TAM 598 Lecture 6 :

RANDOM VECTORS AND MULTIVARIATE GAUSSIANS

Announcements:

- HW 1 covers lectures 1-4 ; due on Feb 12 ; to be submitted via CANVAS
- HW2 covers lectures 4-8 ; due on Feb 26

$\boxed{\text{Random Vectors}}$ – take N random variables $X_1, X_2, \ldots, X_N$ and put them in a vector

$$\underset{\sim}{X} = (X_1, \ldots, X_N)$$

Random vectors are used to model uncertain quantities such as

→ state of a multibody system : vector of coordinates & velocities

→ an unknown function : vector of its function values at N test points

→ an image : vector of pixel values

PDF of a Random Vector is the joint PDF of its components.

$$p(\underset{\sim}{x}) = p(x_1, x_2, \ldots, x_N)$$

⊙ can marginalize $\quad p(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x_1, x_2, \ldots x_N)\, dx_2\, dx_3 \ldots dx_N$

⊙ can integrate out subsets $\quad p(x_1, x_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x_1, x_2, \ldots x_N)\, dx_3 \ldots dx_N$

Expectation of a Random Vector is the vector of expectations of each component

$$\mathbb{E}[\underset{\sim}{X}] = \begin{pmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_2] \end{pmatrix}$$

This expectation is linear:

$$\mathbb{E}[a\underset{\sim}{X} + b\underset{\sim}{Y}] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]$$

Covariance Matrix of Two Random Vectors

Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ be random vectors of dimension N and M respectively. The covariance of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ is the $N \times M$ matrix consisting of all covariances between components of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$, ie

$$\mathbb{C}[\underset{\sim}{X}, \underset{\sim}{Y}] = \left(\mathbb{C}[X_i, Y_j] \right)$$

- sometimes $\mathbb{C}[\underset{\sim}{X}, \underset{\sim}{Y}]$ is called Σ ("sigma")

- we can rewrite the covariance matrix as

$$\mathbb{C}[\underset{\sim}{X}, \underset{\sim}{Y}] = \mathbb{E}\left[(\underset{\sim}{X} - \mathbb{E}[\underset{\sim}{X}])(\underset{\sim}{Y} - \mathbb{E}[\underset{\sim}{Y}])^T\right]$$

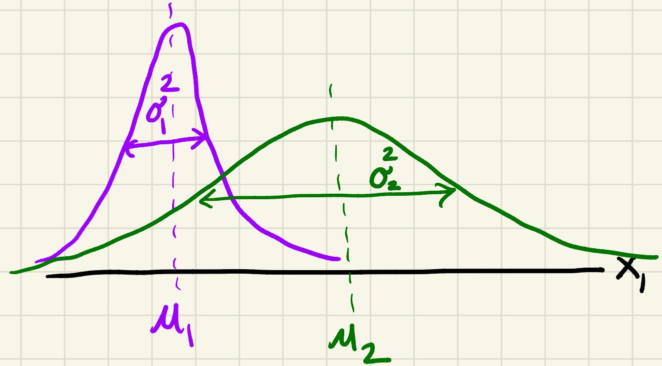
- the $N \times N$ matrix $\mathbb{C}[\underset{\sim}{X}, \underset{\sim}{X}]$ is the self covariance or just the covariance matrix of $\underset{\sim}{X}$.

  diagonal: variance of each component of $\underset{\sim}{X}$

  off-diagonal: covariance of X_i, X_j for $i \neq j$

Multivariate Gaussian Distributions

recall univariate Gaussian distributions $\quad X \sim N(\mu, \sigma^2)$



PDF :

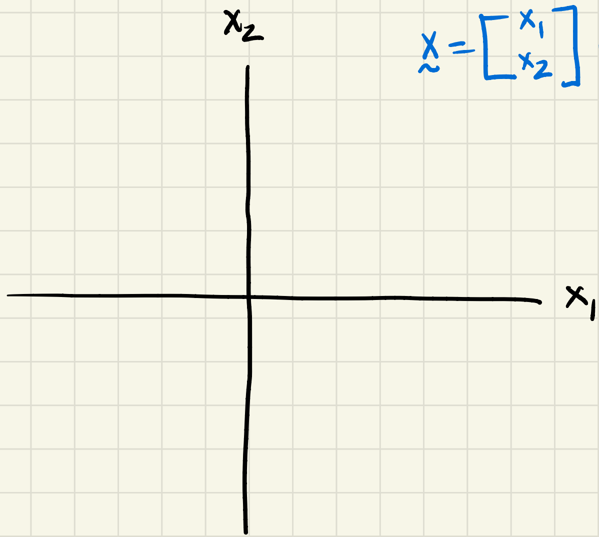
properties of univariate Gaussian distributions :

① closed under linear transformation

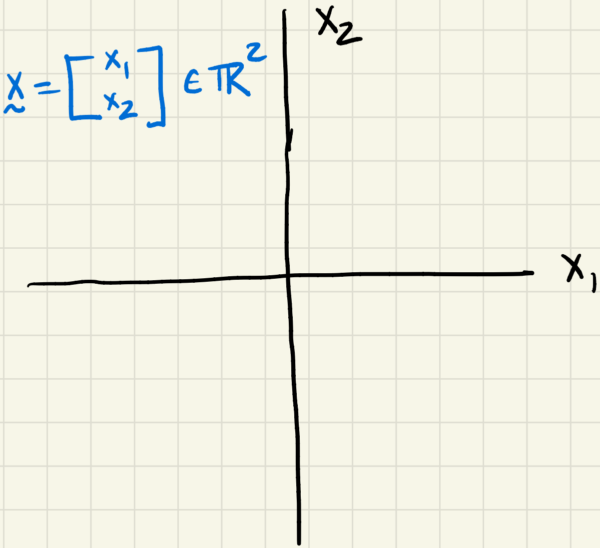
② adding / subtracting Gaussian RVs

Bivariate Gaussian Distributions – two random variables from which we construct a two-component random vector

$$\underset{\sim}{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$



$$\underset{\sim}{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$



recall how we defined covariance of two random variables :

$$\text{cov}(X_1, X_2) = \mathbb{E}\left((X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2]) \right)$$

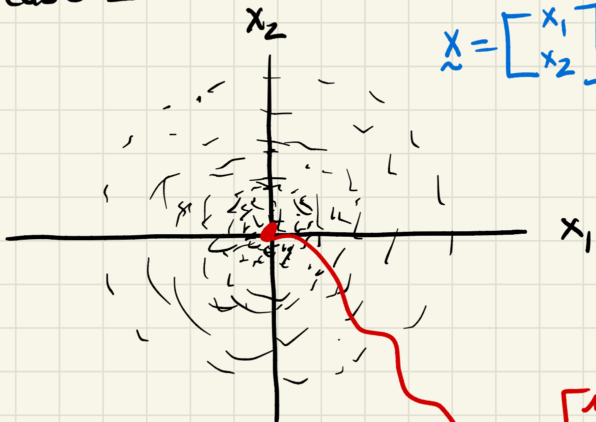
- if $\mathbb{E}[X_1] = 0$, $\mathbb{E}[X_2] = 0$ then

- $$\text{cov}(X_1, X_1) = \mathbb{E}\left((X_1 - \mathbb{E}[X_1])(X_1 - \mathbb{E}[X_1]) \right)$$

  $$=$$

- another way to think of $\text{cov}(X_1, X_2)$

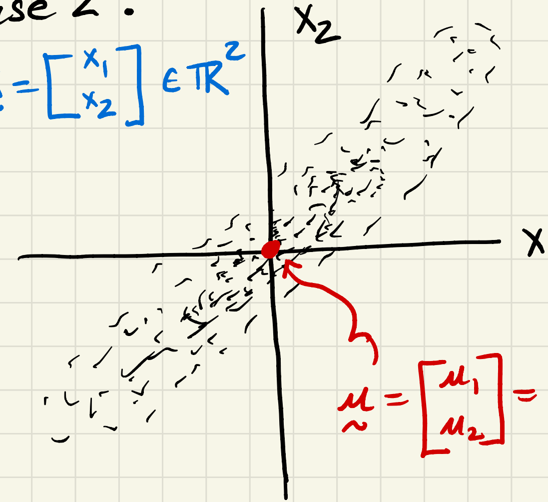
case 1:



$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

$$\underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

case 2:



$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

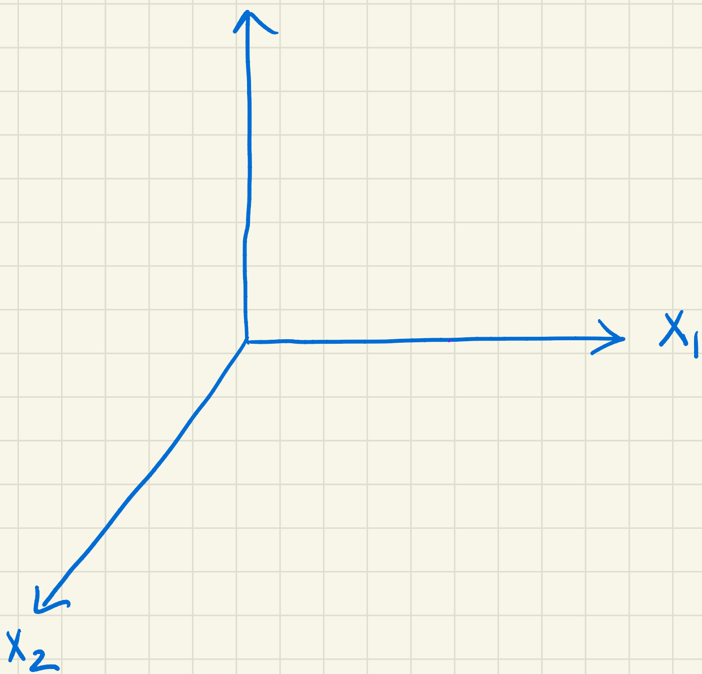
$$\underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right)$$

covariance matrix

$$\boxed{\Sigma_{ij} = \operatorname{cov}(x_i, x_j)}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right)$$



MULTIVARIATE GAUSSIAN THEOREM

text: Kevin Murphy Ch 4
Bishop ch 2.3

(allows us to reason about the marginals & conditionals of a multivariate Gaussian.)

Suppose $\underset{\sim}{X} = (\underset{\sim}{X}_1, \underset{\sim}{X}_2)$ is jointly Gaussian with parameters

$$\underset{\sim}{\mu} = \begin{bmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{bmatrix} \qquad \underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{pmatrix} \qquad \underset{\sim}{\Lambda} = \underset{\sim}{\Sigma}^{-1} = \begin{pmatrix} \underset{\sim}{\Lambda}_{11} & \underset{\sim}{\Lambda}_{12} \\ \underset{\sim}{\Lambda}_{21} & \underset{\sim}{\Lambda}_{22} \end{pmatrix}$$

Then:

the marginals are

$$p(\underset{\sim}{x}_1) = N(\underset{\sim}{x}_1 \mid \underset{\sim}{\mu}_1, \underset{\sim}{\Sigma}_{11})$$

$$p(\underset{\sim}{x}_2) = N(\underset{\sim}{x}_2 \mid \underset{\sim}{\mu}_2, \underset{\sim}{\Sigma}_{22})$$

the conditionals are

$$p(\underset{\sim}{x}_1 \mid \underset{\sim}{x}_2) = N(\underset{\sim}{x}_1 \mid \underset{\sim}{\mu}_{1|2}, \underset{\sim}{\Sigma}_{1|2})$$

where

$$\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$
$$= \mu_1 - \Lambda_{11}^{-1}\Lambda_{12}(x_2 - \mu_2)$$

$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$
$$= \Lambda_{11}^{-1}$$

So if Multivariate Gaussians on $\mathbb{R}^n$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_N \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \cdots & \Sigma_{2n} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \cdots & \Sigma_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \Sigma_{n3} & \cdots & \Sigma_{nn} \end{bmatrix} \right)$$

joint PDF $p(\underset{\sim}{x}) =$

* closure under linear transf :

So if **Multivariate Gaussians** on $\mathbb{R}^n$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \sim N\left(\underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_M \end{bmatrix}}_{\substack{\text{mean} \\ \underset{\sim}{\mu} \in \mathbb{R}^n}} , \underbrace{\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \cdots & \Sigma_{2n} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \cdots & \Sigma_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \Sigma_{n3} & \cdots & \Sigma_{nn} \end{bmatrix}}_{\substack{\text{covariance} \\ \underset{\sim}{\Sigma} \in \mathbb{R}^n \times \mathbb{R}^n}} \right)$$

pos. def.

joint PDF $$p(\underset{\sim}{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\underset{\sim}{\Sigma})}} \exp\left\{ -\frac{1}{2} (\underset{\sim}{x} - \underset{\sim}{\mu})^T \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) \right\}$$

- addition & subtraction :

So if Multivariate Gaussians on $\mathbb{R}^n$

$$\underset{\sim}{X} = \begin{bmatrix} \boxed{X_1} \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} \sim N\left(\begin{bmatrix} \boxed{\mu_1} \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_M \end{bmatrix}, \begin{bmatrix} \boxed{\Sigma_{11}} & \Sigma_{12} & \Sigma_{13} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \cdots & \Sigma_{2n} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \cdots & \Sigma_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \Sigma_{n3} & \cdots & \Sigma_{nn} \end{bmatrix} \right)$$

marginals are easy

So if Multivariate Gaussians on $\mathbb{R}^n$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \sim N\left(\overset{\mu_A}{\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_M \end{bmatrix}}, \overset{\Sigma_A}{\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \cdots & \Sigma_{2n} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \cdots & \Sigma_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \Sigma_{n3} & \cdots & \Sigma_{nn} \end{bmatrix}} \right)$$

marginals are easy :

So if

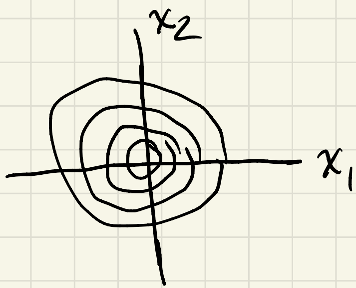
$$\mu_A \qquad \Sigma_A \qquad \Sigma_{AB}$$

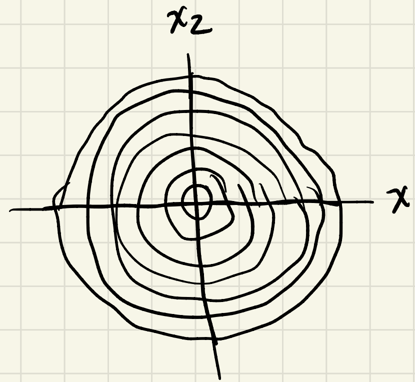
$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_N \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \cdots & \Sigma_{2n} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \cdots & \Sigma_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \Sigma_{n3} & \cdots & \Sigma_{nn} \end{bmatrix} \right.$$

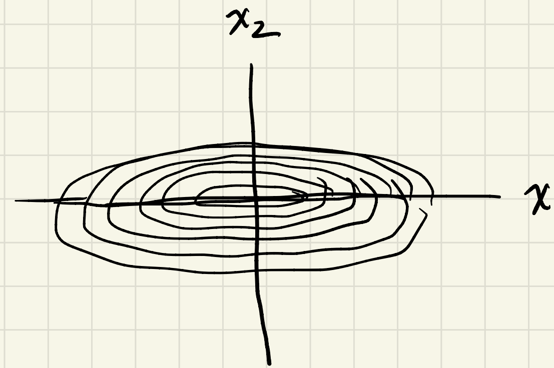
$$\mu_B \qquad \Sigma_{BA} \qquad \Sigma_B$$

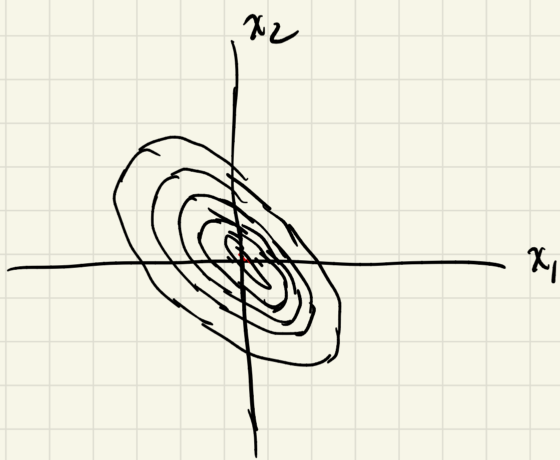
conditionals are also not so bad

Back to bivariate case



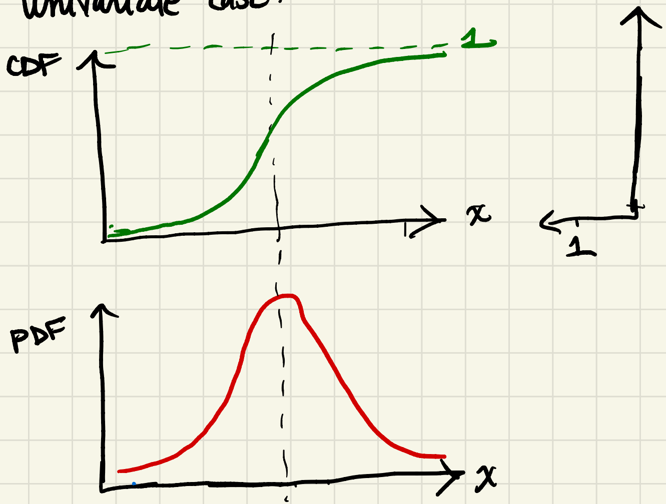






How do we draw samples from an arbitrary multivariate Gaussian?

univariate case:



$$X \sim N(0, 1)$$