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Preconditioning conjugate gradient with symmetric algebraic reconstruction technique (ART) in computerized tomography

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Abstract

We apply the conjugate gradient method to the system

$$A^t C_\omega^{-t} C_\omega^{-1} Ax = A^t C_\omega^{-t} C_\omega^{-1} b,$$

where $C_\omega = (D + \omega L)D^{-1/2}$ and $AA^t = L + D + L^t$. This corresponds to applying CG to the generalized least square problem

$$\min \left\| C_\omega^{-1}(Ax - b) \right\|.$$

We use an efficient way to compute the expressions $A^t C_\omega^{-t} r$ and $C_\omega^{-1} Aw$ using only the rows of A.

Keywords: Conjugate gradient; ART; Computerized tomography; Iterative methods; Ill-posed problems

1. Conjugate gradient to the generalized least square problem

We consider the problem of estimating a solution x of

$$Ax + \varepsilon = b, \tag{1}$$

where A is a real $m \times n$ matrix, b is an m-vector of observations and ε is an error vector. We assume that the components of ε are random with zero mean, and A is ill-conditioned [4].

The algebraic reconstruction technique (ART) method is defined by

$$\begin{cases} x^{(k+1,1)} = x^k, \\ x^{(k+1,i+1)} = x^{(k+1,i)} + \omega \dfrac{(b_i - a_i^t x^{(k+1,i)})}{\|a_i\|^2} a_i, \quad i = 1, \ldots, m, \\ x^{k+1} = x^{(k+1,m+1)}, \end{cases} \tag{2}$$

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where ω is a relaxation parameter in the interval $(0, 2)$, and $A = [a_1, \ldots, a_m]^t = \{a_{ij}\}$. This method was originally devised by Kaczmarz [10] for a square matrix A and $\omega = 1$. Its convergence properties have been studied for general $m \times n$ matrices and $\omega = 1$ by Tanabe [14]. Björck and Elfving [1] pointed out that this method corresponds to applying the SOR method to the system

$$\begin{cases} AA^t y = b, \\ x = A^t y. \end{cases} \tag{3}$$

If a sweep forward in (2) is followed by a sweep backward, we obtain the symmetric ART

$$\begin{cases} x^{(k+1,1)} = x^k, \\ x^{(k+1,i+1)} = x^{(k+1,i)} + \omega \dfrac{(b_j - a_j^t x^{(k+1,i)})}{\|a_j\|^2} a_j, \quad i = 1, \ldots, 2m, \ j = \min(i, 2m + 1 - i), \\ x^{k+1} = x^{(k+1,m+1)}. \end{cases} \tag{4}$$

As pointed out by Björck and Elfving [1], this corresponds to applying the SSOR method to system (3).

The conjugate gradient method (CG) of Hestenes and Stiefel [9,8] applied to the normal equations (CGNR) can be written as: (for a given x^0)

$$r^0 = b - Ax^0, \qquad s^0 = A^t r^0, \qquad w^0 = s^0$$

and for $k = 0, 1, \ldots$

$$p^k = Aw^k$$

$$\alpha^k = \frac{\|s^k\|^2}{\|p^k\|^2}$$

$$x^{k+1} = x^k + \alpha^k w^k$$
$$r^{k+1} = r^k - \alpha^k p^k$$
$$s^{k+1} = A^t r^{k+1}$$

$$\beta^k = \frac{\|s^{k+1}\|^2}{\|s^k\|^2}$$

$$w^{k+1} = s^{k+1} + \beta^k w^k.$$

We consider the CGNR method preconditioned with a symmetric ART method. Let $AA^t = L + D + L^t$, where L is the strictly lower triangular part and D the diagonal of AA^t, respectively. Let

$$C_\omega = (D + \omega L) D^{-1/2}. \tag{5}$$

In [1], Björck and Elfving applied CG to the system

$$C_\omega^{-t} C_\omega^{-1} AA^t y = C_\omega^{-t} C_\omega^{-1} b, \tag{6}$$

in the consistent case.

Here, we apply CG to the system

$$A^t C_\omega^{-t} C_\omega^{-1} Ax = A^t C_\omega^{-t} C_\omega^{-1} b. \tag{7}$$

This corresponds to applying CG to the generalized least squares problem

$$\min \left\| C_\omega^{-1}(b - Ax) \right\|^2.$$

In the ART method, ω is the relaxation parameter. Here it has a different role: it is the weight of the lower triangular part of A in the preconditioning. When ω is zero, this corresponds to a diagonal scaling. The conjugate gradient method applied to (7) can be written as:

$$r^0 = C_\omega^{-1}\left(b - Ax^0\right), \qquad s^0 = A^t C_\omega^{-t} r^0, \qquad w^0 = s^0$$

and for $k = 0, 1, \ldots$

$$p^k = C_\omega^{-1} A w^k$$

$$\alpha^k = \frac{\|s^k\|^2}{\|p^k\|^2}$$

$$x^{k+1} = x^k + \alpha^k w^k$$
$$r^{k+1} = r^k - \alpha^k p^k$$
$$s^{k+1} = A^t C_\omega^{-t} r^{k+1}$$

$$\beta^k = \frac{\|s^{k+1}\|^2}{\|s^k\|^2}$$

$$w^{k+1} = s^{k+1} + \beta^k w^k.$$

Considering that the elements of the matrix L are not explicitly known, it remains to present an efficient way to compute the products $A^t C_\omega^{-t} r$ and $C_\omega^{-1} A w$. In order to do this we describe below a technique taken from [1].

Define $h_m = 0$. For $i = m, m-1, \ldots, 1$ compute the number

$$s_i = d_i^{-1/2} r_i - \omega d_i^{-1} a_i^t h_i, \tag{8}$$

and the vector

$$h_{i-1} = h_i + s_i a_i, \tag{9}$$

obtaining $h_0 = A^t C_\omega^{-t} r$, where $D = \operatorname{diag}(d_1, \ldots, d_m)$.

In the same way, define $g_1 = w$. For $i = 1, 2, \ldots, m$, compute the component

$$t_i = d_i^{-1/2} a_i^t g_i, \tag{10}$$

and the vector

$$g_{i+1} = g_i - \left(\omega d_i^{-1/2} t_i\right) a_i, \tag{11}$$

obtaining $t = C_\omega^{-1} A w$, where $D = \operatorname{diag}(d_1, \ldots, d_m)$. For simplicity's sake we will refer to the above method as PCCGNR (preconditioned conjugate gradient applied to the normal equations).

2. Application to positron emission tomography

The goal of positron emission tomography (PET) is the quantitative determination of the moment-to-moment changes in the chemistry and flow physiology of radioactive labelled components inside the body. The mathematical problem consists of reconstructing a function representing the distribution of radioactivity in a body cross-section from measured data that are the total activity along lines of known location. One of the main differences between this problem and that arising in X-ray tomography [5] is that here measurements tend to be much more noisy, so direct inversion using convolution backprojection (CBP) does not necessarily give the best results (see [16]).

In positron emission tomography (PET) [15], the isotope used emits positrons which annihilate with nearby electrons generating two photons traveling away from each other in (nearly) opposite directions; the number of such photon pairs (detected in time coincidence) for each line or pair of detectors is related to the integral of the concentration of isotope along the line.

Suppose now that we discretize the problem by subdividing the reconstruction region into n small square-shaped picture elements (pixels, for short) and we assume that the activity in each pixel j is a constant, denoted by x_j. If a_{ij} denotes the probability that a photon emitted by pixel j is detected by pair i, then b_i is a sample from a Poisson distribution whose expected value is $\sum_{j=1}^{n} a_{ij}x_j$. Taking this

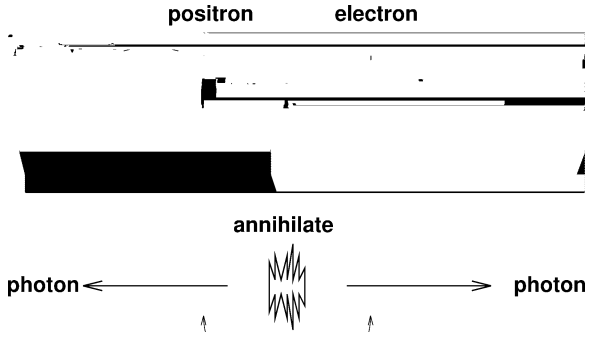

Fig. 1. Positrons which annihilate with nearby electrons generating two photons traveling away from each other in (nearly) opposite directions.

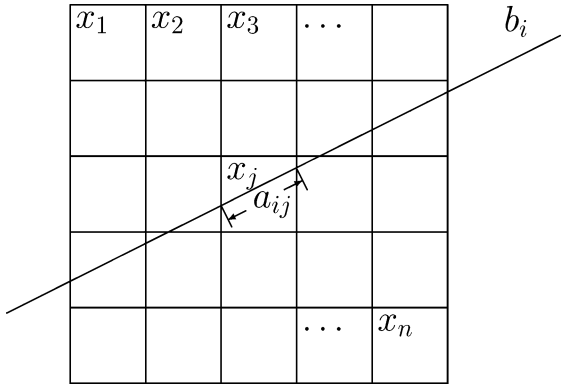

Fig. 2. The discretization.

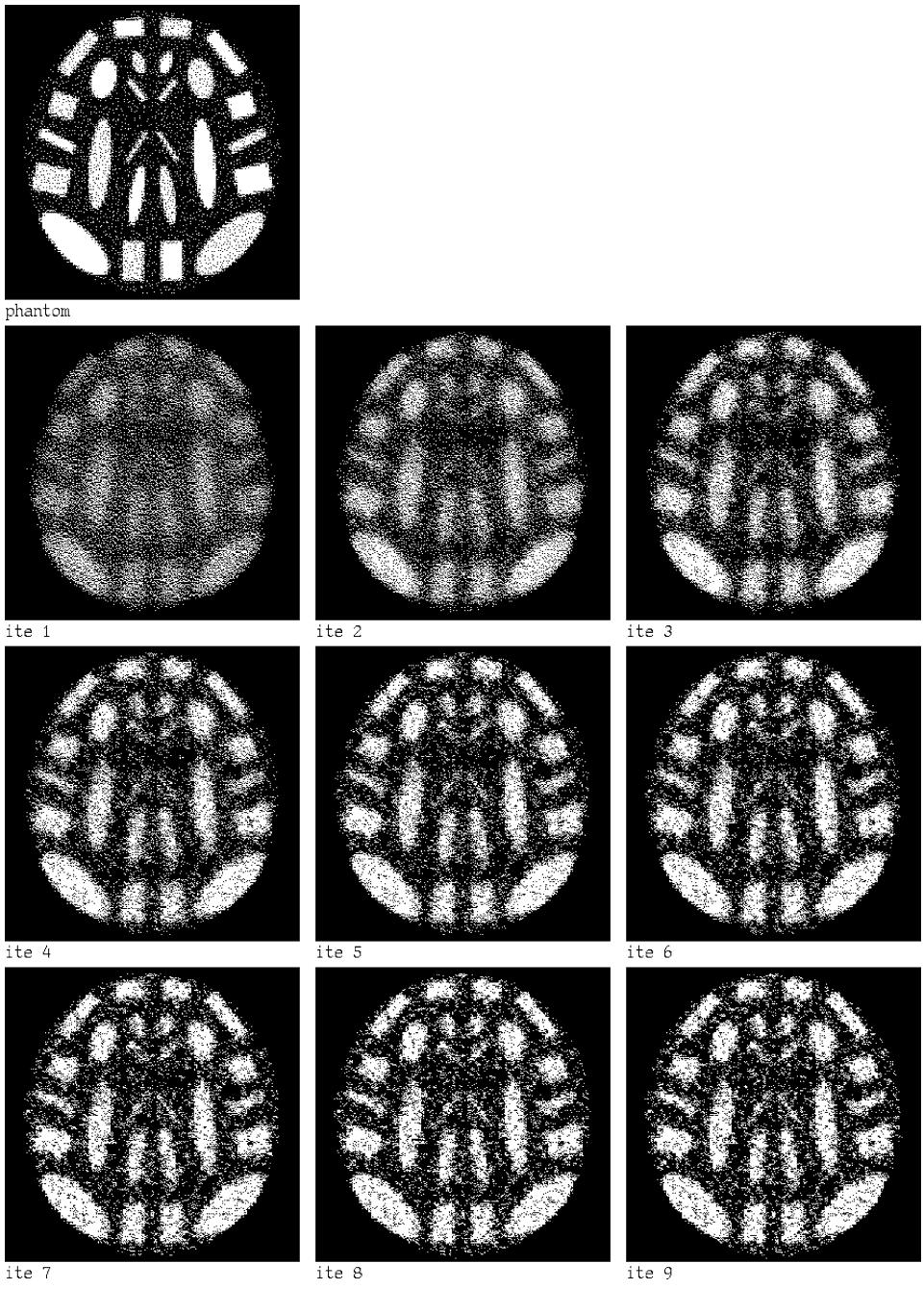

Fig. 3. A 95×95 phantom and its reconstructions using ART with $\omega = 0.025$, PET errors.

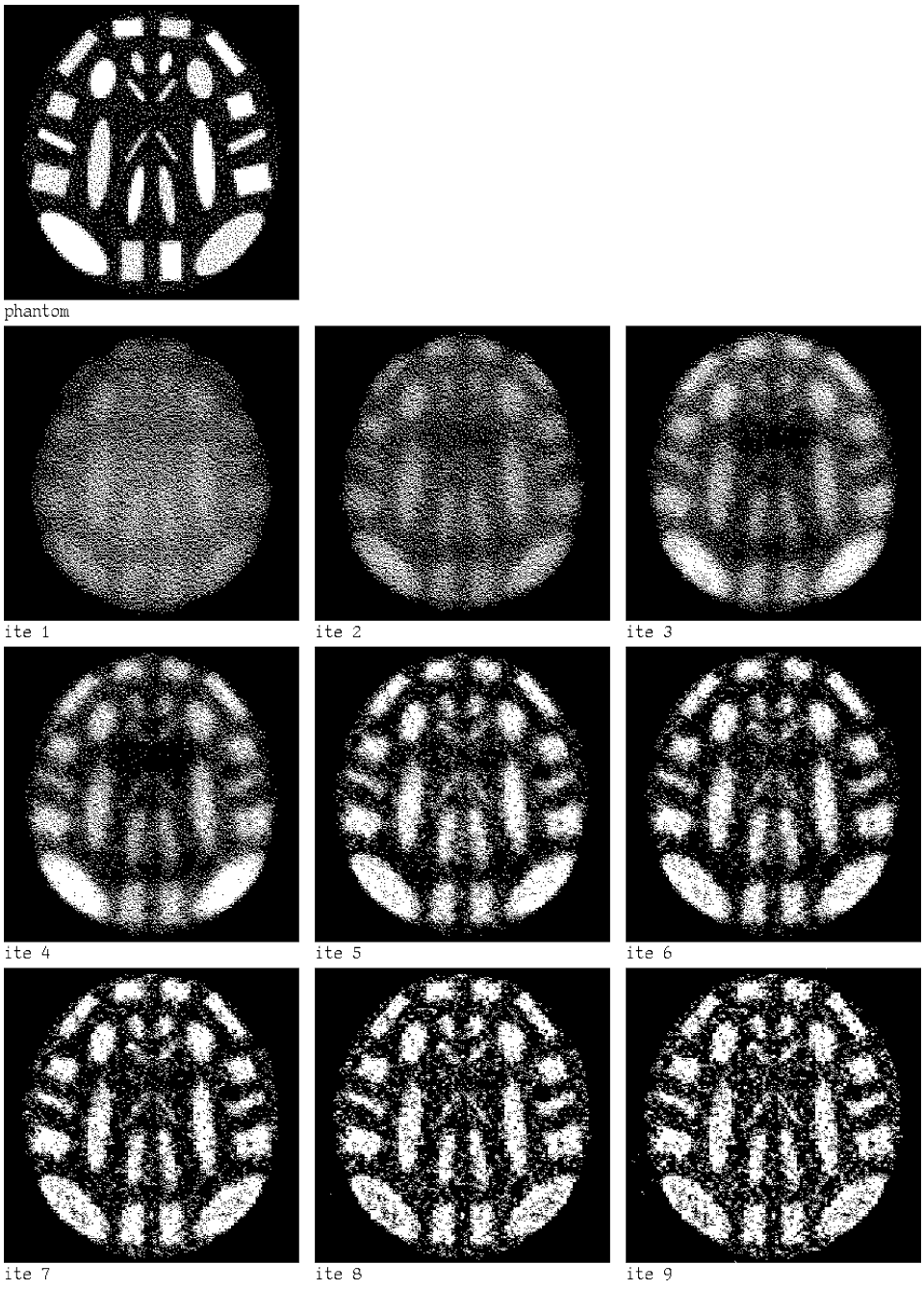

Fig. 4. A 95×95 phantom and its reconstructions using PCCGNR with $\omega = 0.0$ (diagonal scaling), PET errors.

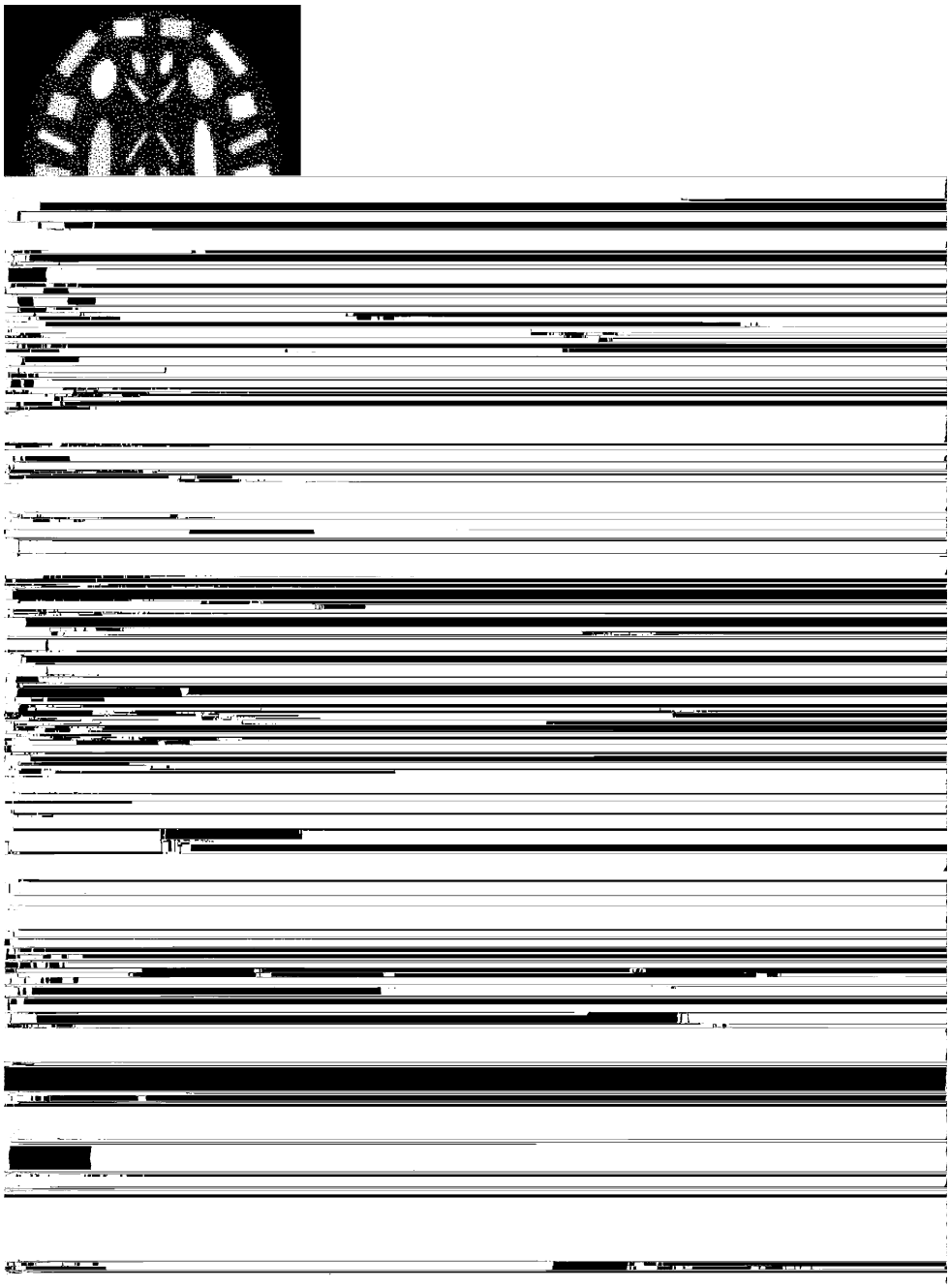

Fig. 5. A 95×95 phantom and its reconstructions using PCCGNR with $\omega = 0.025$, PET errors.

into account, it was suggested in [12] to estimate $x = (x_1, \ldots, x_n)^t$ by maximizing the Poisson likelihood; in [13], Shepp and Vardi suggested the use of the EM algorithm [3] for this maximization, and that has, since then, become very popular in the field. Herman and Meyer [6] proposed the use of ART for PET obtaining a speed-up of at least one order of magnitude with respect to the EM, and equivalent image quality. It is well known that ART, with small relaxation parameters [6], approximates a weighted least squares solution of (1). This fact suggests that the application of the conjugate gradient method to the system (6) could give similar or better results.

In our numerical experiments we used the programming system SNARK93, developed by the Medical Image Processing Group of the University of Pennsylvania [2]. The images to be reconstructed (phantom) were obtained from a computerized atlas based on typical values inside the brain, as in [6]. The data collection geometry was a divergent one simulating a typical PET data acquisition [2]. We used a discretization with $n = 95 \times 95$ pixels and the divergent geometry had 300 views, of 101 rays each, a total number of $m = 30292$ equations (8 rays were not considered because they did not intersect the image region). The starting point was a uniform image $x^0 = (a, \ldots, a)^t$, where a is an approximation of the average density of the phantom given by

$$a = \frac{\sum_{i=1}^{m} b_i}{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}}. \tag{12}$$

The choice of a uniform non-zero starting point was advocated by Kaufman [11] and is widely accepted as the best choice for many researchers in PET. The vector b was taken from a pseudorandom number generator with a Poisson distribution (see [7]). The total photon count was 2 022 085, 991 179, 514 925 and 238 172.

We have done experiments with photon counts 2 022 085, 991 179, 514 925 and 238 172 and with six values of ω, between 0.0 (diagonal scaling) and 0.025. For each value ω and each photon count we repeated the experiments ten times. The ten tests give very similar results.

Fig. 3 shows the images corresponding to several iterations of ART in one typical test. Figs. 4 and 5 show the reconstructions corresponding to several iterations of PCCGNR, with $\omega = 0$ and $\omega = 0.025$. We observed that preconditioning with $\omega > 0$ makes the method faster than with $\omega = 0.0$ (diagonal scaling) and than the ART method.

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