TAM 598 Lecture 19 :

State Space Models – Filtering

Announcements:

- HW 5 covers lectures 17-20 ; due on Fri Apr 18

State Space Models - Filtering

* given a dynamical system, we observe a noisy measurement of the system's state and want to figure out the actual state

Our stochastic dynamical system is described by **state variables** $\underset{\sim}{x}_t$ at discrete time steps $t = 0, 1, 2, \ldots, n$.

Given a closed system (no external perturbations) then

transition probability

$$\boxed{p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{x}_{0:t}) = p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{x}_t)}$$

Markov property - state at $t+1$ depends ONLY on the state at t.

This means:

$$\begin{cases} p(\underset{\sim}{x}_{0:1}) = \underline{p(\underset{\sim}{x}_0)\, p(\underset{\sim}{x}_1 \mid \underset{\sim}{x}_0)} \\ p(\underset{\sim}{x}_{0:2}) = \underline{p(\underset{\sim}{x}_{0:1})}\, \underline{p(\underset{\sim}{x}_2 \mid \underset{\sim}{x}_{0:1})} \\ \qquad = \underline{p(\underset{\sim}{x}_0)\, p(\underset{\sim}{x}_1 \mid \underset{\sim}{x}_0)\, p(\underset{\sim}{x}_2 \mid \underset{\sim}{x}_1)} \\ p(\underset{\sim}{x}_{0:3}) = \underline{p(\underset{\sim}{x}_{0:2})}\, \underline{p(\underset{\sim}{x}_3 \mid \underset{\sim}{x}_{0:2})} \end{cases}$$

$$= \underline{p(\underset{\sim}{x}_0)}\, \underline{p(\underset{\sim}{x}_1 \mid \underset{\sim}{x}_0)\, p(\underset{\sim}{x}_2 \mid \underset{\sim}{x}_1)\, p(\underset{\sim}{x}_3 \mid \underset{\sim}{x}_2)}$$

Joint prob. density over all sequences of states

$$\boxed{p(\underset{\sim}{x}_{0:n}) = p(\underset{\sim}{x}_0) \prod_{t=0}^{n-1} p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{x}_t)}$$

Now we introduce observations - sensors that measure something $\underset{\sim}{y}_t$ at each time step.

Assume $\underset{\sim}{y}_t$ only depends on the state of the system at time t, $\underset{\sim}{x}_t$

$$\boxed{p(\underset{\sim}{y}_t \mid \underset{\sim}{x}_{0:t}) = p(\underset{\sim}{y}_t \mid \underset{\sim}{x}_t)}$$

emission probability

Then the joint probability density of the system state and the observations is

$$\boxed{p(\underset{\sim}{x}_{0:n}, \underset{\sim}{y}_{1:n}) = p(\underset{\sim}{x}_0) \prod_{t=0}^{n-1} p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{x}_t)\, p(\underset{\sim}{y}_t \mid \underset{\sim}{x}_t)}$$

Next we introduce controls - at every timestep, we pass a control command $\underset{\sim}{u}_t$ to the system that affects where the system state goes in the next time step.

For a Markovian system

transition prob. w/ controls

$$\boxed{p\left(\underset{\sim}{x}_{t+1} \,\middle|\, \underset{\sim}{x}_{0:t}, \underset{\sim}{u}_{0:t}\right) = p\left(\underset{\sim}{x}_{t+1} \,\middle|\, \underset{\sim}{x}_t, \underset{\sim}{u}_t\right)}$$

and the joint probability density is

$$\boxed{p\left(\underset{\sim}{x}_{0:n}, \underset{\sim}{y}_{1:n} \,\middle|\, \underset{\sim}{u}_{0:n-1}\right) = p\left(\underset{\sim}{x}_0\right) \prod_{t=0}^{n-1} p\left(\underset{\sim}{x}_{t+1} \,\middle|\, \underset{\sim}{x}_t, \underset{\sim}{u}_t\right) p\left(\underset{\sim}{y}_t \,\middle|\, \underset{\sim}{x}_t\right)}$$

Filtering Problem - estimate the current state given all data $p(\underset{\sim}{x}_n \mid \underset{\sim}{y}_{1:n}, \underset{\sim}{u}_{0:n})$

Smoothing - estimating all states (including the past) given all data

$$p(\underset{\sim}{x}_{0:n} \mid \underset{\sim}{y}_{1:n}, \underset{\sim}{u}_{0:n-1})$$

$$p(A \mid B, C) = p(A, B \mid C) / p(B \mid C)$$

$$= \frac{p(\underset{\sim}{x}_{0:n}, \underset{\sim}{y}_{1:n} \mid \underset{\sim}{u}_{0:n-1})}{p(\underset{\sim}{y}_{1:n} \mid \underset{\sim}{u}_{0:n-1})}$$

$$\propto \; p(\underset{\sim}{x}_0) \prod_{t=0}^{n-1} p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{x}_t, \underset{\sim}{u}_t)\, p(\underset{\sim}{y}_t \mid \underset{\sim}{x}_t)$$

Example: linear transitions and gaussian emission probabilities via equations

↳ initial conditions $\underset{\sim}{x}_0 = \underset{\sim}{\mu}_0 + \underset{\sim}{z}_0$

fixed parameter

$\underset{\sim}{z}_0 \sim N(\underset{-}{0}, \underline{\underline{V}}_0)$

↳ transitions $\underset{\sim}{x}_{t+1} = \boxed{\underline{\underline{A}}\, \underset{\sim}{x}_t + \underline{\underline{B}}\, \underset{\sim}{u}_t} + \underset{\sim}{z}_t$

$\underset{\sim}{z}_t \sim N(0, \underline{\underline{Q}})$ where $\underline{\underline{Q}}$ is process covariance

↳ emissions

$\underset{\sim}{y}_t = \underline{\underline{C}}\, \underset{\sim}{x}_t + \underset{\sim}{w}_t$

$\hookrightarrow N(0, \underline{\underline{R}})$

$\underline{\underline{R}}$ is our measurement covariance

Example: linear transitions and gaussian emission probabilities via probabilistic modeling

↳ initial conditions

$$p(\underset{\sim}{x}_0) = \mathcal{N}\left(\underset{\sim}{x}_0 \,\middle|\, \underset{\sim}{\mu}_0, \underset{=}{V}_0\right)$$

we assume $\underset{=}{A}$, $\underset{=}{B}$, $\underset{=}{C}$, $\underset{=}{V}_0$, $\underset{-}{\mu}_0$, $\underset{=}{Q}$, $\underset{=}{R}$ are known.

↳ transitions

$$p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{x}_t, \underset{\sim}{u}_t) = \mathcal{N}\left(\underset{\sim}{x}_{t+1} \,\middle|\, \underset{=}{A}\,\underset{\sim}{x}_t + \underset{=}{B}\,\underset{-}{u}_t, \underset{=}{Q}\right)$$

↳ emissions

$$p(\underset{\sim}{y}_t \mid \underset{\sim}{x}_t) = \mathcal{N}\left(\underset{\sim}{y}_t \,\middle|\, \underset{=}{C}\,\underset{-}{x}_t, \underset{=}{R}\right)$$