MATH RESEARCH PROJECT

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1.Introduction

Elastic waves are not dispersive in an unbounded homogeneous medium, but they become dispersive under repeated processes occurring at the boundaries of wave guides [7, 10]. Dispersive elastic waves have found many important applications in certain areas such as seismology, geophysics, nondestructive inspection of material surfaces, and electronic signal processing devices. More information about applications and for reviews, we refer to Ewing [7], Achenbach [21], Farnell [1], and Maugin [13].

The effect of constitutional non-linearity on the propagation characteristics of dispersive elastic waves has been studied by many investigators such as Teymur [17, 18, 19], Maugin and Hadouaj[12], Mayer[14], Fu[9], Porubov and Samsonov[15], Ferreira and Boulanger[8], Pucci and Saccomandi[16], Ahmetolan and Teymur [2, 3], Destrade et al.[6], Teymur et al.[20], and Demirkus and Teymur [5], In [5], the propagation of non-linear shear horizontal (SH) waves in a homogeneous, isotropic, and compressible hyper-elastic layer overlying a rigid substratum was investigated. Moreover, the propagation of linear Love waves in a heterogeneous media was discussed by Hudson [11] and Avtar [4].

The aim of this work is to study the propagation of non-linear SH waves in a heterogenous, isotropic, and incompressible hyper-elastic layer overlying a rigid substratum. Heterogeneity is varied with the depth hyperbolically, and uniform is any direction parallel to the boundaries. We use the method of multiple scales and strike a balance between the non-lineratiy and dispersion in the asymptotic analysis, to derive a non-linear Schrödinger (NLS) equation describing a self-modulation of non-linear SH waves.

Subscripts preceded by a comma indicate partial differentiation with respect to material or spatial coordinates. In the absence of body forces, the equations of the motion in the reference state are

$$T_{\Delta\beta,\Delta} + T_{3\beta,3} = 0, \quad T_{\Delta 3,\Delta} + T_{33,3} = \rho_3 \tag{1}$$

where T_{Kk} are the components of the first Piola-Kirchhoff stress tensor field accompanying the deformation field (1), a dot over u_3 indicates partial differentiation with respect to $\imath$, and $\rho = \rho(X_\Delta)$ is the density of the layer. The boundary conditions can be written as

$$T_{2k} = 0 \quad on \quad X_2 = h \quad and \quad u_3 = 0 \quad on \quad X_2 = 0 \tag{2}$$

The components of the deformation gradient tensors $x_{k,K}$ and $X_{K,k}$ are as given in the list below

$$x_{\alpha,\Delta=\delta_{\alpha\Delta}}, \quad x_{\alpha,3} = 0, \quad x_{3,\Delta} = u_{3,\Delta}, \quad x_{3,3} = 1, \tag{3}$$

$$X_{\Delta,\alpha} = \delta_{\Delta\alpha}, \quad X_{\Delta,3} = 0, \quad X_{3,\alpha} = -u_{3,\Delta}\delta_{\Delta\alpha}, \quad X_{3,3} = 1 \tag{4}$$

for the deformation field (1) which is isochoric. *i.e.* $j = \det x_{k,K} = 1$. In addition to using the components (4) and (5), if the relations $t_{kl} = j^{-1}x_{k,K}T_{Kl}$ and $T_{Kl} = jX_{K,k^t,kl}$ are taken into consideration, then the components of the Cauchy stress tensor t_{kl} and of the first Piola-Kirchhoff stress tensor T_{Kl} can be written as

$$t_{\alpha\beta} = \delta_{\alpha\Delta}T_{\Delta\beta}, \quad t_{\alpha 3} = \delta_{\alpha\Delta}T_{\Delta 3},$$

$$t_{3\beta} = u_{3,\Delta}T_{\Delta\beta} + T_{3\beta}, \quad t_{33} = u_{3,\Delta}T_{\Delta 3} + T_{33}, \tag{5}$$

$$T_{\Delta\beta} = \delta_{\Delta\alpha}t_{\alpha\beta}, \quad T_{\Delta 3} = \delta_{\Delta\alpha}t_{\alpha 3},$$

$$T_{3\beta} = -u_{3,\Delta}\delta_{\Delta\alpha}t_{\alpha\beta} + t_{3\beta}, \quad T_{33} = -u_{3,\Delta}\delta_{\Delta\alpha}t_{\alpha 3} + t_{33}, \tag{6}$$

respectively. Therefore, the equations of the motion (2) in terms of t_{kl} are expressed as follows:

$$(\delta_{\Delta\alpha}t_{\alpha\beta})_{,\Delta} + (-u_{3,\Delta}\delta_{\Delta\alpha}t_{\alpha\beta} + t_{3\beta})_{,3} = 0,$$

$$(\delta_{\Delta\alpha}t_{\alpha 3})_{,\Delta} + (-u_{3,\Delta}\delta_{\Delta\alpha}t_{\alpha 3} + t_{33})_{,3} = \rho_3. \tag{7}$$

If the layer consists of hyper-elastic materials, there exists a strain energy function $\sum$ which gives the mechanical properties of the constituent materials, and stress constituve equations can be given by

$$T_{Kk} = \frac{\partial\Sigma}{\partial x_{k,K}}. \tag{8}$$

We consider that the constituve materials are isotropic and heterogeneous, so Σ is the function of the principal invariants of the Finger deformation tensor c^{-1} and X_Δ, as

$$I = trc^{-1}, \quad 2II = (trc^{-1})^2 - tr(c^{-2}), \quad III = detc^{-1} \tag{9}$$

and calculated on the deformation field (1) as

$$I = II = 3 + K^2, \quad III = 1 \tag{10}$$

where $K^2 = u_{3,\Delta}u_{3,\Delta}$.
Let us now assume that the heterogeneity is varied only with the depth and uniform in any direction parallel to the boundaries and consider generalized neo-Hookean materials. Hence, the strain energy function Σ has a form

$$\Sigma = \Sigma(I, X_2) \tag{11}$$

Then the stress constituve equations are

$$t_{kl} = 2\frac{d\Sigma}{dI}(-\delta_{kl} + c_{kl}^{-1}) \tag{12}$$

where $c_{kl}^{-1} = x_{k,K^x l,K}$ are the components of Finger deformation tensor, found for the deformation field(1) as follows:

$$c_{\alpha\beta}^{-1} = \delta_{\alpha\Delta}\delta_{\beta\Delta}, \quad c_{\alpha 3}^{-1} = \delta_{\alpha\Delta}u_{3,\Delta}, \quad c_{3\beta}^{-1} = u_{3,\Delta}\delta_{\beta\Delta}, \quad c_{33}^{-1} = 1 + K^2. \tag{13}$$

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