Ch 13: Electric Field

Main Concepts
- EM Fields - A continuous sheet of forces
- Electrons & nuclei as important to mechanics (not just atoms & bonds)

Electric Charge/Force
- Treat protons/electrons as points b/c little
- EMF (electromagnetic force) calculated by Coulomb's Law

can also be written k_e — tends to be towards each other (center being origin)

$$|\vec{F}| = F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{|Q_1 Q_2|}{r^2}$$

sometimes F_C

where Q_1 & Q_2 electric charges of 2 particles, r is distance b/w 2 particles, $\boxed{\varepsilon_0}$ is electric constant / vacuum permittivity

CONSTANT describing electricity's ability to travel in vacuum ($8.85 \cdot 10^{-12}$ F $\cdot$ m^{-1}) ← Farads (electric capacitance) or $C^2 \cdot N^{-1} \cdot m^{-2}$

* Note: this is only magnitude

Units
- Coulomb (C) - electric charge
  - 1 proton $= 1.6 \cdot 10^{-19}$ C (mass $9 \cdot 10^{-31}$ kg) ~2000×
  - 1 electron $= -1.6 \cdot 10^{-19}$ C (mass $1.7 \cdot 10^{-27}$ kg)
  - * $e = 1.6 \cdot 10^{-19}$ C ← useful shorthand

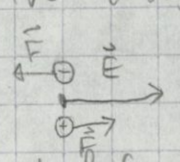
Fun Fact!
Muon: $1.88 \cdot 10^{-28}$ kg, $\pm e$ ($\mu^{\pm}$)
Pion: $2.48 \cdot 10^{-28}$ kg, $\pm e$ ($\pi^{\pm}$) (same)

Electron Cloud 25,000× bigger than nucleus! ($\sim 1 \cdot 10^{-10}$ m vs $\sim 4 \cdot 10^{-15}$ m)

EM Fields
- Why use fields? Easier general understanding & easy changing of charge @ location
- Electric field at location given by

$$\vec{F}_A = q_A \vec{E}_1$$ where $\vec{F}_A$ is force on a particle A, q_A is the charge of particle A, $\vec{E}_1$ is the electric force/field where A is located

- How do we find $\vec{E}$? Experimentation w/ known particle A. Known properties of the EM field producers.
- How is $\vec{E}$ measured? N/C (Newtons per Coulomb)
- Positive charges go w/ field, Negative against



$$\boxed{\begin{array}{l} \text{Coulomb} \Rightarrow \text{EM Field} \\ K_g \Rightarrow \text{Gravitational Field} \end{array}}$$

- Normal Reference:
  A charge q_1 (source charge) creates a field $\vec{E}_1$ & a different charge q_2 is placed in that field. We find the force $\vec{F}_{on\,q_2} = q_2 \vec{E}_1$
- Field: Something w/ a value at every location in space.
  - EM Field is field of vectors
  - Temperature is a field of scalars
  - Field of Cir

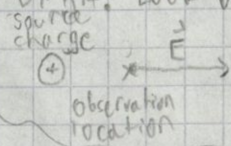
$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0} \frac{|q_1|}{|\vec{r}|^2} \hat{r}$$ where q_1 is source charge, $\vec{r}$ is position vector from source charge to observation location

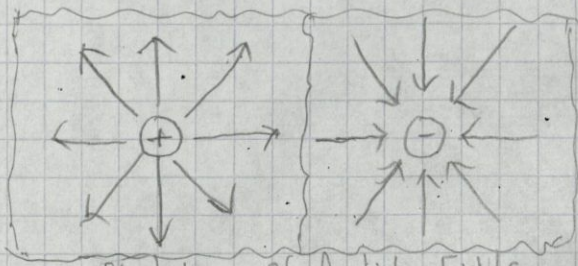
- Electric Field of Single Point Charge

$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{|\vec{r}|^2} \hat{r}$ where q_1 is source charge, $\vec{r}$ is vector from source charge to observation location in physical space (i.e. length is dist. b/w them)

*Note: This is Coulomb's Law w/ a unit charge of +1C & direction (away from +, towards −)

Wait away from? I thought post. charges went w/ $\vec{E}$!
BAKA! Look below, a post. charge will go w/ $\vec{E}$ still.





Sketches of Particle Fields

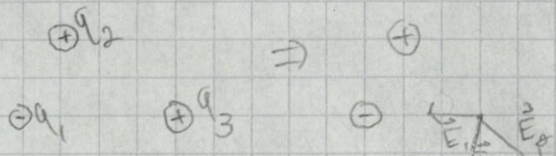
Uniformly Charged Sphere EM Field
- This uses the Superposition Principle not yet covered (see next section & Ch 15)
- For a sphere w/ radius R & charge Q uniformly spread over its surface produces the following charge distributions where $\vec{r}$ is a vector from the center of the sphere to the observation location:

$$\vec{E}_{sphere} = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} & r > R \text{ (i.e. outside sphere)} \\ 0 & r < R \text{ (i.e. inside sphere)} \end{cases}$$

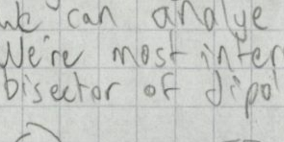
Whoa! This explains why particles don't move themselves & can be treated as points

*Acts like a sphere outside!

Superposition Principle
- The superposition principle describes how to find $\vec{E}$ in the presence of 2+ charges.
- Method: Take the vector sum of all independent fields at the point (i.e. as if it was the only field, unaffected by all other fields)
  - Same as for individual forces
  - Means matter can't block electricity! (kinda)

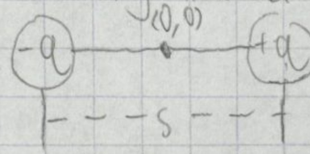


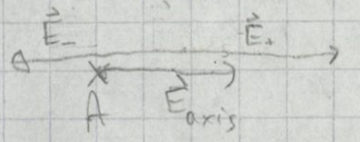
Dipoles & EM Fields
- Dipole: A simplified model of "neutral things" (i.e. things w/ no consistent charge distribution). Model describes 2 equally & oppositely charged particles w/ charge +q & −q separated by distance s.
  - This is most accurate w/ small, ionic molecules (e.g. HCl) but can sometimes scale.
  - We can analyze any point using superposition principle.
  - We're most interested in point along axis of dipole & along perpendicular bisector of dipole, since that gives us nice simplifications/approximations.



Making Math Shortcuts for Dipoles (along Axes)

Along Parallel Axis





Using the superposition principle,

$\vec{E}_A = \vec{E}_+ + \vec{E}_-$

$\vec{E}_+ = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{|\vec{r}_+|^2} \hat{r}_+$

$\vec{r}_+ = \langle x, 0, 0\rangle - \langle s/2, 0, 0\rangle = \langle x - s/2, 0, 0\rangle$

$|\vec{r}| = |x - s/2|$

$\hat{r} = \langle 1, 0, 0\rangle$

$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_+}{(x-s/2)^2} \cdot \langle 1,0,0\rangle = \langle \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_+}{(x-s/2)^2}, 0, 0\rangle$

$\vec{E}_-$ is identical except $\overset{q=-q}{\vee}$ $\vec{r}_- = \langle x + s/2, 0, 0\rangle$ ← (from $\langle x, 0, 0\rangle - \langle -s/2, 0, 0\rangle$)

$\therefore \vec{E}_- = \langle \frac{1}{4\pi\varepsilon_0} \cdot \frac{-q}{(x+s/2)^2}, 0, 0\rangle$

Since we only have an x component, let's rename $\vec{E}_A$ to $E_{\parallel}$ to find force along axis of dipole. Similarly, rename $x \to r$ since dipole can be in any orientation.

$E_{\parallel s} = \overbrace{\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{(r-s/2)^2}}^{E_+} + \overbrace{\frac{1}{4\pi\varepsilon_0} \cdot \frac{-q}{(r+s/2)^2}}^{E_-}$

$= \frac{1}{4\pi\varepsilon_0} \cdot q\left(\frac{1}{(r-s/2)^2} - \frac{1}{(r+s/2)^2}\right)$

$= \boxed{\frac{1}{4\pi\varepsilon_0} \cdot \frac{2qsr}{(r-s/2)^2(r+s/2)^2}}$

What about points REALLY far away from the dipole?

$r >> s \Rightarrow (r - s/2)^2 \approx (r + s/2)^2 \approx r^2$

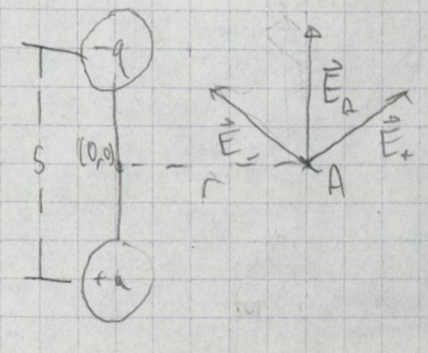
$\therefore E_{\parallel} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2qsr}{r^4}$

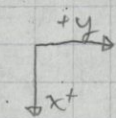
$\boxed{E_{\parallel s} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2qs}{r^3} \text{ when } r >> s}$

Why isn't this $1/r^2$? It's 2 charges, so it doesn't need the same dependence.

*This only give magnitude since we assumed the dipoles are in the same place. Use which side of dipole you're on to determine.

Along Perpendicular Axis





Use these axes for math below

Note: Both sides of the dipole have identical electric fields due to symmetry.

Using the superposition principle,

$\vec{E}_A = \vec{E}_+ + \vec{E}_-$

$\vec{E}_+ = \frac{1}{4\pi\varepsilon_0} \cdot \frac{+q}{|\vec{r}_+|^2} \cdot \hat{r}_+$

$\vec{r}_+ = \langle 0, y, 0\rangle - \langle s/2, 0, 0\rangle = \langle -s/2, y, 0\rangle$

$|\vec{r}_+| = \sqrt{(s/2)^2 + y^2}$

$\hat{r}_+ = \frac{\langle -s/2, y, 0\rangle}{\sqrt{(s/2)^2+y^2}}$

$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{+q}{((s/2)^2+y^2)} \cdot \frac{\langle -s/2, y, 0\rangle}{\sqrt{(s/2)^2+y^2}}$

Since $\vec{E}_-$ is the same as $\vec{E}_+$ but w/ $q = -q$ & $\vec{r} = \langle 0, y, 0\rangle - \langle -s/2, 0, 0\rangle = \langle s/2, y, 0\rangle$, I use the above as a shortcut.

$\vec{E}_- = \frac{1}{4\pi\varepsilon_0} \cdot \frac{-q}{((s/2)^2+y^2)} \cdot \frac{\langle s/2, y, 0\rangle}{\sqrt{(s/2)^2+y^2}}$

I factored negative to front

$\vec{E}_A = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{((s/2)^2+y^2)} \cdot \frac{\langle -s/2, y, 0\rangle}{\sqrt{(s/2)^2+y^2}} - \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{(s/2)^2+y^2} \cdot \frac{\langle s/2, y, 0\rangle}{\sqrt{(s/2)^2+y^2}}$

$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{((s/2)^2+y^2)^{3/2}} \left(\langle -s/2, y, 0\rangle - \langle s/2, y, 0\rangle\right)$

$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{((s/2)^2+y^2)^{3/2}} \left(\langle -s, 0, 0\rangle\right)$

$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{qs}{((s/2)^2+y^2)^{3/2}} \cdot \langle -1, 0, 0\rangle$ ← Hey! This just goes towards the negative side parallel to dipole axis b/c the y's cancelled.

Let's generalize to any dipole orientation by renaming $\vec{E}_a$ to $E_\perp$ & y to r. Let's also just measure magnitude since direction is unambiguously towards negative parallel to dipole:

$$\boxed{E_\perp = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qs}{\left(\left(\frac{s}{2}\right)^2 + r^2\right)^{3/2}}}$$

Note: These approximations are important to give us a good, intuitive understanding of the system.

What if we're really far away again?

$r >> s \Rightarrow (s/2)^2 + r^2 \approx r^2$

Note: the $1/r^3$ fall off holds true off the axes too but is hard to show.

$\therefore \boxed{E_\perp = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qs}{r^3}}$ ← from above & $(r^2)^{3/2} = r^3$

* Neat! These look similar to the parallel ones! Just half the magnitude.

Properties of Dipoles

- Electric Dipole Moment (p): The EM field of a dipole is directly proportional to this quantity. This is also what is measured in real life on H_2O, HCl, etc.

$p = qs$ where q magnitude of charge on single pole, s is separation b/w poles.

* This makes sense b/c having weak charges far apart produces a significant charge, as does having strong charges close together

- Using the electric dipole moment (p):

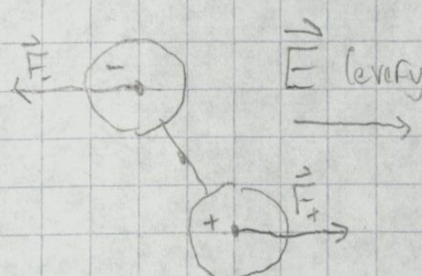
$$E_{\parallel} \approx \frac{1}{4\pi\varepsilon_0} \cdot \frac{2p}{r^3} \quad \text{for } r \gg s.$$

$$E_{\perp} \approx \frac{1}{4\pi\varepsilon_0} \cdot \frac{p}{r^3} \quad \text{for } r \gg s.$$

- You can treat the dipole moment as a vector ($\vec{p}$) in the middle of a dipole pointing from - to + w/ magnitude qs.
  + Points in same direction as $E_{\parallel}$.

- This will be similar to magnetic dipoles in future (Ch 17).

- In a uniform electric field, a dipole aligns itself w/ the field ($\vec{p}$ w/ $\vec{E}$) but does not move. This is b/c the dipole experiences no net force (+ & - forces cancel) but does experience net torque.



Importance of Systems

We broadly divide the universe into
- Sources of the EM field (we consider fixed)
- Things affected by EM field (we consider to not affect EM field)

Importance of EM Field

EM Field concept allows us to understand the electric properties of matter outside their effects. Note: The EM Field is (probably) real & not just a calculation convenie as shown by experiments w/ special relativity (i.e. things can't act instantaneously over a distance).

This means changes in EM field travel at the speed of light.

∴ Coulomb's Law is only an approximation that works at low speeds, as it doesn't consider time or the speed of light. Likewise for electric field of point charge equation.

We will discuss moving & accelerating charges later where this becomes important.

Accounting for Special Relativity (well, not time)
When a charged particle is moving in $+x$ direction, x is unaffected.
However a factor of γ is applied to the y & z component, where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Ch 14: Electric Fields & Matter

Net Charge $= \sum_{part \in Object} \text{charge}(part)$.

Even electrically neutral (i.e. net charge ≈0) things can create EM field (see dipoles).

Net charge is conserved. We say net charge b/c electrons & positrons can & do annihilate each other.

Types of Materials
Broadly, there are two types of materials (electrically):
- Conductors: Charged particles move w/in & thru easily. Has mobile charges
  - Salt water
  - Most metals
    - Copper

Mostly just electrons

- Insulators: Charged particles cannot move w/in & thru easily. Has no mobile charges.
  - Pure water
  - Rubber
  - Skin
  - Wood

Charging
Rubbing or touching insulators (or anything) seem to give them a charge (e.g. baloons, glass rods). This normally yields a charge on order of $10^{-8}C$ to small objects.

In almost all cases, electrons (bonds) move, not protons. Ions also move sometimes.

Polarization of Atoms
Both positive & negative charges attract neutral materials. However, using $\vec{F}=q\vec{E}$ w/ the above seems to suggest neutral atoms are positive & negative.

We can make sense of this with atomic polarization.

well, really a probability distribution b/c quantum mechanics

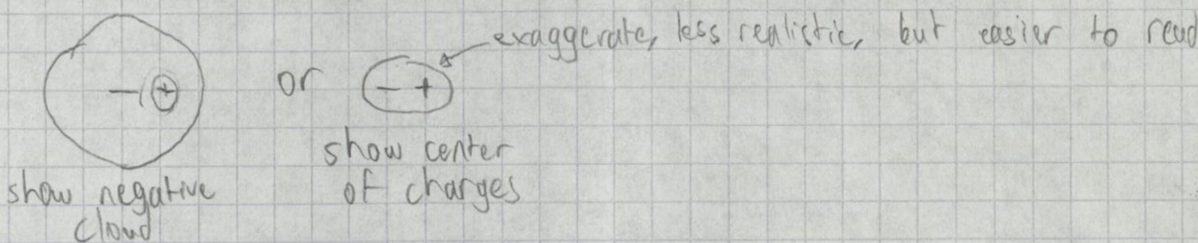
Electrons in an atom are structured as a cloud, normally fairly uniform. However, external EM forces change the structure of this cloud. These forces can (& often do) shift the average position of an electron from the nucleus (its default) to slightly off to one side of the nucleus. This <u>polarizes the atom</u> & <u>induces</u> an <u>atomic dipole</u>.

The dipole is always induced in such a way that the atomic dipole is attracted by (or attracts) the external charge. This can be seen b/c bringing a positive charge near an atom attracts electrons, meaning the negative end points towards the positive charge. Vice versa for negative charges.

Note: Although I talk like only the electron cloud is moving, really they both are.

Since outer electrons are farther from the nucleus, they are more affected by outside charges.

We show polarized atoms / atomic dipoles thus:



The phrases atomic polarization, induced dipole, & atomic dipole all refer to this phenomenon.

Polarizability

Different materials have different affinities to be polarized. We call this polarizability (α) & use the following equation to find the dipole

We use the following to determine the dipole moment ($\vec{p}$) of an induced dipole given the applied EM field ($\vec{E}$) & polarizability of dipole (α):

$$\vec{p} = \alpha \vec{E}$$

* Recall: $|\vec{p}| = qs$ w/ $\vec{p}$ pointing from − to +.

A regular α for atoms is $\alpha \approx 1 \cdot 10^{-40} \frac{C \cdot m}{N/C}$

The earlier dipole moment approximations still work. Recall:

$$\vec{E}_{\parallel} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2\vec{p}}{|\vec{r}|^3} \quad \& \quad \vec{E}_{\perp} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\vec{p}}{|\vec{r}|^3} \quad \text{for } r >> s.$$

We can similarly make shortcuts for the electric field caused by an induced dipole caused by a particle at a distance r.

$$E_{dipole} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2p}{r^3} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2\alpha E_{dipole\text{-}charge}}{r^3}$$

The math here is kinda messy w/ how things are oriented. Basically, forces always attract

$$E_{dipole\text{-}charge} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_{charge}}{r^2}$$

$$E_{dipole} = \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{2\alpha}{r^3}\right)\left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{q_{charge}}{r^2}\right) = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \left(\frac{2\alpha q_{charge}}{r^5}\right)$$

* Neat! This is proportional to $1/r^5$.

Because of $F_{charge\text{-}dipole} = q_{charge} E_{dipole}$

$$\vec{F}_{\substack{charge-\\dipole}} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \left(\frac{2\alpha q^2_{charge}}{r^5}\right)\hat{r}_1$$ ← This always points towards dipole.

* Neat! The force on a charge from an induced dipole on that charge is proportional to $1/r^5$.
* Note! q_{charge} is squared b/c one accounts for dipole polarization & other $\vec{F}=q\vec{E}$

Because of reciprocity, the same relationship exists for the force on an induced dipole from a charge.

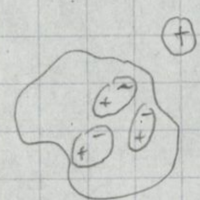
$$\vec{F}_{dipole} = -\vec{F}_{charge}$$ ← i.e. the same & also attractive

Polarization of Insulators

Since, in <u>insulators</u>, electrons cannot move significantly & are bound to their atom or molecule (by definition), each individual atom/molecule in the insulator can be polarized by an outside charge. ← This is also what causes build up of charges!

This individual polarization is the <u>polarization of insulators</u>. And, since all atoms/molecules in an insulator polarize in (mostly) the same way, the affect can be pretty big.

To draw this effect, we normally draw multiple oversized molecules in a blob.



<u>Note</u>: Induced dipoles in an insulator technically induce dipoles in their neighbors. Here, we ignore that effect since it is small relative to the external charge. This is called the "<u>Low Density Approximation</u>"

Polarization of Conductors

<u>Conductors</u> have charged particles that can somehow move around. This means that polarization is much more extreme as the charges can move much farther. However, this polarization is weakened by the like forces repelling each other.

Drift Speed

Unlike you may expect, ions/charges in a solution don't constantly accelerate. Instead they have some terminal <u>drift speed</u> due to drag from the fluid.

This drift speed ($\bar{v}$) is proportional to the EM field (E) by some constant we call the <u>mobility</u> of the ions in the solution (u).

$\bar{v} = uE$ where u has units $\frac{m/s}{N/C}$ ~~duh~~

$\frac{\bar{v}\ m/s}{E\ N/C}$

Drift speed reaches zero when the EM field in a solution reaches zero.
This occurs when the EM field by the dipole of solution is equal & opposite the external one. This is called <u>equilibrium</u>.

<u>Note:</u> This hold for other conductors as well, not just solutions. This means metals have a drift speed!

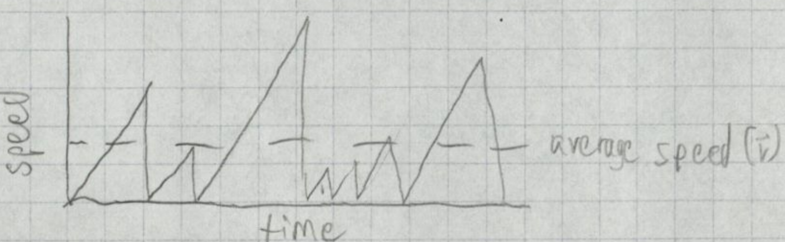
This equilibrium also explains why electricity moves so quickly. The particles don't actually have to move super far to reach equilibrium b/c they only have to "counteract" the external charge.

Motion of Charges in Metal

In metals, outer electrons are very mobile, creating an electron sea within the regular lattice of metal.

Since there is no EM field in a metal by default, the electrons act like an ideal gas!

Here we model electron movement via the <u>Drude Model</u> which explains how electrons move in an EM field. It states electrons move b/c of the EM field until they collide w/ the metal lattice, slowing down & then accelerating again.



Note

This models <u>drift speed</u> which is the same concept for conductors from earlier.

<u>Note:</u> Really, the electrons are moving chaotically in all directions w/ a general tendency to a certain direction. Like air molecules in wind. Understanding this requires quantum mechanics tho, so we'll use the simple Drude model.

Let's find drift speed ($\bar{v}$) using the Drude model.

$\frac{\Delta p}{\Delta t} = F_{net}$ We use the momentum principle (aka 2nd Law or F=ma [not technically correct b/c Lorentz Factor]) w/ discrete time steps (where Δp is change in momentum & Δt is time b/w collisions)

$\Rightarrow \Delta p = F_{net}\Delta t$

$F_{net} = eE_{net}$ (we only look at magnitude so no sign)

$\Delta p = p - 0$ (we assume collisions remove all momentum)

$\Rightarrow p = eE_{net}\Delta t$

$\Rightarrow v = \frac{\Delta p}{m_e} = \frac{eE_{net}\Delta t}{m_e}$ where m_e is mass of electron

$\Rightarrow \bar{v} = \frac{eE_{net}\bar{\Delta t}}{m_e}$ We take the average since we want average velocity. Note that v & Δt are the only variables.

$\overline{\Delta t}$ (average time b/w collisions) is affected by the nature of the metal & temperature (more motion causes more collisions).

To reiterate the drift speed ($\bar{v}$) of electrons in a metal is:

$$\boxed{\bar{v} = \frac{e E_{net} \overline{\Delta t}}{m_e}}$$

* Note: We can't fully in good faith say doubling E_{net} doubles $\bar{v}$ since increasing E_{net} may (& probably does) affect $\overline{\Delta t}$. Here, we normally assume $\overline{\Delta t}$ is unaffected by E_{net}.

Using our earlier equation

$$\bar{v} = u E_{net}$$

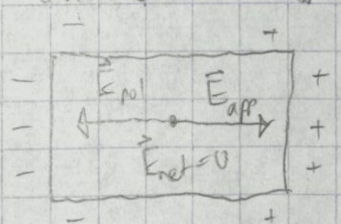
& the above new equation gives us

$$u = \frac{e \overline{\Delta t}}{m_e}$$

* For metals using the Drude model

Much like insulators & conductors, metal polarizes very quickly b/c the sea hardly has to move to create change/equilibrium.

We show the polarization of metal by drawing charges on the surface to show they are on the surface & not inside, as the inside is neutral (unlike an insulator).



Note: "Polarized" deals w/ distribution of charge. "Charged" deals w/ net charge.

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We can understand $\vec{E}_{pol}$ pointing the opposite direction of $\vec{E}_{app}$ b/c $\vec{E}_{pol}$ is only inside the conductor. You can see this force b/c, inside the metal, charges do not want to be pushed to the positive side b/c like charges repel.

This generalizes to all conductors

Conductor vs Insulator Polarization

	Conductor	Insulator
Mobile Charges	Yes	No
Polarization	Sea of mobile charges move	Individual atoms/molecules polarize
Equilibrium	$\vec{E}_{net} = \vec{0}$ inside	$\vec{E}_{net} \neq \vec{0}$ inside
Excess Charge	On surface	Everywhere

Grounding

Grounding is the process of spreading out the charges on a charged object to a larger one. This effectively neutralizes the smaller charged object.

This gets its name from Earth, since Earth is a huge conductor which neutralizes most things that make contact w/ it. People are also large conductors (think sweat)

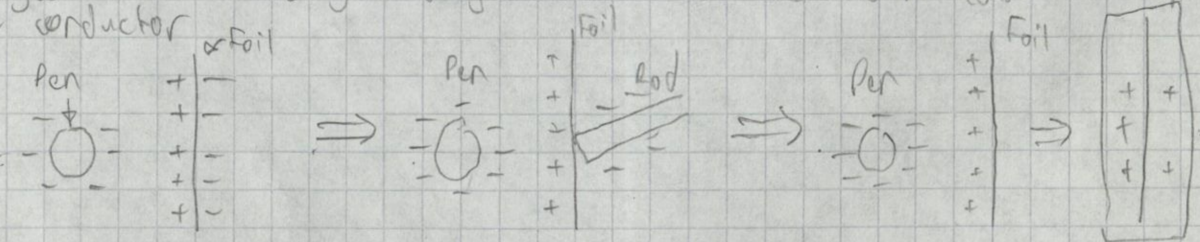
This is trivially done thru contact w/ conductors since their charges move freely.

For insulators, this requires prolonged contact/rubbing w/ a conductor. The conductor then 'donates' mobile charges (either positive or negative) neutralizing the object.

Charging by Induction

We can take advantage of grounding to charge conductors w/out making them.

Basically, polarize a conductor. Touch one side w/ the neutral conductor. All the charges on that side go to ground. Remove the neutral conductor. Stop polarizing the conductor



Molecular Polarization

Some molecules (e.g. H_2O) are permanent dipoles. Water landing on things & dissolving the salt there improves conductivity (even in air). It also contributes some charge, neutralizing things.

Pragmatics of Measuring

Our earlier equation, $\vec{F} = q\vec{E}$ only works for q small enough to not disturb $\vec{E}$. For some fields, this is impossible as q would be much smaller than e

Therefore, it is often more practical to place a neutral particle in a place & measure its polarization to get the EM field at that location. This is b/c, even tho the particle still affects the EM field, it does so very little.

Alternatively, we could test w/ a negative & positive charge of equal size in 2 separate experiments & take the average, since they cancel out.

Ch15 EM Field for Distributed Charges

Here's where we use calculus! We can model a continuous surface using infinitely many infintesimal charges & use integrals to find the charge at a point.

We've only analytically worked out a few simple cases

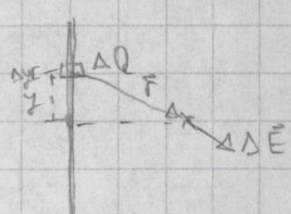
Uniformly Charged Thin Rod

For rod w/ length L & total charge Q, we choose a point in the plane perpendicular to rod @ its midpoint b/c it's easy.

We follow the following steps for the rod (& in general):

1. Divide charged object into smaller pieces.
2. Choose origin & axes.
3. Find EM field contributed by piece ($\Delta\vec{E}$).
4. Sum all $\Delta\vec{E}$'s

First, we divide the rod & choose a representative piece



The Δy is length of slice. This is important for finding charge of slice & will become dy.

We represent this slice w/ a single point charge.
Note: In the full version, we assume a rod thin enough that a slice can be reasonably approximated as a point.

Now, we find $\Delta\vec{E}$ relative to y (our integration variable)

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\Delta Q}{|\vec{r}|^2}\hat{r}$$

$\vec{r} = \langle x, 0, 0\rangle - \langle 0, y, 0\rangle = \langle x, -y, 0\rangle$

$\hat{r} = \frac{\langle x, -y, 0\rangle}{(x^2+y^2)^{1/2}}$

$|\vec{r}|^2 = x^2+y^2$

$\Delta Q = \frac{\Delta y}{L} Q$ ← either fraction of total charge ($\frac{\Delta y}{L}$) or charge density ($\frac{Q}{L}$)

...

$$\Delta \vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\Delta y Q}{L} \cdot \frac{1}{(x^2+y^2)} \cdot \frac{\langle x, -y, 0\rangle}{(x^2+y^2)^{1/2}}$$
$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{L(x^2+y^2)^{3/2}} \langle x, -y, 0\rangle \Delta y$$

$$\Rightarrow \Delta E_x = \frac{1}{4\pi\varepsilon_0} \cdot \frac{xQ}{L(x^2+y^2)^{3/2}} \Delta y$$
$$\Delta E_y = \frac{1}{4\pi\varepsilon_0} \cdot \frac{-yQ}{L(x^2+y^2)^{3/2}} \Delta y$$ ← $-y$ b/c if above you the pushes down
$$\Delta E_z = 0$$

Now, we take limit as number of little pieces goes to infinity & take sum. In other words, we integrate!

$$E_x = \int_{-L/2}^{L/2} \frac{1}{4\pi\varepsilon_0} \cdot \frac{xQ}{L(x^2+y^2)^{3/2}}\, dy$$
$$= \boxed{\frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{L} x \int_{-L/2}^{L/2} \frac{1}{(x^2+y^2)^{3/2}}\, dy}$$

$$E_y = \int_{-L/2}^{L/2} \frac{1}{4\pi\varepsilon_0} \cdot \frac{-yQ}{L(x^2+y^2)^{3/2}}\, dy$$
$$= \boxed{\frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{L} \int_{-L/2}^{L/2} \frac{-y}{(x^2+y^2)}\, dy}$$

Now, let's see if we can simplify for a better understanding.

I'm pretty sure these would require trig sub, which is annoying, so I'm gonna trust the book.

$$E_x = \frac{1}{4\pi\varepsilon_0} \left(\frac{Q}{L} x\right) \left[\frac{y}{x^2\sqrt{x^2+y^2}}\right]_{-L/2}^{L/2}$$
$$= \frac{1}{4\pi\varepsilon_0} \left(\frac{Q}{x\sqrt{x^2+(L/2)^2}}\right)$$

$E_y = 0$ ← in this scenario

*Note: Since a rod is cylindrically symmetrical, we can replace x w/ r. (We're basically switching to cylindrical coordinates)

$$\therefore \boxed{E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r\sqrt{r^2+(L/2)^2}}}$$

for distance r from midpoint w/in plane perpendicular to the rod, intersecting the midpoint. The direction is either directly toward or away from the rod depending on sign.

Special cases

When $r \gg L$ (we're far away from rod),

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r\sqrt{r^2+(L/2)^2}} \approx \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r\sqrt{r^2+0}} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}$$ ← It's like a point!

When $L >> r$ (we have a large rod),

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r\sqrt{r^2+(L/2)^2}} \approx \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r\sqrt{0+(L/2)^2}} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2Q}{rL} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2(Q/L)}{r}$$

We're not going to show it here, but our calculation holds reasonably well whenever we're near the center of a rod, not just right on the center

Calculating General Electric Field

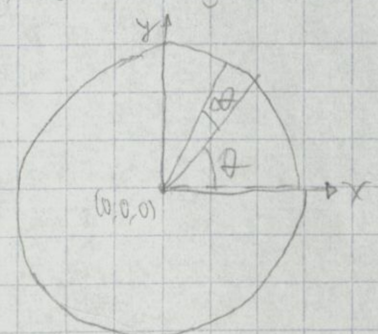
- Divide charge distribution into pieces of known field. Specifically approximate as necessary (point charge!)
- Write expression for EM Field for pieces. (Split into categories as necessary)
- Sum the EM field for all pieces. If possible, do definite integral.

↑ Make sure to include "integration variables"

Uniformly Charged Ring

For this section, suppose we have a ring w/ radius R & charge q.

Here, we analyze the line straight thru the ring.



We describe piece in terms of θ.

$$\vec{r} = \vec{obs} - \vec{src} = \langle 0, 0, z\rangle - \langle R\cos(\theta), R\sin(\theta), 0\rangle = \langle -R\cos(\theta), -R\sin(\theta), z\rangle$$

$$|\vec{r}| = \sqrt{R^2\cos^2(\theta) + R^2\sin^2(\theta) + z^2} = (R^2+z^2)^{1/2}$$

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\Delta q}{|\vec{r}|^2} \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$\Delta q = \frac{\Delta\theta}{2\pi} \cdot q$$

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\Delta\theta q}{2\pi} \cdot \frac{\vec{r}}{|\vec{r}|^3} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\Delta\theta q}{2\pi} \cdot \frac{\langle -R\cos(\theta), -R\sin(\theta), z\rangle}{(R^2+z^2)^{3/2}}$$

$$\therefore d\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{2\pi} \cdot \frac{\langle -R\cos(\theta), -R\sin(\theta), z \rangle}{(R^2+z^2)^{3/2}} \, d\theta$$

We now integrate to find ring

$$\vec{E} = \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{2\pi} \cdot \frac{\langle -R\cos(\theta), -R\sin(\theta), z \rangle}{(R^2+z^2)^{3/2}} \, d\theta$$

$$\Rightarrow E_x = \cancel{\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{2\pi} \cdot \frac{-R}{(R^2+z^2)^{3/2}} \cdot \int_0^{2\pi} \cos(\theta)\, d\theta} = 0$$

$$E_y = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{2\pi} \cdot \frac{-R}{(R^2+z^2)^{3/2}} \cdot \cancel{\int_0^{2\pi} \sin(\theta)\, d\theta} = 0$$

$$E_z = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{2\pi} \cdot \frac{z}{(R^2+z^2)^{3/2}} \int_0^{2\pi} d\theta =$$

$$\boxed{E_z = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qz}{(R^2+z^2)^{3/2}}}$$

The direction is always trivially directly towards or away from ring, depending on sign.

Uniformly Charged Disk

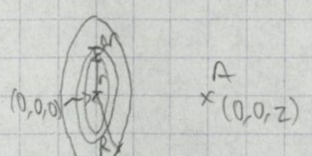
It's important to consider these b/c they are part of capacitor.

Note: Near the center, these produce a nearly uniform EM field & near the edges magnitude still doesn't change much.

Here tho, we'll only consider axis thru disk perpendicular to it. (You can apply the calculation to near center axis too!)

For this section, suppose we have a thin disk of radius R w/ a charge of Q uniformly distributed over it.

Since we just solved for the EM field of rings, we divide the disk into concentric rings.



We describe the field in terms of r (integration variable).

$\vec{r} = \langle 0, 0, z \rangle$

$|\vec{r}| = z$

$\hat{r} = \langle 0, 0, 1 \rangle$

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\Delta q\, z}{(r^2+z^2)^{3/2}} \langle 0, 0, 1 \rangle$$

b/c only z is nonzero

$$\Delta q = \frac{2\pi r \Delta r}{\pi R^2} \cdot Q = \frac{2 r \Delta r}{R^2} \cdot Q$$

$$\Delta\vec{E}_z = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2 r \Delta r}{R^2} \cdot Q \cdot \frac{z}{(r^2+z^2)^{3/2}} \langle 0, 0, 1 \rangle$$

$$\therefore dE_z = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2r}{R^2} \cdot Q \cdot \frac{z}{(r^2+z^2)^{3/2}} \, dr$$

Now, we integrate.

$$E_z = \int_0^R \frac{1}{4\pi\varepsilon_0} \cdot \frac{2r}{R^2} \cdot Q \cdot \frac{z}{(r^2+z^2)^{3/2}} \, dr$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{2Q}{R^2} \cdot z \int_0^R \frac{r}{(r^2+z^2)^{3/2}} \, dr$$

$u = r^2 + z^2$

$du = 2r\,dr$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{2Q}{R^2} \cdot z \cdot \frac{1}{2} \int_{r=0}^{r=R} u^{-3/2} \, du$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{2Q}{R^2} \cdot z \cdot \frac{1}{2} \cdot -2\left[u^{-1/2}\right]_0^R$$

$$= \frac{1}{2\varepsilon_0} \cdot \frac{Q}{\pi R^2} \cdot z \cdot -1\left[\frac{1}{\sqrt{r^2+z^2}}\right]_{0=r}^{R=r}$$

$$= \frac{1}{2\varepsilon_0} \cdot \frac{Q}{\pi R^2} \cdot -z\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{z}\right]$$

$$= \frac{1}{2\varepsilon_0} \cdot \frac{Q}{\pi R^2} \left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]$$

Rewrite πR^2 to A (of disk)

$$\boxed{E_z = \frac{Q/A}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)}$$

For $0 < z << R$ (close to disk but not touching)

$$E_z \approx \frac{Q/A}{2\varepsilon_0}\left(1 - \frac{z}{R}\right) \approx \frac{Q/A}{2\varepsilon_0}$$

This means, at reasonably close distances, distance doesn't matter.

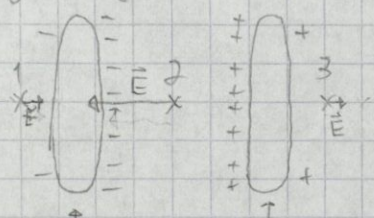
The direction can be inferred trivially by sign.

Capacitors (2 Uniformly Charged Disks)

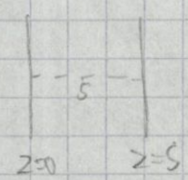
Here, we assume the disks have charge magnitude Q where one disk is positive & the other negative.

Due to the combined push & pull of the oppositely charged metal disks, the inside EM field is nearly uniform (especially when disks are close).

Diagramming the EM field



Showing Coordinates



Now, we describe the EM field along axis of capacitor (we call it z here), in b/w the two disks assuming uniform & equal charge density. We also assume s<<R (aka the disks are very close).

Using $E \approx \frac{Q/A}{2\varepsilon_0}\left(1 - \frac{z}{R}\right)$ from earlier

at z

$$E_{left} = \frac{Q/A}{2\varepsilon_0}\left(1 - \frac{z}{R}\right)$$

$$E_{right} = \frac{Q/A}{2\varepsilon_0}\left(1 - \frac{s-z}{R}\right)$$

we want it to be positive & z<s b/c we assume 0<z<s

Direction can be trivially found by knowing the signs. Due to the nature of capacitors (two oppositely charged disks on opposite sides of the particle), they are guaranteed to have the same direction.

Now, we sum to find the total EM field.

$$E_z \approx E_{left} + E_{right}$$

$$\approx \frac{Q/A}{2\varepsilon_0}\left(1 - \frac{z}{R}\right) + \frac{Q/A}{2\varepsilon_0}\left(1 - \frac{s-z}{R}\right)$$

$$\approx \frac{Q/A}{2\varepsilon_0}\left(1 - \frac{z}{R} + 1 - \frac{s-z}{R}\right)$$

$$\approx \frac{Q/A}{2\varepsilon_0}\left(2 - \frac{s}{R}\right)$$

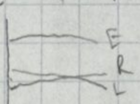
$$\approx \frac{Q/A}{\varepsilon_0}\left(1 - \frac{s/2}{R}\right)$$

We assumed s<<R

$$\therefore \boxed{E \approx \frac{Q/A}{\varepsilon_0}} \text{ inside capacitor}$$

Shockingly, the effect of two plates working together when s<<R is approximately twice what a single plate would be when z<<R. It's also almost constant!

Really, it kinda varies like this:



(this is exaggerated)

For outside

$$E_3 \approx \frac{Q/A}{2\varepsilon_0}\Big(1-\frac{z-s}{R}\Big) - \frac{Q/A}{2\varepsilon_0}\Big(1-\frac{z}{R}\Big)$$

E_{right} (closer) ; b/c $z>s$ & want dist. ; E_{left} (farther) ; $z>0$

$$E_1 \approx \frac{Q/A}{2\varepsilon_0}\Big(1-\frac{z}{R}\Big) - \frac{Q/A}{2\varepsilon_0}\Big(1-\frac{z+s}{R}\Big)$$

$z<0$; $z<0$; E_{left} (closer) ; E_{right} (farther)

Both simplify to this since $z \ll R$.

$$\boxed{E \approx \frac{Q/A}{2\varepsilon_0}\Big(\frac{s}{R}\Big)}$$

added from single disk

outside capacitor (along axis)

We call this field outside of a capacitor the <u>fringe field</u>.

*<u>Note</u>: The <u>fringe field</u> is opposite in direction of the inner/inside field.

*<u>Note</u>: When far from a capacitor, $(z \gg R)$ it looks like a dipole.

*<u>Note</u>: This relation also works reasonably well for non-circular capacitors b/c we only consider center.

*<u>Note</u>: The field doesn't change much w/in capacitor, as we will later show w/ Gauss's law.

Spherical Shell

Suppose we have a spherical shell w/ radius R & charge Q.

We already covered

$$\vec{E}_{sphere} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} \quad ; \quad r>R \text{ (outside sphere)}$$

$$\vec{E}_{sphere} = 0 \quad ; \quad r<R \text{ (inside sphere)}$$

* <u>Note</u>: R doesn't matter! We're just like a point outside the sphere

However, outside of making an EM field like a point charge outside of R, it also reacts like a point charge. (This let's our math work.)

*Note: The EM field inside a charged sphere isn't alway zero. Only uniformly charged ones.

Since uniformly charged shells make no EM field inside, this means that the center of spheres " " w/ a charged surfaces are not polarized or charged. This makes sense in conductors b/c things want to get as far away as possible.

This zero inside field is true for any conductive sphere in equilibrium, as discussed earlier.

Solid Sphere Charged Thruout + like a nucleus! (kinda)

Suppose we have solid sphere w/ radius R & charge Q uniformly distributed thruout volume.

We trivially demonstrate charge outside sphere

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2} \quad ; r > R$$ where dQ is charge of shell & dE is field from shell

$$\Rightarrow E = \int_0^R \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

Inside the sphere, we must only consider the inner shells (basically a smaller sphere w/ same density) since the outer ones contribute 0 field.

$$E = \frac{1}{4\pi\varepsilon_0}\frac{\Delta Q}{r^2}$$ ← our part of Q

$$\Delta Q = Q\frac{(\text{vol inner})}{(\text{vol total})} = Q\frac{\cancel{\frac{4}{3}\pi} r^3}{\cancel{\frac{4}{3}\pi} R^3} = \frac{Q}{R^3}r^3$$

$$= \boxed{\frac{1}{4\pi\varepsilon_0}\cdot\frac{Q\,r}{R^3}} = \underbrace{\frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{R^2}\cdot\frac{r}{R}}_{\text{a bit more intuitive imo}} \quad ; r < R$$

*Note: Field inside sphere directly proportional to R

Spheres, like spherical shells, act like particles & react like particles.

Summary

How to calculate EM field for object:

1) Divide object into pieces you know the EM field for. (approximate if necessary)
2) Write expression for EM field for each "family" of pieces
3) Sum (or integrate if necessary!) the EM field from each piece

Useful Expressions

Point:

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^2}$$

Thin Rod @ Mid-plane:

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{\underset{\text{closest dist}}{r}\underset{\text{farthest dist}}{\sqrt{r^2+(L/2)^2}}}$$

$E \approx \frac{1}{4\pi\varepsilon_0}\frac{2(Q/L)}{r}$ for $L >> r$ $\quad$ $E \approx \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r^2}$ for $r >> L$

Thin Ring along Central Axis:

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qz}{(R^2+z^2)^{3/2}}$$

Disk along Central Axis: ← Really, any surface near center

$$E = \frac{(Q/A)}{2\varepsilon_0}\left(1 - \frac{z}{(R^2+z^2)^{1/2}}\right) \qquad E \approx \frac{Q/A}{2\varepsilon_0}\left(1 - \frac{z}{R}\right) \quad z \ll R$$

$$E \approx \frac{Q/A}{2\varepsilon_0} \quad z \lll R$$

Capacitor:

$$E_{inside} = \frac{Q/A}{\varepsilon_0}$$

opposite direction

$$E_{fringe} = \frac{Q/A}{2\varepsilon_0}\left(\frac{s}{R}\right)$$

Spherical Shell:

$$E = \begin{cases} \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2} & ; \ r > R \\ 0 & ; \ r < R \end{cases}$$

Sphere:

$$E = \begin{cases} \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2} & ; \ r > R \\ \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{R^3} \cdot r & ; \ r < R \end{cases}$$

THESE ONLY DEAL W/ LOCATION OF CHARGE. A solid sphere w/ a charged surface is treated like a spherical shell electrically.

Ch 16: Electric Potential

Electrical Potential is potential energy for EM. This is used in batteries & generators

*Note: The EM field is the gradient of the potential field wrt position.

Potential Energy Review

Particle Energy $= mc^2 + K$ where
where K is kinetic energy
mc^2 is rest energy (mass & speed of light)

Since we normally don't change rest energy we ignore it. ← only occurs during atomic changes

Normally $K \approx \frac{1}{2}mv^2$ for $v \ll c$

Recall $W = \sum(\vec{F} \cdot \Delta\vec{r})$.

Recall:

$$\Delta K_{sys} = W_{surr} + W_{int} + Q$$ ← we ignore heat here

$$\Rightarrow \underbrace{W_{surr}}_{\text{external input}} = \underbrace{\Delta K_{sys} - W_{int}}_{\text{energy change in system}}$$

$-W_{int}$ has a better definition of change in potential energy (ΔU):

$\Delta U_{sys} = -W_{int}$ ← this makes sense b/c internal work "uses up" potential energy.

$$\Rightarrow \boxed{W_{surr} = \Delta K_{sys} + \Delta U_{sys}} \quad \text{when } \Delta(mc^2) = \Delta Q = 0$$

Charged Systems

We normally neglect external forces (like gravity) b/c they're small. Therefore,

$$\Delta K_{sys} + \Delta U = 0$$

This means $\Delta K > 0 \Leftrightarrow \Delta U < 0$ & vice versa.

*Note: You should always go $W_{int} \to \Delta U \to \Delta K$ so you don't get messed up going straight to K in the future.

Potential Difference of Uniform Field

Note:
mass of proton = $1.7 \cdot 10^{-27}$ kg
mass of electron = $9.1 \cdot 10^{-31}$ kg

Potential difference is basically generalized potential energy. (Like EM field w/ forces) We call this potential difference voltage (V) & we measure it in joules per coulomb (J/C) aka volts (V).

$\Delta U = q \Delta V$ See the parallels w/ E? E in N/C & ΔV in J/C
$F = qE$ & $\Delta U = q \Delta V$

Fun Fact! Electron volts (eV) are just the energy required to push an electron up a volt.

We calculate voltage by basically measuring how "against" the field you're going:

$\Delta V = -\vec{E} \cdot \Delta \vec{l}$ ← l for length

You can also do the inverse. *kinda Since there are infinitely many $\vec{E}$ & $\Delta \vec{l}$ pairs, you can only do the special case when you know Δl.

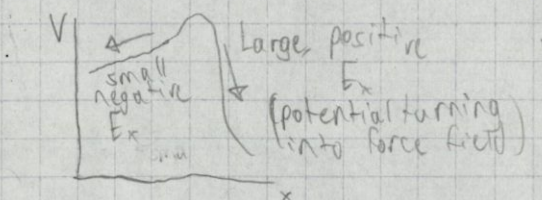
$\vec{E} = -\frac{\Delta V}{|\Delta \vec{l}|} \Delta \hat{l}$ or, as the book shows, $E_x = -\frac{\Delta V}{\Delta x}$ *Note: This shows N/C = V/m

We use the $E_x = -\frac{\Delta V}{\Delta x}$ equation above for simplicity:

$E_x = -\frac{\Delta V}{\Delta x} = -\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = -\frac{dV}{dx}$ ← really $-\frac{\partial V}{\partial x}$

This will be useful when we analyze non-uniform fields in future.

This also gives us intuition in that $\vec{E}$ is a gradient of potential, much like a hill



*Note: Increasing in voltage is hard, like climbing a hill, while decreasing is easy, like going down a hill. This is for positive charges at least. This is reflected by its opposite relationship w/ $\vec{E}$ (hence $\Delta V = -\vec{E} \cdot \Delta \vec{l}$)

More generally

$\vec{E} = -\nabla \vec{V}$

$\Rightarrow E_x = -\frac{\partial V}{\partial x}$ $E_y = -\frac{\partial V}{\partial y}$ $E_z = -\frac{\partial V}{\partial z}$

On a similar note $F = -\nabla U$

Potential Difference in Non-Uniform Field

To go beyond uniform fields we need integrals! This requires we have $\vec{E}$ in terms of $\vec{r}$ (the integration variable).

$\Delta V = \int_i^f \vec{E} \cdot d\vec{l}$ ← this is a line integral

*Note: This is for continuously varying EM fields.

For finitely many uniform EM fields,

$$\Delta V = \sum \vec{E} \cdot \Delta \vec{l}$$

Recall from Calc 3 that independence of path for a line integral means that, as long as the start & end point are the same, the integral is the same.

Potential difference is independent of path, meaning you should choose the simplest path.

We can also apply the "superposition principle" to calculating potential difference.

Potential at One Location

*Note: EM field at location says nothing about potential at location.

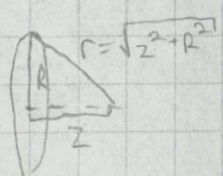
We determine the potential of a location A to be potential difference between location A & a point infinitely far away. We, by convention say V_∞ is 0.

For a point charge A at observation location B, which is a distance r from A:

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_A}{r}$$

The same is true for spheres outside of their radius, for similar reasons as earlier

Potential Along Axis of Ring



For a single point charge dQ

$$dV = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dQ}{r} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dQ}{\sqrt{z^2+R^2}}$$

Now, we integrate

$$V = \int dV = \int \frac{1}{4\pi\varepsilon_0} \cdot \frac{dQ}{\sqrt{z^2+R^2}} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{z^2+R^2}} \cdot \int dQ = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{\sqrt{z^2+R^2}}$$

We could've also integrate EM field of a ring.

Potential Inside Conductor

Suppose we have a metal sphere of charge Q & radius R.

$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r}$ for $r > R$

$\therefore V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{R}$ at the surface of the metal.

Since $V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{R}$ & the potential difference in a conductor is 0, the potential inside a conductor is

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{R}}$$

Shifting Zero Potential

Since potential is all absolutely "relative" i.e. proportions don't matter, we can choose any arbitrary point to be our "zero" or "basis" for measuring voltage, rather than infinite. We normally choose this basis/zero to be convenient, such as ground

Generally Determining Potential

Broadly, we can find potential by adding all the known objects' contributions (normally approximating w/ point charges)

$$V_A = \sum_i \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_i}{r_i}$$

or travel from our "zero" (or very far) location to the location of interest, adding $-\vec{E} \cdot d\vec{l}$

$$V_A = -\int_{\infty}^{A} \vec{E} \cdot d\vec{l}$$

This is similar to finding potential difference using difference of potentials

$$\Delta V = V_B - V_A$$

Or travelling along from A to B, adding $-\vec{E} \cdot d\vec{l}$

$$\Delta V = -\int_A^B \vec{E} \cdot d\vec{l}$$

Potential Difference of Insulator

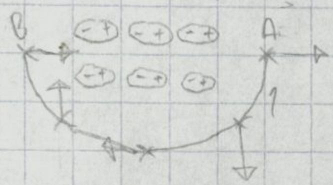
Insulators are complex b/c the EM field inside them isn't 0.

We can try to model insulators as a bunch of super thin capacitors but that doesn't make sense, causes issues, & complicates things.

To simplify this, we use the "round trip" arguement.

Round Trip Argument for Insulators

We want to find dominant field direction in insulator.



Arrows represent EM field at location.
Solid line is path

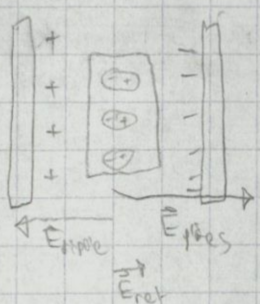
Along path 1, you can see $\vec{E}\cdot d\vec{l} > 0$ always

$$\therefore \Delta V = -\int_A^B \vec{E}\cdot d\vec{l} < 0$$

Therefore, going straight from A to B, due to path independence is also negative

Since $\Delta V = -\vec{E}\cdot\Delta\vec{l}$, this means $\vec{E}$ points to the left inside the insulator, on average.

The EM field w/in a polarized insulator points towards the negative charge. "weakening" the applied field.



Dielectric Constant (K) another word for insulator

We measure the factor by which an insulator "weakens" an applied charge using the dielectric constant (K). This is a measure of polarizability

$$\vec{E}_{\substack{\text{net in}\\ \text{insulator}}} = \frac{\vec{E}_{applied}}{K}$$

Note: You can use the dielectric constant to figure out the non-net field by doing

$$\vec{E}_{\substack{\text{net in}\\ \text{insulator}}} = \frac{\vec{E}_{applied}}{K} = \vec{E}_{applied} - \vec{E}_{\substack{\text{insulator}\\ \text{dipole}}}$$

*Note: The dielectric constant must be at least 1, b/c insulators can't strengthen fields. Larger K ⇒ more weakening

We normally assume the EM field produced by an insulator is negligible outside of the insulator.

Energy Density & EM Field

Until now, we've considered energy to be associated w/ interacting charged particles. You can, however, consider the EM field to store energy.

To illustrate this, suppose we have a capacitor.

$$E_{\text{one plate}} = \frac{Q/A}{2\varepsilon_0} \quad \text{(very small gap)}$$

$$\therefore F_{\text{one plate}} = Q\left(\frac{Q/A}{2\varepsilon_0}\right)$$

If we try to move the plate a distance Δs away from the other plate

$$\Delta U = W = F_{\text{one plate}}\, ds = Q\left(\frac{Q/A}{2\varepsilon_0}\right)\Delta s$$

$$\Rightarrow \Delta U = \frac{1}{2}\varepsilon_0\left(\frac{Q/A}{\varepsilon_0}\right)^2 A\Delta s$$

We make the 2nd, admittedly weird, transformation to get $E_{\text{capacitor}}$ in ΔU.

Notice $A ds$ is the change in occupied volume (ΔV)

$$\Rightarrow \boxed{\frac{\Delta U}{\Delta V} = \frac{1}{2}\varepsilon_0 E^2 \quad [\text{J}/\text{m}^3]}$$

*Note: Although we derived this using capacitors, it is generally true, even for other force fields (e.g.: gravity)

We call the above, the energy density (J/m^3). You can see the above is potential energy change for volume change

Intuitively, pulling apart the capacitor increased the volume in which a sizeable EM field existed. Therefore, we say the energy became "stored in the EM field"

*Note: We do this for a similar reason as the EM field & potential.

This explains why 2 oppositely charged particles accelerating towards each other doesn't break physics. Them moving towards each other reduces/uses up the energy of the EM field.

Ch 17: Magnetic Field

Magnetic Fields ($\vec{B}$) are generated by moving charged particles.

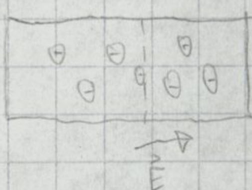
Electron Current

We'll study this quickly since it gives us a simple case which generates a magnetic field.

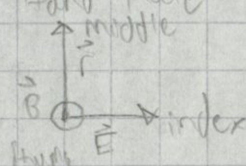
- Electron Current (i): The number of electrons per second moving thru section of a conductor.

In a steady electric circuit, current is the same at every section of wire w/ the same width & composition.

Funnily, measuring current directly is hard, so we normally measure magnetic field as proxy. (We could also do heat of wire but that's more difficult.)



Right Hand Rule



You can also do fingers curling w/ magnetic field & thumb pointing w/ $\vec{E}$

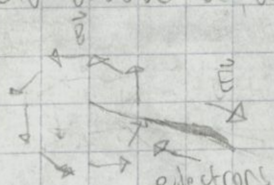
Detecting Magnetic Field

Here, we detect a magnetic field w/ a compass. (Note: Compasses are affected by some neutral metals (e.g. iron) but not by others (e.g copper).)

When you bring a current carrying wire near a compass, the compass deflects to be perpendicular w/ the wire.

Originally discovered by Oersted in 1820. Thus called the "Oersted effect".

This shows the moving charges generate a magnetic field perpendicular to their movement (i.e. the electric field). Additionally, the magnetic field above a wire is opposite that below.



Magnetic field also follows the superposition principle. This means we can do a vector sum to find the resultant field (& thus deflection from Earth's magnetic field.)

Biot-Savart Law: Single Moving Charge
(bīō səvaL)

As we saw earlier w/ a wire, a moving point charge produces both an electric field (parallel to movement) & a magnetic field (curling around movement)

The Biot-Savart law gives us the magnetic field of a point charge (like Coulomb's law gives us electric field). The units are <u>teslas</u> (T)

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{q(\vec{v} \times \hat{r})}{r^2} \quad \text{where} \quad \frac{\mu_0}{4\pi} = 1 \cdot 10^{-7} \frac{T \cdot m^2}{C \cdot m/s} \quad \text{exactly}}$$

$\vec{v}$ is velocity of point charge q
$\hat{r}$ is unit vector in direction from point charge to observation location

Recall: $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin(\theta)$ where θ is the (acute) angle b/w $\vec{A}$ & $\vec{B}$.
$\vec{A} \times \vec{B}$ is perpendicular to plane formed from $\vec{A}$ & $\vec{B}$ following right hand rule.
$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$
We use $\odot$ for out of page & $\otimes$ for into page

The Biot-Savart law implies that two oppositely charged particles moving in the same direction produce opposite magnetic fields. This makes sense & hints us towards the relation b/w the electric field & the magnetic field.

Relatavistic Effects

Frame of Reference

We already know that moving charged particles create magnetic fields & that movement is relative to some frame of reference. Therefore, it makes sense that magnetism is relative to some frame of reference.

As you suspect, this means you should use the velocity relative to your frame of reference.

This makes hints towards (special) relativity & the relation b/w the electric & magnetic field.

Retardation

Like the electric field, the magnetic field travels at the speed of light. This makes Biot-Savart law an approximation only for charges where $v \ll c$, like Coulomb's Law.

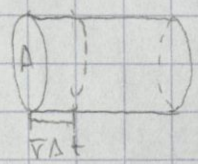
Electron Current & Conventional Current

Since studying single particles is difficult, we normally study flows/currents. To apply the Biot-Savart law tho, we need to know the number of particles.

We can calculate the number of electrons if we now the drift speed, cross-sectional area, & time step.

$n = A\bar{v}\Delta t$

You can also view this as the volume electrons drift, where $\bar{v}\Delta t$ is the length of the cylinder & A the area.



Now, given some (volumetric) electron density n, the number of electrons flowing thru a section of wire of a time Δt is

$$n \cdot \text{volume} = nA\bar{v}\Delta t$$

This means the current (i) or number of electrons flowing over time Δt is

$$\boxed{i = nA\bar{v} = \frac{nA\bar{v}\Delta t}{\Delta t}}$$ where n is mobile electron density (number of mobile electrons per volume
A is cross-sectional area of wire
$\bar{v}$ is drift speed of electrons

*Note: this has units electrons/second

Recall that most of the time, drift speed is unbelievably low.

Conventional Current

Since we started studying current before we understood electrons, we conventionally model current using positive charges moving thru the wire.

Luckily, due to the nature electricity & magnetism, this doesn't change anything & only occasionally makes it more confusing.
(Biot-Savart law

*Note: Sometimes there are actually moving, positive holes of current, works
Sometimes this is an important distinction (Hall effect)

We notate conventional current I & electron current i.

Conventional current is measured in holes per second w/ holes of charge $|q|$.

$$\boxed{I = |q| nA\bar{v} = |q| i \ \ C/s \text{ or } A}$$ ← Amount of charge persecond rather than electrons

In metals $|q| = e$ b/c that's the charge of an electron, but it can be different in ionic charges

Electric Field & Current

Recall: $\bar{v} = u E$ for conductors where u is mobility of conductor, E is electric field

$\therefore I = |q| n A \bar{v} = |q| n A (u E)$ for conductors

Biot-Savart & Currents

We need to tweak the Biot-Savart law to work w/ currents. This does require us assume we have a thin wire.

To do this, we divide the wire into small Δl length sections w/ cross-sectional area A w/ electron density n. Now, we find $\Delta\vec{B}$ assuming all charges are in center of that section.

$$\therefore \boxed{\Delta \vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I \Delta \vec{l} \times \hat{r}}{|\vec{r}|^2}}$$

* Note: The Biot-Savart law applies to conventional current.

This makes sense b/c for section of volume $A\Delta l$ containing $n(A\Delta l)$ moving charges, the magnetic field is $n(A\Delta l)$ times larger than magnetic field of single charge.

$$\Delta \vec{B} = n(A\Delta l)\vec{B}_q = n(A\Delta l)\frac{\mu_0}{4\pi} \cdot \frac{q\vec{v} \times \hat{r}}{|\vec{r}|^2} = \frac{\mu_0}{4\pi} \cdot \frac{nA\Delta l q \vec{v} \times \hat{r}}{|\vec{r}|^2}$$

Using $I = qnA\bar{v}$ & making $\Delta\vec{l}$ point in the direction of $\vec{v}$, we get

handles directions

$$\Delta \vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I \Delta \vec{l} \times \hat{r}}{|\vec{r}|^2}$$ where $\Delta\vec{l}$ is vector in direction of conventional current whose magnitude is length of segment of wire

$\hat{r}$ points from center of wire segment to observation location

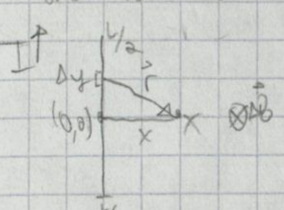
Magnetic Field of Current Distributions

To find the magnetic field of some current distribution, we

1) Segment the distribution into sections we know $\vec{B}$ for
2) Write expression for $\vec{B}$ for each class/family of sections
3) Sum up $\vec{B}$ from every section

Magnetic Field of Large Straight Wire

Suppose we have a wire of length L w/ a conventional current I upwards. We divide the wire into Δy length short sections & find magnetic field a perpendicular distance x away from center.



Now, we find the prerequisites for $\Delta\vec{B}$.

$\vec{r} = \langle x, 0, 0 \rangle - \langle 0, y, 0 \rangle = \langle x, -y, 0 \rangle$ $\quad \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, -y, 0 \rangle}{|\vec{r}|}$

$|\vec{r}| = \sqrt{x^2 + y^2}$

$\Delta\vec{l} = \Delta y \langle 1, 0, 0 \rangle$

Now, we plug in to find $\Delta\vec{B}$

$$\Delta\vec{B} = \frac{\mu_0}{4\pi}\frac{I\Delta\vec{l}\times\hat{r}}{|\vec{r}|^2} = \frac{\mu_0}{4\pi}\cdot\frac{I\Delta y\langle 0,1,0\rangle}{x^2+y^2}\times\frac{\langle x,-y,0\rangle}{\sqrt{x^2+y^2}}$$

$$\langle 0,1,0\rangle\times\langle x,-y,0\rangle = \langle 0,0,-x\rangle$$

$$= \frac{\mu_0}{4\pi}\cdot\frac{I\Delta y}{(x^2+y^2)^{3/2}}\langle 0,0,-x\rangle$$

$$\Delta\vec{B} = -\frac{\mu_0}{4\pi}\cdot\frac{Ix\Delta y}{(x^2+y^2)^{3/2}}\langle 0,0,1\rangle$$

As you can see, the magnetic field is only in the $-z$ direction, so we can just drop it off & find magnitude. This makes it easier to generalize.

$$\Delta B = \frac{\mu_0}{4\pi}\cdot\frac{Ix\Delta y}{(x^2+y^2)^{3/2}}$$

Now, we take a limit & do an integral to get B

$$dB = \frac{\mu_0}{4\pi}\cdot\frac{Ix}{(x^2+y^2)^{3/2}}dy$$

$$B = \int dB = \frac{\mu_0}{4\pi}\cdot Ix\int_{-L/2}^{L/2}\frac{dy}{(x^2+y^2)^{3/2}}$$

Integrating this, we get

$$B = \frac{\mu_0}{4\pi}Ix\left[\frac{y}{x^2\sqrt{x^2+y^2}}\right]_{-L/2}^{L/2}$$

$$= \frac{\mu_0}{4\pi}Ix\left[\frac{L/2}{x^2\sqrt{x^2+(L/2)^2}} - \frac{-L/2}{x^2\sqrt{x^2+(-L/2)^2}}\right)$$

$$= \frac{\mu_0}{4\pi}\cdot\frac{LI}{x\sqrt{x^2+(L/2)^2}}$$

Generalizing to a general r, we get

$$\boxed{B_{wire} = \frac{\mu_0}{4\pi}\cdot\frac{LI}{r\sqrt{r^2+(L/2)^2}}}$$

When $r \ll L$ (long rod),

$$\sqrt{r^2+(L/2)^2} \approx L/2$$

$$\therefore \boxed{B_{wire} \approx \frac{\mu_0}{4\pi} \cdot \frac{2I}{r} \text{ when } r \ll L}$$

Note that this only gives you magnitude & you must find direction yourself.

When $r \gg L$ (far away)

$$\sqrt{r^2+(L/2)^2} \approx r$$

$$\therefore \boxed{B_{wire} \approx \frac{\mu_0}{4\pi} \cdot \frac{LI}{r^2}} \text{ when } r \gg L$$

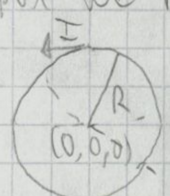
Just Biot-Savart law

Circular Loop of Wire

This situation helps us analyze atomic current loops in magnets

We will only analyze the easy case along the central axis of the ring.

Suppose we have a loop of radius R w/ conventional current I throughout.



We divide the ring into sections of Δl length & notice that $\hat{r}$ is perpendicular to every section.

We could now find $\Delta\vec{B}$ for any segment. However, this is very messy. WLOG we can analyze only ΔB_z since, due to the symmetry of a ring, ΔB_x & ΔB_y will cancel out w/ the opposite section. This simplifies things b/c every section Δl will have the same ΔB_z, again due to symmetry.

Now, since every section Δl has the same ΔB_z, we choose an easy piece, say $\langle 0, R, 0 \rangle$ & find its ΔB_z

$$\vec{r} = \langle 0, 0, z \rangle - \langle 0, R, 0 \rangle = \langle 0, -R, z \rangle$$
$$|\vec{r}| = \sqrt{R^2+z^2}$$

$\Delta \vec{l} = \langle -R\Delta\theta, 0, 0 \rangle$ ← due to our choice of $\langle 0, R, 0 \rangle$

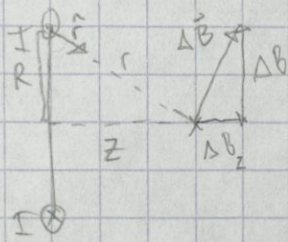
$$\Delta\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I \Delta\vec{l} \times \hat{r}}{|\vec{r}|^2} = \frac{\mu_0}{4\pi} \cdot \frac{I \langle -R\Delta\theta, 0, 0 \rangle \times \langle 0, -R, z \rangle}{(R^2+z^2)^{3/2}}$$
← b/c $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ so we get $|\vec{r}|^3$ on bottom

$$\langle -R\Delta\theta, 0, 0 \rangle \times \langle 0, -R, z \rangle = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -R\Delta\theta & 0 & 0 \\ 0 & -R & z \end{bmatrix} = \langle 0, zR\Delta\theta, R^2\Delta\theta \rangle$$

$$\Delta\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I \langle 0, zR\Delta\theta, R^2\Delta\theta \rangle}{(R^2+z^2)^{3/2}}$$

Since we only need z-component (the other's ($\Delta B_\perp$) cancel!)

$$\Delta B_z = \frac{\mu_0}{4\pi} \cdot \frac{I R^2 \Delta\theta}{(R^2+z^2)^{3/2}}$$



Now, we take the limit & integrate (using θ as our integration variable).

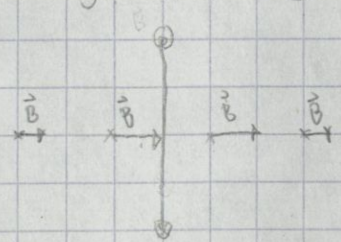
$$B_z = \int dB_z = \int_0^{2\pi} \frac{\mu_0}{4\pi} \cdot \frac{IR^2}{(R^2+z^2)^{3/2}} \, d\theta = \frac{\mu_0}{4\pi} \cdot \frac{IR^2}{(R^2+z^2)^{3/2}} \int_0^{2\pi} d\theta$$

$$\therefore \boxed{B_{loop} = \frac{\mu_0}{4\pi} \cdot \frac{2\pi R^2 I}{(z^2+R^2)^{3/2}}}$$

← where R is radius of loop
I is conventional current
z is distance from center

* We reorganized b/c fuck it I guess

Basically, our ring acts like a book ring!

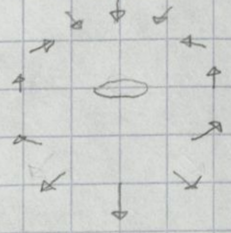


We can see the magnetic field falls off w/ $1/z^3$ b/c

$$\boxed{B_{loop} \approx \frac{\mu_0}{4\pi} \cdot \frac{2\pi R^2 I}{z^3} \quad \text{when} \dots z \gg R}$$

Outside the Loop

We won't analyze this analytically b/c its difficult. But a ring looks like a dipole from outside.



Right Hand Rule

If you're a pleb & don't use the straight-finger right hand rule, you can curl your fingers in the direction of conventional current (I) & your fingers point in the direction of magnetic field (B).

Thin, Long Coils

For coiled wire coiled N times, its magnetic field is N times stronger than that of a single ring/loop of the same size, assuming the wire is thin & coiled tightly.

Later, we'll get into solenoids which is coiled around a tube (i.e. not thin & tightly).

Magnetic Dipole Moment

Recall the equation for an electric field along axis of electric dipole:

$$E_{axis} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2p}{r^3}$$ where p is the dipole moment & $p = qs$

We can do a similar thing for current carrying coil:

$$B_{axis} \approx \frac{\mu_0}{4\pi} \cdot \frac{2\mu}{r^3}$$ where μ is the <u>magnetic dipole moment</u> & $\mu = IA$ where A is the area of the ring ($A = \pi R^2$). If there are N loops, $\mu = NIA$.

Now that we have abstracted the specifics of having a circular loop, we can see this is valid even when the loop isn't circular.

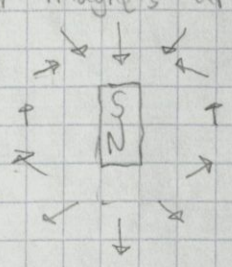
We consider the <u>magnetic dipole moment</u> ($\vec{\mu}$) to be a vector pointing in the direction of the magnetic field along the axis. It has units $A \cdot m^2$.

We will see in the future that magnetic dipole moments act as compasses, aligning themselves w/ the magnetic field.

Bar Magnet

Unlike electric charges, magnets are always dipoles & can be permanent. They also only interact w/ some things (e.g. iron & steel).

Bar magnets are magnetic dipoles, like current loops! (This will make more sense later)



* Note: The magnetic field varies w/ $1/r^3$.

Interestingly, Earth acts like a giant bar magnet where the north pole is an S-type pole & the south pole is a N-type pole. This should make sense if you look at the image above. Also magnetic poles ≠ solar poles.

For bar magnets, we use their magnetic dipole moment (μ) &

$$B_{axis} \approx \frac{\mu_0}{4\pi} \cdot \frac{2\mu}{r^3}$$

Magnetic Monopoles

As far as we can tell magnetic monopoles do not exist & would violate the laws of physics. This is unlike electric charges (which are monopoles). This means dipoles are the fundamental sources of the magnetic field (well, kinda): they can't be broken further. This is why splitting a magnet yields 2 magnets.

Atomic Structure of Magnets

As we have already seen, a loop of current looks & acts like a magnetic dipole. This actually points to the fact that magnetic dipoles are really caused by atomic electron loops!

Single Atom Dipole

As you know, there are electrons in an atom. Thes moving electrons cause the magnetic field we see. There are 3 possible configurations/sources that would cause this magnetic field:

- An electron orbiting the nucleus
- An electron spinning around its own axis
- Rotation/spin of nucleus

All of these have angular momentum (L) we'll use to find the magnetic dipole moment (μ).

$\mu = kL$ where k is some factor

Orbitting Electron

Easiest to understand are orbiting electrons (using Bohr Model). Assume there is a single electron orbiting at a constant speed:

$$\therefore \mu = IA = I(\pi R^2)$$

$$I = \frac{Q}{T} = \frac{e}{\frac{2\pi R}{v}} = \frac{ev}{2\pi r}$$

$$\Rightarrow \mu = \left(\frac{ev}{2\pi r}\right)(\pi R^2) = \tfrac{1}{2} eRv$$

is the magnetic dipole moment (μ) for a single orbiting electron.

Relating Magnetic Dipole Moment & Angular Momentum

Here, we'll be using the simple Bohr model.
As you remember, there is only an electron cloud, not orbits. Also the s orbital is spherical leading to $L=0$

For an orbiting electron

$$|\vec{L}| = |\vec{r}\times\vec{p}| = |\vec{r}||\vec{p}|\sin(\theta)$$

Since $\vec{r}\perp\vec{p}$, $\theta = 90°$

$$L = Rp\sin(\theta) = Rmv$$

We change μ to use L

$$\mu = \tfrac{1}{2} eRv = \tfrac{1}{2} \tfrac{e}{m}(Rmv) = \tfrac{1}{2} \tfrac{e}{m} L$$

This means our earlier k in $\mu = kL$ is

$$k = \frac{e}{2m}$$

We assume this value is valid for orbiting electrons, spinning electrons, & spinning nucleons (protons & neutrons). (It is roughly)

If you calculate it out, the factor for electrons is ~2000 times that of nucleons (b/c high mass of nucleons), meaning most of magnetic field comes from electrons.

Recall that angular momentum is quantized by Planck's constant ($\hbar$)

$$\therefore L = N\hbar \quad N\in\mathbb{Z}^+ \;\&\; \hbar = 1.05\cdot 10^{-34}\ \mathrm{J\cdot s}$$

Assume the angular momentum for an atom's electrons is $L=\hbar$ ← simplifying assumption

$$\Rightarrow \mu = \tfrac{1}{2}\tfrac{e}{m}\hbar = 1\cdot 10^{-23}\ \mathrm{A\cdot m^2} \text{ per atom}$$

Our Simplifications

We used the Bohr model. Really, there is an electron cloud, not orbits. Also, the s orbital is spherical w/ $L=0$.

We assumed all atoms have only 1 electron w/ $L=\hbar$

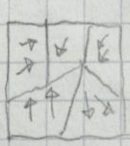
Modern Theory

Most magnetic field is from electron spin. Nuclear Magnetic Resonance (NMR) is magnetism from nucleons & is sometimes important.

Why not Everything is Magnetic

some are naturally ordered!

For most materials, their magnetic field is disorderly & averages to zero. Applied magnetic fields order their fields causing them to be magnetic. (Sometimes they don't reorient, just grow or weaken.)



*Note: The sections here are called domains!

Interestingly, some metals remain magnetic after the applied field is removed. Also a lot of the times the magnetic field of the metal (e.g. iron) becomes stronger than the applied field: this is the multiplier effect.

In terms of why there are magnetic domains & not everything is lined up. It is a battle b/w the short range tendency to be parallel & the long range tendency to be antiparallel.
(electric interaction)
(magnetic interaction)

For short disks, electric interaction wins. For long bars, magnetic interaction wins.

Things with these magnetic domains are normally called ferromagnetic.

Ch 18: Electric Field & Circuits

Circuits Aren't at Equilibrium

Circuits involve moving charges. This makes them, by definition, not at equilibrium. Additionally, they are not at equilibrium for a long time.

We'll analyze circuits microscopically

Fun Fact! Only moving charges create & feel magnetic field

Current at Different Parts of Circuit

Once circuits are set up, they very quickly reach a (mostly) steady state where charges flow at a constant current.

Equilibrium is no drift speed ($\bar{v}=0$).
Steady state is charges are moving ($\bar{v}\neq 0$) but their velocities are constant throuout time.

In a circuit, current at any given location is constant. This may be confusing but realize that electrons/current aren't used up. Instead, realize they are only slowed down. If you bottleneck a certain part of the current it will be equally slow before (no "back up") & after ("no "catch up"). This also makes sense of why it still uses energy. The field (& electrons/charges) must do more work to flow!

Kirchhoff Node Rule

really consequence of conservation of energy & steady state

Kirchhoff's node rule states that, for any node in a circuit, the current flowing in is the same as out.

When using this for finding current, negative means you guessed the wrong direction.

Note: There can be no excess charge at a steady state. (would cause blockage)

Electric Field & Current

Recall $\bar{v}=uE$ where $\bar{v}$ is drift speed & u is electron mobility.

Now, you may think that electrons interact. However, since mobile electrons are always around a positive nucleus, they don't affect others. Like a noble gas!! This means electrons can't continually push each other & their must be an applied Electric Field. This, combined w/ electrons interacting w/ positive lattice, makes superconductors not really exist. (broadly)

Analyzing steady state circuits, we know i must be consistent.

Note: Since electrons in metals are so mobile, they don't need much field to flow.

$\therefore i_1 = i_2$ Recall: $i = nA\bar{v}$

$\Rightarrow n_1 A_1 \bar{v}_1 = n_2 A_2 \bar{v}$

* Note: This only applies in series circuits w/in the circuit.

Direction in Circuit

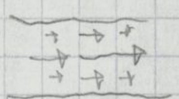
consistent $|q|, n, A,$ & u

Assuming a wire of consistent material & shape, at steady state, E must be constant. This follows from

$I_1 = I_2$ & $I = |q| n A u E$ w/ constant $|q|, n, A,$ & u

From this follows that $\vec{E}$ must be parallel to the wire & consistent w/in wire

We can also prove this by looking at the possible paths along a wire if the field was not constant.



Imagine going thru the center & around edge.

This isn't always true for non-steady-state circuits!

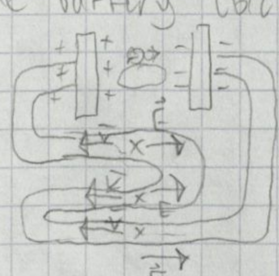
Cause of Electric Field in Wire

We know in a steady state wire, $\vec{E} \neq 0$, $|\vec{E}|$ is constant in wire of uniform material & cross section (b/c I (or i) is constant), & $\vec{E}$ is parallel to the wire. However, what charges are causing this field?

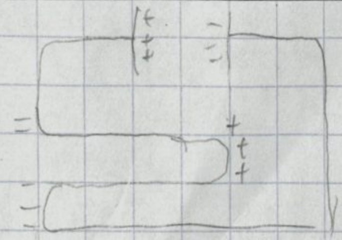
We know they can't be on the interior of the wire, so they are on the surface. You may think they are just in/on the battery but then the electric field wouldn't be constant. This would also mean the battery would be a dipole, meaning moving the parts of a circuit around would significantly change the intensity.

To help us understand this, we'll model a battery as 2 oppositely charged plates (only partly wrong) w/ a "conveyor belt" transferring electrons from positive to negative (semi-analogous to chemical processes)

Imagine we have this simple battery in the circuit below & we consider only fields due to the battery (b/c that's all there is right now).



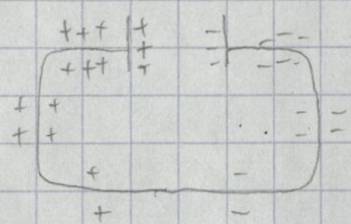
This set up leads to the following build up (b/c electrons are entering some parts & leaving others)



This build up keeps getting more & more intense until electrons start going "the right way" down the wire. Since our battery is constantly replacing the charges, the flow occurs at a constant rate (otherwise it wouldn't be steady & would tend to being steady.)

This is example of <u>negative feedback</u> (keep in same position).

In the end, a circuit will have an electron distribution like so (simplified)



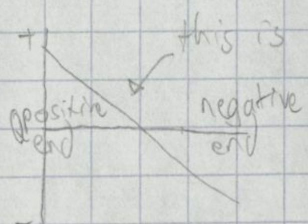
* Note: Doing the same thing for the bendy wire would be more complicated b/c bends have more extreme charge distributions

Surface Charge Distributions

* Note: The amount of surface charge depends on the voltage of the circuit

Broadly, the surface of the positive end of a circuit is positive. The negative end is more negative. More specifically, there is a charge gradient on the surface of a circuit. In a steady state circuit, this gradient is (roughly) linear. (Note: It is truly linear only along straight segments, It's more complex in corners.)

Plotting this out for a linear circuit we get



* Note: The total amount of charge on the surface is negligible & has little effect outside the wire. This is b/c electrons in metal are very mobile.

You can model the simple case of a straight wire using infinitely many uniformly charged rings around the surface of the wire.

General model

Since we haven't solved everything analytically yet, we generally use a computational model by breaking the circuit's surface into many tiny squares.

We then only consider the charge on the battery & evaluate all square's new charge. We then repeat using the new charges until the change becomes sufficiently little.

Conclusions of Model

Most importantly, while the specific charge distribution gets complex, the current stays constant. Basically, everything but current ends up cancelling.

Non-Homogenous Serial Circuits

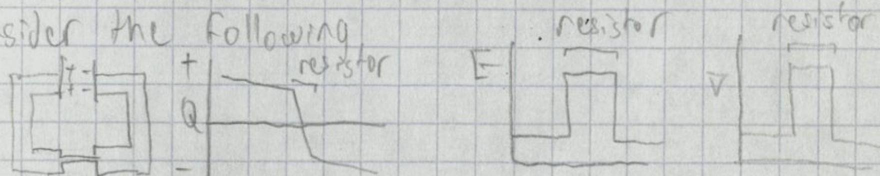
Most useful circuits are not just loops of wire, they have things like resistors, branches, light bulbs, etc.

For now, we'll just analyze resistors in serial circuits. These can either be literal resistors, light bulbs, or different material wires.

As always in serial circuits, i (or I) is constant. Since $i = nA\bar{v}$, if n or A drop (as in resistor), $\bar{v}$ must increase to compensate. Since $\bar{v} = uE$, this also means either u or E grow depending on whether u is constant.

In the case of a thinner wire of the same material, E must increase (since u is constant. This means the surface charge gradient is more intense.

Consider the following

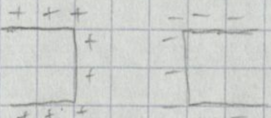


Circuit   Charge Gradient E along wire $\bar{v}$ along wire

Start of a Circuit & the Initial Transient

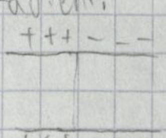
The whole period of a circuit reaching steady state from equilibrium is very quick & is called the initial transient. This is where $\bar{v}$ is not constant.

A wire starts like



Where $\vec{E}_{self} + \vec{E}_{other} = 0$ inside the wire (b/c @ equilibrium)

When we connect the wire, the faces neutralize & the surfaces form a steep charge gradient.



The charge gradient quickly becomes more linear as the surface charges intermingle. (It travels at the speed of light, assuming instant connection.) This is why electronics turn on extremely quickly despite individual electrons being so slow.

Circuit Feedback

During the initial transient, several things are changing. However, the current experiences stabilizing (negative feedback), where the current will slow down if it was too fast & causes a pile up or vice versa.

This is rapid & explains why bending/changing circuits seems to leave current unaffected.

Surface Charge & Resistors

When analyzing resistors, we always use

$I_1 = I_2$ OR $i_1 = i_2$

Then, it is a matter of using the equations to find what's consistent.

- Same Shape $\Rightarrow A_1 = A_2$
- Same Material $\Rightarrow n_1 = n_2$ & $u_1 = u_2$

Otherwise, conceptually, thinner wires of the same material in a circuit have a higher drift speed ($\bar{v}$) to maintain a consistent i w/ their smaller A. This means E in the thinner wires is larger. This means the gradient of surface charges is more intense.

Energy in a Circuit

We know current must be consistent. But what's the current?

The energy principle (aka Kirchoff's loop rule) says a round trip / loop cannot gain energy. Using this & the known voltage across a battery, we can find the change in potential & thus current.

$\Delta V_{bat} + \Delta V_{circ} = 0$ ← Kirchoff's loop rule

$\Rightarrow \Delta V_{circ} = -\Delta V_{bat}$

When using Kirchoff's loop rule, start at any arbitrary point & go on arbitrary direction. As long as you're consistent, it doesn't matter.

How Batteries Work (aside)

Batteries work by having a non-Coulomb force (normally a chemical process), that pushes for the charges to be uneven. This pushes against the Coulomb force of the electrons trying to be stable. When these balance out, you have your battery.

This should explain why longer batteries have more power, there's basically just more behind the force.

We ca imagine our non-Coulomb force to be how hard the conveyor belt was spinning in our earlier mechanical battery analogy.

In symbols

$|\Delta V_{batt}| = E_C s = \frac{F_{NC} s}{e}$ where E_C = is Coulombic electric field
s is separation / length of battery
$F_{NC} = F_C = eE_C$ is non-Coulombic force

We call this quantity emf (historically electromotive force) & is the energy per unit charge of the battery. This is fundamental to battery itself. Not only is it the potential difference across the battery, it is also the amount of energy used to move a charge thru the circuit.

* Note: Altho emf is in volts, it's really just in energy units.

In chemical batteries emf is fairly constant.

Note: No mattery the properties of the circuit, batteries always create consistent electric field, given consistent length of circuit.

Internal Resistance of Batteries

Really, all batteries have some internal resistance. Here, we'll model that a bit & then make a simplifying assumption.

$\bar{v} = u E_{net} = u(E_{NC} - E_C) = u(F_{NC}/e - E_C)$ for some charge e & battery w/ mobility u.

Since the motor's force (F_{NC}) is fixed, we get max drift speed when there's no E_C meaning no charge on the ends of the battery. To simplify so we always have some $\bar{v}$, we assume u is very large (i.e. little resistance)

$\therefore |\Delta V_{bat}| \approx emf.$

Note: This internal resistance limits the current a battery can provide.

We call these batteries ideal batteries.

Electric Field & Current is Simple Circuit

In a circuit, the E in a battery points opposite that of a normal wire. for obvious reasons
Using the energy principle we get

$\Delta V_{bat} + \Delta V_{circ} = 0$

from earlier chapters

$\Rightarrow emf + (-EL) = 0$

$\Rightarrow \boxed{E = \frac{emf}{L}}$

here, we assume $|q| = e$

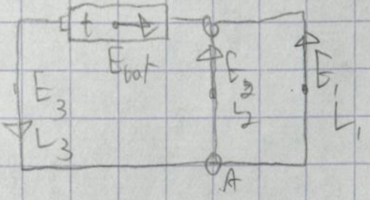
Using the properties of a wire (i.e. A, $|q|$, n, & u), we can find current

$I = enAuE$

Current in Parallel Circuits

Note: Thin wires tend to have high E & $\bar{v}$ to compensate for their low A.

Consider the following



If you follow the path 3 & 2, you get

$$E_3L_3 + E_2L_2 + \overbrace{(-\text{emf})}^{\text{battery's emf in opposite direction}} = 0$$

Note: Not all wires are equal. Some parts of a circuit contribute much more resistance. Duh!

Similarly, for 1 & 3

$$E_3L_3 + E_1L_1 + (-\text{emf}) = 0$$

Using elimination, we get

$$E_2L_2 - E_1L_1 = 0$$

$$\Rightarrow \boxed{E_2L_2 = E_1L_1}$$

This makes sense b/c the wires have the same start & end points

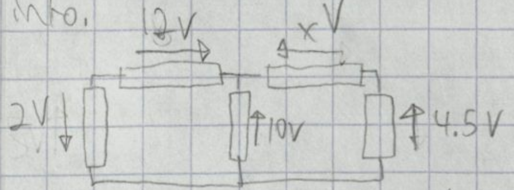
Also, recall the node rule at node A.

$$i_3 = i_1 + i_2$$

Loop Rule

The loop rule states that the total change in potential for any possible loop must be zero. This includes arbitrarily complex circuits

This is useful for determining resistance/potential change using only partial info.

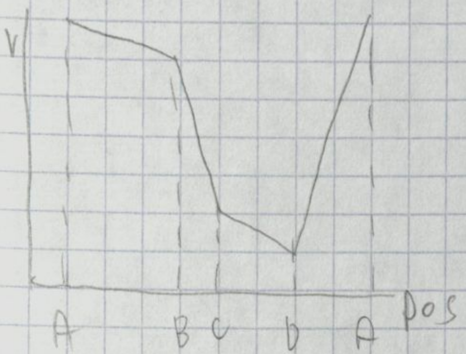


$-x - 4.5 + 10 = 0$
$\Rightarrow x = 5.5$

or

$+x - 4.5 - 2.5 + 12 = 0$
$\Rightarrow x = 5.5$

We can also visualize this graphically by saying completing a loop around a circuit yields a cyclic graph of potential.



Note: Current taking the "path of least resistance" isn't b/c it's smart. It's just dumb REALLY quickly.

Note: Bulbs glow b/c electrons in circuit lose potential & gain kinetic energy. This kinetic energy is then spent colliding w/ the filament causing increase in thermal energy of filament.

Summary

related to conservation of mass

Current Node Rule / Conservation of Charge:

In most circuits higher temperature means lower mobility.

For any node in a circuit

$i_{in} = i_{out}$ Recall: $i = nAuE$

$I_{in} = I_{out}$ Recall: $I = |q|nAuE$

Loop Rule / Conservation of Energy:

For any round trip at an instant in time,

$\Delta V_1 + \Delta V_2 + \ldots = 0$

Quick Review *

*Note: This is done to make our future reasoning & short cuts more intuitive.

Doubling the length of a circuit halves the current

$E = \frac{emf}{2L_{old}}$

$i = nAuE = \frac{nAuE}{2L_{old}} = \frac{1}{2}\left(\frac{nAuE}{L_{old}}\right)$

* Note: This makes sense why multiple lightbulbs reduces current.

Doubling the cross sectional area of a circuit doubles the current

$i = n(2A_{old})uE = 2(nA_{old}uE)$

*Note: This is analogous to putting two bulbs in parallel.

Ch 19: Circuit Elements

Here, we'll cover circuits macroscopically

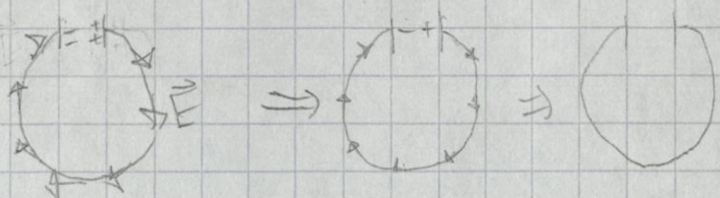
Capacitors ← * Basically, capacitors are just charge resivoirs!

Our capacitors tend to be thin plates of foil separated by an insulator. There is no conducting path & charge does not travel across the capacitor.

Discharging

Very shortly after connecting a charged capacitor, it reaches steady state & begins discharging.

Electrons start moving b/c of the fringe field.



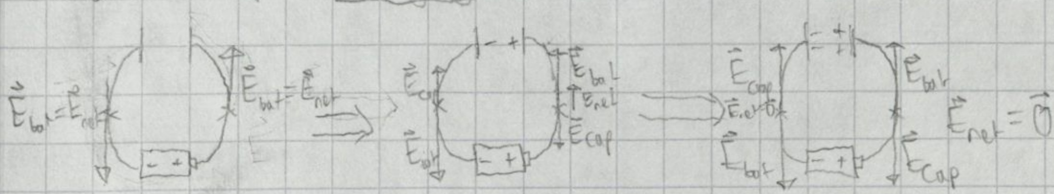
Since the electric field points from the positive to the negative plate, the plates become less charged. This makes the electric field weaker, thus reducing the current. This also reduces the rate at which current decreases.

Eventually, the capacitor fully discharges, meaning current is zero & nothing happens.

Charging

We can analyzi a charging capacitor similarly.

Very shortly after connecting an uncharged capacitor, it reaches steady state & begins charging.



Initially, the battery pushes charges, as you would expect, at steady state. This causes charge to build up on the capacitor, opposing the electric field of the battery. This reduces the current in the circuit. This continues until the electric fields negate each other completely & the current becomes zero.

Effect of Resistors on Capacitors

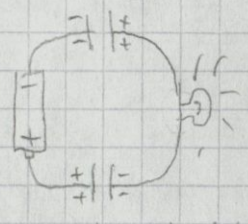
High resistance causes capacitors to charge/discharge more slowly (since current is less). Low resistance causes the opposite.

The resistance does not effect final charge. Duh!

Capacitors in Circuit

When a capacitor is connected in parallel, you can think of it as their area's adding. (If you have 2 identical capacitors connected in parallel, treat it as one capacitor w/ 2x the area.) Unsurprisingly, it takes 2x longer to charge 2 capacitors in parallel (or one capacitor that is 2x the size).

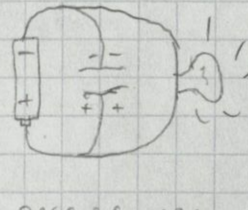
You can also hook up a bulb connected to a battery only thru capacitors & it works! Consider the following circuit



Here, the light bulb is lit b/c the battery causes charge buildup on the 2 capacitors. This causes the opposite charge on the other circuit. It's like the bulb is connected directly to the battery! (thru a buffer.)

If you charge a capacitor in a certain position & then reverse its orientation for discharging, it acts like a temporary battery.

In real life, however, capacitors are often used to smooth power interruptions, using the configuration below where the capacitor is connected in parallel w/ the battery.

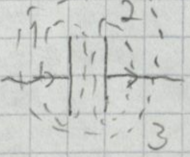


For reasons you can probably see, disconnecting the battery will temporarily leave the bulb powered by the capacitor. This is suitable for most minor blips of power. You'd normally use high-capacity capacitors for this.

Capacitors are also used to filter out AC voltage changes in a DC circuit.

Current Rule for Capacitors

If you look at the plates of a charging/discharging capacitor, they are not in steady state. However, if you look at the whole capacitor, it is at steady state.



1 & 2 are not in steady state.
3 is in steady state.

This means, during charging & discharging, a capacitor as a whole is in steady state even though the plates are not.

Capacitance

In a capacitor, potential difference $|\Delta V|$ across the gap is greater for a larger Q. Likewise for the reverse. Capacitance is a measure of this proportionality.

$$Q = C|\Delta V| \text{ where } C \text{ is capacitance in } C/V \text{ or } F$$

For parallel plate capacitors, recall

Capacitance is just a derived quantity

$$E = \frac{Q/A}{\varepsilon_0}$$

$$\Rightarrow |\Delta V| = Es = \left(\frac{Q/A}{\varepsilon_0}\right)s$$

$$\therefore \cancel{Q} = C\left(\frac{\cancel{Q}/A}{\varepsilon_0}\right)s$$

$$\Rightarrow \boxed{C = \frac{\varepsilon_0 A}{s}}$$

Now, we have just related the geometry of a capacitor w/ its capacitance. This is really useful!

Steady State in Capacitors

Batteries are somewhat similar but more consistent

Capacitors aren't really in steady state. Their current is constantly decreasing as they become more/less charged. However, for small time steps, we consider them in steady state b/c they discharge fairly slowly.

In the future we'll be able to model & thus handle a capacitors non-constant current.

Resistors

Now we'll be modelling a resistor macroscopically. Here's a quick mapping from microscopic topics to macroscopic:

- Electron Current / Electron Drift Speed ⇒ Conventional Current
- Electric Field ⇒ Potential Difference
- Mobility ⇒ Resistance

Conductivity

We want a single number to describe the complex microscopic factors (i.e. electron density 'n', mobility 'u', & unit charge '|q|') that affect how charge flows in a material.

We combine this by finding the conventional current per area. Recall $I = (|q|nu)AE$.
We call this conductivity σ

$$J = \frac{I}{A} = (|q|nu)E = \sigma E$$ where J is current density $[A\,m^{-2}]$
σ is conductivity

We basically just lumped everything that affects current & made it a scalar between $\vec{E}$ & $\vec{J}$. This makes macro analysis easier

$$\boxed{\vec{J} = \sigma \vec{E}}$$

Higher conductivity means that the electric field "doesn't have to work as hard" to get a certain current.

* Note: Conductivity depends on the material in question.

This property elegantly handles cases with multiple different charged particles flowing (e.g. salt water w/ Na^+ & Cl^-), where otherwise we'd have to consider them & their properties separately. Here's the equation for conductivity then.

$$\sigma = |q_1|n_1u_1 + |q_2|n_2u_2 + \ldots$$

Resistance

Resistance is a way of combining the conductivity of a material & its geometry to get a useful quantity.

To get this value, we use the equation for current density:

$$J = \sigma E$$
$$J = \frac{I}{A}$$
$$E = \frac{\Delta V}{L}$$
$$\frac{I}{A} = \sigma \frac{\Delta V}{L}$$

Now, we rearrange this to get resistance (R), a relation b/w current 'I' & potential difference 'ΔV'.

$$\boxed{I = \frac{\Delta V}{L/(\sigma A)} = \frac{\Delta V}{R}}$$

We now have resistance in terms of material (σ) & geometry (L & A)

$$\boxed{R = \frac{L}{\sigma A}}$$ where R is resistance in $[V/A]$ or $[\Omega]$

Sometimes, instead of conductivity, we use resistivity $\rho = 1/\sigma$

Micro-Macro Connection

$$\bar{v} = uE \Longrightarrow J = \sigma E$$

$$i = nA\bar{v} = nAuE \Longrightarrow I = |q|nA\bar{v} = \frac{\Delta V}{R}$$

Ohmic & Non-Ohmic Materials

We call materials with constant conductivity (σ) & therefore (almost) constant resistance (R) <u>ohmic</u>.

No material is truly ohmic b/c temperature always affects conductivity somewhat.

Resistors made of ohmic materials ohmic resistors.

Metals are broadly approximately ohmic. They do get lower conductivity at high temperatures. This makes sense if you imagine the electrons colliding more w/ the more vibrating lattice.

Semiconductors

Semiconductors increase their current exponentially w/ voltage. Applying a small voltage gives you almost no current, but applying a large voltage gives you a huge amount of current.

This is because, at low E in a semiconductor, the electrons are bound to an atomic core. However, at larger E they begin to break free & move. In metals, the electrons are always free

$\therefore E\uparrow \Rightarrow n\uparrow$ in semiconductors

The same increase in n occurs w/ temperature.

Non-Ohmic Circuit Elements

Not all parts of a circuit ohmic. In fact basically only resistors are.

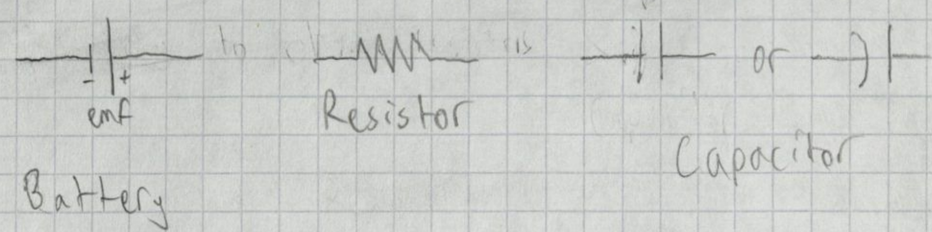
Capacitors aren't ohmic b/c $\Delta V = Q/C$

actually, it slightly decreases!

Batteries aren't ohmic, as changes in current don't change voltage.

In summary, remember $I = \Delta V / R$ applies only to ohmic resistors & is in no way a fundamental part of physics.

Circuit Symbols



Resistance Equations

Since we often deal w/ ohmic resistors, we can make a few shortcuts for them.

Series Resistors

Suppose we have 3 resistors in series in a circuit. We want to find their combined resistance.

Here, we'll use the energy-conservation loop rule / Kirchoff's loop rule.

$\sum_{loop} \Delta V = 0$

$emf - \Delta V_1 - \Delta V_2 - \Delta V_3 = 0$

$\Delta V_n = I R_n$ ← since current is constant, there is no subscript.

$emf - IR_1 - IR_2 - IR_3 = 0$

$emf - I(R_1 + R_2 + R_3) = 0$

$emf - IR_{eq} = 0$

We have just shown that resistors series can be replaced by a single resistor w/ the sum of each resistor's resistance as it's resistance. In other words

$\boxed{R_{eq} = R_1 + R_2 + \ldots \quad \text{for resistors in series}}$

This also implies that, if each resistor has the same conductivity & cross-sectional area, it's like you added their lengths b/c

$R = \frac{L}{\sigma A}$

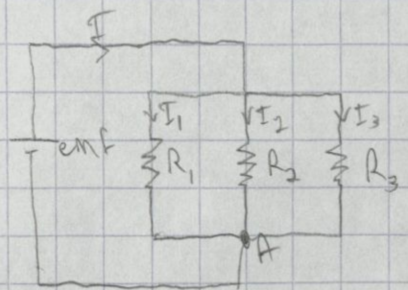
$R_{eq} = R_1 + R_2 + \ldots$

$\frac{L_{eq}}{\sigma A} = \frac{L_1}{\sigma A} + \frac{L_2}{\sigma A} + \ldots$

$L_{eq} = L_1 + L_2 + \ldots$

Parallel Resistors

Suppose we have a circuit like below. We want to find the resistance of a resistor we could use to replace the branching.



Here, we'll use the current node rule for A & $I = \Delta V / R$

$$I = \frac{emf}{R_1} + \frac{emf}{R_2} + \frac{emf}{R_3} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) emf = \frac{emf}{R_{eq}}$$

This shows that ohmic resistors in parallel follow this pattern

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots}$$

If all resistors are the same length & conductivity (material), this is like summing their areas b/c

$$R = \frac{L}{\sigma A} \Rightarrow \frac{1}{R} = \frac{\sigma A}{L}$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots$$

$$\frac{\sigma A_{eq}}{L} = \frac{\sigma A_1}{L} + \frac{\sigma A_2}{L} + \ldots$$

$$A_{eq} = A_1 + A_2 + \ldots$$

Work & Power

Recall that moving a charge results in a change of potential ΔU

$$\Delta U = q \Delta V$$

If we want to find power, we divide by change in time Δt

$$\text{Power} = \frac{\Delta U}{\Delta t} = \frac{q\Delta V}{\Delta t}$$

However, note that $q/\Delta t$ is the definition of I

$$\therefore \boxed{\text{Power} = I\Delta V}$$

This is a general result which you should memorize.

Shortcuts for Resistors

Since, for resistors, $I = \Delta V/R$, we can simplify the power equation to be

$$\text{Power} = \frac{\Delta V}{R} \cdot \Delta V = \boxed{\frac{\Delta V^2}{R}}$$

Alternatively, using $\Delta V = IR$

$$\text{Power} = I \cdot IR = \boxed{I^2 R}$$

Batteries

Until now, we have been treat batteries as ideal devices w/ no internal resistance. This meant we could always say the potential difference across the battery was its emf.

To handle this here, we'll be treating the battery as an ideal battery connected in series to its resistance.

Recall that charges move in a battery due to non-coulombic forces F_{NC} (chemical forces encouraging potential difference) & coulombic forces F_C (electric forces discouraging potential difference). In general $F_{NC} > F_C$ in a battery.

The drift speed of ions thru a battery is proportional to $F_{NC} - eE_C$. If σ is the conductivity for moving ions thru the battery & A is cross-sectional area, the current density $J = I/A$ is given not by $J = \sigma E$ but by

$$J = \frac{I}{A} = \sigma\,(\text{force per unit charge}) = \sigma\left(\frac{F_{NC}}{e} - E_C\right)$$

This makes sense b/c conductivity is a garbage-can scalar that holds a bunch of meaning.

Rearranging this we get

$$E = \frac{F_{NC}}{e} - \frac{I/A}{\sigma}$$

We assume that our battery of length s has a uniform electric field so $\Delta V = E_C s$

$$\therefore \Delta V = \frac{F_{NC}s}{e} - \frac{s}{\sigma A} I$$

Note that $\frac{F_{NC}s}{e}$ is the emf of the battery, or work (force times distance) per unit charge (e). $\boxed{\text{emf} = \frac{F_{NC}s}{e}}$

This makes $\frac{s}{\sigma A}$ our <u>internal resistance</u>

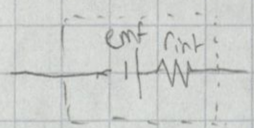
$$\boxed{r_{int} = \frac{s}{\sigma A}}$$

This gives us, for a battery,

$$\Delta V_{bat} = emf - r_{int} I$$

If you analyze this, it should make sense. Essentially, we have some ideal battery hooked up in series to a resistor w/ the real battery's internal resistance. Also, since r_{int} is normally small, our earlier $\Delta V_{bat} \approx emf$ approximation makes sense.

In a circuit diagram, this looks like



Example

Resistor	Ideal Battery ($r_{int}=0$)	Real Battery ($r_{int}=0.25\Omega$)
$100\ \Omega$	0.015A	0.01496 A
$10\ \Omega$	0.15A	0.146 A
$1\ \Omega$	1.5A	1.2 A
$0\ \Omega$	N/A	6 A

This explains why real batteries have a maximum current

The following formula was used to generate this table

$$(emf - r_{int} I) - RI = 0$$ ← This is just Kirchoff's loop rule

$$I = \frac{emf}{r_{int} + R}$$

More Notes

Chemical batteries don't have constant internal resistance. it increases over time w/ use.

Ammeter, Voltmeter, & Ohmmeter

Modern voltmeters measure voltage by time taken to discharge capacitor

- Ammeter: Measures current. In series w/ circuit, + in - out.
- Voltmeter: Measures potential difference. In parallel w/ circuit, + high - low.

Voltmeters must have very high resistance to not affect circuit. Ammeters must have very low resistance.

Ammeters just measure current flowing through them.

Voltmeters are, in principle, ammeters w/ a resistor of known, high resistance. to not affect circuit
We then just use the standard formula

$\Delta V = IR$

Standard multimeters/voltmeters use $10 M\Omega$ resistors, which are good most of the time. The difficulty of using high resistance resistors is that you need a sensitive ammeter for the small amount of current.

Ohmmeter

Ohmmeters are meant to be connected to independent parts of a circuit whose resistance you want to measure.

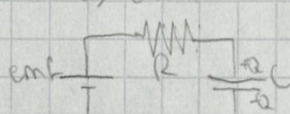
Ohmmeters work by producing a small potential difference (e.g. 50mV) & then measuring the current using a very sensitive ammeter. They then use the familiar

$R = \frac{\Delta V}{I}$

Because of the way they work, Ohmmeters must have very small resistance in their wires

Analysis of RC Circuits

In general, analyzing abstract circuits is very difficult. However, we can easily analyze simple circuits. In this class, the only ones we'll formally analyze are so-called RC circuits, circuits with a resistor & a capacitor alone.



Recall:

- Potential Difference across Capacitor

  $\Delta V = Q/C$ where Q is charge on single plate
  C is capacitance of capacitor

- Capacitance for Parallel Plate Capacitor

  $C = \frac{K\varepsilon_0 A}{s}$ where A is the area of one of the plates
  s is the separation of the plates
  K is the dielectric constant of material filling gap

The energy equation / loop rule for an RC circuit is

$$\Delta V_{round\_trip} = emf - RI - Q/C = 0$$

Final State of RC Circuit

We know that the final state for an RC circuit is when the capacitor is fully charged. This means current (I) is 0.

$$\Delta V_{round\ trip} = emf - \cancel{RI} - Q/C = 0$$

$$\therefore \boxed{emf = Q/C}$$

Solving for the charge on the capacitor, we get

$$Q = C\,emf$$

It should make sense that the final charge on the capacitor doesn't depend on R. R only slows it down or speeds it up.

Initial State

Initially, the capacitor is uncharged. This means charge (Q) is 0. Solving for current, we get

$$emf - IR - Q/C = 0$$

$$I = \frac{emf - Q/C}{R}$$ ← Remember this. It'll be used in a bit

Since $Q = 0$

$$\boxed{I = \frac{emf}{R}}$$

Transient State

In the transient state charge & current are changing & somewhere in between the initial & final states. We can formally describe how this change occurs.

Recall the definition of current

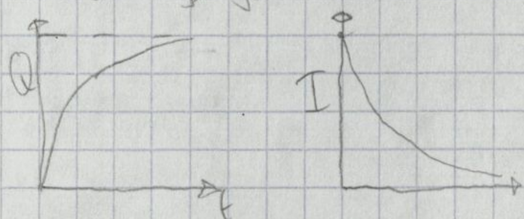
$$I = \frac{dQ}{dt}$$

We can use Q as the amount of charge on the capacitor. This gives us

$$I = \frac{dQ}{dt} = \frac{emf - Q/C}{R}$$

$$\therefore dQ = \left(\frac{emf - Q/C}{R}\right)dt$$ ← shhh little math head. it's okay

We can now integrate dQ, knowing the initial point of $Q=0$ at $t=0$. If you do this numerically, you'll see something like this



* These should make sense!

Let's try to solve this analytically.

$$I = \frac{emf}{R} - \frac{Q}{RC}$$ ← rewritten to see math more easily

$$\Rightarrow \frac{dI}{dt} = 0 - \frac{1}{RC}\frac{dQ}{dt}$$ ← since Q depends on t

All this analytical stuff is actually only an approximation, since R changes w/ temperature which changes w/ I & material & surroundings

Recall $\frac{dQ}{dt} = I$

$$\therefore \frac{dI}{dt} = -\frac{1}{RC} I$$

This derivative then gives us

$$I = I_0 e^{-at} \quad \text{where } a = +\frac{1}{RC}$$

$$I_0 = \frac{emf}{R}$$ ← from earlier

$$\therefore \boxed{I = \left(\frac{emf}{R}\right) e^{-t/RC}}$$

We can now use this I to find Q, since $Q = \int_0^t I\,dt$ (from $I = dQ/dt$). We won't do the integral right now but,

$$\boxed{Q = C(emf)\left(1 - e^{-t/RC}\right)}$$

This equation now gives us, for $t \to \infty$, $Q = C(emf)$. This means $I=0$, which we know to be true at $t \to \infty$.

RC Time Constant

It would be great if we could get a sense of how long it takes to reach equilibrium w/o all that math. Luckily, since the equation always changes w/ $e^{-t/RC}$, we can just find RC to see how long it will take to reach equilibr

* Note: this doesn't give you the actual time, just a sense of it. Really, it's how long it takes to reach $e^{-RC/RC} = e^{-1} = 1/e = 0.37$ it's original.

Ch 20: Magnetic Force

Recall the Biot-Savart Law

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}$$

derive using definition of current

$$\Delta \vec{B} = \frac{\mu_0}{4\pi} \frac{I \Delta \vec{l} \times \hat{r}}{r^2}$$

When we analyze magnetic force, we consider only the magnetic fields made by other charges & we analyze them independently & then sum the results. In other words, the superposition principle applies & charges don't self-affect.

Magnetic Force on a Moving Charge

For point charges, the direction of the magnetic field, velocity of the charge, & sign of the charge determine the direction of the magnetic force. The force is given by

$$\boxed{\vec{F}_{mag} = q\vec{v} \times \vec{B}}$$

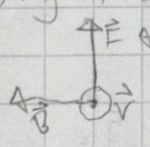
where q is the charge (including sign) of moving charge
$\vec{v}$ is velocity of moving charge
$\vec{B}$ is applied magnetic field (tesla)

Teslas are the unit of magnetic field

$$T = \frac{N/C}{m/s}$$

* Note: Magnetic fields only affect moving charges

Using the right hand rule for $\vec{F}_{mag}$:



Magnitude of Force

A particle's acceleration is proportional to q/m

$$\left|\frac{d\vec{p}}{dt}\right| = q \underbrace{v B \sin(\theta)}_{|\vec{v} \times \vec{B}|}$$

$\therefore \left|\frac{d\vec{v}}{dt}\right| = \frac{q}{m} vB\sin(\theta)$ for $v \ll c$

Motion

In a constant magnetic field, a charged particle experiences helical/circular motion, which you can see by the right hand rule & the force only being exerted perpendicular to motion. In the case where $\vec{v}$ & $\vec{B}$ are fully perpendicular, it moves in a circle.

Recall the general form

$$\left|\frac{d\vec{p}}{dt}\right| = p\left(\frac{v}{R}\right) \quad \text{where } p = \underbrace{\gamma}_{\text{the Lorentz factor}} mv = \underbrace{\frac{1}{\sqrt{1-(v/c)^2}}}_{\text{the Lorentz factor}} mv$$

Recall v/R is the angular speed (ω) in rad/s.

$$\therefore p\left(\frac{v}{R}\right) = \omega p = \omega \gamma mv$$

Recall Newton's 2nd Law / the Momentum principle

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Combing all this with the equation for magnetic force $\vec{F} = q\vec{v} \times \vec{B}$, we get

assuming $\perp$ b/c circular motion

$$\omega \gamma mv = |q\vec{v} \times \vec{B}| = |q| mv\sin(90^\circ)$$

$$\therefore \boxed{\omega = \frac{|q|B}{\gamma m}} \quad \text{for any speed of circular motion}$$

At low speeds, we can ignore the Lorentz factor (γ) since $\gamma \approx 1$

$$\therefore \boxed{\omega \approx \frac{|q|B}{m} \quad v \ll c}$$

This is the non-relativistic form of the equation derived from the non-relativistic momentum principle. Here's a quick breakdown

$$\left|\frac{d\vec{p}}{dt}\right| = p\left(\frac{v}{R}\right) \approx mv\left(\frac{v}{R}\right) = \frac{mv^2}{R}$$

$$\frac{d\vec{p}}{dt} \approx m\frac{d\vec{v}}{dt} = \vec{F}_{net}$$

$$\frac{mv^2}{R} = |q| vB\sin(90^\circ)$$

$$\omega = \frac{|q|B}{m}$$

Orbital Periods

$\omega = \frac{2\pi}{T}$, by definition of ω,

$\therefore T = \frac{2\pi}{\omega}$

Using earlier ω

$\omega \approx \frac{|q|B}{m}$ for $v \ll c$

$\Rightarrow \boxed{T = 2\pi \frac{m}{|q|B}}$

This may seem odd b/c the period doesn't depend on velocity $\vec{v}$ radius R. However, this independence is the basis of a cyclotron.

Momentum of Particle

$\left|\frac{d\vec{p}}{dt}\right| = p\left(\frac{v}{R}\right) = |q|vB$

$\Rightarrow p = |q|BR$

Basically, we know momentum if we know how hard something is being pulled ($|q|B$) & its radius/energy level (R).

Cyclotron

Cyclotrons are used to accelerate protons into the nucleuses of other atoms to study them. They can create enormous potential difference (million volt+) in a small size.

Basically you have 2 D-shaped boxes (dees), which are placed in a very strong magnetic field. Inside the dees, the magnetic field makes them curve but there's no electric field. Outside the dees, there's an electric field caused by the potential difference of the dees, this speeds up the proton. We the flip the charge whenever the proton switches sides, to always gain energy.

This is way easier & more economical than huge potential differences

* Note: Since period (T) doesn't depend on velocity ($\vec{v}$), we can have it constantly oscillate. This only works when $v \ll c$, otherwise

relativity starts mattering.

Magnetic Force & Current Carrying Wire

Our method here is to sum the contributions of many electrons.

To count how many electrons are in a small volume, take the length Δl, cross-sectional area A, & electron density. The number of electrons is then

$$\#\text{ electrons} = nA\Delta l$$

Using this to get something useful

$$\vec{F} := (\#\text{ electrons})\,\vec{F}_{electron} = (nA\Delta l)(q\vec{v} \times \vec{B})$$

Make $\Delta\vec{l}$ point in the same direction as $\vec{v}$.

$$\vec{F} = (nA\bar{v})(q\Delta\vec{l} \times \vec{B})$$

This is just derived from micro!

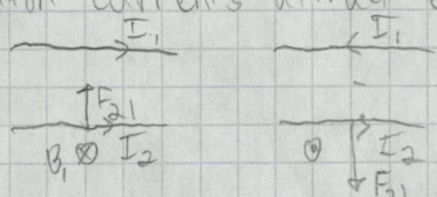
Recall $I = qnA\bar{v}$

$\therefore$ $\boxed{\vec{F} = I\Delta\vec{l} \times \vec{B}}$ where $\Delta\vec{l}$ is a tiny length in the direction of the wire
I is current
$\vec{B}$ is applied magnetic

*Note: This works whether you consider positive charges moving w/ I or negative charges against.

Currents in Parallel Wires

It's pretty trivial to show qualitatively, but for parallel wires, same direction currents attract each other & opposite direction currents repel.



Since the positive metal atomic cores in a wire are stationary, they do not experience a magnetic force from the wire.

Long Sections of Wire

For long sections of wire

$\boxed{|\vec{F}| = ILB\sin(\theta)}$ ← basically just above rewritten

Electric + Magnetic Forces

moving charged particles are subject to a magnetic & electric force, if present.

$$\Sigma\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{v} \times \vec{B}$$

This net force is called the Lorentz Force.

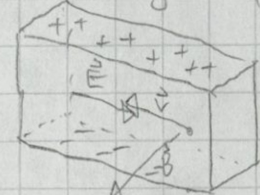
It is always possible to arrange $\vec{E}$ & $\vec{B}$ in such a way that $\Sigma\vec{F} = \vec{0}$ for any $\vec{v}$. In general choose $\vec{F}_m$ & $\vec{F}_e$ to be equal & opposite.

$$|\vec{F}_e| = |\vec{F}_m|$$
$$qE = qvB\sin(90°)$$
$$\boxed{E = vB}$$

If we know $\vec{F}_e$ & $\vec{F}_m$ are opposite in direction, we can use this to solve for their magnitudes.

Hall Effect

Magnetic forces on moving charged particles can cause it to polarize. This is the Hall effect. This doesn't affect the current as the surface charges quickly change to oppose it & cancel it out.



As you can see, the moving electrons are deflected towards the bottom of the conductor. The original gradient is still there but omitted for clarity.

This charge difference on the top & bottom sides of the wire creates a $\vec{E}_\perp$ which opposes the magnetic force. This $\vec{E}_\perp$ is called the transverse electric field. Thus $\vec{F}_{e\perp}$ & $\vec{F}_{mag}$ are equal & opposite.

$$\therefore |\vec{F}_{e\perp}| = |\vec{F}_m|$$

Recall $I = |q|nA\bar{v}$
$\therefore \bar{v} = \frac{I}{|q|nA}$

$$qE_\perp = q\bar{v}B\sin(\theta)$$

$$\Rightarrow \boxed{E_\perp = \bar{v}B}$$ ← This is similar to earlier!

If you connect a voltmeter to the top & bottom of the wire, you should get

$\Delta V = hE_\perp = h\bar{v}B$ where h is height of wire.

This voltage is called the Hall voltage. It is normally very small.

Density of Mobile Charges from Hall Effect

If you know Hall voltage (ΔV_{hall}) & conventional current (I), you can find mobile charge density (n).

$$\Delta V_{hall} = E_\perp h = \bar{v} B h = \left(\frac{I}{|q| n A}\right)(Bh)$$

$$\Rightarrow n = \frac{I}{|q| \Delta V_{hall} A} Bh$$

Sometimes n metals only give up a single electron per atom to the sea. These are called monovalent & n is the same as atomic density.

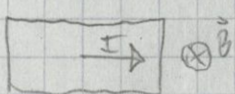
This is a useful way to physically determine.

Sign of Mobile Charges from Hall Effect

The way it was determined that electrons are the mobile charges in metals was thru the Hall effect (& the Hall voltage).

When really measuring the Hall effect, it's good practice to remove the magnetic field & make sure you get no voltage

Consider the following case



Imagine if the mobile charges were positive. Then $\bar{v}$ would be to the right & $\vec{F}_{mag}$ would point up. This means the mobile charges would accumulate on top.

Imagine if the mobile charges were negative. Then $\bar{v}$ would be to the left & $\vec{F}_{mag}$ would point up. This means the mobile charges would accumulate on top.

In both cases the mobile charges accumulated on top. However, in one case they were negative & the other positive. This means the Hall voltage (ΔV_{Hall}) would have opposite signs. This means <u>we can determine the charge of the mobile charges using the Hall effect.</u>

Additional Applications

You can also use a known material to measure magnetic field by measuring its Hall effect.

Explaining Why Current Carrying Wire Moves

Recall that for the moving charges in a conductor, $\vec{F}_m$ & $\vec{F}_e$ cancel out.
Recall that for the stationary atomic cores $\vec{F}_m = \vec{0}$. However, they still experience $\vec{F}_e$ from the Hall voltage / surface charge build up.

This means that the positive charges get pulled to one side by the balancing electric field for mobile charges.

* <u>Note:</u> This means that the motion of a wire in a magnetic field is a side effect of the surface charges rearranging in response to the magnetic force on the moving charges.

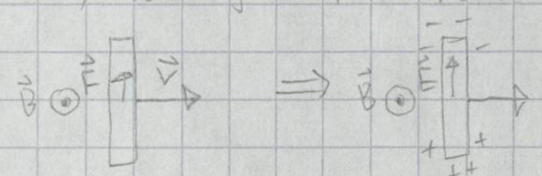
Alternatively, the mobile charges get pulled to one side. Since the mobile charges can't escape the wire, they pull it w/ them.

Motional EMF

We just saw that the magnetic field exerts a force on a current carrying wire. We can get the same effect by moving the wire instead of it having a current. This again is split into our familiar steps of initial transient & steady state.

Initial Transient

If you analyze the following situation using our earlier results, we get the following b/c $\vec{F}$ points up.



During the initial transient, there is no electric field $\vec{E}$ but some $\vec{F}$. Steady state occurs when enough of an $\vec{E}$ appears to counteract $\vec{F}$.

Steady State

Since the bar is in steady state, $\vec{E}$ is constant in the bar. If the bar has length L & is in magnetic field $\vec{B}$, this means

$$\Delta V = EL = vBL \quad (\text{since } E = vB)$$

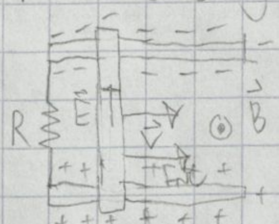
The electrons in the metal are at equilibrium & thus stationary

We call this *motional emf*.

Motional EMF & Current

Electrons act like a fluid. When you start moving a bar, they lag behind for an incredibly small amount of time.

Bar in Steady State:



$\vec{F}_{NC}$ is the force pulling the bar

hence "emf"

The above should look like a battery. You have some non-coulombic force maintaining a charge separation at equilibrium w/ a coulombic force. This charge separation causes current through the resistor.

Additionally, note that since electrons are moving up during the initial transient (& would continue up if being discharged), the bar starts to experience a resistive magnetic force from the electrons moving up. This means, the faster the bar moves, the faster the electrons move up, the stronger the magnetic field opposes the motion.

This means, for a certain force, the charge separation & thus voltage is limited. Putting this quantitatively, below is the work by the bar.

$$F_{NC}L = ev_{bar}BL \quad (\text{since } F_{NC} = q|\vec{v} \times \vec{B}| = qvB)$$

The magnetic field does no work! The work by vert. part of $\vec{F}_{mag}$ on electrons does positive work. The horizontal component does negative work. This cancels out, as can be seen by the power equation

$$\vec{F}_{mag} \cdot \vec{v}_{electron} = 0 \quad (\text{b/c } \vec{F}_{mag} \perp \vec{v}_{electron})$$

Converting this to work per unit charge

$$\frac{F_{NC}L}{e} = \frac{ev_{bar}BL}{e} = v_{bar}BL$$

Assuming negligible internal resistance ($r_{int} \approx 0$),

$$\boxed{\Delta V = emf = v_{bar}BL}$$

Assuming non-negligible internal resistance (r_{int})

$$\boxed{\Delta V = emf - r_{int}I}$$

Detecting Motional EMF

Motional EMF is really hard to detect, since it is so small. You can't see it by just waving a wire & a light bulb in front of a magnet.

You need very high velocity & magnetic field w/ a large length to do anything significant.

Analyzing Force on Bar

We know the bar is in steady state. We want to find the force being applied to maintain this steady state.

We know the magnetic force in the horizontal is

$F_{mag} = NevB$ where N is number of electrons, the rest are clear

$\Rightarrow F_{mag} = (nAL)evB$ where A is cross-sectional area of bar, L is length of bar

$\Rightarrow \underline{F_{mag} = (enAv)LB = ILB}$

This means

$$\boxed{F_{applied} = ILB}$$

This means our power input & thus power output (of the battery) is

$$\boxed{P = F_{app}\frac{\Delta x}{\Delta t} = F_{app}v_{bar}}$$

$$\Rightarrow P = ILBv_{bar}$$

$$\Rightarrow \boxed{P = I(emf)} \quad \Rightarrow emf = LBv_{bar}$$

Applications

This is essentially how all electric power generation works, except you move the bar in a circle. This converts mechanical to electric energy.

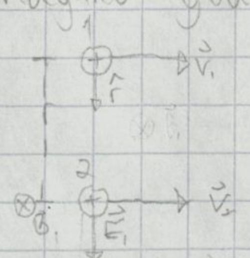
Magnetic Force in a Moving Reference Frame

Since magnetic field depends on velocity of a charged particles, different reference frames may see the magnetic field differently. (Imagine being still vs moving w/ a particle.)

Basically, "stationary particles" produce just an electric field while "moving particles" produce both an electric & magnetic field. The "hand off" of electric to magnetic field must be such that it handles moving & stationary reference frames.

Ratio of Magnetic & Electric Forces

Imagine you have two protons like so analyzing only proton 2



$$\vec{F}_{el} = q_2\vec{E}_1 = e\left(\frac{1}{4\pi\epsilon_0}\cdot\frac{e}{r^2}\right) = \frac{1}{4\pi\epsilon_0}\cdot\frac{e^2}{r^2}$$

$$\vec{F}_{mag} = q_2\vec{v}_2\times\vec{B}_1$$

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\cdot\frac{q_1\vec{v}_1\times\hat{r}}{r^2} \quad b/c\ \vec{v}_1\perp\hat{r}$$

$$B_1 = \frac{\mu_0}{4\pi}\cdot\frac{q_1 v}{r^2}$$

We made the math easier by making everything perpendicular. However, the ratio works out in general

$$F_{mag} = q_2 v_2 B_1 \quad (b/c\ \vec{v}_2\perp\vec{B}_1)$$

$$F_{mag} = \frac{\mu_0}{4\pi}\cdot\frac{e^2v^2}{r^2}$$

If you now take the ratio, you get

$$\frac{F_{mag}}{F_{el}} = \frac{\left(\frac{\mu_0}{4\pi} \cdot \frac{\cancel{q^2} v^2}{\cancel{r^2}}\right)}{\left(\frac{1}{4\pi\varepsilon_0} \cdot \frac{\cancel{q^2}}{\cancel{r^2}}\right)}$$

$$= \frac{v^2}{\left(\frac{1}{4\pi\varepsilon_0} \Big/ \frac{\mu_0}{4\pi}\right)}$$

$$= \frac{v^2}{(9 \cdot 10^9 / 1 \cdot 10^{-7})}$$

$$= \frac{v^2}{9 \cdot 10^{16}}$$

$$\boxed{\frac{F_{mag}}{F_{el}} = \frac{v^2}{c^2}}$$

We observe relativistic effects here. The details of the math aren't fully correct but they arrive at the same general concept. We missed the relativistic effects on the field. However, it cancels out.

Wack! The speed of light showed up (w/ appropriate units).

Well, this does show that magnetic forces are significantly weaker than electric force when $v \ll c$ for two charged particles.

Although this is generally true, circuits are an exception b/c they can experience significant magnetic field, due to high current wires, but low electric field, due to the small size of the wire.

Using this ratio, we can get the following net force:

$$F_{net} = F_{el} - F_{mag} = F_{el} - F_{el} \cdot \frac{F_{mag}}{F_{el}} = F_{el}\left(1 - \frac{F_{mag}}{F_{el}}\right)$$

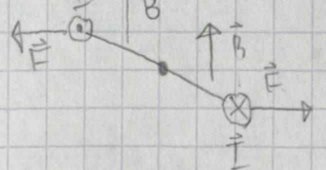
$$\Rightarrow \boxed{F_{net} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q^2}{r^2}\left(1 - \frac{v^2}{c^2}\right)}$$

This is actually a relativistic effect! It shows that if you run beside two extremely fast particles, they seem to separate more quickly than if you were still. This is because you are moving thru time at a faster rate & you are experiencing time at a slower rate.
This exact phenomenon / apparent paradox is exactly what led Einstein to develop his theory of relativity.

Magnetic Torque

We've already seen coils & compasses twist in a magnetic field. Studying magnetic torque gives us a way to quantify this.

Consider a rectangular current loop unaligned w/ the magnetic field, viewed from the side.

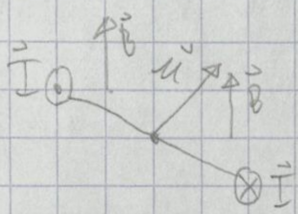


If the current was flipped, the forces would flip, & then the coil would flip.

You can see this causes a counter-clockwise torque. However, as you can see, when the loop is aligned w/ the magnetic field, there is no torque.

This gets easier to reason about when we consider the magnetic dipole moment ($\vec{\mu}$) of the coil. $\vec{\mu}$ points in the direction of the magnetic field generated by the coil & has magnitude $\mu = IA$ (describing strength of moment).

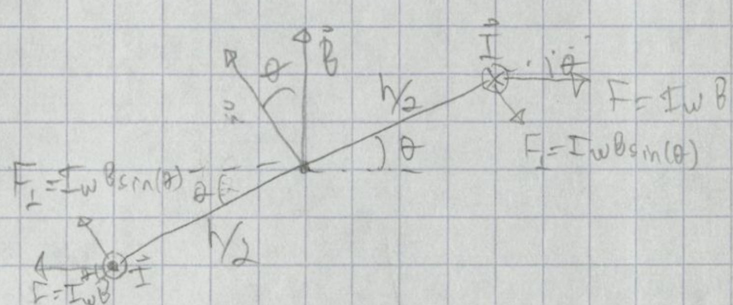
A in $\mu = IA$ is the area of the loop center.



You can see that $\vec{\mu}$ is not aligned w/ $\vec{B}$. However, it "wants" to be.

Calculations

Recall that torque is the distance from the axel (<u>lever arm</u>) times the perpendicular component of force ($\perp$ to lever arm).



$$\tau = F_\perp \overbrace{h/2}^{\text{lever arm}} + F_\perp h/2 = F_\perp h$$

$$\Rightarrow \tau = IwB\sin(\theta)h = (Iwh)B\sin(\theta)$$

$\mu = IwB$

$\tau = \mu B\sin(\theta)$ } we choose a direction such that the right
$\vec{\tau} = \vec{\mu} \times \vec{B}$ & hand rule for torque is followed.

Even though we just did this for a rectangular loop, this applies to any loop b/c we convert any arbitrary loop into a (potentially infinite) number of rectangular loops, where touching internal parts cancel.

$\therefore \boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$ where $\vec{\mu}$ is the magnetic dipole of the loop
B is the applied magnetic field

Potential Energy of Magnetic Dipole

When free to rotate, a magnetic dipole always aligns w/ the magnetic field. This means that this is a lower potential energy than unaligned.

To find the amount of work done rotating the dipole from θ_i to θ_f, we must integrate. The force acts a small distance $\frac{h}{2}d\theta$ w/ a force based off the current angle.

$$work = \Delta U_m = \int_{\theta_i}^{\theta_f} 2IwB\sin(\theta)\left(\tfrac{h}{2}d\theta\right) = IwhB\int_{\theta_i}^{\theta_f}\sin(\theta)d\theta$$

$$\Delta U_m = -IwhB\left(\cos(\theta_f) - \cos(\theta_i)\right)$$

$$\mu = IA = Iwh$$

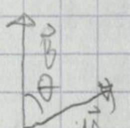
$$\Rightarrow \Delta U_m = \Delta(-\mu B\cos(\theta))$$

We define potential energy where $\theta = 0$ is lowest energy

$$\therefore \boxed{U_m = -\mu B\cos(\theta)}$$

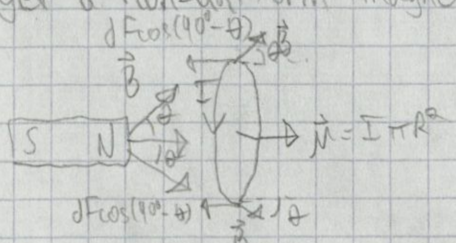
$$\text{or } \boxed{U_m = -\vec{\mu}\cdot\vec{B}}$$

We get θ from



Force on Magnetic Dipole

When we have a bar magnet & a magnetic dipole (i.e. current loop), we get a non-uniform magnetic field affecting the dipole.



All parts of the ring experience $d\vec{F} = I\Delta\vec{l}\times\vec{B}$. The vertical parts of the magnetic field cancel, so the horizontal component

$$dF\cos(90° - \theta) = dF\sin(\theta)$$

This makes the net force

$$F_{net} = IB\sin\theta\oint dl = IB\sin(\theta)(2\pi R)$$

$$\mu = IA = \pi R^2$$

$$\boxed{F_{net} = \frac{\mu(2B\sin(\theta))}{R}}$$

θ is angle from magnet to ring

Force on Dipole

We can find the magnetic force on an object by analyzing it's change in energy in a field.

$$F_{mag}\Delta x = \Delta U_m = \mu \Delta B$$

$$\Rightarrow F_{mag} = \mu \frac{\Delta B}{\Delta x}$$

$$\Rightarrow F_{mag} = \mu \frac{dB}{dx}$$

In general,

$$F_x = -\frac{dU}{dx}$$ ← you should understand this

When we apply our specific $dU = d(-\vec{\mu}\cdot\vec{B})$, we get

$$\boxed{F_x = -\frac{d(-\vec{\mu}\cdot\vec{B})}{dx} = \mu\frac{dB}{dx}}$$

Individual particles have quantized angular momentums.

Magnetic Force & Work

Because $\vec{F}_{mag} = q\vec{v}\times\vec{B}$ is always $\perp$ to $\vec{v}$, the magnetic force changes the direction of a particles momentum but not magnitude.

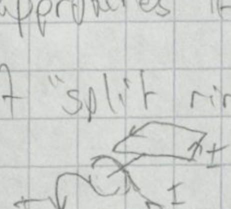
However, the magnetic force can give dipoles rotational kinetic energy, but this does no work since displacement & force are parallel.

Motors & Generators

You know coils twist to be at the lowest potential in a magnetic field. If we constantly change where this lowest potential point is, we can get the coil to rotate continuously, making a motor.

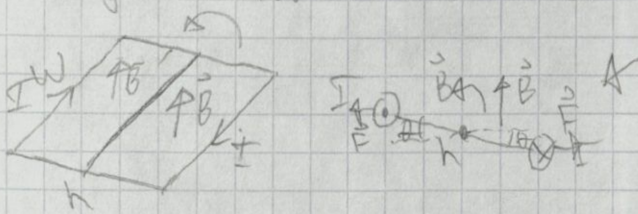
We do this by flipping/changing the stable point as the coil approaches it.

A "split ring commutator" like below is the simplest



#Rotating loop in Magnetic Field

Rotating a loop in a magnetic field causes a current, just like current in a loop in a Magnetic Field makes it rotate.



This makes sense if you follow where the forces must point to rotate in the given direction & how the current must flow to support this.

We can find the motional emf (non-coulombic work per unit charge)

$$emf_{left} = \frac{\overbrace{(qvB\sin(\theta))}^{force}\overbrace{w}^{length}}{q} = vBw\sin(\theta)$$

This is just the left, if we want to combine both sides (assuming symmetry), we get

$$\boxed{emf = 2vBw\sin(\theta)}$$

Recall all this $IwB\sin(\theta)$ is just $L \cdot IB$ (magnetic force on rod w/ current I & length L) where you only consider the $\perp$ part to the rotating loop.

Using $v = \omega(h/2)$ & $\sin(\theta) = \sin(\omega t)$, we get

$$emf = \omega B(hw)\sin(\omega t)$$

$$A = hw$$

$$\boxed{emf = \omega BA\sin(\omega t)}$$

Since, for a circuit w/ a single resistors,

$$I = \frac{emf}{R}$$

$$\boxed{I = \frac{\omega BA\sin(\omega t)}{R}}$$

This means that if we drive/rotate the loop at a constant rate, we get constant EMF & I.
*assuming constant R

We can see the emf from rotating the loop is proportional to $\sin(\omega t)$, giving us sinusodial / alternative current (AC).

Power of Generator

To power a generator we must provide $F = IwB\sin(\theta)$ to counteract the $F = IwB\sin(\theta)$ of the magnetic field. This means ΔW over small distance $h/2\,\Delta\theta$ is

$$\Delta W = \underset{\text{2 b/c we have force on left \& right}}{2}(IwB\sin(\theta))(h/2\,\Delta\theta)$$

$$\therefore \frac{dW}{dt} = 2(IwB\sin(\theta))\left(\frac{h}{2}\frac{d\theta}{dt}\right) = (IwB\sin(\theta))(h\omega) = I(Bwh\omega\sin(\theta))$$

Recall

$$I = \frac{B\omega h \omega \sin(\theta)}{R}$$

This means we can rewrite dW/dt

$$\boxed{\frac{dW}{dt} = I(IR) = RI^2}$$

This is the power disapated in the motor/generator in any instant.

Ch 21: Patterns of Field in Space

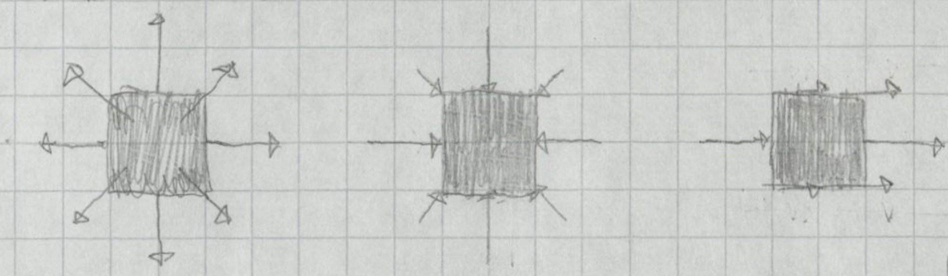
Here's the good calculus shit. Fields & charges can get very complicated. How can we simplify it using surfaces (Gauss's Law) & loops (Ampere's Law)?

Gauss's Law ↑ Like Coulomb's Law

Ampere's Law ↑ Like Biot-Savart Law

Gauss's Law & Surfaces

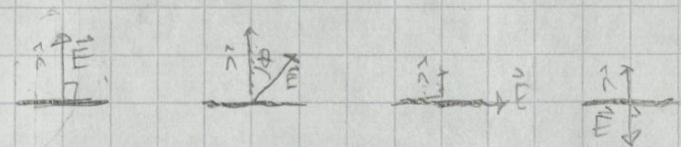
Look at the following 2D surfaces & their surrounding electric fields. It's pretty easy to qualitatively see what the charge distribution of the box could be.



(observation @ head)

Gauss's Law & Coulomb's Law are equivalent for stationary charges! (They differ significantly when relativity matters b/c they don't account for field propagation)

We can quantitatively analyze the charge entering/exiting these boxes using flux (Φ). Flux is essentially the flow of a field thru a surface.



$\Phi = E$ $\quad \Phi = E\cos(\theta)$ $\quad \Phi = 0$ $\quad \Phi$

See! Electric flux has sign based off the "orientation" of the surface ($\hat{n}$)

We quantify this by finding the amount of electric field ($\vec{E}$) pointing in the direction of the surface ($\hat{n}$) w/ area A.

$\boxed{\Phi_{el} = \vec{E} \cdot \hat{n} A}$ ← Only works for uniform surfaces & fields

$\boxed{\Phi_{el} = \int \vec{E} \cdot \hat{n}\, dA}$ ← Works for non-uniform surfaces & fields. You have to parameterize of course.

* Note: You can also use $E_\perp = \vec{E} \cdot \hat{n}$ or $d\hat{n} = d\vec{A} = \hat{n}\, dA$

* Recall: Use $\oint$ over $\int$ for integrals of loops/closed surfaces.

Gauss's Law relates flux (Φ_{el}) w/ charge (q).

$$\boxed{\Phi_{el} = \oint \vec{E}\cdot\hat{n}\,dA = \frac{\sum q_{inside}}{\varepsilon_0}}$$

This only works for closed integrals ($\oint$). Otherwise, we might have outside influences (think leaks).

Why use Gauss's Law? It can make lengthy/ impossible calculations trivial! One of the main ways is by allowing us to choose arbitrary, simple surfaces, called Gaussian Surfaces.

Unlike Coulomb's Law, Gauss's Law is always correct, even w/ relativity.

Gauss's Law & Magnets

Recall that magnetic monopoles don't exist. Magnets fundamentally exist as dipoles. This implies, that you can never have non-zero magnetic flux, since it is always just pushed/pulled, never created/ destroyed. In other words,

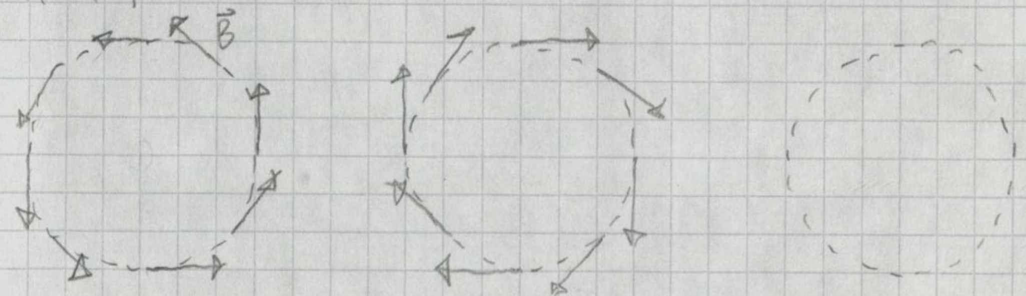
$$\boxed{\Phi_{mag} = \oint \vec{B}\cdot\hat{n}\,dA = 0}$$

Ampere's Law

Like Gauss's Law is relativistically correct (unlike Coulomb's Law), Ampere's Law is relativistically correct (unlike Biot-Savart's Law]

Much like Gauss's law deals w/ closed surfaces, Ampere's law deals w/ closed loops.

Ampere's Law helps us translate b/w magnetic field & current (in both directions). For example you can qualitatively tell something about the current in the given loops



We quantify the amount of current w/in the loop by drawing some arbitrary surface w/ the loop as its edges. Then you measure the amount of current pentrating into this surface. To handle signs, we give the surface an arbitrary orientation using the right hand rule.

Since, here, we'll be going from magnetic field along the loop/path, we just represent that current stuff above using I.

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 \sum I_{inside}}$$

Maxwell's Equations

James Clerk Maxwell extended Gauss's Law & Ampere's Law to form "Maxwell's Equations," a collection of 4 equations (which we'll leave incomplete for now).

- $\oint \vec{E} \cdot \hat{n}\, dA = \frac{\sum q_{inside}}{\varepsilon_0}$ Gauss's Law for electricity
- $\oint \vec{B} \cdot \hat{n}\, dA = 0$ Gauss's law for magnetism
- $\oint \vec{E} \cdot d\vec{l} = 0$ Incomplete Faraday's Law
- $\oint \vec{B} \cdot d\vec{l} = \mu_0 \sum I_{inside}$ Ampere's Law (& incomplete Ampere-Maxwell Law)

The last 2 are incomplete b/c of time effects we haven't gotten to yet.

Ampere's Law & Electric Charges

For stationary electric charges

$$\oint \vec{E} \cdot d\vec{l} = 0$$

Ch 22 Faraday's Law

So far we've dealt w/ magnetic fields constant in time. However, time has important effects on our analysis.

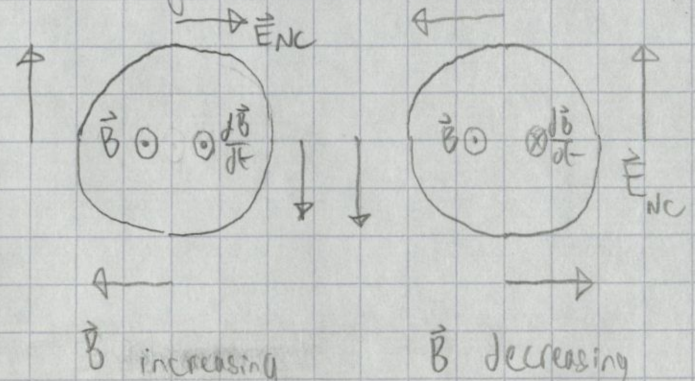
Most importantly, the magnetic & electric fields themselves are connected by time through Faraday's Law.

Curly Electric Fields

When a magnetic field varies in time, it produces an electric field that curls around it. Wack! This electric field is proportional to the rate of change of the magnetic field ($d\vec{B}/dt$)

The electric fields produced by varying magnetic fields are called non-Coulomb electric fields.

Experimentally, this electric field's direction is determined by the right hand rule & $-d\vec{B}/dt$. Where your thumb is w/ $-d\vec{B}/dt$ & your fingers curl w/ $\vec{E}_{NC}$.



You get the picture...

We can use this $\vec{E}_{NC}$ to drive current like a battery (or a generator)!

* Note: $\vec{E}_{NC}$ is NOT caused by a surface charge gradient. That's what $\vec{E}_C$ is caused by.

We can determine the emf caused by this changing magnetic field by finding the force per unit charge (E_{NC}) multiplied by path length.

$$\text{emf} = \oint \vec{E}_{NC} \cdot d\vec{l} = E_{NC}(2\pi r)$$ assuming circle/coil as above

* Note: This emf doesn't change if r changes b/c if r doubles, E halves.

Faraday's Law

From performing experiments, we know $E_{NC} \propto -dB/dt$ & A (cross-sectional) We can combine $\vec{B}$ & A like we did w/ $\vec{E}$ to get magnetic flux (Φ_{mag})

$$\Phi_{mag} = \int \vec{B} \cdot \hat{n}\, dA = \int B_{\perp}\, dA$$

Using this, we can quantitatively summarize the "curly, non-coulombic electric field effect" into Faraday's Law

$$\boxed{emf = -\frac{d\Phi_{mag}}{dt}} \quad \text{where} \quad emf = \oint \vec{E}_{NC} \cdot d\vec{l}$$

$$\Phi_{mag} = \int \vec{B} \cdot \hat{n}\, dA$$

Faraday's Law is a fundamental fact & cannot be derived (as far as we know).

* This means emf around a round trip path is proportional to the rate of change of magnetic flux.

More formally,

$$\boxed{\oint \vec{E}_{NC} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n}\, dA}$$

Where do we get the path & surface tho? Just pick the path which you're finding the emf of. Then imagine a soap film stretched across that path. That soap film is the surface.

Coulombic Electric Field in Faraday's Law

Above, we wrote

$$emf = \oint \vec{E}_{NC} \cdot d\vec{l}$$

However, since coulombic electric field ($\vec{E}_C$) is conservative, we could instead write

$$emf = \oint \vec{E} \cdot d\vec{l}$$

$$\vec{E} = \vec{E}_{NC} + \vec{E}_C$$

$$= \oint (\vec{E}_{NC} + \vec{E}_C) \cdot d\vec{l} = \oint \vec{E}_{NC} \cdot d\vec{l} + \oint \vec{E}_C \cdot d\vec{l} = \oint \vec{E}_{NC} \cdot d\vec{l} + 0$$

This makes real life analysis much easier, since we can't separate $\vec{E}_C$ & $\vec{E}_{NC}$.

emf & Circuits

We can use the emf generated by Faraday's Law just like any other emf. It is easiest to treat the emf as a battery.

This means you can hook a voltmeter to itself around a varying magnetic field & it will read a voltage like it's connected to a battery. This is why you must be aware of AC currents w/in voltmeters affecting the reading. The varying currents producing varying magnetic fields which produce emf.

Maxwell's Equations

Now we can update our earlier Maxwell's equations listings w/ Faraday's Law.

- Gauss's Law: $\oint \vec{E} \cdot \hat{n} dA = \frac{1}{\epsilon_0} \Sigma q_{inside}$ — Charges act like sinks of electric field
- Gauss's Law of Magnetism: $\oint \vec{B} \cdot \hat{n} dA = 0$ — Magnetic field can only be stirred
- Faraday's Law: $\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n} dA$ — Changing magnetic field stirs electric field
- Ampere's Law: $\oint \vec{B} \cdot d\vec{l} = \mu_0 \Sigma I_{inside}$ ← incomplete — Current stirs magnetic field

Superconductors

Superconductors achieve zero resistance at some low temperature, unlike normal conductors. This means

- Current runs forever in a superconductor.
- Magnetic flux through a superconducting ring is constant.
  - Any change in Φ_{mag} would cause some emf in the ring ($emf = -\frac{d\Phi_{mag}}{dt}$), which would cause undefined current, which can't happen.
  - This means electrons always move in a way to counteract any $\Delta\Phi_{mag}$ causing some finite current.
- Magnetic flux in a superconducting disk is zero. ← Meissner Effect
  - Currents on the disk's surface ensure this.
  - Hard to explain w/o quantum mechanics.
  - Also causes levitation!

This is like the electrons "orbiting" an atom's nucleus.

Inductance

Changing the emf in one coil can induce an emf in another. Coils also self-induce an emf when their emf changes. This self-induced emf opposes the change, causing a "sluggishness" to changes.

We now find the self-inductance of a solenoid of radius R, length $d \gg R$, wound N times. A current I runs through the solenoid producing a magnetic field B.

Recall the Magnetic Field inside a Solenoid:

$$|\vec{B}_{solenoid}| = \mu_0 \frac{NI}{d}$$

We now find the emf in one loop of the solenoid when its current changes at a rate dI/dt.

$$emf_{one} = \left|\frac{d\Phi_{mag}}{dt}\right| = \frac{d}{dt}(BA) = \overset{\text{area constant}}{A}\frac{dB}{dt} = A\frac{d}{dt}\left(\mu_0\frac{NI}{d}\right) = (\pi R^2)\left(\frac{\mu_0 N}{d}\cdot\frac{dI}{dt}\right)$$

$$\boxed{emf_{one} = \frac{\mu_0 N}{d}\cdot \pi R^2 \cdot \frac{dI}{dt}}$$

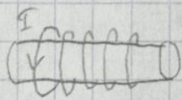
For the whole solenoid

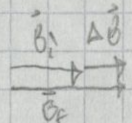
$$emf = N(emf_{one}) = N\left(\frac{\mu_0 N}{d}\cdot \pi R^2 \cdot \frac{dI}{dt}\right)$$

$$\boxed{emf = \frac{\mu_0 N^2}{d}\cdot \pi R^2 \cdot \frac{dI}{dt}}$$

This should make sense. Imagine if it assisted!

Earlier, we said that the emf opposed change. Let's illustrate that now!





$\therefore -\frac{dB}{dt}$ $I_{induced}$ $I_{induced}$ opposes change.

Suppose I is increasing

*Note: This applies to other changes. Inductors always oppose the changing current.

For wires w/ very little resistance, the induced electric field is nearly equal to the applied ($\vec{E}_C \approx \vec{E}_{NC}$). This means current changes very slowly. This also means there is a voltage drop across the solenoid (ΔV_{sol}).

Normally, we lump all the coefficients of $\frac{dI}{dt}$ together, giving us inductance (L)

$$\boxed{|emf| = L\left|\frac{dI}{dt}\right|}$$ where L is inductance

Recall: AC Current is given by $\Delta V = \Delta V_{max}\cos(\omega t)$ where ω is the frequency of the current.

For a solenoid, this means its (self) inductance is

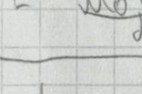
$$\boxed{L = \frac{\mu_0 N^2}{d}\cdot \pi R^2}$$

Inductance has units V·s/A or henries (H).

AC-Transformers

Transformers are important for energy transmission. They work by having a primary coil & a secondary coil. The primary coil induces emf in the secondary coil.

Suppose we have a primary coil 1 & a secondary coil 2. We assume areas & lengths are the same & I is the applied current. Since transformers are kind of like electrical gears, we only want to get a ratio between them. We assume coil 1 has the initial current for the sake of analysis.



Since coil 1 is powered/given current & coil 2 isn't

$$emf_1 = \frac{\mu_0 N_1^2}{d}\cdot A \cdot \frac{dI_1}{dt} \qquad emf_2 = N_2\frac{d\Phi_{mag}}{dt} = N_2 A\frac{dB}{dt}$$ ← magnetic field shared

We can already find emf_1, since we know all factors there already.
We want to find emf_2 by first finding dB/dt, which requires we find B.

$$B = \mu_0 \frac{N_1 I_1}{d} \leftarrow \text{magnetic field caused by coil 1}$$

$$emf_2 = N_2 A \cdot \frac{dB}{dt}$$

$$\frac{dB}{dt} = \frac{d}{dt}\left(\mu_0 \frac{N_1 I_1}{d}\right) = \frac{\mu_0 N_1}{d} \cdot \frac{dI_1}{dt}$$

$$emf_2 = N_2 \cdot \frac{\mu_0 N_1}{d} \cdot A \cdot \frac{dI_1}{dt}$$

We now form a ratio of $emf_1 : emf_2$ to find their relationship.

$$\frac{emf_1}{emf_2} = \frac{\frac{\mu_0 N_1^2}{d} \cdot A \cdot \frac{dI_1}{dt}}{N_2 \cdot \frac{\mu_0 N_1}{d} \cdot A \cdot \frac{dI_1}{dt}} = \frac{N_1}{N_2}$$

$$\therefore \boxed{emf_1 = {}^{N_1}/_{N_2}\, emf_2}$$
$$\therefore \boxed{emf_2 = {}^{N_2}/_{N_1}\, emf_1}$$

Assuming power is conserved (i.e. $emf_1 I_1 = emf_2 I_2$), ~~this means~~

I think it's easier just to remember the emf & derive the current

$$\frac{emf_1}{emf_2} I_1 = I_2$$

$$\Rightarrow \frac{N_1}{N_2} I_1 = I_2$$

$$\therefore \boxed{I_1 = {}^{N_2}/_{N_1}\, I_2}$$
$$\therefore \boxed{I_2 = {}^{N_1}/_{N_2}\, I_1}$$

Suppose coil 1 has 100 loops w/ an emf of 500 V & a current of 1 A.
Suppose coil 2 has 20 loops. Using the above equations, coil 2 would have 100 V ($500 \cdot \frac{20}{100} = 100$) & 5 A ($1 \cdot \frac{100}{20} = 5$)

* <u>The one w/ more loops has higher voltage but lower current.</u>

Energy Density

Recall the energy density of an electric field is so, which we found using a capacitor.

$$\boxed{\frac{\text{electric energy}}{\text{volume}} = \frac{1}{2} \varepsilon_0 E^2}$$

We can perform a similar procedure to find the energy density of a magnetic field using an inductor.

The power going into an inductor is

$$P = I\Delta V = I(\text{emf}) = I\left(L\frac{dI}{dt}\right)$$

When we integrate power to get energy, we get

just show difference

$$\Delta\text{Energy} = \int P\,dt = \int_{t_i}^{t_f} I\,L\frac{dI}{dt}\,dt = L\int_{I_i}^{I_f} I\,dI = L\left[\tfrac{1}{2}I^2\right]_{I_i}^{I_f} = \Delta\left(\tfrac{1}{2}LI^2\right)$$

As always for energy, we assume initial energy is zero, so

$$\text{Energy} = \tfrac{1}{2}LI^2$$

Using $L = \frac{\mu_0 N^2}{d}\pi R^2$ & $B = \frac{\mu_0 N I}{d}$, we get

$$\text{Energy} = \tfrac{1}{2}\left(\frac{\mu_0 N^2}{d}\cdot\pi R^2\right)\left(\frac{Bd}{\mu_0 N}\right)^2$$

$$\text{Energy} = \tfrac{1}{2}\frac{(\pi R^2 d)B^2}{\mu_0}$$

Since $\pi R^2 d$ is the volume, this gives us an energy density of

$$\frac{\text{Energy}}{\text{Volume}} = \tfrac{1}{2}\cdot\tfrac{1}{\mu_0}B^2 \ [\text{J/m}]$$

*Note: Although we calculated this w/ a solenoid, the result generalizes.

Energy Density of Electricity & Magnetism

$$\frac{\text{Energy}}{\text{Volume}} = \tfrac{1}{2}\varepsilon_0 E^2 + \tfrac{1}{2}\cdot\tfrac{1}{\mu_0}B^2$$

Inductors in Circuits

RL Circuits

RL circuits contain a battery, resistor (R), and inductor (L). We will analyze the current over time in an RL circuit.

From KLR (energy conservation) we get the following immediately after connecting circuit

$$\Delta V_{bat} + \Delta V_R + \Delta V_L = 0$$

$$\text{emf} - RI - L\frac{dI}{dt} = 0 \quad \leftarrow \text{b/c inductor resists rising emf}$$

By solving the above differential equation, we get

$$\boxed{I = \frac{\text{emf}}{R}\left(1 - e^{-(R/L)t}\right)}$$

← means $t=0 \Rightarrow I=0$, which should make sense.
$\frac{\text{emf}}{R}$ is final current in steady state (i.e. inductor is no longer opposing anything). ($t=\infty \Rightarrow I = \frac{\text{emf}}{R}$)

This means that the change in current depends on the resistor (R) & inductor (L), not the battery! We can compute the "time constant" of the RL circuit to quantify this. It's an arbitrary measure. It determines how long it takes the current to reach ~63% its final value.

$\boxed{\text{time constant} = L/R}$

Really, all circuits have some self-inductance!

LC Circuit

An LC circuit consists only of a charged capacitor (C) & inductor (L); no battery. In such a circuit, current oscillates forever. (In reality, it would slowly stop due to resistance damping the oscillation)

This oscillation occurs b/c the capacitor initially drains slowly b/c of the inductor. When the capacitor has finished draining, there is significant current. The inductor "wants" to keep this current, causing it to charge the capacitor in the opposite of the initial direction. This repeats.

Analyzing this thru KLR (energy conservation), we get

$$\Delta V_C + \Delta V_L = Q/C - L\frac{dI}{dt} = 0$$

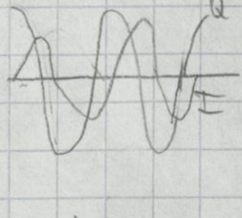
Since the only source of current is from the capacitor, we know $I = -\frac{dQ}{dt}$.

$$\therefore Q/C + L\frac{d^2Q}{dt^2} = 0$$

Solving this differential equation we get

$$\boxed{\begin{aligned} Q &= Q_i \cos\left(\frac{t}{\sqrt{LC}}\right) \\ I &= \frac{Q_i}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right) \end{aligned}}$$

← since $I = -\frac{dQ}{dt}$



This gives us a period & frequency, if you remember trig, of

$$T = \frac{2\pi}{\left(\frac{1}{\sqrt{LC}}\right)} = 2\pi\sqrt{LC} \qquad f = 1/T = \frac{1}{2\pi\sqrt{LC}}$$

LC Circuit Energy

We can also analyze LC circuits using energy.

Recall the energy of a capacitor: $U_{cap} = \frac{1}{2} Q^2/C$
Recall the energy of an inductor: $U_{ind} = \frac{1}{2} L I^2$ ← solenoid magnetic energy from earlier

Initially, all energy is in the capacitor. When the capacitor discharges, all energy is in the inductor. We can use these facts to find I_{max}.

Initially, $U = U_{cap} + U_{ind} = U_{cap} = \frac{1}{2} Q_i^2/C$

When $I = I_{max}$, $Q = 0$

$\therefore U = U_{cap} + U_{ind} = \frac{1}{2} Q^2/C + \frac{1}{2} L I_{max}^2 = \frac{1}{2} L I_{max}^2$

$\therefore \frac{1}{2} L I_{max}^2 = \frac{1}{2} Q_i^2/C$

$\therefore \boxed{I_{max} = \frac{Q_i}{\sqrt{LC}}}$ ← This is consistent w/ our differential equations

AC circuits show similar behavior here, where maximum current & maximum emf are 90° out of phase.

Internally, AC circuits don't have 0 round trip potential b/c there is a non-zero $d\vec{B}/dt$, causing some $\vec{E}_{nc}$. This makes AC currents only have significant current on outside where $\vec{E}_{nc}$ doesn't slow anything down. Skin effect.

Differential Faraday's Law

Using more formal vector calculus, we get Faraday's Law as

$$\text{curl}(\vec{E}) = \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Lenz's Rule

Lenz's Rule is an alternative to the right hand rule & $-d\vec{B}/dt$. It states $\vec{E}_{nc}$ points in a direction that would make a $\vec{B}$ which tries to keep flux constant (but fails)

Chapter 23 - Electromagnetic Radiation

Here we'll discuss classical EM radiation.

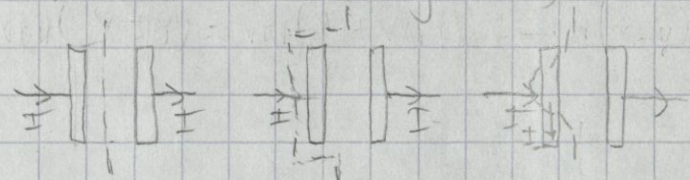
Ampere-Maxwell Law & the final missing piece of Maxwell's equations!

Ampere-Maxwell "Law" deals w/ changing electric field & resolves the contradiction when you stretch the Amperian surface in the middle of a charging capacitor & such.

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0\left(I_{inside} + \varepsilon_0 \frac{d\Phi_{el}}{dt}\right) \qquad \Phi_{el} = \int \vec{E}\cdot\hat{n}\,dA}$$

can be thought of as "current from flux" since $\frac{d\Phi_{el}}{dt} = \frac{d}{dt}\left(\frac{Q}{\varepsilon_0}\right) = \frac{I}{\varepsilon_0}$ (for capacitor)

This handles the following cases for a capacitor



$I_{inside} = 0$, $\frac{d\Phi_{el}}{dt} \neq 0$

$I_{inside} = I$, $\frac{d\Phi_{el}}{dt} = 0$

$I_{inside} = I - I_0$, $\frac{d\Phi_{el}}{dt} \neq 0$

Maxwell actually guessed this & got it right!

* Note: Total must be $\mu_0 I$ in all cases.

Maxwell's Equations

Now that we've nailed down the final Ampere-Maxwell Law, we get the following.

- Gauss's Law: $\oint \vec{E}\cdot\hat{n}\,dA = \frac{1}{\varepsilon_0} Q_{inside}$
- Gauss's Law of Magnetism: $\oint \vec{B}\cdot\hat{n}\,dA = 0$
- Faraday's Law: $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot\hat{n}\,dA = -\frac{d\Phi_{mag}}{dt}$
- Ampere-Maxwell Law: $\oint \vec{B}\cdot d\vec{l} = \mu_0\left(I_{inside} + \varepsilon_0 \frac{d}{dt}\int \vec{E}\cdot\hat{n}\,dA\right) = \mu_0\left(I_{inside} + \varepsilon_0 \frac{d\Phi_{el}}{dt}\right)$

These are the <u>integral</u> forms.

Lorentz Force

The Lorentz force defines the affect of the electric & magnetic field on particles.

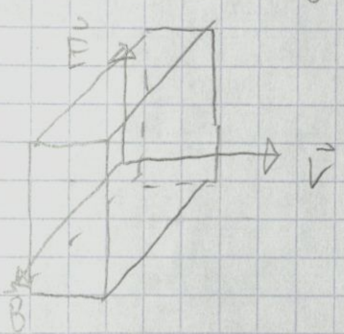
$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \quad \text{OR} \quad d\vec{F} = I\,d\vec{l} \times \vec{B} \text{ (for currents)}$$

Fields thru Space

We now will finally analyze electromagnetic fields which we will show is produced by accelerated charges. Here (& most places), we analyze electromagnetic waves as "moving" or propagating.

To understand this, we will construct an arbitrary system & show it's consistent w/ Maxwell's equations.

Suppose we have a moving imaginary box w/ the given electric & magnetic fields. the fields are zero everywhere outside the box.



Gauss's Law

In the box, since there is equal are on the top & bottom, there is not net charge in box (true).

Gauss's Law for Magnetism

There is zero magnetic flux for the same reason above

Faraday's Law

Since the box is moving, we do have to consider a curly electric field.

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_{mag}}{dt}$$

$$\frac{d\Phi_{mag}}{dt} = B\frac{dA}{dt} = Bvh$$

only part w/ parallel component of E

$$\oint \vec{E} \cdot d\vec{l} = |emf| = Eh\cos(0) = Eh$$

$$Eh = Bvh$$

$$E = vB$$

We have shown Faraday's Law holds if $\underline{E = vB}$.

Ampere-Maxwell Law

Here, we draw a horizontal Amperian surface perpendicular to $\vec{E}$. there are no currents, so we're only dealing w/ Φ_{el}.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{inside} + \varepsilon_0 \frac{d\Phi_{el}}{dt} \right) = \mu_0 \varepsilon_0 \frac{d\Phi_{el}}{dt}$$

$$\frac{d\Phi_{el}}{dt} = E \frac{dA}{dt} = Evh$$

$$\oint \vec{B} \cdot d\vec{l} = Bh\cos(0) = Bh$$ ← same reasoning as earlier.

$$Bh = \mu_0 \varepsilon_0 Evh$$

$$B = \mu_0 \varepsilon_0 v E$$

We have shown the Ampere-Maxwell law holds if $B = \mu_0 \varepsilon_0 v E$.

What is v Anyway?

We've shown our system of moving electric & magnetic waves only works for a specific B where

$$E = vB \quad \text{AND} \quad B = \mu_0 \varepsilon_0 v E$$

If we relate the above (by multiplying them together for simplicity), we get

$$\cancel{E}\cancel{B} = (v\cancel{B})(\mu_0 \varepsilon_0 v \cancel{E})$$

$$v^2 = \frac{1}{\mu_0 \varepsilon_0}$$

If we evaluate this numerically, we get the speed of light (c)

$$\boxed{c = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \approx 3 \cdot 10^8 \text{ m/s}}$$

The speed of light was actually known at the time of Maxwell's analysis. He just happened to bump into it here & conclude electromagnetic waves were light.

This gives us the following relation of E & B in an electromagnetic wave.

$$\boxed{E = cB}$$

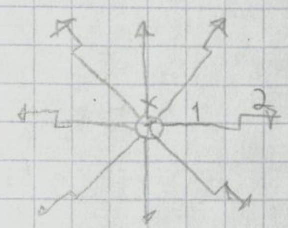
This velocity is in the same direction as $\vec{E}\times\vec{B}$. (i.e. use right hand rule).

Additional Notes

This configuration of waves, where $\vec{E}$ & $\vec{B}$ are perpendicular & travelling at $c \approx 3\cdot10^8$ m/s in the direction $\vec{E}\times\vec{B}$ where $E=cB$ is the <u>only valid</u> configuration

Accelerated Charges Producing EM Radiation

The major (only?) way to produce the above configuration of fields is by accelerating charges. This occurs b/c charges don't immediately update their fields, causing kinks.



B/w 2 & 1 the proton was "kicked"

Since $E=cB$, we normally only analyze E b/c it's easy to extend to B.

As the "kink" passes over you you get a pulse of transverse wave.

Direction of Transverse Electric Field

When kicking a positive charge, the transverse electric field points opposite of $\vec{a}_\perp$, parallel to a sphere. For a negative charge, the transverse electric field points w/ $\vec{a}_\perp$. The magnetic field in both cases is also parallel the sphere as the transverse wave / sphere expands.

Magnitude of Transverse Electric Field

We won't derive the transverse electric field magnitude right now, since it's difficult. However, it should make sense that it is relative to charge (q) & acceleration (a).

$$\boxed{\vec{E}_{rad} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{-q\vec{a}_\perp}{c^2 r}}$$

where $\vec{a}_\perp$ is "projected" acceleration (stronger if acceleration is $\perp$ to your line of sight)

* Note: This falls off w/ $1/r$ (not $1/r^2$) ← This means energy outflow (based on surface area of EM radiation is constant as it propagates, decreasing as $1/r^2$ in intensity as area increases as r^2

* <u>Note</u>: The field is opposite $q\vec{a}_\perp$.

Summary of Fields made by Charges

- Stationary: $1/r^2$ electric field & no magnetic field.
- Moving at Constant Velocity: $1/r^2$ electric field & $1/r^2$ magnetic field.
- Accelerated Charge: $1/r$ electric field & $1/r$ magnetic field.

Sinusoidal EM Radiation ← super common b/c electrons on atoms act as springs!

Recall the general sinusoidal wave

$$y = y_{max}\sin(\omega t)$$

We already know the transverse electric field ($\vec{E}_{rad}$), which corresponds to y_{max}, So now we just need the angular frequency (ω [rad/s]).

Review of Waves

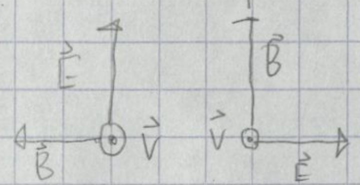
$$f = \frac{1}{T} = \frac{\omega}{2\pi} \qquad v = \frac{\lambda}{T} = \lambda f$$

*Note: The time b/w crests of EM radiation & the distance b/w crests ($v = \frac{\lambda}{T}$) won't always give you the speed of light (c) b/c of relativity. However, the speed of wave propagation will always be c.

is this right?

Polarization

EM radiation can go in the same direction but rotated differently. This is called polarization



Two different polarizations of EM waves.

Normally, light produced from several vibrating sources is unpolarized (Different polarizations present).

Energy & momentum of EM Radiation

We can do some quick math to find how much energy is in an EM pulse.

Suppose we have a stationary charge being affected by an EM pulse that is d meters long.

$$p = F\Delta t$$
$$F = qE$$
$$\Delta t = \frac{d}{c}$$
$$p = (qE)\left(\frac{d}{c}\right)$$

This gives us an energy of (ΔK = Energy right after pulse)

$$\Delta K = \frac{p^2}{2m} = \left(qE\frac{d}{c}\right)^2\left(\frac{1}{2m}\right)$$

*** Note:** This assumes the ball stays slow during the interaction.

We have show Energy of EM Radiation $\propto E^2$.

Energy Density

Recall the energy density of electric & magnetic fields.

$$\frac{\text{Energy}}{\text{Volume}} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0} B^2 \quad [J/m^3]$$

Since $E = cB$ for EM radiation, we can simplify the above to

$$\frac{\text{Energy}}{\text{Volume}} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0}\left(E/c\right)^2 = \frac{1}{2}\varepsilon_0 E^2 \left(1 + \frac{1}{\varepsilon_0 \mu_0 c^2}\right)$$

$$\mu_0 \varepsilon_0 = 1/c^2$$

$$\boxed{\frac{\text{Energy}}{\text{Volume}} = \varepsilon_0 E^2}$$ ← means magnetic field carries same energy as electric!

Poynting Vector

We often want to measure the flow of energy for EM radiation (energy flux). The poynting vector ($\vec{S}$) allows us to quantify this. We define the poynting vector to have a magnitude of the energy flux & the direction of $\vec{E} \times \vec{B}$.

$$\text{Energy} = \left(\frac{\text{Energy}}{\text{Volume}}\right)(\text{Volume})$$

$$\text{Energy} = (\varepsilon_0 E^2)(A \overbrace{c\Delta t}^{\text{thickness}})$$

$$\text{Energy Flux} = \frac{\text{Energy}}{\text{Area} \cdot \text{Time}}$$

$$\text{Energy Flux} = \frac{\varepsilon_0 E^2 A c \Delta t}{(A)(\Delta t)} = \varepsilon_0 E^2 c$$

Since $E = cB$ & $\mu_0 \varepsilon_0 = 1/c^2$, we can the above to

$$\boxed{\text{Energy Flux} = \varepsilon_0 E^2 c = \varepsilon_0 E B c^2 = \frac{EB}{\mu_0}} \quad [W/m^2]$$

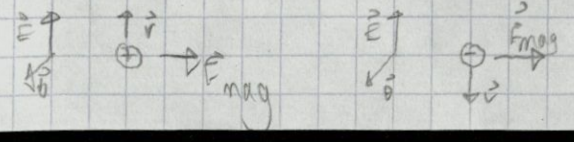
Combining this w/ the earlier direction we get

$$\boxed{\vec{S} = \frac{EB}{\mu_0}(\hat{E} \times \hat{B}) = \frac{\vec{E} \times \vec{B}}{\mu_0}}$$ ← you can also figure out the direction qualitatively

*** Note:** This is in W/m^2

Pressure of EM Radiation

We already discussed the electric field in EM radiation moving things. However, the magnetic field also has a slight effect once things start moving



As you can see, opposite charges exposed to the same radiation experience a magnetic force in the same direction. We call this radiation pressure (even neutral matter experiences it b/c it's made up of charges!)

We can estimate the effect here, where charge q acquires speed v.

$$\boxed{\frac{F_{mag}}{F_{el}} = \frac{q v_{avg} B}{qE} = \frac{(v/2)(E/c)}{c} = \frac{1}{2}\frac{v}{c}}$$

Since radiation pressure is so week, we almost always neglect it.

This means F_{mag} is almost always negligible.

Momentum of EM Radiation

Recall the relativistic momentum equation (E is energy)

$$E^2 = (pc)^2 + (mc^2)^2$$

For light, $m=0$, so

$$E = pc$$

This means we can find the momentum of light. It's easier & more useful to analyze its momentum flux using the poynting vector ($\vec{S}$) tho.

$$\boxed{\text{Momentum Flux: } \frac{\vec{S}}{c} = \frac{1}{\mu_0 c}\vec{E}\times\vec{B} \quad [N/m^2]}$$

← this is where we get radiation pressure (N/m^2) from!

If you recall the conservation of momentum, this means the EM radiation must lose some of its energy (momentum) after transferring it to matter. Luckily, this happens as re-radiation/scattering.

When a charge is hit, the EM radiation keeps going. However, the charge begins to oscillate itself, cancelling out the radiation that made it move. Since the charge produces EM radiation in all directions, this is called scattering.

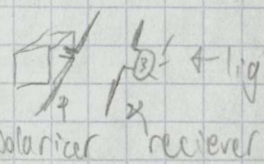
*Note: The cancelling out completely accounts for the transferred momentum.

EM Radiation of Neutral Atoms

Neutral atoms are made up of charged particles (proton & electron), so EM radiation still affects them.

The main effect of EM radiation is it polarizes the atom by temporarily kicking its electron significantly to the side. This is most significant for metal rods where this kick can result in a current & a potential difference.

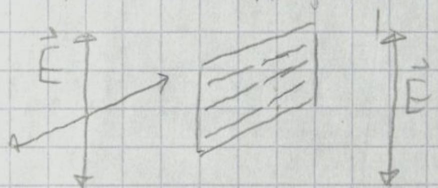
We can take advantage of this to build a polarizer which can wirelessly transmit electricity. (Radio antenna) We have some metal rod w/ AC current. The vibrating electrons cause EM radiation. This radiation can then interact w/ another rod to produce similar radiation.



← lightbulb on!

* Note: Moving the reciever further away dims it (recall $1/r$ drop off). Rotating the reciever vertical turns off the light b/c it becomes unaligned.

We can also polarize light using plastic filters w/ long, aligned molecules. EM radiation aligned w/ the long molecules transfer most of their energy to the filter. Radiation unaligned is mostly unaffected.



Resonance

Molecules have a certain resonance frequency, what they'll vibrate at when coming to rest. If molecules experience EM radiation at roughly this resonance frequency, they'll vibrate much more violently.

Our eyes light detecting molecules have resonance frequencies at visible light's frequencies. This is why/how we see them!

This resonance is how we tune radios & other radio equipment!

Why is the Sky Blue?

Randomly vibrating air molecules reradiate the sunlight received. They also polarize it! Since the only light that is reradiated down happens to be vibrating perpendicular to that coming from the sun.

Since the acceleration is relative to frequency

$$x = A\cos(\omega t) \Rightarrow \vec{a} = -\omega^2 A\cos(\omega t)$$

Higher frequencies accelerate more & reradiate more. This means blue reradiates more! (Why isn't it violet then? There's still the lower frequencies, so the bring the average down.)

This is the same reason mountains are blue.

Light Propogating in a Medium

We have yet to account for the medium through which light propagates which impacts some of its properties (through reradiation & superposition).

Wavefronts & Plane Waves

It's useful though to imagine radiation as discrete rays.

When radiation is emitted, it is emitted in waves. A wave front is a set of radiation peaks all emitted at the same time, forming an imaginary surface.

Normally, these wavefronts form a sphere b/c they're emitted by an oscillating point charge. However, far away these form a plane, called a plane wave, which is perpendicular to the direction of propogation.

Reradiation & Superposition

Although radiation puts the same (magnitude of) force on protons & electrons, low-mass electrons contribute most to re-radiation.

We've already discussed how charged particles re-radiate EM radiation. Well, matter (& thus all matter) is made up of charged particles! This reradiation significantly affects the net EM field.

If the reradiation destructively interferes w/ the original, the material is opaque. This opacity depends on the incoming radiation's wavelength.

If the reradiation does not destructively interfere w/ the original, the material is transparent & may bend/refract the light. Normally, this creates a resultant wave moving slower than c but w/ the same f.

Refraction & Reflection

* The index of refraction differs for different wavelengths!

We can measure the intensity of refraction using an index of refraction (n)

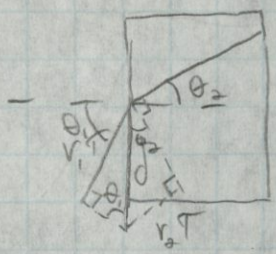
$$n = \frac{c}{v}$$ ← This should make sense. The more you slow down ($v\downarrow$), the more you bend when entering at an angle ($n\uparrow$).

The molecules of new medium oscillate in such a way they produce light in the opposite direction. This is called reflection.

Snell's Law

Snell's Law gives you a quantitative way to measure how things refract. W

Let the wavefront speed in air be v_1 & in glass be v_2. We measure angles based on the "normals" to the surface (i.e. lines $\perp$ to the surface



* They share a common hypotenuse d.

$$\sin(\theta_1) = \frac{v_1 T}{d} \quad \& \quad \sin(\theta_2) = \frac{v_2 T}{d}$$

We can rewrite & combine these into

$$\boxed{\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}}$$

Rewriting these using the index of refraction we get

$$\frac{\sin(\theta_1)}{c/n_1} = \frac{\sin(\theta_2)}{c/n_2}$$

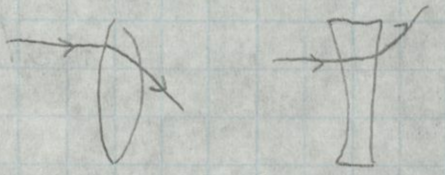
$$\boxed{n_1 \sin(\theta_1) = n_2 \sin(\theta_2)}$$ ← should make some intuitive sense

Total Internal Reflection

For certain angles & indexes of refraction, it is possible to get an invalid equation. When this occurs, no light escapes & you get total internal reflection

This is used for fiberoptics!

Lenses



Converging Lense   Diverging Lense

We'll cover focal length in a bit, but eyeglasses are measured in diopters, which is the inverse of focal length in meters.

This this class, we'll only study thin lenses b/c we can make some simplifying assumptions w/ them.

Small Angle Approximations

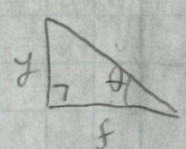
For $\theta \approx 0$ $\tan\theta \approx \theta$, $\sin\theta \approx \theta$, $\cos\theta \approx 1$, $x \approx R \approx r$

Thin Lenses (Small Angles / $\theta \approx 0$)

this is when the light has an infinitely distant source ($d \approx \infty$)

For thin lenses, all rays of light perpendicular to the lense intersect a common point f away from the lense. f is called the focal length & the point the focal point.

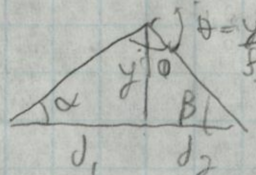
This means we can calculate deflection angle (θ) so



$\tan\theta = \frac{y}{f} \Rightarrow \boxed{\theta \approx \frac{y}{f}}$

For converging lenses we use positive focal length. For diverging we us negative. This should make sense.

We can apply this deflection angle to any rays, not just parallel ones.



$\alpha \approx \tan\alpha = \frac{y}{d_1}$ $\beta \approx \tan\beta = \frac{y}{d_2}$

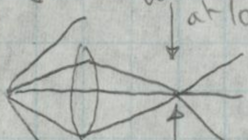
Since $\phi = 180 - (\alpha + \beta)$ & $\phi = 180 - \theta$, $\alpha + \beta = \theta$

We can rewrite this in terms of distances to get

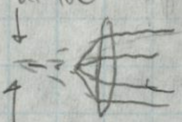
$$\frac{y}{f} = \frac{y}{d_1} + \frac{y}{d_2} \Rightarrow \boxed{\frac{1}{f} = \frac{1}{d_1} + \frac{1}{d_2}}$$

This generalizes to rays not not along axis

Images



Real Images are where observers see a "new" image caused by light seeming to emanate from a new point. Occurs when solved for distance is positive.



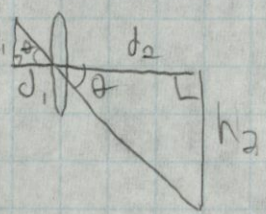
Virtual Images are where observers see light emanating from a point where no light actually emanates from. Occurs when solved for distance is negative.

Abberations occur when light rays don't intersect exactly at the focal point b/c the thin lense approximation is imperfect. This is why you must focus cameras.

Magnification

When you want to magnify something, always recall the lense should be placed where the top can reach each other w/o reflection. this is b/c, if they couldn't reach each other in a straight line, you'd have refraction & a messed up image.

Mirrors' angles of reflection depends only on geometry.



you can use this to find ideal focal length

$$\tan\theta = \frac{h_1}{d_1} \quad \& \quad \tan\theta = \frac{h_2}{d_2} \Rightarrow \frac{h_1}{d_1} = \frac{h_2}{d_2}$$

Since diverging lenses don't focus light to a point, they can never make real images, only virtual.

Sign Conventions

We always consider light travelling right.

Left of lense is negative & right is positive.

Converging lenses have positive focal lengths; diverging negative.