TAM 598 Lecture 15 :

Advanced Topics in Bayesian Linear Regression

Announcements:

- HW 4 covers lectures 13-16; due on Fri Mar 4

updated!

last time: Bayesian Linear Regression

(1) Max Likelihood Estimation (MLE)

(2) Max A Posteriori Estimation (MPE)

(3) Bayesian Linear Regression

Topics

(1) Evidence Approximation to estimate (non-weight) parameters like noise variance σ^2 and other hyperparameters α

(2) Automatic Relevance Determination to select important basis functions → $l, \ldots$

(3) Model quality assessment using standardized errors and quantile-quantile charts

Need to test on previously unseen data

(1) Evidence Approximation

recall: Bayesian linear regression - can separate out epistemic and aleatoric uncertainty, but requires us to choose several parameters by hand:

- precision α
- noise variance σ^2
- basis function parameters ℓ

call these **hyperparameters**

$\theta = \{\alpha, \sigma^2, \ell, \dots\}$

$$[w_0\ w_1\ w_2]\begin{bmatrix}\phi_0(x)\\ \phi_1(x)\\ \phi_2(x)\end{bmatrix}$$

our model: $y(\underline{x};\underline{w}) = \underline{w}^T\,\underline{\phi}(\underline{x})$

data likelihood: $\underline{y}_{1:n} \mid \underline{x}_{1:n}, \underline{w}, \underline{\theta} \sim p(\underline{y}_{1:n} \mid \underline{x}_{1:n}, \underline{w}, \underline{\theta})$

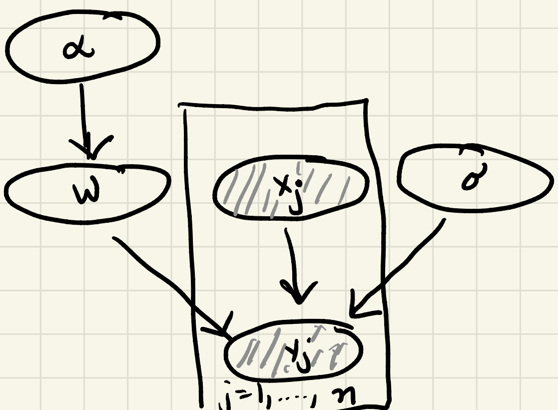
weight prior: $\underline{w} \mid \underline{\theta} \sim p(\underline{w} \mid \underline{\theta})$

* hyper-prior: $\underline{\theta} \sim p(\underline{\theta})$

BAYES: $P(A|B)\,p(B) = p(B|A)\,p(A)$

$A = \underline{w}, \underline{\theta}$

$B = \underline{x}_{1:n}, \underline{y}_{1:n}$



The full posterior

$$p(\underline{w},\underline{\theta} \mid \underline{x}_{1:n}, \underline{y}_{1:n}) \propto p(\underline{y}_{1:n}, \underline{x}_{1:n} \mid \underline{w}, \underline{\theta})\, p(\underline{w}, \underline{\theta})$$

$$\boxed{p(\underline{y}_{1:n} \mid \underline{x}_{1:n} \mid \underline{w}, \underline{\theta})\, p(\underline{w} \mid \underline{\theta})\, p(\underline{\theta})}$$

consider the marginal posterior of $\underline{\theta}$:

$\int p(w, \theta \mid x, y)\, dw$

$$p(\underline{\theta} \mid \underline{x}_{1:n}, \underline{y}_{1:n}) \propto \int p(\underline{y}_{1:n} \mid \underline{x}_{1:n} \mid \underline{w}, \underline{\theta})\, p(\underline{w} \mid \underline{\theta})\, p(\underline{\theta})\, d\underline{w}$$

Assume a flat hyperprior $p(\underline{\theta}) \propto 1$.

Use a max a posteriori estimate for $\underline{\theta}$:

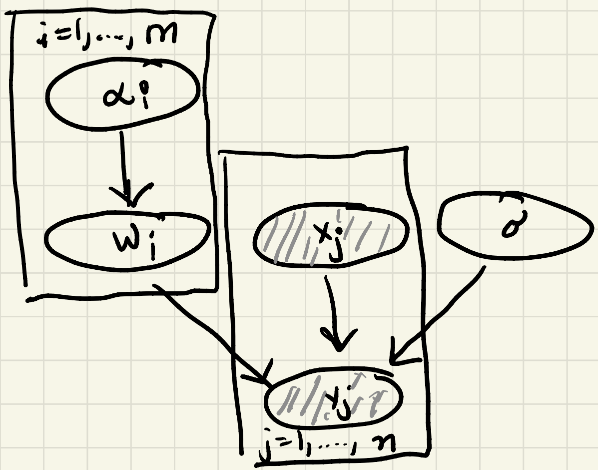
$$\theta_{EV} = \underset{\theta}{\operatorname{argmax}} \int p(\underline{y}_{1:n} \mid \underline{x}_{1:n} \mid \underline{w}, \underline{\theta})\, p(\underline{w}, \underline{\theta})\, d\underline{w}$$

$\rightarrow$ is analytic for Gaussian likelihood and prior

(2) **Automatic Relevance Determination** - how to select which basis functions to keep and which not.

approach: use a different precision α_i for each weight w_i corresponding to each basis function ϕ_i

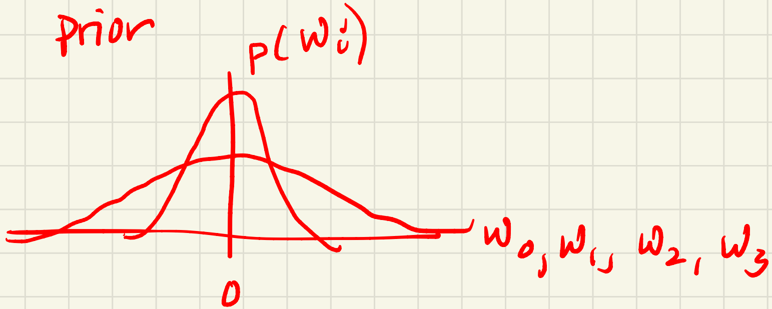
$$p(w_i \mid \alpha_i) \propto \exp(-\alpha_i w_i^2)$$



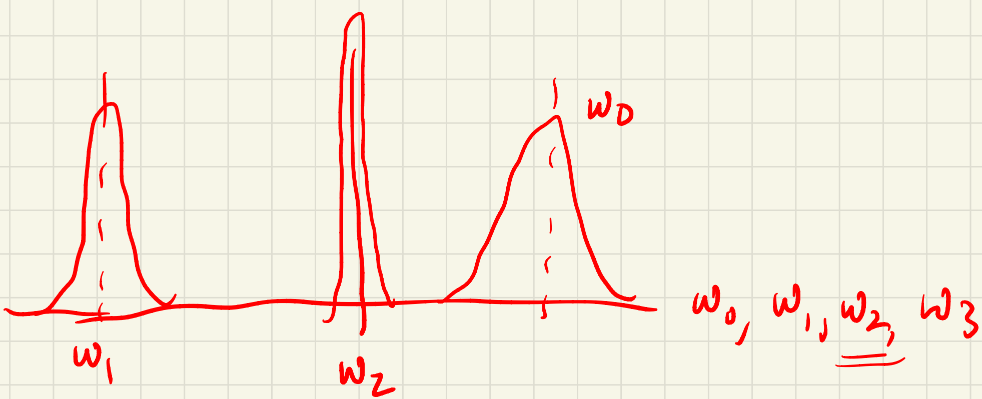
in practice:

- evidence approx to find all hyperparameters $\theta = \{\alpha_i, \sigma^2, \ell, \dots\}$
- the precisions α_i of the basis functions that are not needed become very large
- consequently, posteriors of basis function weights collapse to delta function centered at zero

prior



posterior



(3) Model Quality Assessment - need a validation dataset of inputs $\underline{x}^v_{1:n} = (\underline{x}^v_1, \dots, \underline{x}^v_n)$ and outputs $\underline{y}^v_{1:n} = (y^v_1, y^v_2, \dots y^v_n)$

statistical diagnostics to compare the predictive distribution to the distribution of the validation dataset

given our gaussian predictive distribution with posterior predictive mean $\underline{m}(\underline{x})$ and posterior predictive variance $s^2(\underline{x})$, define

standardized errors $$e_i = \frac{y^v_i - m(\underline{x}^v_i)}{s(\underline{x}^v_i)}$$

If our model is correct, these errors should be distributed as $N(0, 1)$ since (if model is correct) $y^v_i \sim N(m(\underline{x}^v_i), \sigma^2(\underline{x}^v_i))$

Approach: calculate $\{e_i\}$, see if $\{e_i\} \sim N(0, 1)$