A COMPARISON BETWEEN DIFFERENT BASES IN IMAGE RECONSTRUCTION FROM NONUNIFORM SAMPLES

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Abstract. In this article we present two methods based on the application of Conjugate Gradient method that use wavelets and shift invariant spaces bases. We give an efficient way of computing the products Ax and A^*y for each basis. We compare, experimentally, our approach for image reconstruction from nonuniform samples when using Fourier, D6 wavelets and uniform cubic splines bases.

Key words. D6 wavetes, splines, image reconstruction, nonuniform samples

AMS subject classifications. 65-05, 65Fxx, 65Yxx, 68Wxx

1. Introduction.

We are concerned with the problem of reconstructing images from nonuniform samples. In [8] Gröchenig and Strohmer presented a method using the Fourier Basis and least squares (see also [10]), but no comparison with other methods was shown.

The problem of reconstructing images from nonuniform samples using Fourier bases has been well studied (see for example ([7, 4, 5, 10, 11, 8]). Attractive methods are based on the application of conjugate gradient method to the normal equations using the fact that A^*A is block Toeplitz with Toeplitz blocks and some nonuniform FFT algorithm like the ones from Beylkin ([2]) or Rokhlin ([3]). CG can not be implemented in the same way for wavelets bases.

In [8] Gröchenig and Strohmer pointed out that it is worth developing other methods based on local Fourier bases or wavelets to handle the restoration of large areas of missing pixels. We show here how we can apply conjugate gradient method with wavelets bases in a simple manner.

In [1], it is proposed a method for image reconstruction using splines, implemented by means of a multigrid algorithm obtained by using classical iterators (damped-Jacobi or Gauss-Seidel) as building blocks and they presented no comparison with others methods. Here we present a different and more efficient implementation that uses conjugate gradient applied to least squares using bases of shift invariant spaces, like uniform splines.

An important observation is that we are dealing with an ill posed problem. Therefore some kind of regularization is required. This can be done either with an explicit penalization term (as in [1]) or in an implicit way, using an appropriate stopping criterion as in [9].

In what follows we present our new implementations of the reconstruction methods using D6 Wavelets §2 and cubic splines §3. In §4 we compare the methods visually as well as their convergence rates.

2. Reconstruction using D6 Wavelets.

Let us consider the image function

$$f : \{0, \ldots, N_1\} \times \{0, \ldots, N_2\} \to \mathbb{C}$$

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given by

$$f(i,j)=\sum_{m=N_1-M_1+1}^{N1}\sum_{n=N_2-M_2+1}^{N2} c_{mn}w_{mn}(i,j),$$

where $\{w_{mn}\}$ is a two-dimensional discrete wavelet basis of $\mathbb{C}^{N_1\times N_2}$ (see [6]). On the other hand we are given a nonuniform sample of f

$$\{f(x_1,y_1), f(x_2,y_2), \ldots, f(x_r,y_r)\},$$

for $\Lambda=\{(x_1,y_1),(x_2,y_2),\ldots,(x_r,y_r)\}$ that is a subset of

$$\{(0,0),(0,1),\ldots,(0,N_2-1),\ldots,(N_1-1,0),\ldots,(N_1-1,N_2-1)\}.$$

With $N_1\geq M_1$, $N_2\geq M_2$ and $N_1N_2>r\geq M_1M_2$.

Replacing (i,j) by (x_k,y_k) in the above expression for f, for $k=1,\ldots,r$, we obtain

$$f(x_k,y_k)=\sum_{m=-M_1}^{M_1}\sum_{n=-M_2}^{M_2} c_{mn}w_{mn}(x_k,y_k) \quad \text{for } k=1,\ldots,r$$

or, in matrix form,

$$b=A\,c$$

where

$$b=(f(x_1,y_1),f(x_2,y_2),\ldots,f(x_r,y_r))\in\mathbb{C}^r,$$

and

$$A=(w_{mn}(x_k,y_k))_{r\times M_1M_2} \quad \text{and} \quad c=(c_{mn})\in\mathbb{C}^{M_1M_2}.$$

We use Conjugate Gradient method to solve the least squares problem

$$\min||AX-b||.$$

In order to compute the product AX we proceed as follows:

(i) Apply the two-dimensional inverse discrete wavelet transform to

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_{(N_1-M_1+1)(N_2-M_2+1)} & \cdots & x_{(N_1-M_1+1)(N_2-1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & x_{(N_1-1)(N_2-M_2+1)} & \cdots & x_{(N_1-1)(N_2-1)} \end{bmatrix}_{N_1\times N_2}$$

(ii) Take only the components such that the indexes belong to Λ.

For the computation of the product $X = A^*Y$ the procedure is the following:

(i) If $Y = (y_1, \ldots, y_r)$ define $\tilde{Y} = (y_{kl})_{N_1 \times N_2}$ where

$$y_{kl} = \begin{cases} y_{k'}, & \text{if } (k,l) = (x_{k'}, y_{k'}) \in \Lambda \\ 0, & \text{if } (k,l) \notin \Lambda \end{cases}$$

(ii) Apply the two-dimensional discrete wavelet transform to $\tilde{Y}$. If

$$\begin{bmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ & \cdots & * & * & \cdots & * \\ & \cdots & * & x_{(N_1-M_1+1)(N_2-M_2+1)} & \cdots & x_{(N_1-M_1+1)(N_2-1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ & \cdots & * & x_{(N_1-1)(N_2-M_2+1)} & \cdots & x_{(N_1-1)(N_2-1)} \end{bmatrix}_{N_1 \times N_2}$$

is the result, then $X = A^*Y$ is the vector which components are in the matrix

$$\begin{bmatrix} x_{(N_1-M_1+1)(N_2-M_2+1)} & \cdots & x_{(N_1-M_1+1)(N_2-1)} \\ \vdots & & \vdots \\ x_{(N_1-1)(N_2-M_2+1)} & \cdots & x_{(N_1-1)(N_2-1)} \end{bmatrix}_{M_1 \times M_2}.$$

3. Reconstruction Using Cubic Splines. Consider the function

$$f : [0, N_1] \times [0, N_2] \to \mathbb{R}$$

given by

$$f(x,y) = \sum_{m=0}^{N_1/a_1-1} \sum_{n=0}^{N_2/a_2-1} c_{mn} \phi\left(\frac{x}{a_1} - m, \frac{y}{a_2} - n\right),$$

for a_1, a_2 positive integers and $\phi(x,y)$ is a given function. Let

$$\{f(x_1,y_1), f(x_2,y_2), \ldots, f(x_r,y_r)\},$$

be a and a nonuniform sample of the function f for $\Lambda = \{(x_1,y_1),(x_2,y_2),\ldots,(x_r,y_r)\}$, a subset of

$$\{(0,0),(0,1),\ldots,(0,N_2-1),\ldots,(N_1-1,0),\ldots,(N_1-1,N_2-1)\},$$

with $r \geq \frac{N_1 N_2}{a_1 a_2}$.

Evaluating f at the point (x_k, y_k), for $k = 1, \ldots, r$, we obtain

$$f(x_k,y_k) = \sum_{m=0}^{N_1/a_1-1} \sum_{n=0}^{N_2/a_2-1} c_{mn} \phi\left(\frac{x_k}{a_1} - m, \frac{y_k}{a_2} - n\right) \quad \text{for } k = 1, \ldots, r \tag{3.1}$$

or

$$b = A\,c$$

where

$$A = \left(\phi\left(\frac{x_k}{a_1} - m, \frac{y_k}{a_2} - n\right)\right)_{r\times\frac{N_1N_2}{a_1a_2}}, \quad b = \mathrm{vec}(\mathrm{f}(\mathrm{x_k}, \mathrm{y_k})) \quad \text{and}$$

$$c = \mathrm{vec}(\mathrm{c_{mn}}) \in \mathbb{R}^{\frac{\mathrm{N_1N_2}}{\mathrm{a_1a_2}}}.$$

We can write (3.1) as

$$f(x_k, y_k) = \sum_{m=0}^{N_1-1}\sum_{n=0}^{N_2-1} c_{mn\uparrow a_1a_2}\phi\left(\frac{x_k - m}{a_1}, \frac{y_k - n}{a_2}\right) \quad \text{para } k = 1, \ldots, r$$

where

$$c_{mn\uparrow a_1a_2} = \begin{cases} c_{m'n'}, & \text{if } m = a_1m',\ n = a_2n' \\ 0, & \text{otherwise.} \end{cases}$$

The matrix A is a submatrix of (see [12])

$$\mathrm{bttb}\left(\phi\left(\frac{\mathrm{m}}{\mathrm{a_1}}, \frac{\mathrm{n}}{\mathrm{a_2}}\right)\right)_{\mathrm{m}=-\mathrm{N_1},\ldots,\mathrm{N_1},\mathrm{n}=-\mathrm{N_2},\ldots,\mathrm{N_2}}$$

Now we apply Conjugate Gradient method to solve the least squares problem

$$\min ||AX - b||.$$

To calculate the product AX we proceed as follows:

(i) If $X = \mathrm{vec}(\mathrm{x_{mn}})$ define X^{ext} by

$$X^{\mathrm{ext}}_{mn} = \begin{cases} x_{m'n'}, & \text{if } m = a_1m',\ n = a_2n' \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Compute the product

$$\mathrm{bttb}\left(\phi\left(\frac{\mathrm{m}}{\mathrm{a_1}}, \frac{\mathrm{n}}{\mathrm{a_2}}\right)\right)_{\substack{\mathrm{m}=-\mathrm{N_1},\ldots,\mathrm{N_1} \\ \mathrm{n}=-\mathrm{N_2},\ldots,\mathrm{N_2}}} \mathrm{vec}(\mathrm{X^{ext}}).$$

(iii) Take only the components such that the indexes belong to Λ.

To calculate the product $X = A^tY$ we proceed in the following way:

(i) If $Y = (y_1, \ldots, y_r)$ define $\tilde{Y} = (y_{kl})_{N_1\times N_2}$ by

$$y_{kl} = \begin{cases} y_{k'}, & \text{if } (k, l) = (x_{k'}, y_{k'}) \in \Lambda \\ 0, & \text{if } (k, l) \notin \Lambda \end{cases}$$

(ii) Compute the product

$$\mathrm{bttb}\left(\phi\left(\frac{\mathrm{m}}{\mathrm{a_1}}, \frac{\mathrm{n}}{\mathrm{a_2}}\right)\right)_{\substack{\mathrm{m}=\mathrm{N_1},\ldots,-\mathrm{N_1} \\ \mathrm{n}=\mathrm{N_2},\ldots,-\mathrm{N_2}}} \mathrm{Y^{ext}}.$$

(iii) Take only the components with indexes m, n such that $m = a_1m'$ and $n = a_2n'$.

4. Comparison. Our example is the 512×512 image of Lena that appears in the top left corner of Figure 5.1. To the right in the top there is Lena's image but with 39 % missing points. The rest of the images in Figure 5.1 are reconstructions obtained by Conjugate Gradient method applied to the least squares problem $\min ||AX - b||$ using the Fourier basis with band width $2M_1 + 1 = 2M_2 + 1 = 257$. The matrix A is 158860×66049. The results for one, two, five and ten iterations are shown.

Figure 5.2 shows, in the same order as before the original image, the one with 39 % missing points and the reconstructions obtained by Conjugate Gradient method applied to the least squares problem $\min ||AX - b||$ using D6 wavelets basis with "band width" $M_1 = M_2 = 256$. The matrix A is 158860×65536.

Figure 5.3 shows, in the same order as before the original image, the one with 39 % missing points and the reconstructions obtained by Conjugate Gradient method applied to the least squares problem $\min ||AX - b||$ using **Cubic Splines** with $a_1 = a_2 = 2$ for the reconstructions. The matrix A is 158860×65536. The function $\phi(x, y)$ is given by

$$\phi(x, y) = \varphi(x)\varphi(y)$$

$$\varphi(t) = \begin{cases} \frac{2}{3} - |t|^2 + \frac{|t|^3}{2}, & 0 < |t| < 1 \\ \frac{(2-|t|)^3}{6}, & 1 \le |t| < 2 \\ 0, & |t| \ge 2 \end{cases}$$

For the Fourier, the D6 wavelets and the cubic splines bases we compute for each iteration the normalized least squares error (NLSE) between the original image f and its approximation f_n via

$$e(n) = \frac{||f_n - f||}{||f||}.$$

Figure 5.4 shows the graphics of the NLSE as a function of the number of iterations. From there, it easy to observe that the least error is achieved in 5 iterations when using splines, around 9 iterations for the Fourier basis and more than 15 iterations when using D6 wavelets. This is consistent with the visual evaluation in Figures 1 to 3. The 5th iteration using Fourier basis or D6 wavelets still presents missing points and this is not the case for splines.

5. Conclusion. We have presented how in a simple manner it is possible to apply the Conjugate Gradient method to reconstruction problems with different bases such as wavelets or bases of shift invariant spaces like splines. A visual comparison was also presented as well as for the convergence rates. These preliminary results favor the use of cubic splines. Future work includes the use of preconditioning.

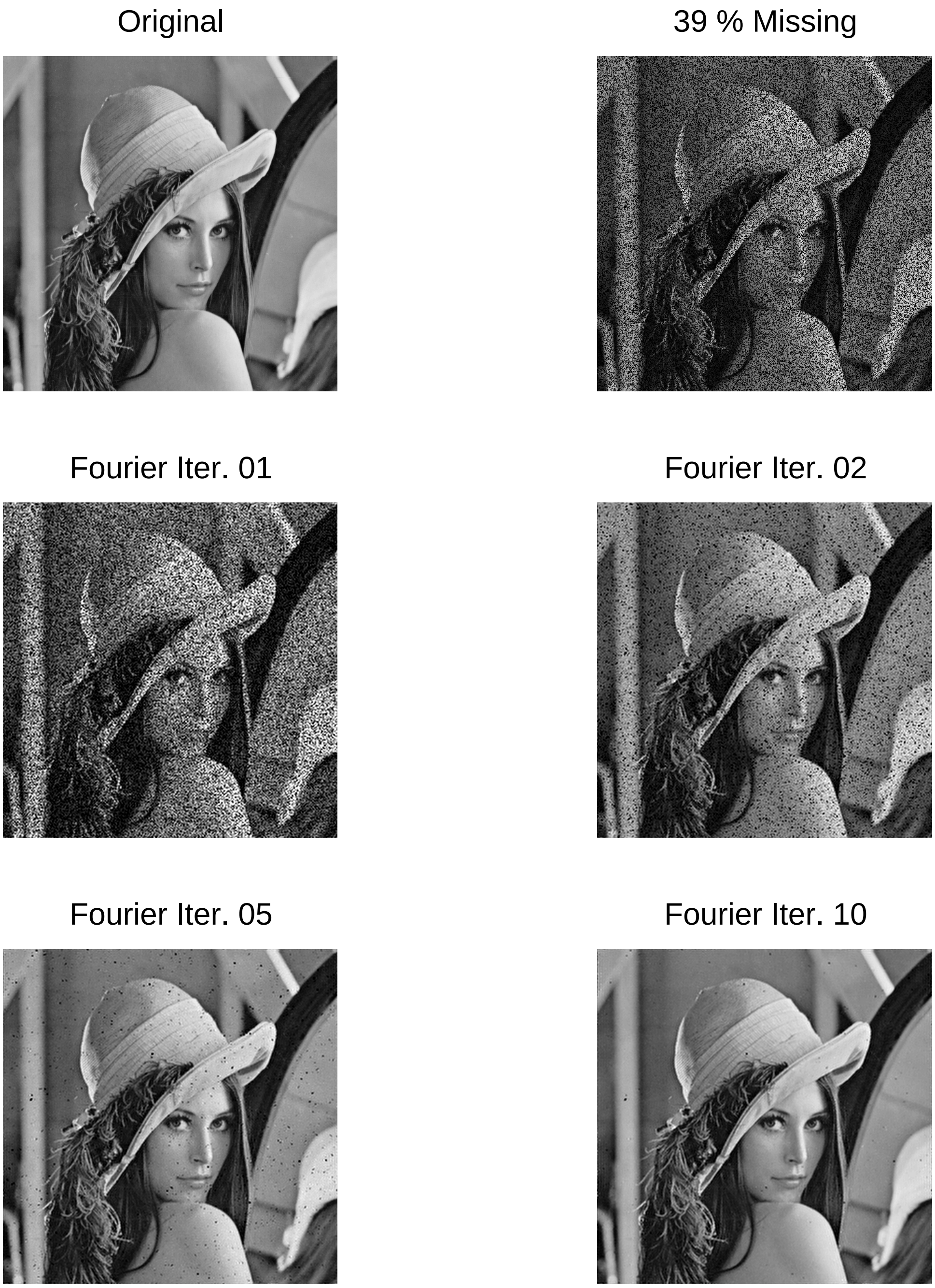

FIG. 5.1. 512×512 *image with 39 % missing samples and reconstructions using a* ***Fourier Basis*** *with band width* $2M_1 + 1 = 257 = 2M_2 + 1 = 257$ *obtained by applying Conjugate Gradient method. The system* 158860×66049.

FIG. 5.2. 512×512 *image with 39 % missing samples and the reconstructions using a* ***D6 Wavelets*** *with* $M_1 = M_2 = 256$ *obtained by applying Conjugate Gradient method. The system* 158860×65536.

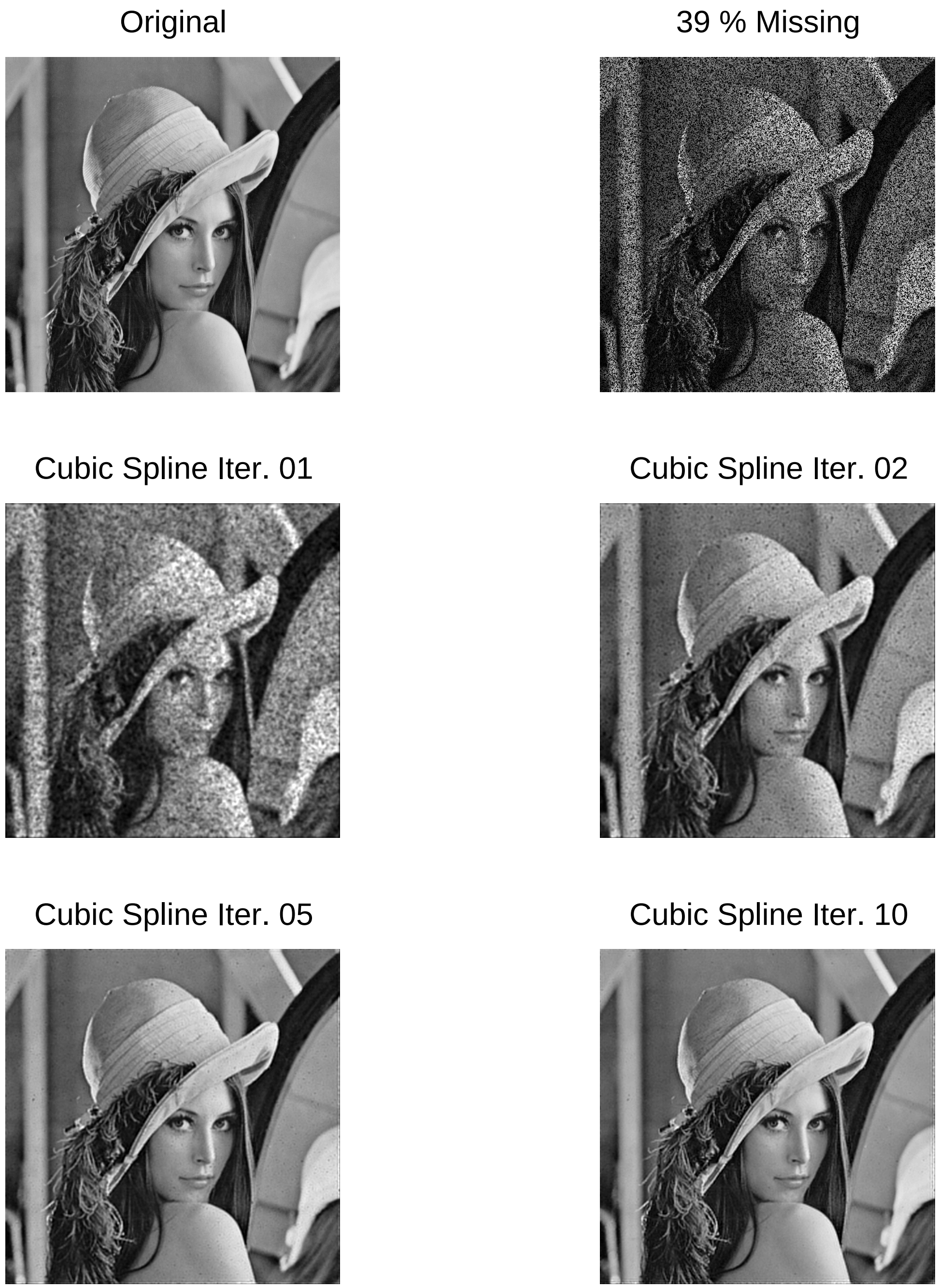

FIG. 5.3. 512×512 *image with 39 % missing samples and reconstructions using a* ***Cubic Splines*** *with* $a_1 = a_2 = 2$ *obtained by applying Conjugate Gradient method. The system* 158860×65536.

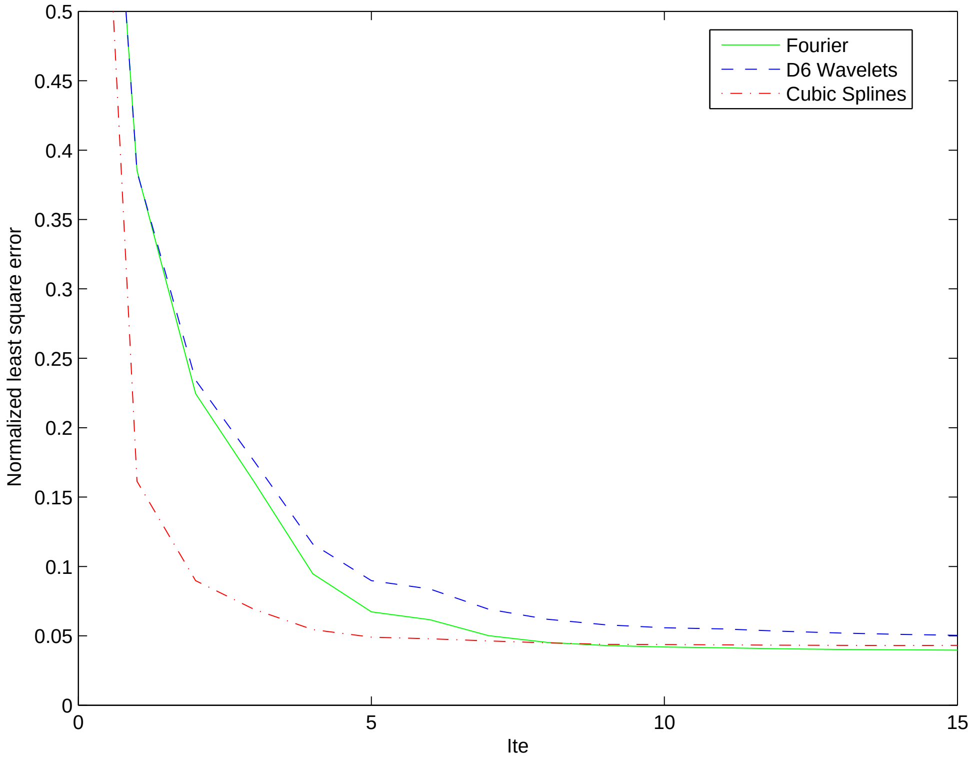

Fig. 5.4. *Convergence rates*

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