CS-526-1 MIDTERM EXAM October 21, 2021

This exam is **no book, no gadgets**. You may use **one sheet of notes** (letter size, two sides), which you submit with the exam. There are 30 questions, worth 62 points. Marks will be curved.

This exam is my own work. I understand it is governed by the ***Emory Honor Code.***

Signature: Solutions (ignoring partial credit cases)

Fill in the Blank: fill each blank appropriately. An answer could be a word, a phrase, or a formula.

Recurrences, Divide and Conquer

(2 pts) 1. Solve the recurrence $T(n) = 7\,T(n/2) + n^3$. (big-Oh)

1. $O(n^3)$

(2 pts) 2. Consider the recurrence $T(n) = aT(n/b) + O(n^d)$, with constants $a > 0$, $b > 1$, and $d \geq 0$. What condition on a, b, d implies $T(n) = \Theta(n^d \log n)$?

2. $\log_b a = d$, $b^d = a$

(2 pts) 3. Recall QuickSelect. It resembles QuickSort, but we only want to output one value (the kth). What *recurrence* did we find bounding $T(n)$, the expected running time of QuickSelect on an input of size n?

3. $T(n) = T(\frac{3}{4}n) + O(n)$

(2 pts) 4. What algorithm did we associate with the recurrence $T(n) = 3\,T(n/2) + O(n)$?

4. integer mult, Karatsuba

(2 pts) 5. Given n numbers $r_1, r_2, \ldots, r_n$, how much time did we need to compute the polynomial $P(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$? (homework, big-Oh)

5. $O(n \lg^2 n)$

(2 pts) 6. In the discrete Fourier transform, we are given a number ω (with $\omega^n = 1$) and a polynomial $A(x) = a_0 + a_1x^1 + a_2x^2 + \cdots a_{n-1}x^{n-1}$. We want to compute the vector $\vec{y} = (y_0, y_1, \ldots y_{n-1})$, where y_j equals what?

6. $A(\omega^j)$

(2 pts) 7. In the FFT circuit, a single "butterfly" computes two values from two other values (it looks like). How many butterflies are in an FFT circuit with n inputs and n outputs ($n = 2^k$)?

7. $nk/2$, $\frac{1}{2} n \lg n$

Dynamic Programming (DP)

(2 pts) 8. To compute the LCS (longest common subsequence) of two sequences of lengths m and n, we first spend $O(mn)$ time computing a matrix of numbers. How much additional time do we need for backtracking, to find the actual sequence? (big-Oh)

8. $O(m+n)$

(2 pts) 9. Recall the algorithm for the chain matrix multiplication problem, finding the fastest way to multiply n matrices. What is the total space (words of memory) used by the algorithm? (big-Oh)

9. $O(n^2)$

(2 pts) 10. In our DP algorithm for the TSP, a subproblem is to compute $C(S, j)$, the cost of a shortest tour starting at 1, ending at j, and visiting the points of $S \subseteq \{1, 2, \ldots, n\}$ (so $\{1, j\} \subseteq S$). d_{kj} is the distance from k to j. For $|S| \geq 3$, give the formula for $C(S, j)$ in terms of smaller subproblems.

10. $\min_{k \in S - \{1,j\}} \left(C(S - \{j\}, k) + d_{kj} \right)$

(2 pts) 11. We are given graph G with n vertices, and a tree decomposition of G with treewidth k, and N tree nodes. We want to decide whether G has a 3-coloring. In our dynamic programming approach, how many subproblems are there? (big-Oh, homework)

11. $O(N \cdot 3^k)$, $O(n \cdot 3^k)$

Greed and Approximation

(2 pts) 12. Suppose a boolean formula is an AND of clauses, each clause is an OR of literals, and each literal is either a positive variable or a negated variable. What additional property would make it a Horn formula?

12. at most one positive per clause

(2 pts) 13. In the Set Cover problem, suppose there are n points to cover, and an optimal solution uses k sets. After t iterations of the greedy algorithm, at most how many points are still not covered?

13. $n(1 - \frac{1}{k})^t$

(2 pts) 14. "PTAS" is an acronym for what?

14. polynomial time approx. scheme

(2 pts) 15. Baker's algorithm reduces the $(1 - \varepsilon)$-approximate MIS problem in planar graphs, to the exact MIS problem in bounded treewidth graphs. What was the treewidth bound in terms of ε? (Include leading constant.)

15. $3/\varepsilon \pm O(1)$

(2 pts) 16. We saw two different 2-approximation algorithms for VC (minimum vertex cover). What is an advantage of the algorithm using LP relaxation?

16. allows weights

(2 pts) 17. What is a minimization problem, other than vertex cover, for which we saw a 2-approximation algorithm?

17. k-center, TSP

MaxFlow: Here G is a flow network with $n = |V|$ vertices, $m = |E|$ edges, and each edge e has capacity $c(e)$.

(2 pts) 18. Suppose we run Ford-Fulkerson, always picking a shortest augmenting path from s to t. What was our bound on the number of augmenting paths needed? (big-Oh, slides)

18. $O(nm)$, $O(VE)$

(2 pts) 19. Suppose Ford-Fulkerson has stopped. We have a flow f, and we cannot find a path from s to t in the residual network G_f. How can we find a min-cut S? (It should have $s \in S$ and $t \in V - S$.)

19. $\{v : s \rightsquigarrow v \text{ in } G_f\}$

(2 pts) 20. Suppose f is a max-flow and $(S, V - S)$ is a min-cut ($s \in S$, $t \in V - S$). What can we say about $f(e)$, the flow on an edge e from S to $V - S$?

20. $f(e) = c(e)$

(2 pts) 21. Given a flow f in G, we can decompose f into at most how many path flows? (homework)

21. m, $|E|$

Duality: Suppose we have dual LP's: MAX=$\max\{c \cdot x : Ax \le b, x \ge 0\}$ and MIN=$\min\{b \cdot y : A^t y \ge c, y \ge 0\}$.

(2 pts) 22. Suppose x and y are feasible points for the two LP's (not necessarily optimal points). What relation do we know between $c \cdot x$ and $b \cdot y$?

22. $c \cdot x \le b \cdot y$

(2 pts) 23. Suppose feasible vectors x^* and y^* achieve MAX=MIN. What condition on the constraints of x^* would imply that y_1^* (the first value in y^*) must be zero?

23. first not tight, $(Ax^*)_1 < b_1$

(2 pts) 24. Suppose we modify the MAX LP, allowing its first variable x_1 to be any real value (possibly negative). How should we modify the MIN LP, to maintain duality?

24. First becomes equality, $(A^t y^*)_1 = c_1$

Other Stuff

(2 pts) 25. We saw a simple reduction from the boolean circuit value problem (CVP) to linear programming (LP). What did this imply about LP?

25. P-complete, probably not in NC

(2 pts) 26. What algorithm (a named algorithm from linear algebra) did we use as a "separation oracle" for the convex domain of the MAXCUT' problem?

26. Cholesky (factorization or decomposition)

(2 pts) 27. Suppose $\vec{v}_1$ and $\vec{v}_2$ are two unit vectors in our MAXCUT' solution, with angle α between them (so $0 \le \alpha \le \pi$). What is the probability that edge $\{1, 2\}$ is cut, when we randomly round to a MAXCUT solution?

27. α/π

(2 pts) 28. To do branch-and-bound for a minimization problem, the book required three routines: "chooose" (picking the next subproblem from a list), "expand" (replacing a subproblem by smaller subproblems), and what?

28. lowerbound

(2 pts) 29. We want to find s minimizing cost(s). We try simulated annealing, at temperature T. In one iteration, we first pick a random a neighbor s' of our current s. What is the probability that we then replace s with s'?

29. $\min(1, e^{-\underbrace{(\text{cost}(s') - \text{cost}(s))}_{\text{or } \Delta}/t})$

Short Answer (you may use the back!)

(4 pts) 30. State the "minimax theorem".

In a two-player matrix game with mixed strategies, it does not matter which player announces strategy first.

(P.C. for other "min/max" theorems.)