BELİRSİZ İNTEGRAL

$f(x)$ tanımlı olduğunda $F'(x)=f(x)$ bağıntısını sağlayan bir F fonksiyonu var ise, bu F fonksiyonuna f nin antitürevi denir.

$$F'(x)=f(x) \Longrightarrow (F(x)+c)'=f(x)$$

f nin tüm antitürevlerinin sınıfına f nin belirsiz integrali denir. ve $\int f(x)dx$ ile gösterilir. Burada, $\int$ integral sembolü, $f(x)$ integrant ve x integrasyon değişkenidir.

Örnek: $\int 6x^5 dx = ?$

Çözüm: $(x^6)'=6x^5$ olduğundan $\int 6x^5 dx = x^6+c$ olur.

İntegral Formülleri:

1) $\int (af(x)+bg(x))dx = a\int f(x)dx + b\int g(x)dx$

2) $\int x^n dx = \dfrac{x^{n+1}}{n+1}+c \quad , \quad (n\neq -1)$

3) $\int \dfrac{dx}{x} = \ln|x|+c$

4) $\int a^x dx = \dfrac{a^x}{\ln a}+c$

5) $\int e^x dx = e^x+c$

6) $\int \sin x\, dx = -\cos x+c$

7) $\int \cos x\, dx = \sin x+c$

8) $\int \dfrac{dx}{\cos^2 x} = \tan x+c$

9) $\int \dfrac{dx}{\sin^2 x} = -\cot x+c$

10) $\int \frac{dx}{1+x^2} = \arctan x + c$

11) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$

12) $\int \sinh x \, dx = \cosh x + c$

13) $\int \cosh x \, dx = \sinh x + c$

14) $\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|x+\sqrt{x^2+a}\right| + c$

Örnekler:

1) $\int \left(3\cos x + \sqrt{x^7} - \frac{5}{x}\right)dx = ?$

Çözüm: $\int \left(3\cos x + x^{7/2} - \frac{5}{x}\right)dx = 3\sin x + \frac{x^{9/2}}{9/2} - 5\ln|x| + c$

2) $\int \left(2e^x - \frac{5}{\sin^2 x} + \frac{10}{1+x^2}\right)dx = ?$

Çözüm: $\int \left(2e^x - \frac{5}{\sin^2 x} + \frac{10}{1+x^2}\right)dx = 2e^x + 5\cot x + 10\arctan x + c$

3) $\int \left(\frac{2x^2-5x+4}{\sqrt{x}} + 3^x - \frac{5}{\sqrt{1+x^2}}\right)dx = ?$

Çözüm: $\int \left(\frac{2x^2}{\sqrt{x}} - \frac{5x}{\sqrt{x}} + \frac{4}{\sqrt{x}} + 3^x - \frac{5}{\sqrt{1+x^2}}\right)dx$

$= \int \left(2x^{3/2} - 5\sqrt{x} + 4x^{-1/2} + 3^x - \frac{5}{\sqrt{1+x^2}}\right)dx$

$= 2 \cdot \frac{x^{5/2}}{5/2} - 5 \cdot \frac{x^{3/2}}{3/2} + 4\frac{x^{1/2}}{1/2} + \frac{3^x}{\ln 3} - 5\ln\left|x+\sqrt{1+x^2}\right| + c$

$= \frac{4}{5}x^{5/2} - \frac{10}{3}x^{3/2} + 8\sqrt{x} + \frac{3^x}{\ln 3} - 5\ln\left|x+\sqrt{1+x^2}\right| + c$

4) $\int \left(5\cosh x - \frac{4}{\sqrt{x^2-1}} + \frac{x^2-1}{x}\right)dx = ?$

Çözüm: $\int \left(5\cosh x - \frac{4}{\sqrt{x^2-1}} + x - \frac{1}{x}\right)dx = 5\sinh x - 4\ln\left|x+\sqrt{x^2-1}\right|$

$+ \frac{x^2}{2} - \ln|x| + c$

İntegral Alma Yöntemleri:

1. Değişken Değiştirme Yöntemi:

g, sürekli türevlere sahip bir fonksiyon olmak üzere $x=g(t)$ dönüşümü yardımıyla $dx=g'(t)dt$ olacağından

$$\int f(x)dx = \int f(g(t)).g'(t)dt$$

olur. İntegral hesaplandıktan sonra tekrar x değişkenine dönülür.

Örnekler:

1) $\int (1-2x)^{2020} dx = \int t^{2020}.\left(-\frac{1}{2}\right)dt = -\frac{1}{2}.\frac{t^{2021}}{2021}+c$

$t=1-2x$
$dt=-2dx$

$= -\frac{1}{2}.\frac{(1-2x)^{2021}}{2021}+c$

2) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = ?$

$t=\sqrt{x}$
$dt=\frac{dx}{2\sqrt{x}}$

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int e^t.2dt = 2e^t+c = 2e^{\sqrt{x}}+c$

3) $\int \tan x\, dx = ?$

Çözüm: $\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx = \int \frac{-dt}{t} = -\ln|t|+c$

$t=\cos x$
$dt=-\sin x\, dx$

$= -\ln|\cos x|+c$

4) $\int \cot x\, dx = ?$

Çözüm: $\int \cot x\, dx = \int \frac{\cos x}{\sin x}dx = \int \frac{dt}{t} = \ln|t|+c$

$t=\sin x$
$dt=\cos x\, dx$

$= \ln|\sin x|+c$

5) $\int \frac{x^5\,dx}{1+x^{12}} = ?$

Çözüm: $\int \frac{x^5\,dx}{1+x^{12}} = \int \frac{\frac{1}{6}dt}{1+t^2} = \frac{1}{6}\arctan t + c$

$t = x^6$

$dt = 6x^5\,dx$

$= \frac{1}{6}\cdot\arctan(x^6) + c$

6) $\int \frac{\sin x - \cos x}{\cos x + \sin x}\,dx = ?$

Çözüm: $\int \frac{\sin x - \cos x}{\cos x + \sin x}\,dx = \int \frac{-dt}{t} = -\ln|t| + c$

$= -\ln|\cos x + \sin x| + c$

$t = \cos x + \sin x$

$dt = (-\sin x + \cos x)\,dx$

7) $\int \sqrt{\cos^7 x}\,\sin x\,dx = ?$

Çözüm: $\int \sqrt{\cos^7 x}\cdot\sin x\,dx = \int t^{7/2}\cdot(-1)\,dt = -\frac{t^{\frac{9}{2}}}{9/2} + c$

$= -\frac{2}{9}\sqrt{\cos^9 x} + c$

$t = \cos x$

$dt = -\sin x\,dx$

8) $\int \cos(ax)\,dx = ?$

Çözüm: $\int \cos(ax)\,dx = \int \cos t\cdot\frac{1}{a}\,dt = \frac{1}{a}\sin t + c$

$= \frac{1}{a}\sin(ax) + c$

$t = ax$

$dt = a\,dx$

9) $\int \sin(ax)\,dx = ?$

Çözüm: $\int \sin(ax)\,dx = \int \sin t\cdot\frac{1}{a}\,dt = \frac{1}{a}(-\cos t) + c$

$= -\frac{1}{a}\cos(ax) + c$

$t = ax$

$dt = a\,dx$

10) $\int e^{ax}\,dx = \int e^t\cdot\frac{1}{a}\,dt = \frac{1}{a}e^t + c = \frac{1}{a}e^{ax} + c$

$t = ax$

$dt = a\,dx$

11) $\int \frac{dx}{a^2+x^2} = ?$

Çözüm: $\int \frac{dx}{a^2+x^2} = \int \frac{a\,dt}{a^2+a^2t^2} = \frac{1}{a}\int \frac{dt}{1+t^2} = \frac{1}{a}\arctan t + c$

$x = at$

$dx = a\,dt$

$= \frac{1}{a} \cdot \arctan\left(\frac{x}{a}\right) + c$

12) $\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\,dt}{\sqrt{a^2-a^2t^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + c$

$x = at$

$dx = a\,dt$

$= \arcsin\left(\frac{x}{a}\right) + c$

13) $\int \frac{dx}{\sqrt{3-2x-x^2}} = ?$

Çözüm: $\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(x+1)^2}} = \int \frac{dt}{\sqrt{4-t^2}} = \arcsin\left(\frac{t}{2}\right) + c$

$3-2x-x^2 = -(x^2+2x-3)$
$= -[(x+1)^2-4]$
$= 4-(x+1)^2$

$t = x+1$

$dt = dx$

$= \arcsin\left(\frac{x+1}{2}\right) + c$

14) $\int x^2\sqrt{x-2}\,dx = ?$

Çözüm: $\int x^2\sqrt{x-2}\,dx = \int (t+2)^2 \cdot \sqrt{t}\,dt = \int (t^2+4t+4)\sqrt{t}\,dt$

$t = x-2$

$dt = dx$

$= \int \left(t^{5/2} + 4t^{3/2} + 4t^{1/2}\right) dt$

$= \frac{t^{7/2}}{7/2} + 4\frac{t^{5/2}}{5/2} + 4\frac{t^{3/2}}{3/2} + c$

$= \frac{2}{7}(x-2)^{7/2} + \frac{8}{5}(x-2)^{5/2} + \frac{8}{3}(x-2)^{3/2} + c$

15) $\int \frac{2^x}{\sqrt{1-4^x}}\,dx = ?$

Çözüm: $\int \frac{2^x\,dx}{\sqrt{1-4^x}} = \int \frac{2^x\,dx}{\sqrt{1-(2^x)^2}} = \int \frac{\frac{1}{\ln 2}\,dt}{\sqrt{1-t^2}} = \frac{1}{\ln 2}\arcsin t + c$

$t = 2^x$

$dt = 2^x \ln 2\,dx$

$= \frac{1}{\ln 2}\arcsin(2^x) + c$

Bazı Özel Değişken Değiştirme Yöntemleri:

1. $\sqrt{a^2-x^2}$ den başka köklü ifade içermeyen fonksiyonların integrali

$x = a\sin t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$ dönüşümü uygulanır.

Örnek: $\int \frac{x+5}{\sqrt{4-x^2}}\,dx = ?$

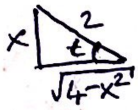
Çözüm: $\int \frac{x+5}{\sqrt{4-x^2}}\,dx = \int \frac{2\sin t+5}{\cancel{\sqrt{4-4\sin^2 t}}}\,\cancel{2\cos t}\,dt$

$\boxed{\begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \end{array}}$

$= \int (2\sin t+5)\,dt = -2\cos t + 5t + c$

$= -2\cdot\frac{\sqrt{4-x^2}}{2} + 5\arcsin\left(\frac{x}{2}\right) + c$

$= -\sqrt{4-x^2} + 5\arcsin\left(\frac{x}{2}\right) + c$



2. $\sqrt{x^2-a^2}$ den başka köklü ifade içermeyen fonksiyonların integrali

$x = a\sec t, \quad 0 < t < \frac{\pi}{2}$ dönüşümü uygulanır.

Örnek: $\int \frac{dx}{x\sqrt{x^2-16}} = ?$

Çözüm: $\int \frac{dx}{x\sqrt{x^2-16}} = \int \frac{\frac{4\sin t}{\cos^2 t}\,dt}{\frac{4}{\cos t}\sqrt{\frac{16}{\cos^2 t}-16}} = \int \frac{\cancel{\frac{4\sin t}{\cos^2 t}}\,dt}{\frac{4}{\cancel{\cos t}}\cdot\frac{\cancel{4\sin t}}{\cancel{\cos t}}}$

$x = \frac{4}{\cos t}$

$dx = \frac{4\sin t}{\cos^2 t}\,dt$

$\cos t = \frac{4}{x}$

$= \frac{1}{4}\int dt = \frac{t}{4} + c$

$= \frac{1}{4}\arccos\left(\frac{4}{x}\right) + c$

Örnek: $\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sqrt{\frac{1}{\cos^2 t}-1}}{\frac{1}{\cancel{\cos t}}}\cdot\frac{\sin t}{\cos^{\cancel{2}} t}\,dt = \int \frac{\sin t}{\cos t}\cdot\frac{\sin t}{\cos t}\,dt$

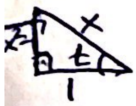
$x = \frac{1}{\cos t}$

$dx = \frac{\sin t}{\cos^2 t}\,dt$

$\cos t = \frac{1}{x}$

$= \int \tan^2 t\,dt = \int\left[(1+\tan^2 t) - 1\right]dt$

$= \tan t - t + c = \sqrt{x^2-1} - \arccos\left(\frac{1}{x}\right) + c$

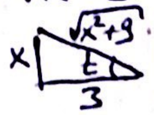


$\tan t = \sqrt{x^2-1}$

3. $\sqrt{x^2+a^2}$ den başka köklü ifade içermeyen fonksiyonların integrali:

$x = a\tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$ dönüşümü uygulanır.

Örnek: $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+9}} = \int \frac{\frac{3}{\cos^2 t}dt}{\frac{9\sin^2 t}{\cos^2 t}\cdot\sqrt{9\tan^2 t+9}}$



$x = 3\tan t$

$dx = \frac{3}{\cos^2 t}dt$

$u = \sin t$

$du = \cos t\, dt$

$$= \int \frac{dt}{3\sin^2 t\cdot\frac{3}{\cos t}} = \int \frac{\cos t\, dt}{9\sin^2 t}$$

$$= \int \frac{du}{9u^2} = \frac{-1}{9u} + c = \frac{-1}{9\sin t} + c$$

$$= \frac{-1}{9\cdot\frac{x}{\sqrt{x^2+9}}} + c = \frac{-\sqrt{x^2+9}}{9x} + c$$

4. $\sqrt[n_i]{ax+b}$ biçiminde ifadeler bulunduran fonksiyonların integrali:

n_i kök kuvvetlerinin en küçük ortak katı p olmak üzere $ax+b = t^p$ değişken değiştirmesi yapılır.

Örnek: $\displaystyle\int \frac{\sqrt[4]{x+1}+2}{\sqrt[6]{x+1}}dx = ?$

Çözüm: $\displaystyle\int \frac{\sqrt[4]{x+1}+2}{\sqrt[6]{x+1}}dx = \int \frac{t^3+2}{t^2}\cdot 12t^{11}dt = 12\int (t^{12}+2t^9)dt$

EKOK(4,6) = 12

$x+1 = t^{12}$

$dx = 12t^{11}dt$

$$= 12\left(\frac{t^{13}}{13} + 2\cdot\frac{t^{10}}{10}\right) + c$$

$$= \frac{12}{13}(x+1)^{\frac{13}{12}} + \frac{12}{5}(x+1)^{\frac{5}{6}} + c$$

Örnek: $\displaystyle\int \frac{\sqrt{2x-3}}{1+\sqrt[3]{2x-3}}dx = ?$

Çözüm: $\displaystyle\int \frac{\sqrt{2x-3}}{1+\sqrt[3]{2x-3}}dx = \int \frac{t^3}{1+t^2}\cdot 3t^5 dt = 3\int \frac{t^8}{t^2+1}dt$

EKOK(2,3) = 6

$2x-3 = t^6$

$dx = 3t^5 dt$

$$\begin{array}{l|l} t^8 & t^2+1 \\ \hline t^8+t^6 & t^6-t^4+t^2-1 \\ -t^6 & \\ -t^6-t^4 & \\ t^4 & \\ t^4+t^2 & \\ -t^2 & \\ -t^2-1 & \end{array}$$

$$= 3\int \left(t^6 - t^4 + t^2 - 1 + \frac{1}{t^2+1}\right)dt$$

$$= 3\left(\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^3}{3} - t + \arctan t\right) + c$$

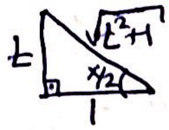
$$= \frac{3}{7}(2x-3)^{7/6} - \frac{3}{5}(2x-3)^{5/6} + (2x-3)^{1/2} - 3(2x-3)^{1/6} +$$

$$+ 3\arctan(2x-3)^{1/6} + c$$

5. Trigonometrik fonksiyonların rasyonel ifadesi olan fonksiyonların integrali:

$\tan\left(\frac{x}{2}\right)=t$ dönüşümü yapılır.

$$\frac{x}{2}=\arctan t \Rightarrow x=2\arctan t \Rightarrow dx=\frac{2dt}{1+t^2}$$



$$\sin\left(\frac{x}{2}\right)=\frac{t}{\sqrt{t^2+1}} \quad , \quad \cos\left(\frac{x}{2}\right)=\frac{1}{\sqrt{t^2+1}}$$

$$\sin x=2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)=\frac{2t}{1+t^2}$$

$$\cos x=\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)=\frac{1-t^2}{1+t^2}$$

Örnek: $\int \frac{dx}{\sin x}=?$

Çözüm: $\int \frac{dx}{\sin x}=\int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}}=\int\frac{dt}{t}=\ln|t|+c$

$\tan\left(\frac{x}{2}\right)=t$

$dx=\frac{2dt}{1+t^2}$

$\sin x=\frac{2t}{1+t^2}$

$$=\ln\left|\tan\left(\frac{x}{2}\right)\right|+c$$

Örnek: $\int \frac{1+\sin x}{(1+\cos x)\sin x}dx=?$

Çözüm: $\int \frac{1+\sin x}{(1+\cos x)\sin x}dx=\int \frac{1+\frac{2t}{1+t^2}}{\left(1+\frac{1-t^2}{1+t^2}\right)\cdot\frac{\cancel{2}t}{\cancel{1+t^2}}}\cdot\frac{\cancel{2}dt}{\cancel{1+t^2}}$

$$=\int \frac{\frac{t^2+2t+1}{\cancel{1+t^2}}}{\frac{2t}{\cancel{1+t^2}}}dt=\frac{1}{2}\int\left(t+2+\frac{1}{t}\right)dt$$

$$=\frac{1}{2}\left(\frac{t^2}{2}+2t+\ln|t|\right)+c$$

$$=\frac{1}{4}\cdot\tan^2\left(\frac{x}{2}\right)+\tan\left(\frac{x}{2}\right)+\frac{1}{2}\cdot\ln\left|\tan\left(\frac{x}{2}\right)\right|+c$$

Örnek: $\int \frac{1-\cos x}{1+\cos x}dx=\int \frac{1-\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2}=\int 2\frac{t^{2+1-1}}{1+t^2}dt$

$$=2\int\left(1-\frac{1}{1+t^2}\right)dt=2(t-\arctan t)+c=2\tan\left(\frac{x}{2}\right)-x+c$$

2. Kısmi İntegrasyon Yöntemi

$$d(u.v) = v.du + u.dv$$

$$u.dv = d(u.v) - v\,du$$

$$\boxed{\int u\,dv = u.v - \int v\,du}$$ Kısmi integrasyon formülü

L A P T Ü

L → Logaritma, A → Arc, P → Polinom, T → Trigonometri, Ü → Üstel

Örnekler:

1. $\int x e^{5x} dx = ?$

Çözüm: $\int x e^{5x} dx = \frac{1}{5} x e^{5x} - \int \frac{1}{5} e^{5x} dx = \frac{1}{5}\left(x e^{5x} - \frac{1}{5} e^{5x}\right) + c$

$u = x \qquad dv = e^{5x} dx$

$du = dx \qquad v = \frac{1}{5} e^{5x}$

2. $\int x \sin 7x\, dx = ?$

Çözüm: $\int x \sin 7x\, dx = \frac{-x}{7}\cos 7x - \int \left(-\frac{1}{7}\right)\cos 7x\, dx$

$= \frac{1}{7}\left(-x\cos 7x + \frac{1}{7}\sin 7x\right) + c$

$u = x \qquad dv = \sin 7x\, dx$

$du = dx \qquad v = \frac{-1}{7}\cos 7x$

3. $\int x^8 . \ln x\, dx = ?$

Çözüm: $\int x^8 \ln x\, dx = \frac{x^9}{9}\ln x - \int \frac{x^9}{9} . \frac{dx}{x} = \frac{x^9}{9}\ln x - \frac{x^9}{81} + c$

$u = \ln x \qquad dv = x^8 dx$

$du = \frac{dx}{x} \qquad v = \frac{x^9}{9}$

4. $\int \arctan x\, dx = ?$

Çözüm: $\int \arctan x\, dx = x \arctan x - \int \frac{x\,dx}{1+x^2}$

$= x\arctan x - \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$

$= x\arctan x - \frac{1}{2} . \ln(1+x^2) + c$

$u = \arctan x \qquad dv = dx$

$du = \frac{dx}{1+x^2} \qquad v = x$

5) $\int e^x \cos 5x\,dx = ?$

Çözüm: $\int e^x \cos 5x\,dx = e^x\cos 5x - \int (-5)e^x \sin 5x\,dx$

$u=\cos 5x$ $\quad dv = e^x dx$	$u=\sin 5x$ $\quad dv = e^x dx$
$du = -5\sin 5x\,dx$ $\quad v=e^x$	$du = 5\cos 5x$ $\quad v = e^x$

$$= e^x\cos 5x + 5\left[e^x \sin 5x - \int 5e^x \cos 5x\,dx\right]$$

$$\int e^x\cos 5x\,dx = e^x\cos 5x + 5e^x \sin 5x - 25\int e^x \cos 5x\,dx$$

$$\Rightarrow 26\int e^x\cos 5x\,dx = e^x(\cos 5x + 5\sin 5x)$$

$$\Rightarrow \int e^x \cos 5x\,dx = \frac{e^x}{26}(\cos 5x + 5\sin 5x) + c$$

6. $\int \frac{\sin 3x}{e^x}dx = ?$

Çözüm: $\int \frac{\sin 3x}{e^x}dx = \int e^{-x}\cdot\sin 3x\,dx = -e^{-x}\sin 3x - \int(-3)e^{-x}\cos 3x\,dx$

$u=\sin 3x$ $\quad dv = e^{-x}dx$	$u=\cos 3x$ $\quad dv = e^{-x}dx$
$du = 3\cos 3x\,dx$ $\quad v = -e^{-x}$	$du = -3\sin 3x\,dx$ $\quad v = -e^{-x}$

$$\int e^{-x}\sin 3x\,dx = -e^{-x}\sin 3x + 3\left[-e^{-x}\cos 3x - \int 3e^{-x}\sin 3x\,dx\right]$$

$$\int e^{-x}\sin 3x\,dx = -e^{-x}\sin 3x - 3e^{-x}\cos 3x - 9\int e^{-x}\sin 3x\,dx$$

$$\int e^{-x}\sin 3x\,dx = -e^{-x}\sin 3x - 3e^{-x}\cos 3x$$

$$\Rightarrow 10\int e^{-x}\sin 3x\,dx = -e^{-x}\sin 3x - 3e^{-x}\cos 3x$$

$$\Rightarrow \int \frac{\sin 3x}{e^x}dx = -\frac{1}{10e^x}(\sin 3x + 3\cos 3x) + c$$

7. $\int x^2 e^{-x}dx = ?$

Çözüm: $\int x^2 e^{-x}dx = -x^2e^{-x} + 2\int x e^{-x}dx = -x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}dx\right]$

$= (-x^2-2x)e^{-x} - e^{-x} + c$

$= -(x^2+2x+1)e^{-x} + c$

$u=x^2$ $\quad dv = e^{-x}dx$	$u=x$ $\quad dv = e^{-x}dx$
$du = 2x\,dx$ $\quad v = -e^{-x}$	$du = dx$ $\quad v = -e^{-x}$

8. $\int \frac{x\,dx}{\cos^2 x} = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + c$

$u = x \quad dv = \frac{dx}{\cos^2 x}$

$du = dx \quad v = \tan x$

Basit Kesirlere Ayırma Yöntemi:

$P(x)$ ve $Q(x)$, iki polinom olsun. $\int \frac{P(x)}{Q(x)}dx$ şeklindeki integraller için basit kesirlere ayırma yöntemi kullanılır. $P(x)$ in derecesi $Q(x)$ in derecesinden küçük olmalıdır. $P(x)$ in derecesi büyük ise, polinom bölmesi yapılarak $\frac{P(x)}{Q(x)} = K(x) + \frac{R(x)}{Q(x)}$ $(\text{der}R(x) < \text{der}Q(x))$ şeklinde yazılır. $\frac{R(x)}{Q(x)}$ kesri, $\frac{M}{ax+b}$, $\frac{N}{(ax+b)^n}$, $\frac{Ax+B}{ax^2+bx+c}$, $\frac{Cx+D}{(ax^2+bx+c)^n}$ biçimindeki kesirlerin toplamı haline getirilerek integrali kolayca hesaplanır.

$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$ şeklinde ise,

$$\frac{K(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

şeklinde yazılır.

$Q(x) = (ax+b)^k$ şeklinde ise,

$$\frac{K(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

şeklinde yazılır.

$Q(x) = (a_1x+b_1)^{k_1}(a_2x+b_2)^{k_2}$ şeklinde ise,

$$\frac{K(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_{k_1}}{(a_1x+b_1)^{k_1}} + \frac{B_1}{a_2x+b_2} + \frac{B_2}{(a_2x+b_2)^2} + \cdots + \frac{B_{k_2}}{(a_2x+b_2)^{k_2}}$$

şeklinde yazılır.

$Q(x) = (ax^2+bx+c)^k$ şeklinde ise,

$$\frac{K(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

şeklinde yazılır.

$$Q(x)=(px+q)(ax^2+bx+c)^k \text{ şeklinde ise,}$$

$$\frac{K(x)}{Q(x)}=\frac{A_0}{px+q}+\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\dots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

şeklinde yazılır.

Örnekler:

1. $\int \frac{4dx}{x^2-4}=?$

Çözüm: $\int \frac{4dx}{x^2-4}=\int\frac{4dx}{(x-2)(x+2)}$

$$\frac{4}{(x-2)(x+2)}=\underset{(x+2)}{\frac{A}{x-2}}+\underset{(x-2)}{\frac{B}{x+2}} \Longrightarrow \frac{4}{(x-2)(x+2)}=\frac{Ax+2A+Bx-2B}{(x-2)(x+2)}$$

$$\Longrightarrow 4=(A+B)x+2A-2B$$

$$\Longrightarrow \begin{cases} A+B=0 \\ 2A-2B=4 \end{cases} \Longrightarrow A=1,\quad B=-1$$

$$\int\frac{4dx}{x^2-4}=\int\left(\frac{1}{x-2}-\frac{1}{x+2}\right)dx=\ln|x-2|-\ln|x+2|+c$$

$$=\ln\left|\frac{x-2}{x+2}\right|+c$$

2. $\int\frac{2x+1}{(x-1)^2}dx=?$

Çözüm: $\frac{2x+1}{(x-1)^2}=\underset{(x-1)}{\frac{A}{x-1}}+\frac{B}{(x-1)^2}$

$$\Longrightarrow 2x+1=Ax-A+B \Longrightarrow \begin{cases} A=2 \\ -A+B=1 \end{cases}$$

$$\Longrightarrow A=2,\quad B=3$$

$$\int\frac{2x+1}{(x-1)^2}dx=\int\left(\frac{2}{x-1}+\frac{3}{(x-1)^2}\right)dx=\int\left(\frac{2}{t}+\frac{3}{t^2}\right)dt$$

$$\left.\begin{array}{l} t=x-1 \\ dt=dx \end{array}\right| \quad =2\ln|t|-\frac{3}{t}+c=2.\ln|x-1|-\frac{3}{x-1}+c$$

3. $\displaystyle\int \frac{-2x+4}{(1+x^2)(x-1)^2}\,dx = ?$

Çözüm: $\displaystyle\frac{-2x+4}{(1+x^2)(x-1)^2} = \underset{(x^2-2x+1)}{\frac{Ax+B}{1+x^2}} + \underset{(1+x^2)(x-1)}{\frac{C}{x-1}} + \underset{(1+x^2)}{\frac{D}{(x-1)^2}}$

$$-2x+4 = Ax^3+(-2A+B)x^2+(A-2B)x+B+Cx^3-Cx^2+Cx-C+D+Dx^2$$

$$-2x+4 = (A+C)x^3+(-2A+B-C+D)x^2+(A-2B+C)x+(B-C+D)$$

$$\left.\begin{aligned} A+C&=0\\ -2A+B-C+D&=0\\ A-2B+C&=-2\\ B-C+D&=4 \end{aligned}\right\} \Rightarrow A=2,\ B=1,\ C=-2,\ D=1$$

$$\int \frac{-2x+4}{(1+x^2)(x-1)^2}\,dx = \int\left(\frac{2x+1}{1+x^2}+\frac{(-2)}{x-1}+\frac{1}{(x-1)^2}\right)dx$$

$$= \int\frac{2x\,dx}{1+x^2}+\int\frac{dx}{1+x^2}-2\int\frac{dx}{x-1}+\int\frac{dx}{(x-1)^2}$$

$$= \ln(1+x^2)+\arctan x-2\ln|x-1|-\frac{1}{x-1}+c$$

$$= \ln\left(\frac{1+x^2}{(x-1)^2}\right)+\arctan x-\frac{1}{x-1}+c$$

4. $\displaystyle\int \frac{2x^4-6x^3+7x^2-2x-2}{x^3-3x^2+3x-1}\,dx = \int\left(2x+\frac{x^2-2}{(x-1)^3}\right)dx$

$$\begin{array}{l|l} 2x^4-6x^3+7x^2-2x-2 & x^3-3x^2+3x-1 \\ \cline{2-2} \underline{2x^4-6x^3+6x^2-2x} & 2x \\ \qquad\qquad x^2-2 & \end{array}$$

$\displaystyle\frac{x^2-2}{(x-1)^3} = \underset{(x-1)^2}{\frac{A}{x-1}} + \underset{(x-1)}{\frac{B}{(x-1)^2}} + \frac{C}{(x-1)^3}$

$$x^2-2 = Ax^2-2Ax+A+Bx-B+C$$

$$x^2-2 = Ax^2+(-2A+B)x+A-B+C$$

$$\left.\begin{aligned} A&=1\\ -2A+B&=0\\ A-B+C&=-2 \end{aligned}\right\} A=1,\ B=2,\ C=-1$$

$$\int \frac{2x^4-6x^3+7x^2-2x-2}{x^3-3x^2+3x-1}\,dx = \int\left(2x+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{(x-1)^3}\right)dx$$

$$= x^2+\ln|x-1|-\frac{2}{x-1}+\frac{1}{2(x-1)^2}+c$$

5. $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)} = ?$

<u>Çözüm:</u> $\displaystyle\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)} = \int \frac{x^2\,dx}{(x-1)(x+1)^2}$

$$\frac{x^2}{(x-1)(x+1)^2} = \underset{(x+1)^2}{\frac{A}{x-1}} + \underset{(x^2-1)}{\frac{B}{x+1}} + \underset{(x-1)}{\frac{C}{(x+1)^2}}$$

$$x^2 = Ax^2+2Ax+A+Bx^2-B+Cx-C$$

$$x^2 = (A+B)x^2+(2A+C)x+A-B-C$$

$$\left.\begin{aligned} A+B&=1\\ 2A+C&=0\\ A-B-C&=0 \end{aligned}\right\} \Rightarrow A=\frac{1}{4},\ B=\frac{3}{4},\ C=\frac{-1}{2}$$

$$\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)} = \int\left(\frac{\frac{1}{4}}{x-1}+\frac{\frac{3}{4}}{x+1}-\frac{\frac{1}{2}}{(x+1)^2}\right)dx$$

$$= \frac{1}{4}\cdot\ln|x-1|+\frac{3}{4}\ln|x+1|+\frac{1}{2}\cdot\frac{1}{x+1}+c$$

6. $\displaystyle\int \frac{x\,dx}{(x+1)(x^2+1)} = ?$

<u>Çözüm:</u> $\displaystyle\frac{x}{(x+1)(x^2+1)} = \underset{(x^2+1)}{\frac{A}{x+1}} + \underset{(x+1)}{\frac{Bx+C}{x^2+1}}$

$$x = Ax^2+A+Bx^2+(B+C)x+C$$

$$x = (A+B)x^2+(B+C)x+A+C$$

$$\left.\begin{aligned} A+B&=0\\ B+C&=1\\ A+C&=0 \end{aligned}\right\} \Rightarrow A=\frac{-1}{2},\ B=\frac{1}{2},\ C=\frac{1}{2}$$

$$\int \frac{x\,dx}{(x+1)(x^2+1)} = \int \left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x+\frac{1}{2}}{x^2+1} \right) dx$$

$$= \frac{1}{2} \int \left(\frac{-1}{x+1} + \frac{1}{2} \cdot \frac{2x}{x^2+1} + \frac{1}{x^2+1} \right) dx$$

$$= \frac{1}{2} \left(-\ln|x+1| + \frac{1}{2} \cdot \ln(x^2+1) + \arctan x \right) + c$$

7. $\displaystyle\int \frac{x+10}{2x^2+5x-3}\,dx = ?$

Çözüm: $\displaystyle\int \frac{x+10}{2x^2+5x-3}\,dx = \int \frac{x+10}{(2x-1)(x+3)}\,dx$

$$\frac{x+10}{(2x-1)(x+3)} = \underset{(x+3)}{\frac{A}{2x-1}} + \underset{(2x-1)}{\frac{B}{x+3}}$$

$x+10 = Ax+3A+2Bx-B$

$x+10 = (A+2B)x+3A-B$

$$\left.\begin{array}{l} A+2B=1 \\ 3A-B=10 \end{array}\right\} \Longrightarrow A=3\ ,\ B=-1$$

$$\int \frac{x+10}{2x^2+5x-3}\,dx = \int \left(\frac{3}{2x-1} - \frac{1}{x+3} \right) dx = \frac{3}{2}\ln|2x-1| - \ln|x+3| + c$$

8. $\displaystyle\int \frac{2x^3+5x^2+8x+4}{(x^2+2x+2)^2}\,dx = ?$

Çözüm: $$\frac{2x^3+5x^2+8x+4}{(x^2+2x+2)^2} = \underset{(x^2+2x+2)}{\frac{Ax+B}{x^2+2x+2}} + \frac{Cx+D}{(x^2+2x+2)^2}$$

$2x^3+5x^2+8x+4 = Ax^3+(2A+B)x^2+(2A+2B+C)x+2B+D$

$$\left.\begin{array}{l} A=2 \\ 2A+B=5 \\ 2A+2B+C=8 \\ 2B+D=4 \end{array}\right\} \Longrightarrow A=2,\ B=1,\ C=2,\ D=2$$

$$\int \frac{2x^3+5x^2+8x+4}{(x^2+2x+2)^2}\,dx = \int \left(\frac{2x+1}{x^2+2x+2} + \frac{2x+2}{(x^2+2x+2)^2} \right) dx$$

$$= \int \left(\frac{2x+2}{x^2+2x+2} - \frac{1}{(x+1)^2+1} + \frac{2x+2}{(x^2+2x+2)^2} \right) dx$$

$u = x^2+2x+2 \qquad t = x+1$

$du = (2x+2)dx \qquad dt = dx$

$$\int \frac{2x^3+5x^2+8x+4}{(x^2+2x+2)^2}dx = \int \frac{du}{u} - \int \frac{dt}{t^2+1} + \int \frac{du}{u^2}$$

$$= \ln|u| - \arctan t - \frac{1}{u} + c$$

$$= \ln(x^2+2x+2) - \arctan(x+1) - \frac{1}{x^2+2x+2} + c$$

9. $\int \frac{x^2+1}{x^3+2x^2+x}dx = ?$

<u>Çözüm:</u> $\int \frac{x^2+1}{x^3+2x^2+x}dx = \int \frac{x^2+1}{x(x+1)^2}dx$

$$\frac{x^2+1}{x(x+1)^2} = \underset{(x+1)^2}{\frac{A}{x}} + \underset{x(x+1)}{\frac{B}{x+1}} + \underset{(x)}{\frac{C}{(x+1)^2}}$$

$x^2+1 = Ax^2+2Ax+A+Bx^2+Bx+Cx$

$x^2+1 = (A+B)x^2+(2A+B+C)x+A$

$$\left.\begin{array}{l} A+B=1 \\ 2A+B+C=0 \\ A=1 \end{array}\right\} \Rightarrow A=1,\ B=0,\ C=-2$$

$$\int \frac{x^2+1}{x^3+2x^2+x}dx = \int\left(\frac{1}{x} - \frac{2}{(x+1)^2}\right)dx = \ln|x| + \frac{2}{x+1} + c.$$

10. $\int \frac{1+\ln x}{x(3+2\ln x)^2}dx = ?$

<u>Çözüm:</u> $t = \ln x \Rightarrow dt = \frac{dx}{x}$

$$\int \frac{(1+\ln x)dx}{x(3+2\ln x)^2} = \int \frac{(1+t)dt}{(3+2t)^2} = \int\left(\frac{\frac{1}{2}}{3+2t} - \frac{\frac{1}{2}}{(3+2t)^2}\right)dt$$

$u = 3+2t$

$du = 2dt$

$$\frac{1+t}{(3+2t)^2} = \underset{(3+2t)}{\frac{A}{3+2t}} + \frac{B}{(3+2t)^2}$$

$1+t = 2At+3A+B$

$$\left.\begin{array}{l} 2A=1 \\ 3A+B=1 \end{array}\right\} \Rightarrow A=\frac{1}{2},\ B=\frac{-1}{2}$$

$$= \int\left(\frac{\frac{1}{4}}{u} - \frac{\frac{1}{4}}{u^2}\right)du$$

$$= \frac{1}{4}\cdot\ln|u| + \frac{1}{4}\cdot\frac{1}{u} + c$$

$$= \frac{1}{4}\cdot\left(\ln|3+2t| + \frac{1}{3+2t}\right) + c$$

$$= \frac{1}{4}\left(\ln|3+2\ln x| + \frac{1}{3+2\ln x}\right) + c$$

11. $\displaystyle\int \frac{e^x\,dx}{e^{2x}+3e^x+2} = ?$

Çözüm: $t=e^x \Longrightarrow dt=e^x\,dx$

$$\int \frac{e^x\,dx}{e^{2x}+3e^x+2} = \int \frac{dt}{t^2+3t+2} = \int \frac{dt}{(t+1)(t+2)}$$

$$\frac{1}{(t+2)(t+1)} = \underset{(t+2)}{\frac{A}{t+1}} + \underset{(t+1)}{\frac{B}{t+2}}$$

$$1=(A+B)t+2A+B$$

$$\left.\begin{array}{r} A+B=0 \\ 2A+B=1 \end{array}\right\} \Longrightarrow A=1\,,\; B=-1$$

$$\int \frac{e^x\,dx}{e^{2x}+3e^x+2} = \int \left(\frac{1}{t+1}-\frac{1}{t+2}\right)dt = \ln|t+1|-\ln|t+2|+c$$

$$= \ln\left|\frac{t+1}{t+2}\right|+c$$

$$= \ln\left|\frac{e^x-1}{e^x+2}\right|+c$$

12. $\displaystyle\int \frac{\cos t}{\sin^2 t-\sin t-6}\,dt = ?$

Çözüm: $x=\sin t \Longrightarrow dx=\cos t\,dt$

$$\int \frac{\cos t\,dt}{\sin^2 t-\sin t-6} = \int \frac{dx}{x^2-x-6} = \int \frac{dx}{(x-3)(x+2)}$$

$$\frac{1}{(x-3)(x-2)} = \underset{(x-2)}{\frac{A}{x-3}} + \underset{(x-3)}{\frac{B}{x-2}}$$

$$1=(A+B)x-2A-3B$$

$$\left.\begin{array}{r} A+B=0 \\ -2A-3B=1 \end{array}\right\} \Longrightarrow A=1,\; B=1$$

$$= \int \left(\frac{1}{x-3}-\frac{1}{x-2}\right)dx$$

$$= \ln|x-3|-\ln|x-2|+c$$

$$= \ln\left|\frac{x-3}{x-2}\right|+c$$

$$= \ln\left|\frac{\sin t-3}{\sin t-2}\right|+c$$