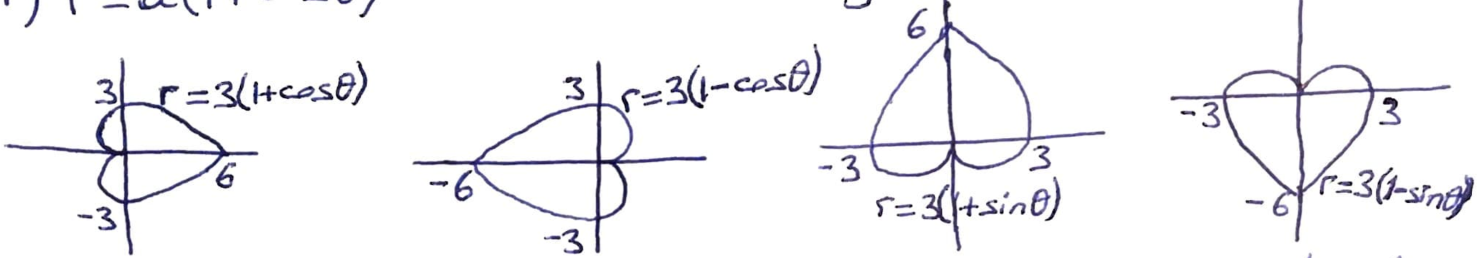
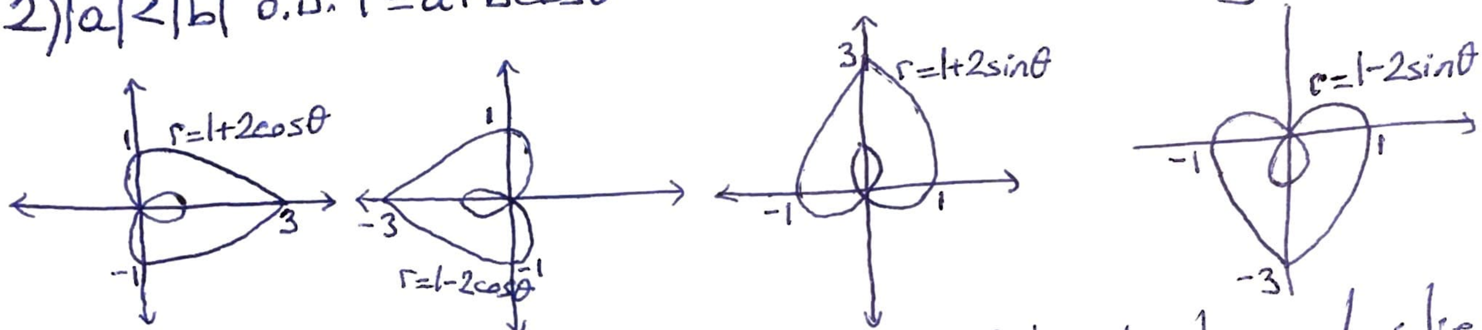
Bazı Temel Eğriler:

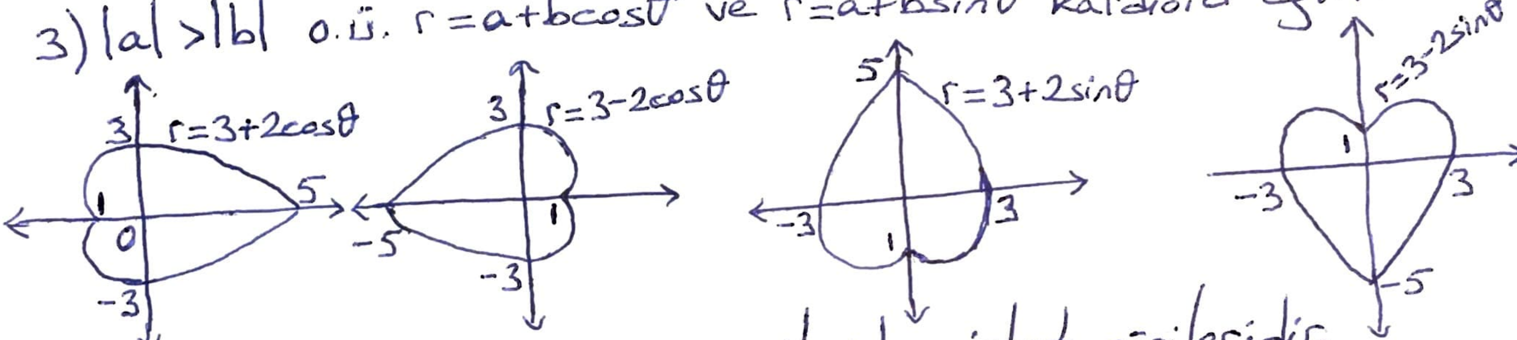
1) $r=a(1\mp\cos\theta)$ ve $r=a(1\mp\sin\theta)$ eğrileri kardioid eğrileridir.



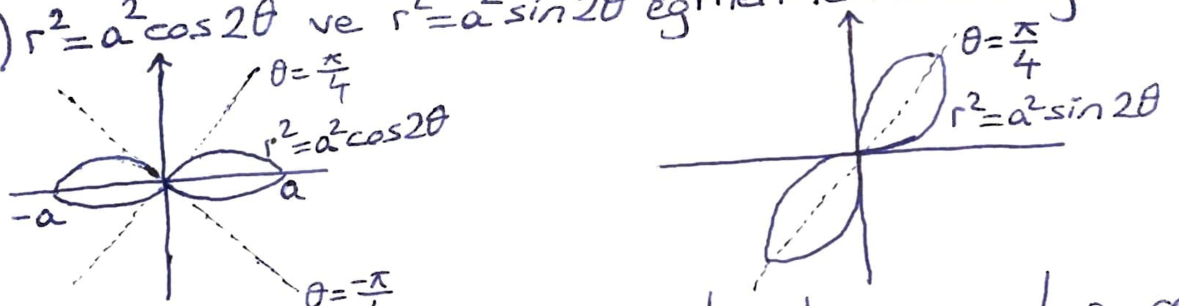
2) $|a|<|b|$ o.ü. $r=a+b\cos\theta$ ve $r=a+b\sin\theta$ limaçon eğrileridir.



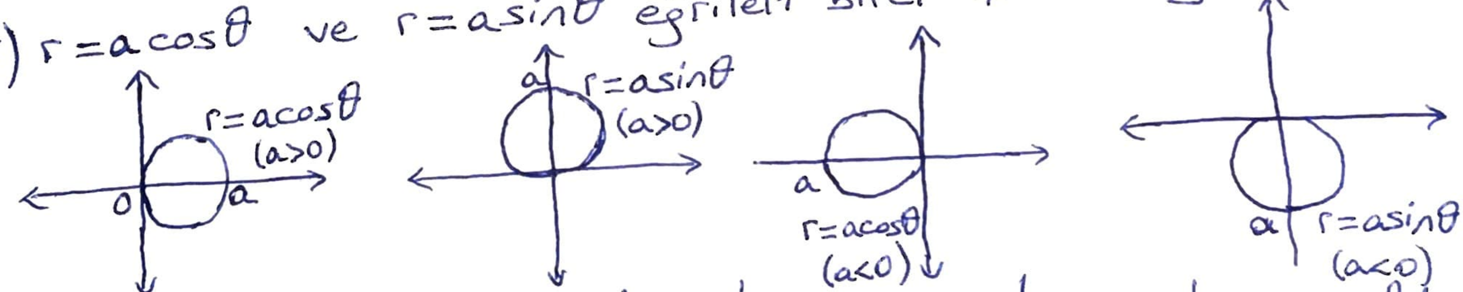
3) $|a|>|b|$ o.ü. $r=a+b\cos\theta$ ve $r=a+b\sin\theta$ kardioid eğrileridir.



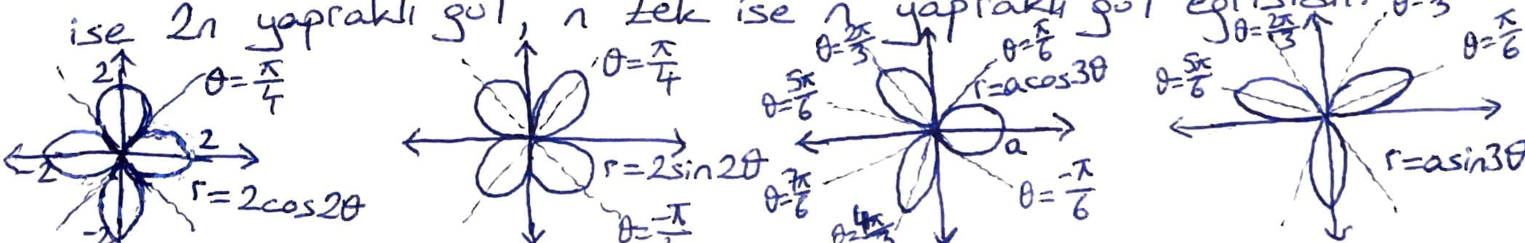
4) $r^2=a^2\cos 2\theta$ ve $r^2=a^2\sin 2\theta$ eğrileri lemniskat eğrileridir.



5) $r=a\cos\theta$ ve $r=a\sin\theta$ eğrileri birer çember gösterir.

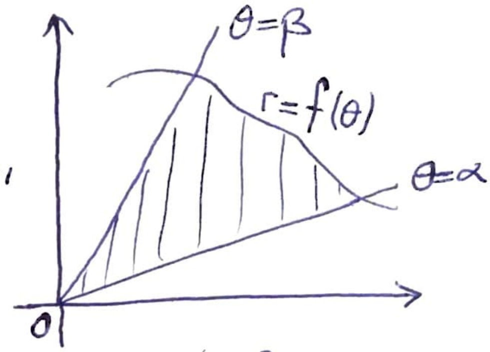


6) $r=a\cos(n\theta)$ ve $r=a\sin(n\theta)$ eğrileri birer gül eğrisidir. n çift ise $2n$ yapraklı gül, n tek ise n yapraklı gül eğrisidir.



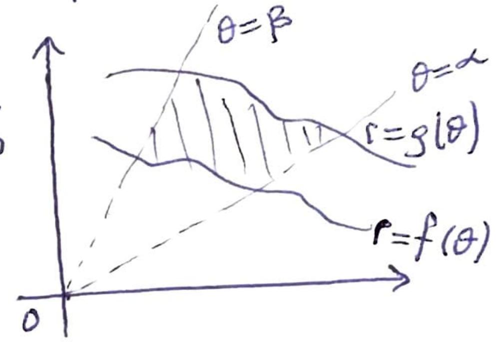
Kutupsal Koordinatlarda Alan Hesabı:

f sürekli bir fonksiyon olmak üzere $r=f(\theta)$ eğrisi, $\theta=\alpha$ ve $\theta=\beta$ doğruları ile sınırlı bölgenin alanı



$$A=\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta=\frac{1}{2}\int_{\alpha}^{\beta}\left(f(\theta)\right)^2 d\theta$$

olur. Ayrıca, $r=f(\theta)$ ve $r=g(\theta)$ eğrileri, $\theta=\alpha$ ve $\theta=\beta$ doğruları tarafından sınırlanan bölgenin alanı



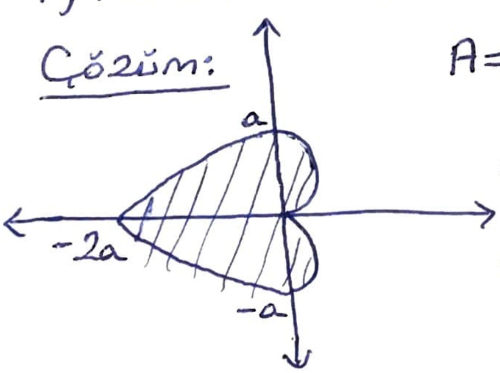
$$A=\frac{1}{2}\int_{\alpha}^{\beta}\left|(g(\theta))^2-(f(\theta))^2\right| d\theta=\frac{1}{2}\int_{\alpha}^{\beta}\left(r_{uzak}^2-r_{yakın}^2\right)d\theta$$

olur.

Örnekler:

1) $r=a(1-\cos\theta)$ kardioidinin sınırladığı bölgenin alanını bulunuz.

Çözüm:



$$A=\frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2\,d\theta$$

$$=2\cdot\frac{a^2}{2}\int_0^{\pi}\left(\cos^2\theta-2\cos\theta+1\right)d\theta$$

$$=a^2\int_0^{\pi}\left[\frac{1}{2}(1+\cos 2\theta)-2\cos\theta+1\right]d\theta$$

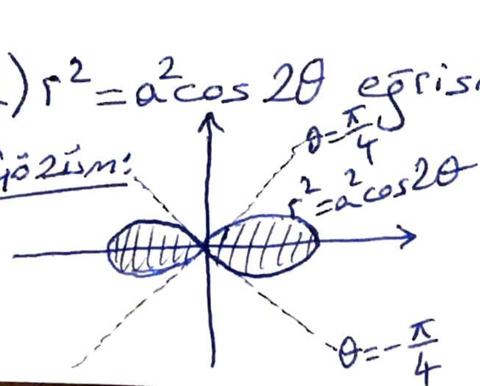
$$=a^2\int_0^{\pi}\left(\frac{1}{2}\cos 2\theta-2\cos\theta+\frac{3}{2}\right)d\theta$$

$$=a^2\left(\frac{1}{4}\sin 2\theta-2\sin\theta+\frac{3\theta}{2}\right\Big|_0^{\pi}$$

$$=\frac{3\pi a^2}{2}\ \text{br}^2$$

2) $r^2=a^2\cos 2\theta$ eğrisinin içinde kalan bölgenin alanını bulunuz.

Çözüm:

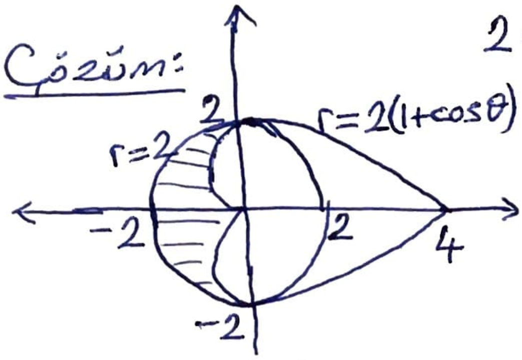


$$A=4\cdot\frac{1}{2}\int_0^{\pi/4} a^2\cos 2\theta\,d\theta$$

$$=2a^2\cdot\frac{1}{2}\sin 2\theta\Big|_0^{\pi/4}=a^2\ \text{br}^2$$

3) $r=2$ çemberinin içinde $r=2(1+\cos\theta)$ kardioidinin dışında kalan bölgenin alanını bulunuz.

Çözüm:



$$2=2+2\cos\theta \Rightarrow \cos\theta=0 \Rightarrow \theta=\frac{\pi}{2},\ \theta=\frac{3\pi}{2}$$

$$A=2\cdot\frac{1}{2}\int_{\pi/2}^{\pi}\left[2^2-\left(2(1+\cos\theta)\right)^2\right]d\theta$$

$$=\int_{\pi/2}^{\pi}\left(-8\cos\theta-4\cos^2\theta\right)d\theta$$

$$=\int_{\pi/2}^{\pi}\left(-8\cos\theta-4\cdot\frac{1}{2}(1+\cos 2\theta)\right)d\theta$$

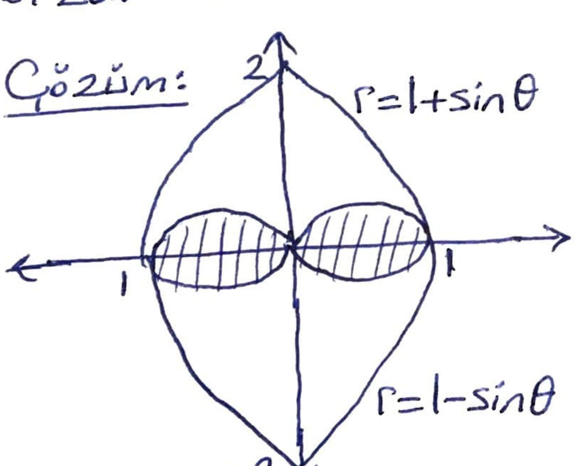
$$=-8\sin\theta-2\theta-\cancel{2}\cdot\frac{1}{\cancel{2}}\sin 2\theta\Big|_{\pi/2}^{\pi}$$

$$=-2\pi-(-8-\pi)$$

$$=8-\pi \quad br^2$$

4) $r=1+\sin\theta$ ve $r=1-\sin\theta$ eğrilerinin iç bölgelerinin ortak noktalarından oluşan bölgenin alanını bulunuz.

Çözüm:



$$A=2\cdot\frac{1}{2}\left[\int_{-\pi/2}^{0}(1+\sin\theta)^2d\theta+\int_{0}^{\pi/2}(1-\sin\theta)^2d\theta\right]$$

$$A=\int_{-\pi/2}^{0}\Big(\underbrace{\sin^2\theta}_{\frac{1}{2}(1-\cos2\theta)}+2\sin\theta+1\Big)d\theta+\int_{0}^{\pi/2}\Big(\underbrace{\sin^2\theta}_{\frac{1}{2}(1-\cos2\theta)}-2\sin\theta+1\Big)d\theta$$

$$A=\int_{-\pi/2}^{0}\left(-\frac{1}{2}\cos2\theta+2\sin\theta+\frac{3}{2}\right)d\theta+\int_{0}^{\pi/2}\left(\frac{-1}{2}\cos2\theta-2\sin\theta+3/2\right)d\theta$$

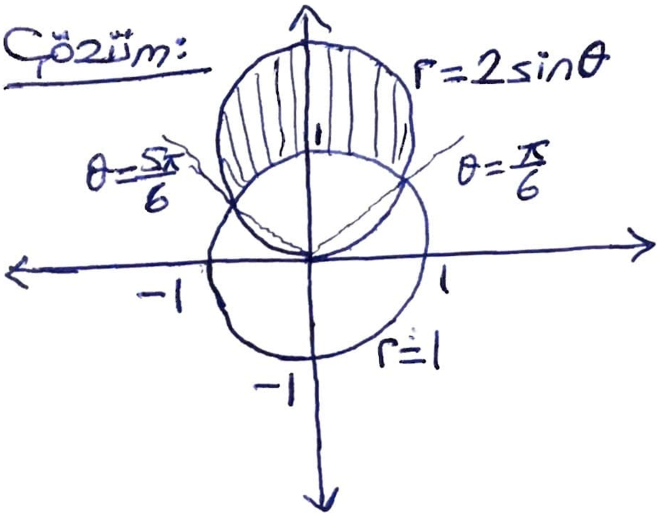
$$A=\left(-\frac{1}{4}\sin2\theta-2\cos\theta+\frac{3\theta}{2}\Big|_{-\pi/2}^{0}\right)+\left(\frac{-1}{4}\sin2\theta+2\cos\theta+\frac{3\theta}{2}\Big|_{0}^{\pi/2}\right)$$

$$=\left[-2-\left(-\frac{3\pi}{4}\right)\right]+\left(\frac{3\pi}{4}-2\right)$$

$$=\frac{3\pi}{2}-2 \quad br^2$$

5) $r=1$ çemberinin dışında $r=2\sin\theta$ çemberinin içinde kalan bölgenin alanını bulunuz.

Çözüm:



$2\sin\theta=1 \Rightarrow \theta=\frac{\pi}{6},\ \theta=\frac{5\pi}{6}$

$$A=\frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[(2\sin\theta)^2-1\right]d\theta$$

$$=\frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[4\cdot\frac{1}{2}(1-\cos 2\theta)-1\right]d\theta$$

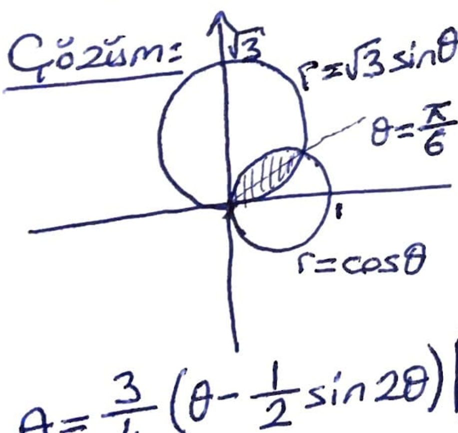
$$=\frac{1}{2}\int_{\pi/6}^{5\pi/6}(1-2\cos 2\theta)\,d\theta$$

$$A=\frac{1}{2}(\theta-\sin 2\theta)\Big|_{\pi/6}^{\frac{5\pi}{6}}=\frac{1}{2}\left[\left(\frac{5\pi}{6}+\frac{\sqrt{3}}{2}\right)-\left(\frac{\pi}{6}-\frac{\sqrt{3}}{2}\right)\right]$$

$$=\frac{\pi}{3}+\frac{\sqrt{3}}{2}\ \text{br}^2$$

6) $r=\cos\theta$ ve $r=\sqrt{3}\sin\theta$ çemberlerinin her ikisininde içinde kalan bölgenin alanını bulunuz.

Çözüm:



$\cos\theta=\sqrt{3}\sin\theta \Rightarrow \tan\theta=\frac{1}{\sqrt{3}} \Rightarrow \theta=\frac{\pi}{6}$

$$A=\frac{1}{2}\int_0^{\pi/6}(\sqrt{3}\sin\theta)^2\,d\theta+\frac{1}{2}\int_{\pi/6}^{\pi/2}\cos^2\theta\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/6}\frac{3}{2}(1-\cos 2\theta)\,d\theta+\frac{1}{2}\int_{\pi/6}^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

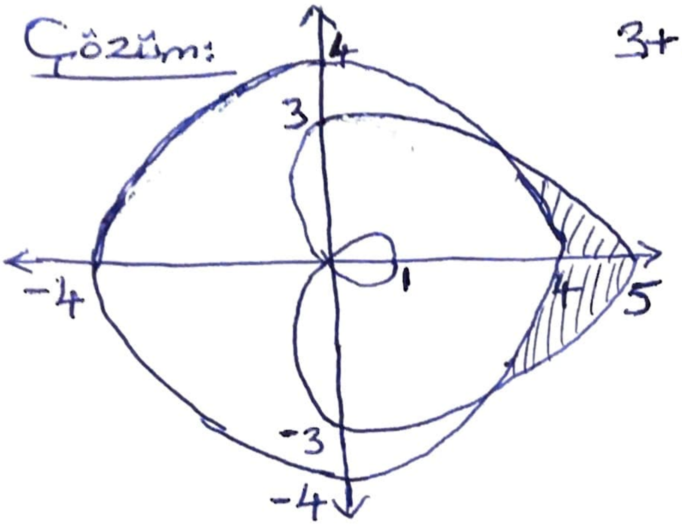
$$A=\frac{3}{4}\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/6}+\frac{1}{4}\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_{\pi/6}^{\pi/2}$$

$$=\frac{3}{4}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)+\frac{1}{4}\left[\frac{\pi}{2}-\left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)\right]$$

$$=\frac{5\pi}{24}-\frac{\sqrt{3}}{4}\ \text{br}^2$$

7) $r=3+2\cos\theta$ limaçonunun içinde, $r=4$ çemberinin dışında kalan bölgenin alanını bulunuz.

Çözüm:



$3+2\cos\theta=4 \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\mp\frac{\pi}{3}$

$$A=2\cdot\frac{1}{2}\int_0^{\pi/3}\left[(3+2\cos\theta)^2-4^2\right]d\theta$$

$$=\int_0^{\pi/3}(4\underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos2\theta)}+12\cos\theta-7)\,d\theta$$

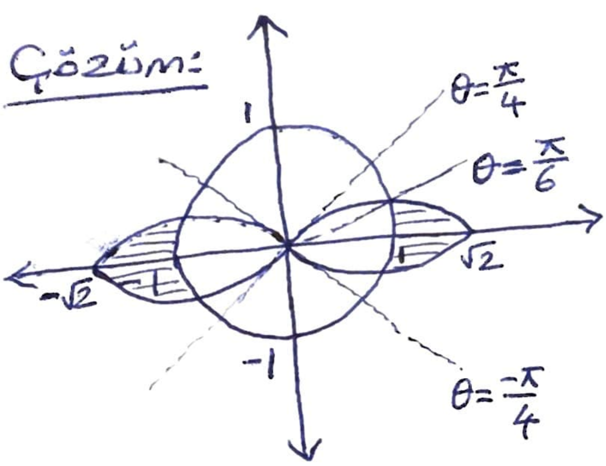
$$=\int_0^{\pi/3}(2\cos2\theta+12\cos\theta-5)\,d\theta$$

$$A=2\cdot\frac{1}{2}\sin2\theta+12\sin\theta-5\theta\Big|_0^{\pi/3}=\frac{\sqrt{3}}{2}+12\cdot\frac{\sqrt{3}}{2}-\frac{5\pi}{3}$$

$$=\frac{13\sqrt{3}}{2}-\frac{5\pi}{3}\ br^2$$

8) $r^2=2\cos2\theta$ lemniskatının içinde $r=1$ çemberinin dışında kalan bölgenin alanını bulunuz.

Çözüm:



$1=2\cos2\theta \Rightarrow \theta=\mp\frac{\pi}{6}$

$$A=4\cdot\frac{1}{2}\int_0^{\pi/6}(2\cos2\theta-1)\,d\theta$$

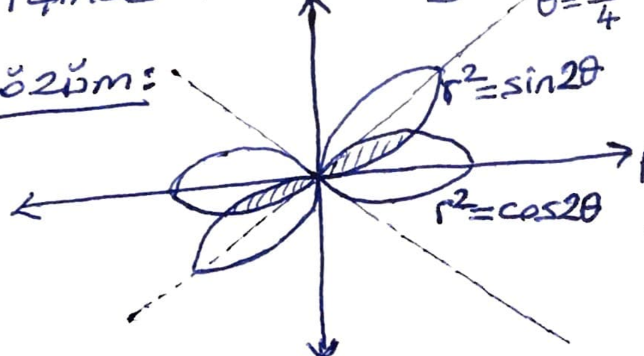
$$=2(\sin2\theta-\theta)\Big|_0^{\pi/6}$$

$$=2\left(\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)$$

$$=\sqrt{3}-\frac{\pi}{3}\ br^2$$

9) $r^2=\cos2\theta$ ve $r^2=\sin2\theta$ lemniskatlarının her ikisininde içinde kalan bölgenin alanını bulunuz.

Çözüm:



$\cos2\theta=\sin2\theta \Rightarrow \tan2\theta=1 \Rightarrow \theta=\frac{\pi}{8}$

$\theta=\frac{9\pi}{8}$

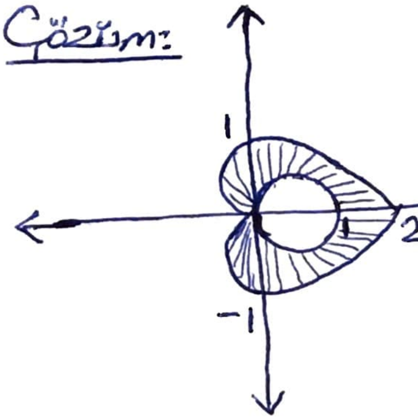
$$A=2\cdot\frac{1}{2}\left[\int_0^{\pi/8}\sin2\theta\,d\theta+\int_{\pi/8}^{\pi/4}\cos2\theta\,d\theta\right]$$

$$=\left(-\frac{1}{2}\cos2\theta\right)\Big|_0^{\pi/8}+\left(\frac{1}{2}\sin2\theta\right)\Big|_{\pi/8}^{\pi/4}$$

$$=1-\frac{\sqrt{2}}{2}\ br^2$$

10) $r=1+\cos\theta$ kardioidinin içinde $r=\cos\theta$ çemberinin dışında kalan bölgenin alanını bulunuz.

Çözüm:



$$A=2\cdot\frac{1}{2}\int_0^{\pi/2}\left[(1+\cos\theta)^2-\cos^2\theta\right]d\theta+2\cdot\frac{1}{2}\int_{\pi/2}^{\pi}(1+\cos\theta)^2 d\theta$$

$$=\int_0^{\pi/2}(1+2\cos\theta)d\theta+\int_{\pi/2}^{\pi}\Big(\underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos 2\theta)}+2\cos\theta+1\Big)d\theta$$

$$=\int_0^{\pi/2}(1+2\cos\theta)d\theta+\int_{\pi/2}^{\pi}\left(\frac{1}{2}\cos 2\theta+2\cos\theta+\frac{3}{2}\right)d\theta$$

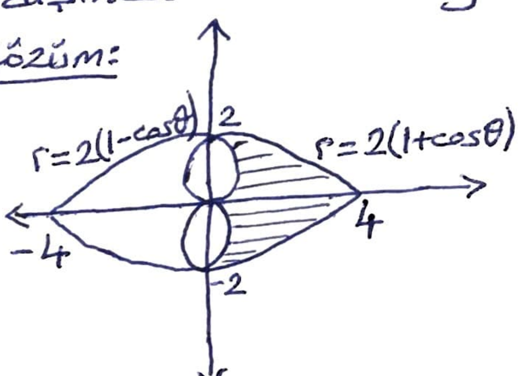
$$=(\theta+2\sin\theta)\Big|_0^{\pi/2}+\left(\frac{1}{4}\sin 2\theta+2\sin\theta+\frac{3\theta}{2}\right|_{\pi/2}^{\pi}$$

$$=\frac{\pi}{2}+2+\left[\frac{3\pi}{2}-\left(2+\frac{3\pi}{4}\right)\right]$$

$$=\frac{5\pi}{4}\ br^2$$

11) $r=2(1+\cos\theta)$ kardioidinin içinde $r=2(1-\cos\theta)$ kardioidinin dışında kalan bölgenin alanını bulunuz.

Çözüm:



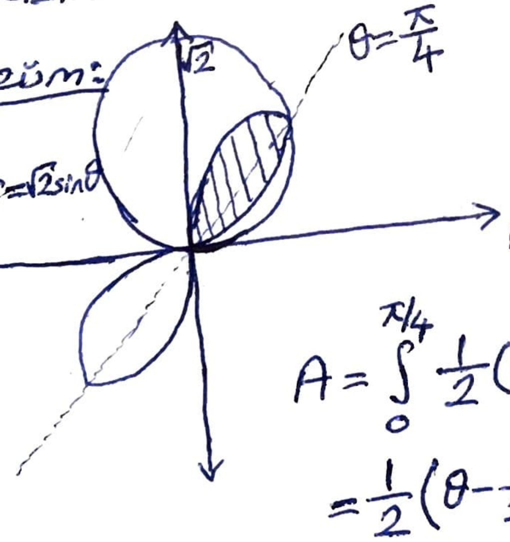
$$A=2\cdot\frac{1}{2}\int_0^{\pi/2}\left[\left(2(1+\cos\theta)\right)^2-\left(2(1-\cos\theta)\right)^2\right]d\theta$$

$$=4\int_0^{\pi/2}4\cos\theta\,d\theta=16\sin\theta\Big|_0^{\pi/2}$$

$$=16\ br^2$$

12) $r=\sqrt{2}\sin\theta$ çemberi ile $r^2=\sin 2\theta$ lemniskatının her ikisininde içinde kalan bölgenin alanını bulunuz.

Çözüm:



$$2\sin^2\theta=\sin 2\theta\Longrightarrow 2\sin^2\theta=2\sin\theta\cos\theta$$

$$\Longrightarrow\tan\theta=1\Longrightarrow\theta=\frac{\pi}{4}$$

$$A=\frac{1}{\cancel{2}}\int_0^{\pi/4}\cancel{2}\sin^2\theta\,d\theta+\frac{1}{2}\int_{\pi/4}^{\pi/2}\sin 2\theta\,d\theta$$

$$A=\int_0^{\pi/4}\frac{1}{2}(1-\cos 2\theta)d\theta+\frac{1}{2}\left(\frac{-1}{2}\cos 2\theta\Big|_{\pi/4}^{\pi/2}\right)$$

$$=\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/4}-\frac{1}{4}(-1-0)$$

$$=\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)+\frac{1}{4}=\frac{\pi}{8}\ br^2$$