  Assume that we want

  "forward" direction to be the y axis,

  "right" to be the x axis

  "up" to be the z axis

  To construct the frame matrix $\mathbf{F}(t)$, fill the $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}, \mathbf{d}$ columns

$\mathbf{d} = \mathbf{x}(t)$ -- origin of the frame is at t along the curve

$\vec{\mathbf{b}} = \dfrac{\vec{\mathbf{x}}'(t)}{|\vec{\mathbf{x}}'(t)|}$ -- y axis of frame is tangent, normalized

$\vec{\mathbf{a}} = \dfrac{\vec{\mathbf{b}} \times \vec{\mathbf{u}}}{|\vec{\mathbf{b}} \times \vec{\mathbf{u}}|}$ -- x axis of frame is perpendicular to forward and global up

$\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ -- z axis of frame is perpendicular to forward and right

Specifying a roll for the frame

- The “lookat” method tries to keep the frame upright
- But what if we want to lean, e.g. for a banked corner?
- Introduce a roll parameter
  - Gives the angle to roll about the “forward” axis.
- Want to be able to vary the roll along the path
  - Specify a roll value per control point

Computing the roll on the path

- How to compute the roll at any value of t?
  - just use the roll values as (1D) spline points!

Given points: $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{N-1}$

and rolls: $r_0, r_1, \ldots, r_{N-1}$

Compute position, tangent:

$$\mathbf{x}(u) = Bez(Nu - i, \mathbf{p}_{3i}, \mathbf{p}_{3i+1}, \mathbf{p}_{3i+2}, \mathbf{p}_{3i+3}) \text{ or}$$
$$= \mathbf{G}_i \mathbf{B} \mathbf{T}_{Nu-i}$$
$$\vec{\mathbf{x}}'(u) = Bez'(Nu - i, \mathbf{p}_{3i}, \mathbf{p}_{3i+1}, \mathbf{p}_{3i+2}, \mathbf{p}_{3i+3}) \text{ or}$$
$$= \mathbf{G}_i \mathbf{B} N \mathbf{T}'_{Nu-i}$$

Compute roll:

$$r(u) = Bez(Nu - i, r_{3i}, r_{3i+1}, r_{3i+2}, r_{3i+3}) \text{ or}$$
$$= \begin{bmatrix} r_i & r_{i+1} & r_{i+2} & r_{i+3} \end{bmatrix} \mathbf{B} \mathbf{T}_{Nu-i}$$

where:

$$i = \lfloor Nu \rfloor$$

Rolling the frame

- To compute the final object-to-world matrix for the frame, compose the upright frame with a rotation matrix:

$$\mathbf{W}(t) = \mathbf{F}(t)\ \mathbf{R}_y(r(t))$$

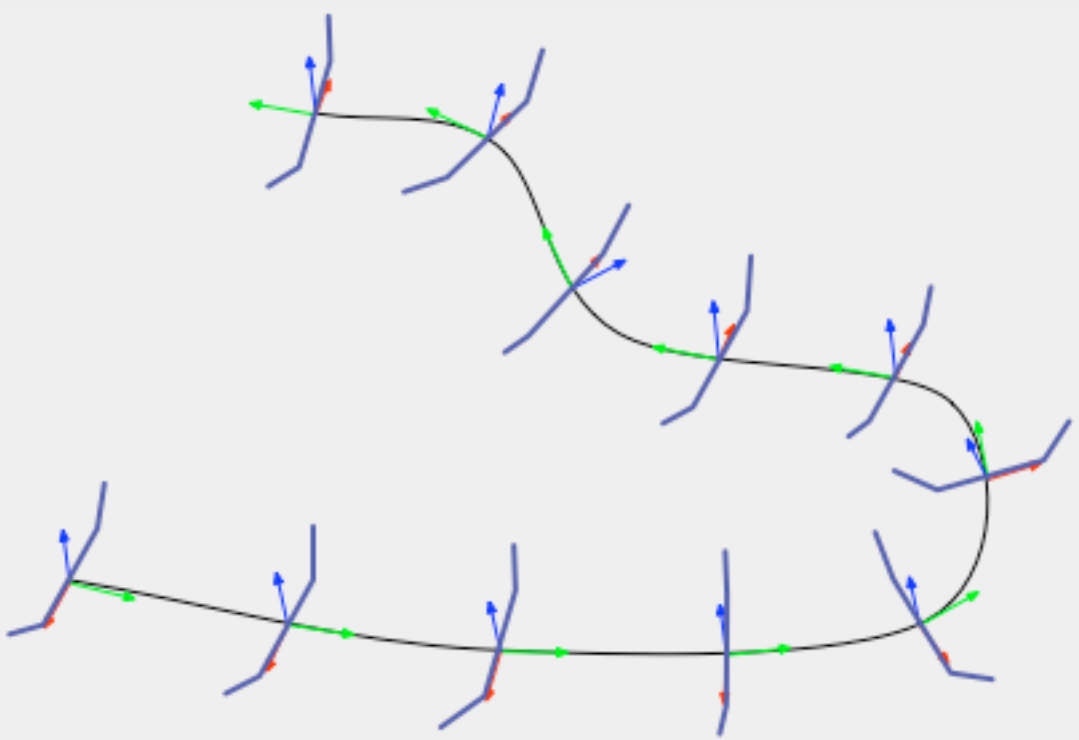
Curve class with frames

- The class can include:

```
class Curve {
    Curve(int N, Point3 points[], float roll[]);
    void SetUp(Vector3 up);
    Point3 Eval(float t);
    Vector3 Tangent(float t);
    Matrix Frame(float t) { return UpFrame(t)*Roll(t); }
 private:
    Matrix UpFrame(float t);
    Matrix Roll(float t);
 }
```

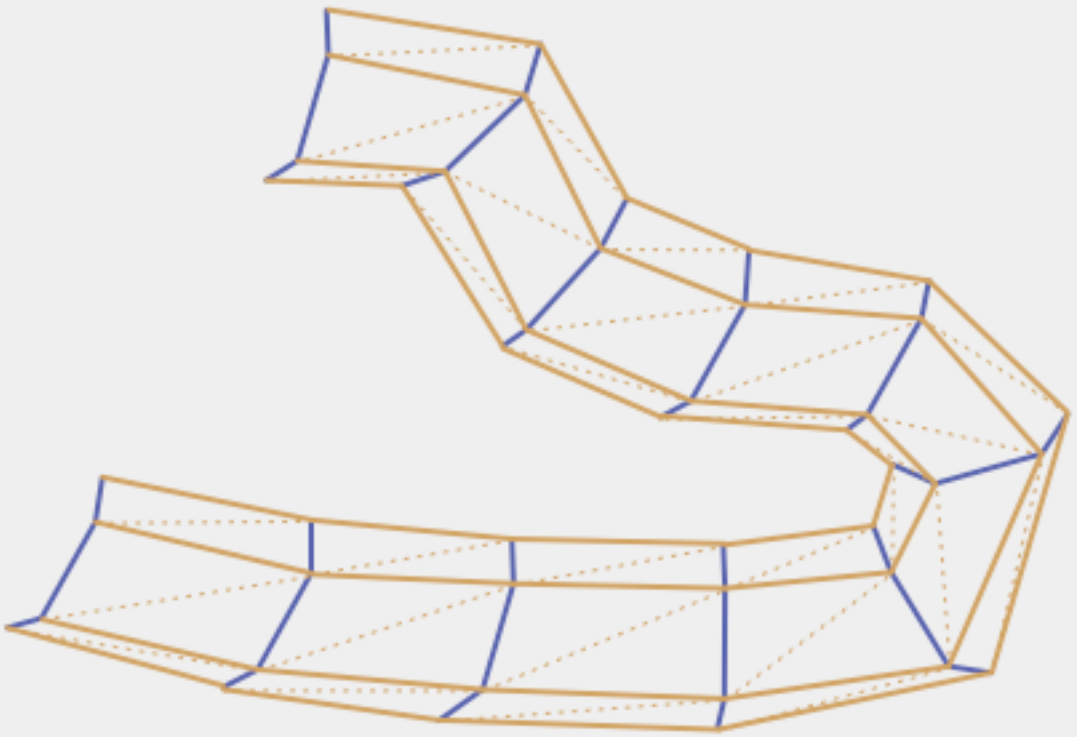
Step 3a. Sweep a cross section

- Define a cross section curve (piecewise-linear is OK)
- Sweep it along the path, using the frames



3b. Tessellate

- Construct triangles using the swept points
- (sweep more finely than this drawing, so it'll be smoother)

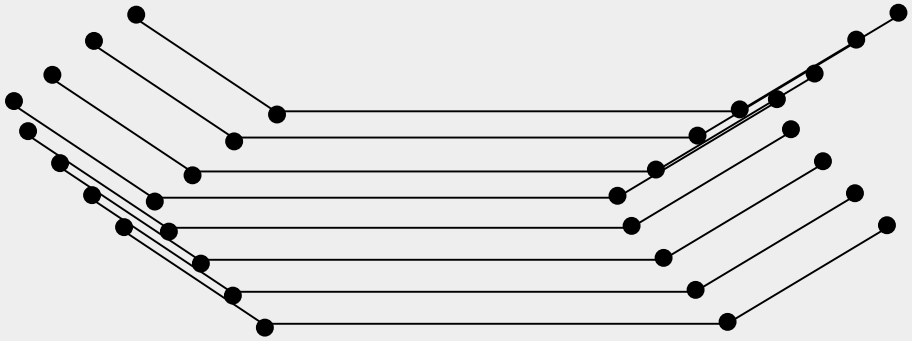


The cross section

- Say we have a cross section curve with N points
  - AKA the *profile curve*
  - For simplicity, assume it's piecewise linear
  - e.g., N=4

Sweep the profile curve

- choose M how many rows to have
  - make this a parameter or value you can change easily
  - too few: not smooth enough
  - too many: too slow
- place M copies of the cross section along the path
  - e.g. M=8 (for a full track you might have M=100's or more)



Sweep the profile curve

- We can get a coordinate frame for any t along the curve
  - i.e. we have an object-to-world matrix $\mathbf{W}(t)$ for t in $0 \ldots 1$

- Distribute the profile curve copies uniformly in t
  - For each row, transform the profile curve using the world matrix:

For each row i in 0 to $M - 1$

For each profile point $\mathbf{q}_j$ for j in 0 to $N - 1$

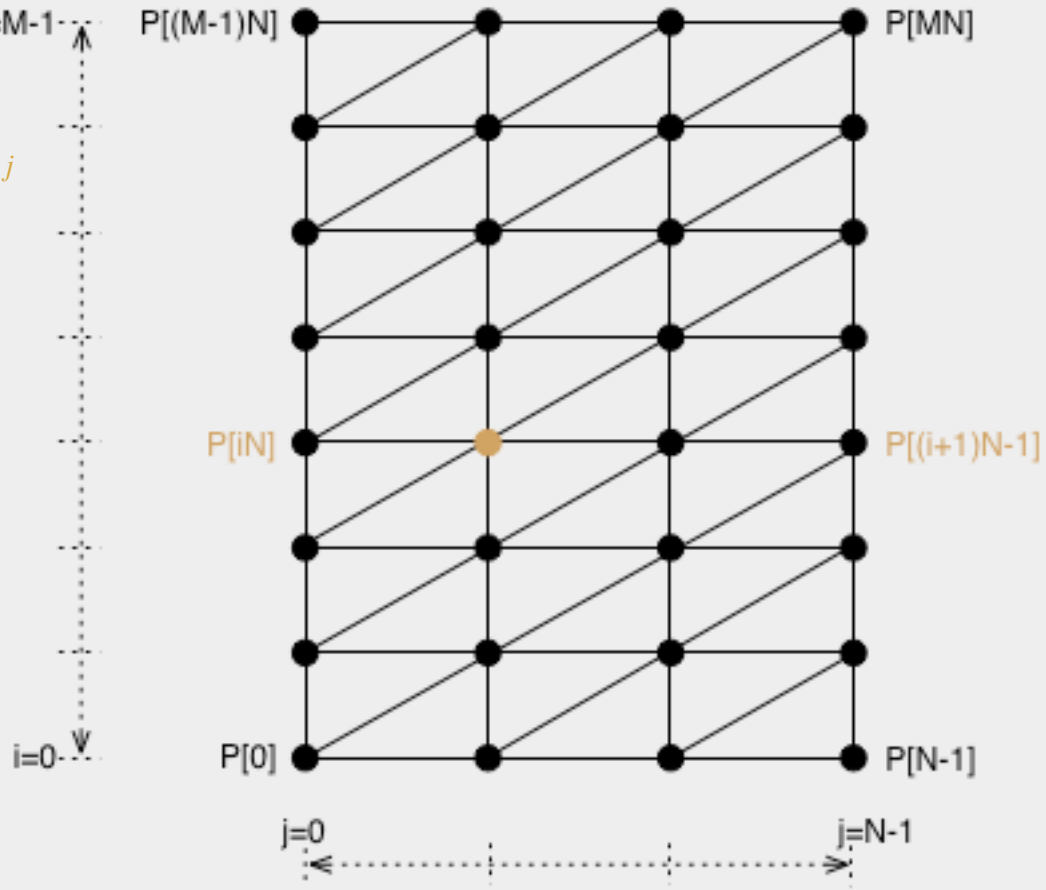
$$t = \frac{i}{M - 1}$$

$$\mathbf{p}_{ij} = \mathbf{W}(t)\mathbf{q}_j$$

Topology

- Simple rectangular topology, M*N points:

$$\mathbf{P}[iN + j] = \mathbf{W}(i/(M-1))\ \mathbf{q}_j$$

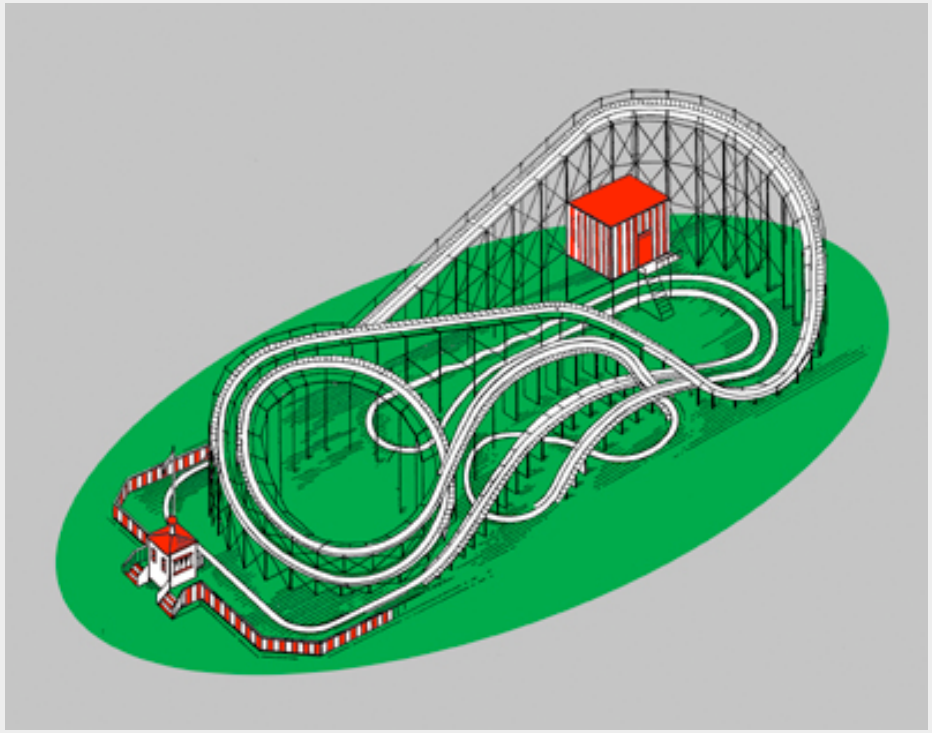


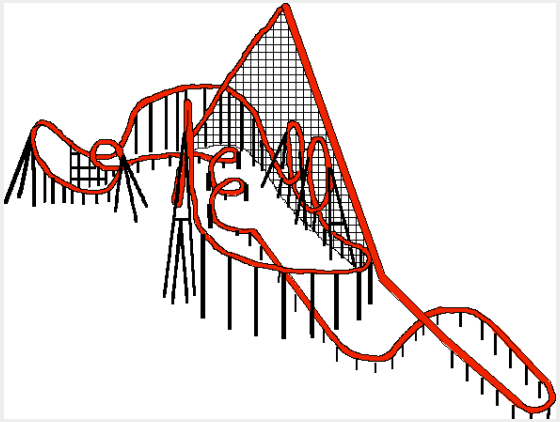
Putting a car on the track

- Again, we can use the path's frame matrix $\mathbf{W}(t)$
  - Probably with a local offset $\mathbf{M}$ if needed:
    - Local-to-world = $\mathbf{W}(t)\mathbf{M}$

- t goes from 0 to 1 and keeps cycling
- Wheels rotate as function of t
  - Will probably slide. Try to tweak so they look reasonable.
  - More complex to really have wheels roll on the track

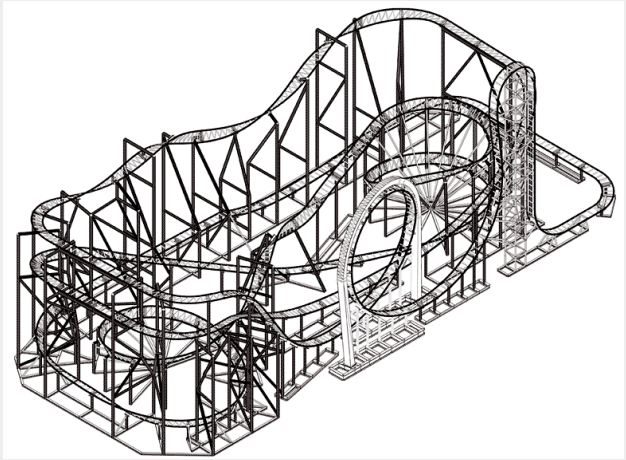
Designing a roller coaster

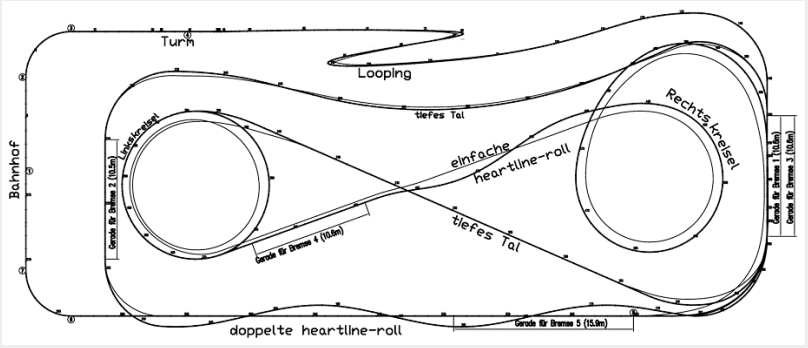


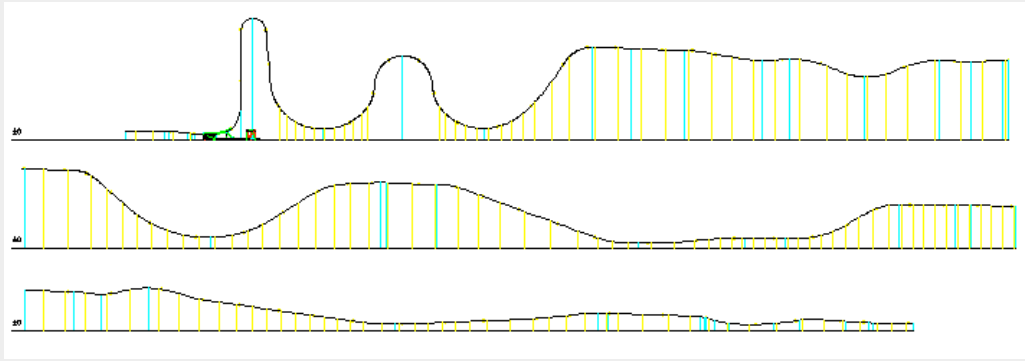


Designing a roller coaster track

- Suggestions…
  - plan it out on (graph) paper first!
  - top view & “unwound” elevation view
  - number and label the control points
    - use a scheme that will let you go back and add more points later!
    - control point coordinates can be hard-wired in C++ source
    - when typing them in, comment them with your labels







Designing a roller coaster

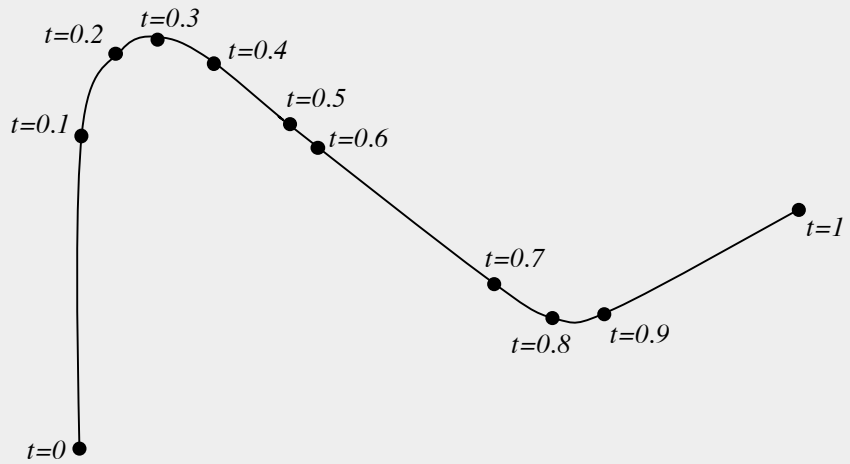
- The path can be fairly simple.
  - start with something simple
  - if you have time at the end, go wild!
- The track could be…
  - a trench, like a log flume that the car slides in
  - a monorail that the car envelops
  - a pair of rails (two cross sections) for a railroad-style car
  - whatever else you come up with
- The car could be…
  - a cube with simple wheels
  - a penguin
  - A sports car that you download off the web and figure out how to read
  - whatever else you come up with
- Trestles/Supports not needed… but could be cool
- Ticket booth & Flags & Fence etc. not needed… but could be cool
- Have fun! Be creative!

Advanced Roller Coaster Topics

- (Not required, but handy for some bells & whistles)
- Arc-Length Parametrization
  - Better control over speed of car
- Parallel transport of frames
  - Make it easier to do loops

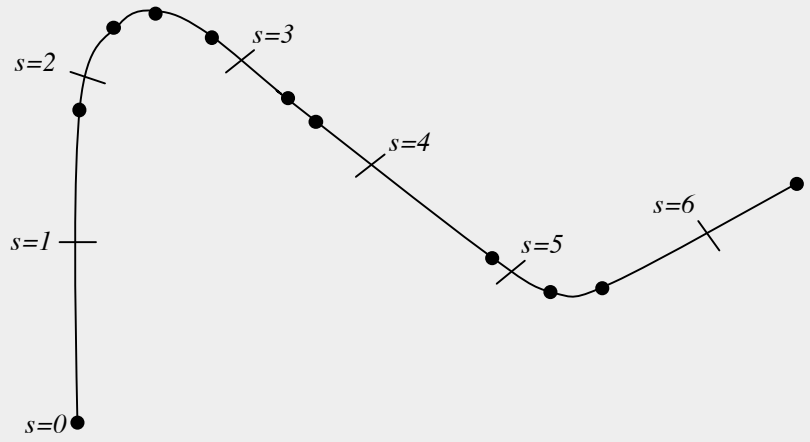
Arc-length Parametrization

- In our current formulation:
  - Position along the curve is based on arbitrary t parameter
  - We animate by increasing t uniformly:
    - Speed of the car isn't constant
    - Speed of the car isn't easily controllable
    - Car spends the same amount of time in each segment:
      - in areas where the control points are far apart, the car goes fast
      - in areas where the control points are close together, the car goes slow



Arc-length parametrization

- Specify position along the curve based on distance s
  - s=0: at start of curve
  - s=1: 1 unit (foot or meter or whatever) along the track
  - s=2: 2 units along the track
- Animate by increasing s uniformly: constant speed
- Adjust speed by adjusting change in s per frame
- Unaffected by where control points happen to be



Arc-length parametrization

- *Arc length* just means length along the curve
- Really, arc length *re*-parametrization
  - We have a parameter t
  - We want to express t in terms of length s
- No simple closed form expression
- A bit of math in principle
- Then we'll do a simple approximation of it

Arc length

- Define $S(t) =$ length along the curve $\mathbf{x}(t)$ from the start until t

$$ds = \sqrt{dx^2 + dy^2 + dz^2} \quad \text{-- differential distance}$$

$$S(t) = \int_0^t \left(\frac{ds}{dt} \right) dt$$

$$= \int_0^t \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2} \, dt$$

$$= \int_0^t \left| \vec{\mathbf{x}}'(t) \right| \, dt$$

Reparameterize

We have:

$\mathbf{x}(t)$ = our curve parametrized by t

$S(t)$ = arc length along $\mathbf{x}(\mathbf{t})$ up to t

We want:

$\mathbf{x}_{arc}(s)$ = our curve re-parametrized by s

$\mathbf{x}_{arc}(s) = \mathbf{x}(t(s))$, where $t(s) = S^{-1}(s)$

We need:

the inverse length function, $S^{-1}(s)$

Arc-length reparameterization

- Exact computation of length & inverse is too hard

- Simple approximation for arc length: *chord length*
  - Sample *M* times, uniformly (sufficiently large, e.g. *M=10N*)
  - Accumulate incremental length into array of length values:

$$S_i = S_{i-1} + \left|\mathbf{x}\left(\tfrac{i}{M}\right) - \mathbf{x}\left(\tfrac{i-1}{M}\right)\right|$$

- Simple approximation to invert:

  Given s, where $0 \le s \le S_M$

  Find neighboring samples: i such that $S_i < s < S_{i+1}$

  Use linear interpolation between the samples:

$$t = \frac{1}{M}\left(i + \frac{s - S_i}{S_{i+1} - S_i}\right)$$

Parallel transport of frames

- Using the “lookat” method with a global “up” vector:
  - Trouble if the “forward” direction lines up with the “up” direction
    - No solution when it lines up
    - Solution tends to spin wildly when forward is very close to up
    - e.g. if track goes straight up or straight down
      - such as would happen going into or out of a loop
- Possible solution:
  - Specify an “up” vector per point
    - Replaces of the roll vector
    - Can compute spline on per-point up vectors
  - This would work, but is cumbersome for the user
- Alternative:
  - Start with a good frame, carry it with you along the path

Parallel Transport

- User specifies initial “up” vector for t=0 (or s=0)
  - Construct an initial frame using the lookat method
- Compute subsequent “up” vectors incrementally
  - Directly for each sample when sampling curve for tesselation; or
  - In advance when defining the curve:
    - Compute and store “up” vector at each control point
    - Use the stored “up” vectors as control points for an “up” vector spline
- Given the tangent and an “up” vector at each point
  - Can construct a frame using the lookat method
  - None of our stored “up” vectors will be parallel to the tangent.

Parallel Transport

- Given at $t = t_i$:

  tangent $\vec{\mathbf{v}}_i = \dfrac{\vec{\mathbf{x}}'(t_i)}{\left|\vec{\mathbf{x}}'(t_i)\right|}$

  "up" vector $\vec{\mathbf{u}}_i$

- For next sample $t = t_{i+1}$:
  - Compute $\vec{\mathbf{v}}_{i+1} = \dfrac{\vec{\mathbf{x}}'(t_{i+1})}{\left|\vec{\mathbf{x}}'(t_{i+1})\right|}$
  - Compute rotation that takes $\vec{\mathbf{v}}_i$ to $\vec{\mathbf{v}}_{i+1}$:

    axis $\vec{\mathbf{a}} = \dfrac{\vec{\mathbf{v}}_i \times \vec{\mathbf{v}}_{i+1}}{\left|\vec{\mathbf{v}}_i \times \vec{\mathbf{v}}_{i+1}\right|}$

    angle $\theta = \tan^{-1}\left(\dfrac{\vec{\mathbf{v}}_i \times \vec{\mathbf{v}}_{i+1}}{\vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_{i+1}}\right)$
  - Use that rotation to take $\vec{\mathbf{u}}_i$ to $\vec{\mathbf{u}}_{i+1}$:

    $\vec{\mathbf{u}}_{i+1} = \mathbf{R}(\vec{\mathbf{a}}, \theta)\ \vec{\mathbf{u}}_i$

- Works if curve doesn't turn 180° between samples
- Can also include (incremental) roll value at each sample:

  $\vec{\mathbf{u}}_{i+1} = \mathbf{R}(\vec{\mathbf{v}}_{i+1}, r_{i+1})\ \mathbf{R}(\vec{\mathbf{a}}, \theta)\ \vec{\mathbf{u}}_i$

Done

- Enjoy your projects!
- Next class: guest lecture on game design