This exam is **no book, no gadgets**. You may use **one sheet of notes** (letter size, two sides), which you submit with the exam. There are 28 questions, worth 58 points. Marks will be curved.

This exam is my own work. I understand it is governed by the ***Emory Honor Code.***

Signature: Solutions

Fill in the Blank: fill each blank appropriately. Each answer could be a word, a phrase, or a formula.

(2 pts) 1. Consider the recurrence $T(n) = aT(n/b) + O(n^d)$, with constants $a > 0$, $b > 1$, and $d \geq 0$. What condition on a, b, d implies $T(n) = \Theta(n^d \log n)$?

1. $d = \log_b a$, $b^d = a$

(2 pts) 2. One of our first examples was the Karatsuba algorithm to multiply two n-bit integers. What recurrence did we use to analyze its running time?

2. $T(n) = 3 \cdot T(n/2) + O(n)$

(2 pts) 3. Recall the randomized Select algorithm. What recurrence did we use to bound $T(n)$, its expected running time on an input of size n?

3. $T(n) = T(\frac{3}{4}n) + O(n)$

(2 pts) 4. Given n complex numbers $r_1, r_2, \ldots, r_n$, how much time did we need to compute the coefficients of the polynomial $(x - r_1)(x - r_2)\cdots(x - r_n)$? (homework, big-Oh)

4. $O(n \lg^2 n)$

(2 pts) 5. In the discrete Fourier transform, we are given ω (with $\omega^n = 1$) and the coefficients $(a_0, a_1, \ldots, a_{n-1})$ of $A(x) = \sum_{i=0}^{n-1} a_i x^i$. We want to output the vector $(y_0, y_1, \ldots, y_{n-1})$, where y_j is what?

5. $A(\omega^j)$

(2 pts) 6. In the FFT circuit, a "butterfly" is a subcircuit computing two outputs from two inputs (it looks like ⊠). How many butterflies are in an FFT circuit with n inputs and n outputs ($n = 2^k$)?

6. $\frac{1}{2} n \lg n$, $\frac{1}{2} nk$

(2 pts) 7. To find the longest common subsequence (LCS) of two sequences of lengths m and n, we first spend $O(mn)$ time computing a matrix of numbers. How much more time is needed to find the actual LCS? (big-Oh)

7. $O(m + n)$

(2 pts) 8. Recall the algorithm for chain matrix multiplication, finding the fastest way to multiply n rectangular matrices. What is the total space (words of memory) used by the algorithm? (big-Oh)

8. $O(n^2)$

(2 pts) 9. In our DP algorithm for the TSP, a subproblem is to compute $C(S, j)$, the minimum length of a path starting at 1, ending at j, and visiting exactly the points of $S \subseteq \{1, 2, \ldots, n\}$. Note we require $\{1, j\} \subseteq S$. For $|S| \geq 3$, give our formula for $C(S, j)$ in terms of smaller subproblems. (Use d_{ij} to denote the distance from i to j.)

9. $\min_{k \in S - \{1, j\}} C(S - \{j\}, k) + d_{kj}$

(2 pts) 10. We have a graph G with n vertices, and a tree decomposition of G with treewidth k and $\Theta(n)$ tree nodes. In homework we saw how to decide whether G has a 3-coloring. How many subproblems are there? (Both types A and B, big-Oh)

10. $O(3^k n)$

(2 pts) 11. Suppose a boolean formula is an AND of clauses, each clause is an OR of literals, and each literal is either a positive variable or a negated variable. What additional property makes it a Horn formula?

11. at most one positive per clause

(2 pts) 12. Baker's algorithm reduces the $(1-\varepsilon)$-approximate MIS problem in planar graphs, to the exact MIS problem in bounded treewidth graphs. What was our bound on their treewidth, in terms of ε? (Include leading constant.)

12. $3/\varepsilon$

(2 pts) 13. We saw two different 2-approximation algorithms for minimum vertex cover. What is an advantage of the second algorithm, which uses LP relaxation?

13. allows vertex weights

(2 pts) 14. Recall our 2-approximate solution to the k-cluster (or k-center) problem, in a metric space (X, d). After picking the first $k-1$ centers $\mu_1, \mu_2, \ldots, \mu_{k-1}$ in X, how do we pick the last center μ_k?

14. Furthest, maximize $\min_{i<k} d(\mu_i, \mu_k)$

(2 pts) 15. In lecture we defined APX, a class of NP optimization problems. Name a problem that is not in APX (as far as we know), unless P=NP.

15. SC, MIS, general TSP

(2 pts) 16. We saw an FPTAS for what problem?

16. Knapsack

(2 pts) 17. Suppose Ford-Fulkerson has finished. We have a max-flow f, and we cannot find an augmenting path in the residual network G_f. Using G_f, how can we define a min-cut set $S \subseteq V$? (You want $s \in S$ and $t \notin S$.)

17. $S = \{v : s \text{ can reach } v \text{ in } G_f\}$

(2 pts) 18. Given a flow f in network $G = (V, E)$, we can decompose f into a sum of path flows, plus maybe a circulation (a flow of value 0). At most how many path flows do we need? (homework)

18. $|E|$

(2 pts) 19. We have dual LP's MAX=$\max\{c \cdot x : \ Ax \leq b, x \geq 0\}$ and MIN=$\min\{b \cdot y : \ A^t y \geq c, y \geq 0\}$. Suppose MAX$= +\infty$. What does this imply about feasible solutions y of the second LP?

19. none, it is infeasible

(2 pts) 20. We have dual LP's MAX=$\max\{c \cdot x: \ Ax \leq b, x \geq 0\}$ and MIN=$\min\{b \cdot y: \ A^t y \geq c, y \geq 0\}$. We have feasible vectors x^* and y^* with $c \cdot x^* = b \cdot y^*$. What condition on x^* implies y_1^* (the first component of y^*) is 0?

20. $(Ax)_1 < b_1$

(2 pts) 21. We have dual LP's MAX=$\max\{c \cdot x: \ Ax \leq b, x \geq 0\}$ and MIN=$\min\{b \cdot y: \ A^t y \geq c, y \geq 0\}$. We modify the MAX LP, making the first constraint an equality constraint: $(Ax)_1 = b_1$. How should we modify the MIN LP, to maintain duality?

21. make y_1 free, remove "$y_1 \geq 0$"

(2 pts) 22. What would be the surprising consequence if linear programming is in NC? (NC is "Nick's class": problems solvable by boolean circuits of poly(n) size and poly($\log(n)$) depth.)

22. P = NC, CVP $\in$ NC

"Fast parallel" algos for all of P

(2 pts) 23. What algorithm (from numerical linear algebra) did we use as a "separation oracle" for the convex domain in the MAXCUT' problem?

23. Cholesky

(2 pts) 24. Suppose $\vec{v}_1$ and $\vec{v}_2$ are two unit vectors in our MAXCUT' solution, with angle α between them (so $0 \leq \alpha \leq \pi$). What is the probability that edge $\{1, 2\}$ is cut, when we randomly round to a MAXCUT solution?

24. α/π

(2 pts) 25. To do branch-and-bound for a minimization problem, the book required three routines: "choose" (picking the next subproblem from a list), "expand" (replace a subproblem by smaller subproblems), and what?

25. lowerbound

(2 pts) 26. Suppose we want to find s minimizing cost(s), by simulated annealing at temperature T. In one iteration, we first pick a random a neighbor s' of our current s. What is the probability that we replace s with s'?

26. $\min\left(1, e^{-(\text{cost}(s')-\text{cost}(s))/T}\right)$

Short Answer (Please indicate if you write on the back.)

(3 pts) 27. State the planar separator theorem, also defining what is a separator.

If G is planar, we can partition V into $A \cup S \cup B$, so $|S| = O(\sqrt{n})$, $|A|, |B| \leq \frac{2}{3}n$, and there is no edge between A and B.

Such an S (of any size) is called a separator.

<u>Optional</u>: weights, time.

(3 pts) 28. Consider a 2-player game with payoff matrix $G = \left(\begin{smallmatrix} 6 & 1 \\ 2 & 4 \end{smallmatrix}\right)$. If you pick row i (1 or 2), and your opponent picks column j (1 or 2), then you pay G_{ij} dollars to your opponent. For example, $G_{21} = 2$.

Suppose you must first announce a mixed strategy (p_1, p_2), and then your opponent gets to pick column j to maximize the expected payoff P. Write down an LP to find p_1 and p_2 minimizing P. (You do not need to solve the LP, just state it.)

minimize P
such that

$$p_1 + p_2 = 1$$
$$P \geq 6p_1 + 2p_2$$
$$P \geq 1p_1 + 4p_2$$
$$p_1, p_2 \geq 0$$