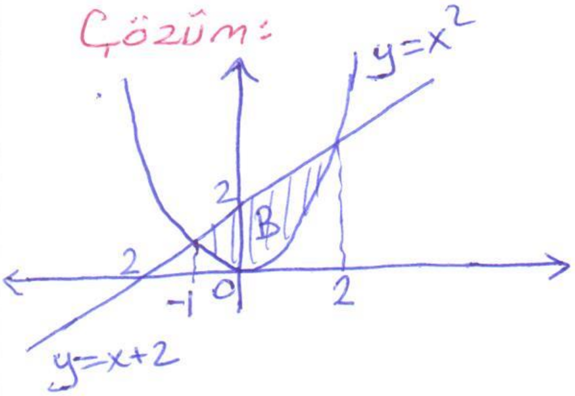
# İki Katlı İntegral Uygulamaları

Kartezyen koordinatlarda bir $B$ bölgesinin alanı $\iint_B dx\,dy$ formülü ile hesaplanır.

Tabanı $B$ bölgesi üzerinde olan $z=f(x,y)$ yüzeyinin hacmi $\iint_B f(x,y)\,dx\,dy$ formülü ile hesaplanır.

**Örnek:** $y=x^2$ parabolü ile $y=x+2$ doğrusu arasında kalan bölgenin alanını bulunuz.

**Çözüm:**



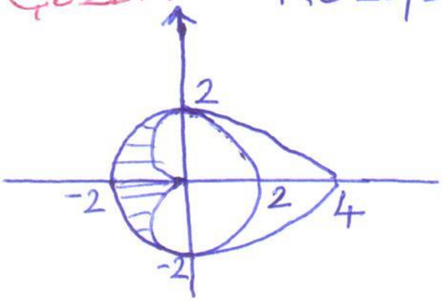

$$A=\iint_B dx\,dy=\int_{-1}^{2}\left(\int_{x^2}^{x+2} dy\right)dx$$

$$=\int_{-1}^{2}\left(x+2-x^2\right)dx$$

$$=\frac{9}{2}\ br^2$$

**Örnek:** $r=2(1+\cos\theta)$ kardioidinin dışında ve $r=2$ çemberinin içinde kalan bölgenin alanını bulunuz.

**Çözüm:** Kutupsal koordinatlarda $J=r$ olduğundan




$$A=\iint_B dx\,dy=\iint_D r\,dr\,d\theta$$

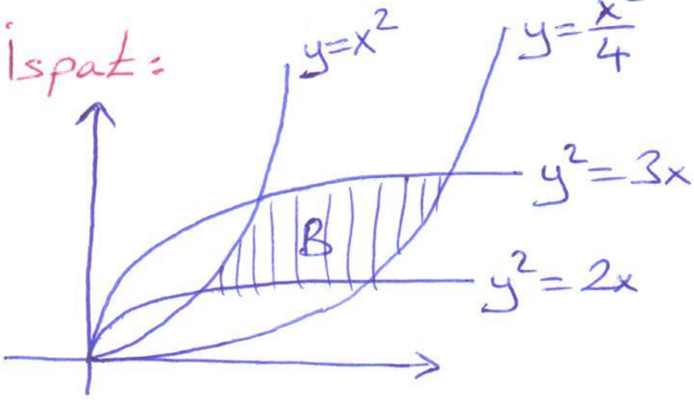
$$=\int_{\pi/2}^{3\pi/2}\left(\int_{2(1+\cos\theta)}^{2} r\,dr\right)d\theta$$

$$=\int_{\pi/2}^{3\pi/2}\left[\frac{r^2}{2}\Big|_{2(1+\cos\theta)}^{2}\right]d\theta$$

$$=\frac{1}{2}\int_{\pi/2}^{3\pi/2}\left[4-4(1+\cos\theta)^2\right]d\theta=8-\pi\ br^2$$

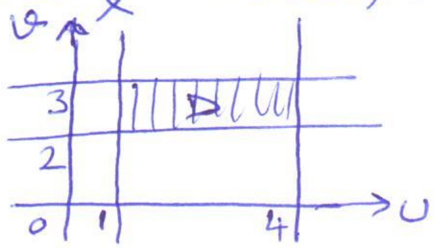
**Örnek:** $y=x^2$, $x^2=4y$, $y^2=2x$, $y^2=3x$ parabolleri tarafından sınırlanan bölgenin alanını bulunuz.

**İspat:**




$$u=\frac{x^2}{y} \Rightarrow u=1,\ u=4$$

$$v=\frac{y^2}{x} \Rightarrow v=2,\ v=3$$

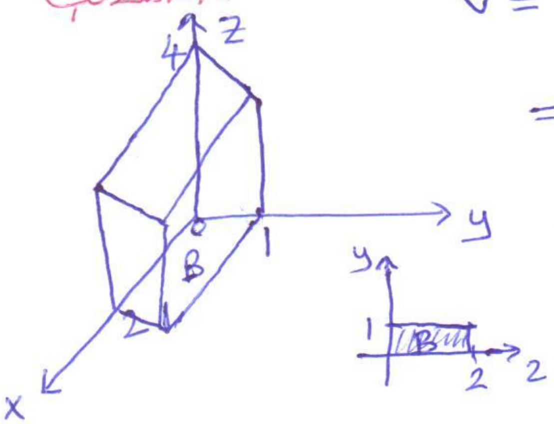



$$j=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}}=\frac{1}{\begin{vmatrix} \frac{2x}{y} & \frac{-x^2}{y^2} \\ \frac{-y^2}{x^2} & \frac{2y}{x} \end{vmatrix}}=\frac{1}{3}$$

$$A=\iint_B dx\,dy=\iint_D \frac{1}{3}\,du\,dv=\int_2^3\left(\int_1^4 \frac{1}{3}\,du\right)dv=\int_2^3 dv=1 \text{ br}^2$$

**Örnek:** xy – düzlemindeki $0\leq x\leq 2$, $0\leq y\leq 1$ bölgesinin üzerinde $z=4-x-y$ düzleminin altındaki hacmi hesaplayınız.

**Çözüm:**




$$V=\iint_B (4-x-y)\,dx\,dy$$

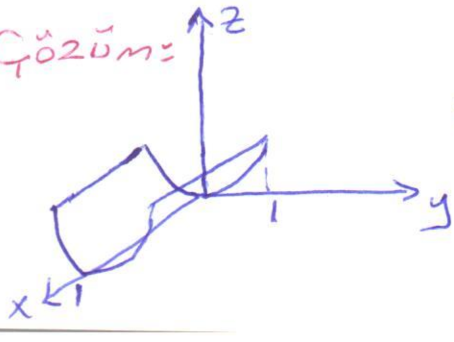
$$=\int_0^1\left(\int_0^2 (4-x-y)\,dx\right)dy$$

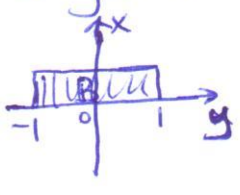
$$=\int_0^1\left[4x-\frac{x^2}{2}-xy\Big|_0^2\right]dy$$

$$=\int_0^1 (6-2y)\,dy=6y-y^2\Big|_0^1=5 \text{ br}^3$$

**Örnek:** $z=y^2$ silindiri ile $x=0$, $z=0$, $y=-1$, $x=1$, $y=1$ düzlemleri arasında kalan bölgenin hacmini hesaplayınız.

**Çözüm:**



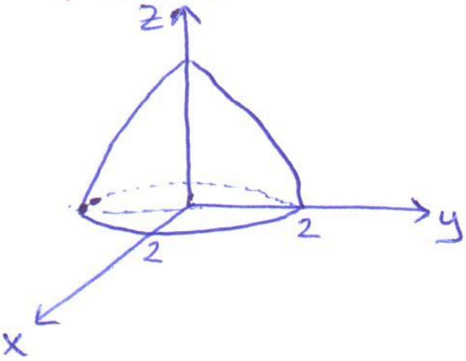

$$V=\iint_B y^2\,dx\,dy$$

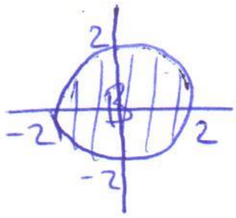
$$=\int_0^1\left(\int_{-1}^1 y^2\,dy\right)dx$$

$$=\int_0^1\left(\frac{y^3}{3}\Big|_{-1}^1\right)dx=\int_0^1 \frac{2}{3}\,dx=\frac{2}{3} \text{ br}^3$$

**Örnek:** $z=4-x^2-y^2$ paraboloidi ile XOY düzlemi arasında kalan bölgenin hacmini bulunuz.

**Çözüm:**

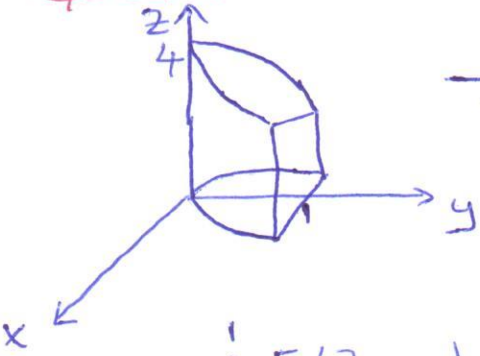


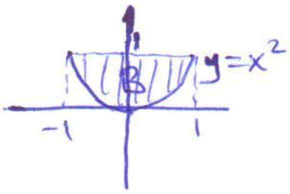




$$V=\iint_B (4-x^2-y^2)\,dx\,dy$$
$$=\int_0^{2\pi}\left(\int_0^2 (4-r^2)\,r\,dr\right)d\theta$$
$$=\int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\Big|_0^2\right]d\theta$$
$$=\int_0^{2\pi} 4\,d\theta$$
$$=8\pi \text{ br}^3$$

**Örnek:** $y=x^2$, $y=1$, $x+y+z=4$, $z=0$ yüzeyleri ile sınırlı bölgenin hacmini bulunuz.

**Çözüm:**

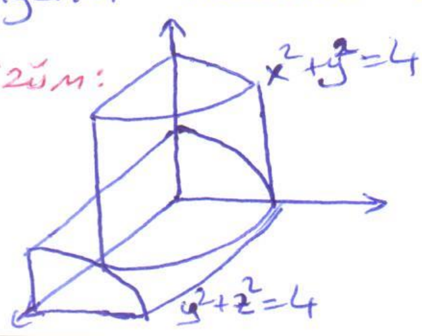


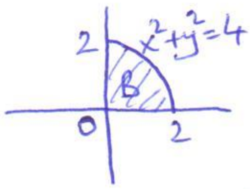




$$V=\iint_B (4-x-y)\,dx\,dy$$
$$=\int_{-1}^{1}\left(\int_{x^2}^{1}(4-x-y)\,dy\right)dx$$
$$=\int_{-1}^{1}\left[4y-xy-\frac{y^2}{2}\Big|_{x^2}^{1}\right]dx$$
$$=\int_{-1}^{1}\left[\left(\frac{7}{2}-x\right)-\left(4x^2-x^3-\frac{x^4}{2}\right)\right]dx=\frac{7x}{2}-\frac{x^2}{2}-\frac{4x^3}{3}+\frac{x^4}{4}+\frac{x^5}{10}\Big|_{-1}^{1}$$
$$=\frac{68}{15}\text{ br}^3$$

**Örnek:** $x^2+y^2=4$ ve $y^2+z^2=4$ silindirleri arasında kalan bölgenin hacmini bulunuz.

**Çözüm:**




$$\frac{V}{8}=\iint_B \sqrt{4-y^2}\,dx\,dy$$

$$\Rightarrow V=8\iint_B \sqrt{4-y^2}\,dx\,dy$$

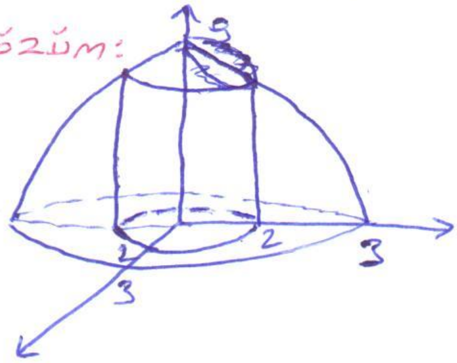
$$V = 8\int_0^2 \left[ \int_0^{\sqrt{4-y^2}} \sqrt{4-y^2}\, dx \right] dy$$

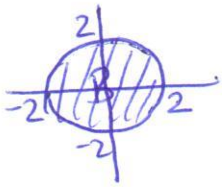
$$= 8\int_0^2 \left[ x\sqrt{4-y^2} \Big|_0^{\sqrt{4-y^2}} \right] dy$$

$$= 8\int_0^2 (4-y^2)\, dy = 8\left(4y - \frac{y^3}{3}\Big|_0^2\right) = 8\left(8-\frac{8}{3}\right) = \frac{128}{3}\ br^3$$

Örnek: $z = 9 - x^2 - y^2$ paraboloidi, XOY düzlemi ve $x^2+y^2=4$ silindiri tarafından sınırlanan bölgenin hacmini bulunuz.

Çözüm:



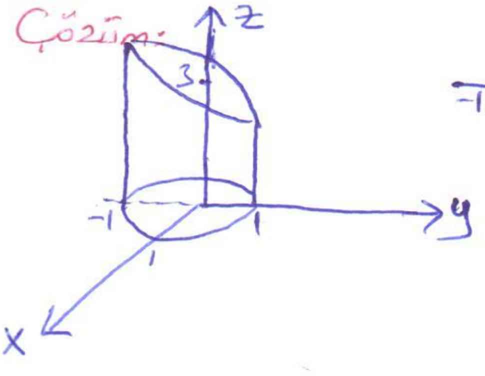

$$V = \iint_B (9-x^2-y^2)\, dx\, dy$$

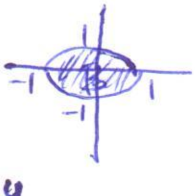
$$= \int_0^{2\pi} \left( \int_0^2 (9-r^2) r\, dr \right) d\theta$$

$$= \int_0^{2\pi} \left[ \frac{9r^2}{2} - \frac{r^4}{4} \Big|_0^2 \right] d\theta = \int_0^{2\pi} 14\, d\theta = 28\pi\ br^3$$

Örnek: $x+y+z=3$, $x^2+y^2=1$, $z=0$ yüzeyleri tarafından sınırlanan bölgenin hacmini bulunuz.

Çözüm:




$$V = \iint_B (3-x-y)\, dx\, dy$$

$$= \int_0^{2\pi} \left( \int_0^1 (3 - r\cos\theta - r\sin\theta) r\, dr \right) d\theta$$

$$= \int_0^{2\pi} \left[ \frac{3r^2}{2} - \frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta \Big|_0^1 \right] d\theta$$

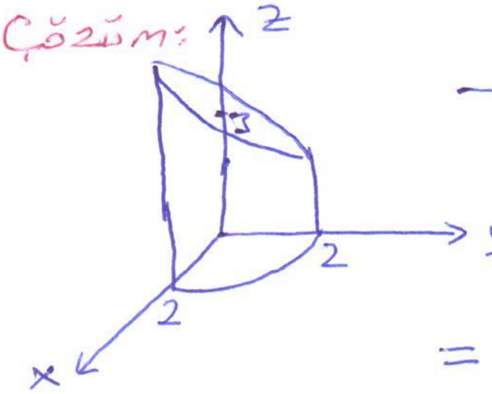
$$= \int_0^{2\pi} \left( \frac{3}{2} - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta \right) d\theta$$

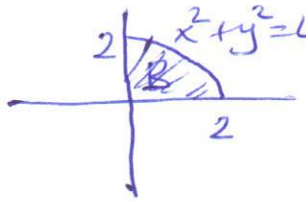
$$= \frac{3\theta}{2} - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta \Big|_0^{2\pi}$$

$$= 3\pi\ br^3$$

**Örnek:** Birinci sekizde birlik bölgede koordinat düzlemleri, $x^2+y^2=4$ silindiri ve $z+y=3$ düzlemi ile sınırlanan cismin hacmini bulunuz.

**Çözüm:**




$$V=\iint_B (3-y)\,dx\,dy$$

$$=\int_0^{\pi/2}\left(\int_0^2 (3-r\sin\theta)\,r\,dr\right)d\theta$$

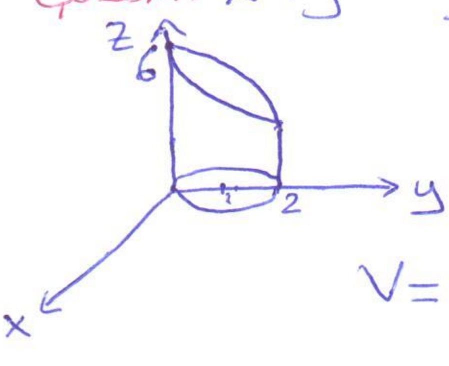
$$=\int_0^{\pi/2}\left[\frac{3r^2}{2}-\frac{r^3}{3}\sin\theta\Big|_0^2\right]d\theta$$

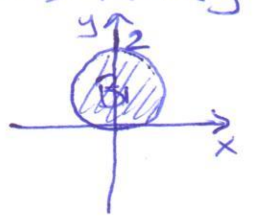
$$=\int_0^{\pi/2}\left(6-\frac{8}{3}\sin\theta\right)d\theta=6\theta+\frac{8}{3}\cos\theta\Big|_0^{\pi/2}$$

$$=3\pi-\frac{8}{3}\ br^3$$

**Örnek:** $x^2+y^2=2y$ silindiri ile $z=0$ ve $x+y+z=6$ düzlemleri arasında kalan bölgenin hacmini bulunuz.

**Çözüm:** $x^2+y^2=2y \Rightarrow x^2+(y-1)^2=1$




$$\left.\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array}\right\} \Rightarrow r^2=2r\sin\theta$$

$$r=0,\ r=2\sin\theta$$

$$V=\iint_B (6-x-y)\,dx\,dy=\int_0^{\pi}\left(\int_0^{2\sin\theta}(6-r\cos\theta-r\sin\theta)\,r\,dr\right)d\theta$$

$$=\int_0^{\pi}\left[3r^2-\frac{r^3}{3}\cos\theta-\frac{r^3}{3}\sin\theta\Big|_0^{2\sin\theta}\right]d\theta$$

$$=\int_0^{\pi}\left[12\sin^2\theta-\frac{8}{3}\sin^3\theta\cos\theta-\frac{8}{3}\sin^4\theta\right]d\theta$$

$$=\int_0^{\pi}12\cdot\frac{1-\cos 2\theta}{2}\,d\theta-\frac{8}{3}\int_0^{\pi}\sin^3\theta\cos\theta\,d\theta-\frac{8}{3}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$=6\left[\theta-\frac{1}{2}\sin 2\theta\Big|_0^{\pi}\right]-\left(\frac{8}{3}\cdot\frac{\sin^4\theta}{4}\Big|_0^{\pi}\right)-\frac{2}{3}\int_0^{\pi}\left(\underbrace{\cos^2 2\theta}_{=\frac{1+\cos 4\theta}{2}}-2\cos 2\theta+1\right)d\theta$$

$$=5\pi\ br^3$$