BOUNDARY STRATA IN MODULI SPACES OF GENUS-ONE CURVES

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Abstract. The moduli space of genus-zero curves has boundary strata that record the possible ways in which marked points can be distributed across the components of a curve. Known results [1] describe the combinatorics of the number of these boundary strata. We generalize this result to the moduli space of genus-one curves using sign-reversing involutions.

1. Automorphisms of Projective Space

In general, one may consider n-dimensional projective space $\mathbb{P}^n$. Here, we only consider the case where $n = 1$. We begin this section by defining the alternative definitions of $\mathbb{P}^1$, and understanding their relations via automorphisms.

Definition 1.1. Let the 1-dimensional projective space be denoted as $\mathbb{P}^1$. Then following definitions are equivalent,

(1) $\frac{\mathbb{C}^2\setminus\{(0,0)\}}{\sim}$ where $(z_1, z_2) \sim (w_1, w_2)$ if there exists $\lambda \in \mathbb{C}$ such that $(z_1, z_2) = \lambda(w_1, w_2)$, and we denote their equivalance class $[z_1 : z_2]$.
(2) The unit sphere in $\mathbb{R}^3$, defined as $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$
(3) The complex numbers union infinity, $\mathbb{C} \cup \{\infty\}$

Moving forward, unless otherwise specified, if we write $\mathbb{P}^1$ we are referring to perspective (1) (as this is the definition that generalizes to higher dimension). In $\mathbb{P}^1$, note that $[z_1 : z_2] = [\frac{z_1}{z_2} : 1]$ provided that $z_2 \neq 0$. We briefly cover the isomorphism between $\mathbb{P}^1$ and $\mathbb{C} \cup \{\infty\}$ as it will be helpful later. Define a function $f : \mathbb{P}^1 \to \mathbb{C} \cup \{\infty\}$ piecewise,

$$f([z_1 : z_2]) = \begin{cases} \frac{z_1}{z_2} \text{ if } z_2 \neq 0 \\ \infty \text{ if } z_2 = 0 \end{cases}.$$

Its inverse $f^{-1} : \mathbb{C} \cup \{\infty\} \to \mathbb{P}^1$ is given by

$$f^{-1}(z) = \begin{cases} [z : 1] \text{ if } z \neq \infty \\ [1 : 0] \text{ if } z = \infty \end{cases}.$$

As stated, we will not carefully prove this is an isomorphism, but the maps f, f^{-1} will be useful later for going between the two spaces.

We will now consider automorphisms of $\mathbb{P}^1$ to better understand the space and its varying perspectives. We begin with a definition.

Definition 1.2. A *morphism* of is a is a function $F : \mathbb{P}^1 \to \mathbb{P}^1$ defined by

$$F([a : b]) = [(f(a, b) : g(a, b)],$$

where $f, g \in K[x, y]$ and homogeneous polynomials of same degree. An *automorphism* is a morphism F whose inverse is also a morphism.

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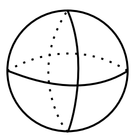

Figure 1. Perspective two, S^2

A polynomial f is homogeneous with degree d if every (non-zero) term has degree d. For example, $f = x_1x_2^2 + x_1^3$ is homogeneous of degree 3, while $g = x_1 + x_2x_1$ is not homogeneous. Consider the following example of automorphisms of $\mathbb{P}^1$.

Example 1.3. Consider the function $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ defined by

$$f(z) = \begin{cases} \frac{z}{2} & \text{if } z \neq \infty \\ \infty & \text{if } z = \infty \end{cases}$$

Viewing this from the equivalence class point of view we have

$$f([z_1 : z_2]) = \begin{cases} f(\frac{z_1}{z_2}) = [\frac{z_1}{2z_2} : 1] & \text{if } z_2 \neq 0 \\ f(\infty) = [1 : 0] & \text{if } z_2 = 0 \end{cases}$$

As one might hope, the isomorphism briefly detailed earlier allows us to easily interpret functions defined on $\mathbb{C} \cup \{\infty\}$ instead on $\mathbb{P}^1$. We assert without proof that if F is an automorphism, then f, g are linear polynomials. Intuitively, polynomials of even degree are not injective, and over $\mathbb{C}$ we may factor any polynomials of deg > 1 into at least one term of even degree. With this, we have that automorphisms of $\mathbb{P}^1$ are of the form $F([z : 1]) = [az + b : cz + d]$. From the $\mathbb{C} \cup \{\infty\}$ perspective this means automorphisms are of the form $f(z) = \frac{az+b}{cz+d}$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$$

is an invertible matrix. This follows from the definition the fact that

$$\mathrm{Aut}(\mathbb{P}^1) = \frac{GL_2(\mathbb{C})}{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}.$$

The following proof gives us a strong tool to better understand our forthcoming discussion of moduli spaces. For this proof, we consider $\mathbb{P}^1$ from the $\mathbb{C} \cup \{\infty\}$ perspective.

Theorem 1.4. *Let $p_0, p_1, p_2 \in \mathbb{P}^1$ be distinct. Then there exists a unique $f \in Aut(\mathbb{P}^1)$ such that $f(p_0) = 0$, $f(p_1) = 1$ and $f(p_2) = \infty$.*

Proof. If none of p_0, p_1, p_2 equal infinity, then

$$f(z) = \frac{(z - p_0)(p_1 - p_2)}{(z - p_2)(p_1 - p_0)}$$

works as one may verify. To show uniqueness, suppose there was another such function, g. Let $h = g \circ f^{-1}$, then h is an automorphism with $h(0) = g(f^{-1}(0)) = g(p_0) = 0$, and similarly, $h(1) = 1$ and $h(\infty) = \infty$. Since h is an automorphism, it is of the form $\frac{az+b}{cz+d}$. Thus,

$$h(0) = \frac{a(0) + b}{c(0) + d} = \frac{b}{d} = 0$$

implies $b = 0$. Then this together with $h(1) = 1$ implies that $a = c + d$. Similarly, $h(\infty) = \lim_{z\to\infty} h(z) = \infty$ implies $c = 0$, that is, $h(z) = z$ and hence the identity. In other words $g \circ f^{-1} = \mathrm{id}$, which implies $g = f$. If p_0 is infinity, then the function

$$f(z) = \frac{z - p_1}{p_2 - p_1}$$

suffices. Similarly, $f(z) = \frac{z-p_2}{p_0-p_2}$ and $f(z) = \frac{z-p_0}{p_1-p_0}$ suffice for when p_1 and p_2 equal infinity respectively. The argument of uniqueness of these functions is the same. $\square$

2. Moduli Space of Smooth n-Pointed Curves

Definition 2.1. We define moduli space of smooth genus-zero n-pointed curves, denoted $\mathcal{M}_{0,n}$ by

$$\mathcal{M}_{0,n} = \frac{\{(p_1, \ldots, p_n) \mid p_i \in \mathbb{P}^1 \text{ distinct } \}}{\sim},$$

where $(p_1, \ldots, p_n) \sim (q_1, \ldots q_n)$ if there exists an $f \in \mathrm{Aut}(\mathbb{P}^1)$ such that $f(p_i) = q_i$ for all $i \in [n]$.

If $n = 3$, then $\mathcal{M}_{0,3} = \{(p_1, p_2, p_3) \in \mathbb{P}^1\}/\sim$. Recall, by theorem 1.4 there exists a unique isomorphism f such that $f(p_1) = 0, f(p_2) = 1$ and $f(p_3) = \infty$ for all $(p_1, p_2, p_3) \in \mathbb{P}^1$. In other words, $\mathcal{M}_{0,3} = \{[0 : 1 : \infty]\}$ and hence not very interesting. Similarly, in $\mathcal{M}_{0,4}$, this automorphisms sends its first three coordinates to $0, 1, \infty$ respectively. In other words we have $\mathcal{M}_{0,4} = \{[0 : 1 : \infty : p_4] \mid p_4 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}\}$. In fact, $\mathcal{M}_{0,4}$ is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Recall the distinctness condition on our marked points. This is why we exclude $\{0, 1, \infty\}$. The requirement that p_i be distinct is necessary; Let $C_t = [0, 1, \infty, t]$ and $C'_t = [0, \frac{1}{t}, \infty, 1]$, then $C_t = \frac{1}{t} C'_t$ so $C_t \sim C'_t$.

Returning to our second example in the paragraph above, $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and thus the space is not compact. The following definition continues our discussion and fixes this problem.

Definition 2.2. We define the compactification of $M_{0,n}$, denoted $\overline{\mathcal{M}}_{0,n}$, by

$$\overline{\mathcal{M}}_{0,n} = \frac{\left\{ (C; p_1, \ldots p_n) \mid \begin{array}{c} p_i \in \mathbb{P}^1 \text{ distinct} \\ C \text{ a stable, genus-zero curve} \end{array} \right\}}{\sim}$$

where $\sim$ is defined as it was in definition 2.1.

A stable curve C in $\overline{\mathcal{M}}_{0,n}$ is a genus-zero curve such that every component has three "special points" on it. A special point can either be a node (where two components of a curve meet) or a marked point. See figure 2 below.

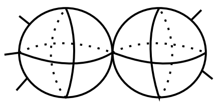

Figure 2. A stable curve $C \in \overline{\mathcal{M}}_{0,5}$

Example 2.3. The process of finding all elements of $\overline{\mathcal{M}}_{0,4}$ can be done very systematically. First, begin with a curve with a single component containing 4 marked points. This corresponds to our original non-compact set. We then can consider all stable degenerations of our curve. A stable degeneration arises when two marked points coincide, and thus a new component splits off. This degeneration is stable if the resulting curve is stable. Then, you can continue this process until there are no stable degenerations left of a given curve. The picture above may not

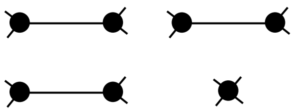

Figure 3. Elements of $\overline{\mathcal{M}}_{0,4}$

illustrative to the fact that there are 2 additional curves; Both appearing to be the same as the one on the right. If we call the first component C_1, and the second component C_2, then curves where p_1, p_2 are on C_1 are different from those where p_1, p_3 are on C_1 for instance.

3. Dual Graphs of Curves and $\mathcal{X}_{k,n}(t)$

We may identify elements of $\overline{\mathcal{M}}_{0,n}$ with their respective dual graphs. Given a curve C construct the dual graph G by adding a vertex for each component, legs (half-edges) for each marked point, and an edge for each node. Note that this notion of a dual graph is different from that of planar graphs. Consider figure three (and the two curves not depicted there), their dual graphs would look like figure 4 below. In the language of graphs, elements of $\overline{M}_{0,4}$ are

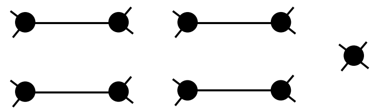

Figure 4. Dual graphs of elements of $\overline{\mathcal{M}}_{0,4}$

connected, acyclic graphs such that each vertex has at least 3 of either marked points or edges. More concisely, a tree with $\mathrm{val}(v) \geq 3$ for all $v \in V$. Dual graphs are elements of what we call a boundary stratum. If G is a dual graph, then S_G is the set of elements who have G as a dual graph. This language is not used too much in our work but good to know.

Definition 3.1. Let $G \in \overline{\mathcal{M}}_{k,n}$. Then define

$$\mathcal{X}_{k,n}(t) = \sum_{G} t^{e(G)}$$

This function nicely encodes information about the types of graphs (and hence curves) in our space. Below we calculate the function for $2 \leq n \leq 5$ in genus 0.

n	$\chi(-1)$	$\chi(t)$
3	1!	t^0
4	$-2!$	$t^0 + 3t$
5	3!	$t^0 + 10t^1 + 15t^2$

Table 1. Calculating $\mathcal{X}_{0,n}$

One may be able to glean that we are interested in the value of $\chi(t)$ when $t = -1$. Indeed we are. Moreover, our odd way of writing 1 and 1!, and -2 and $-2!$ is for a specific reason; As proved by [1] using a sign-reversing involution, $\chi(-1) = \pm(n-2)!$ for $n \geq 3$. Before going further, we should interpret what plugging in $t = -1$ does to our function. In $\overline{\mathcal{M}}_{k,n}$, a given term $a_j t^j$ means we have a_j graphs with j edges. So, plugging in -1 means j and $j+1$ terms would have differing parity (for $0 \leq j \leq \deg(\chi(t))$. Perhaps we could define a sign function in terms of this, and, from there, devise a sign-reversing involution. This is (roughly) how the aforementioned theorem was proved. Naturally the question of generalization arises. In the following chapters we transition our focus to the moduli space of genus-one curves, and prove a fairly similar result.

4. Moduli Space of Genus-One Curves

Much of our discussion translates very easily to genus-one curves. We have not mentioned this until now, but the genus of a curve roughly refers to the number of holes present on the curve. Thus, a genus-one curve has one hole, and unlike $\overline{\mathcal{M}}_{0,n}$, where all the objects are spheres, there are three irreducible components, as seen in figure 5. The torus is a new geometric shape, while the glued banana (middle of figure 5) and the self-loop (right of figure 5) are just $\mathbb{P}^1$'s glued in some particular way; This becomes apparent when considering their dual-graphs.

Prior to this considering their duals we must consider an additional stability condition on genus-one curves. Firstly, every curve must have a hole. That is, it must either include a loop of $\mathbb{P}^1$'s (genus-zero components) or (exactly) one of the objects in figure 5. Moreover, while

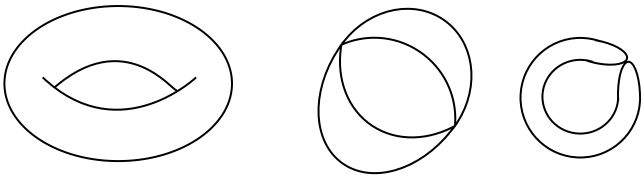

FIGURE 5. The three types genus-one components

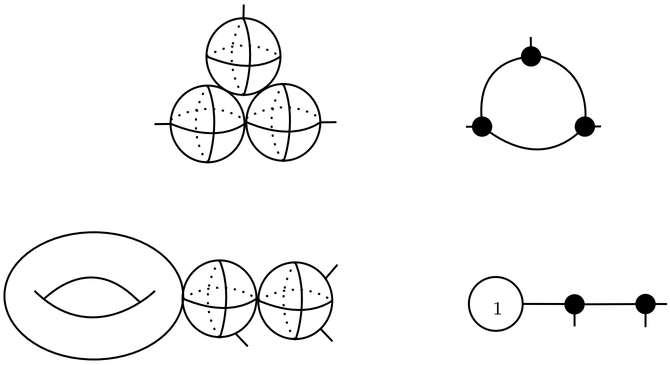

FIGURE 6. Two elements of $\overline{\mathcal{M}}_{1,3}$ and their dual graphs

the stability condition on the $\mathbb{P}^1$'s remain the same, the torus only needs to have one-edge or marked point attached to it. With this we can formally define elements of $\overline{\mathcal{M}}_{1,n}$ using graph language.

Definition 4.1. A stable genus-one n-pointed curve is a curve satisfying the aforementioned stability conditions and whose dual-graph has either a genus-one component or one cycle.

5. The Sign-Reversing Involution

We have now developed what is needed for our goal of proving theorem 6.2 using sign-reversing involutions. Our method is similar to that of [1]. In this section we use the terms "push out" and "contraction" without definition. For an intuitive idea, see the figures in this section and for a more a formal definition of these terms (and assurance that these operations maintain stability) see section 8.

Let $f : S \to S$ be our involution defined piecewise, on graphs with a genus-one component and those without one.

Let G be a graph with a genus-one component v. If $\mathrm{val}(v) = 1$, then v belongs to an edge; Contract this edge. If $\mathrm{val}(v) > 1$, push out the edge and leave a vertex in its place. These operations are inverse of each other and hence f is an involution on such graphs.

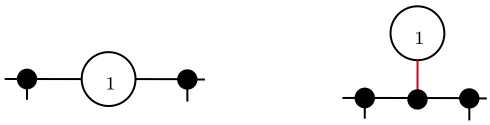

FIGURE 7. Involution on graph with genus-one component, contracted/pushed out edge in red

Now suppose G is a graph without a genus-one component. We say a vertex v is a "single-leg vertex" if $\mathrm{val}(v) = 3$ and it only has one marked point. By stability conditions, there exists exactly one loop (cycle) on G. If every vertex along the loop is a single leg vertex we fix G. Otherwise, ignoring all single-leg vertices along this loop, choose the vertex v on the loop containing a vertex in $\mathrm{span}(v, E)$ of smallest index, say p_i, where E is the set of non-loop edges.

If $\mathrm{val}(v) > 3$, push out an edge. If $\mathrm{val}(v) = 3$, then it must contain an edge not belonging to the cycle (since we are disregarding on single leg vertices), so contract that edge. If $\mathrm{val}(v) = 3$, then after contracting an edge, $\mathrm{val}(v) > 3$ so we would push out an edge. We are guaranteed to choose the same v by our marked point of smallest index condition. Thus the function is an involution on such graphs.

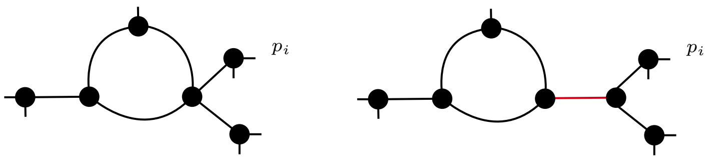

FIGURE 8. Involution on graph, contracted/pushed out edge in red

Our sign-function $\sigma : S \to \{\pm 1\}$ is defined by $\sigma(G) = (-1)^{e(G)}$. Our involution adds or subtracts an edge on every graph not belonging to the fixed set so, $\sigma(f(G)) = -\sigma(G)$ and thus it is a sign-reversing involution. Finally, our function f fixes all graphs with no genus-one component such that every vertex along the loop is single-leg, but this can only happen if our graph has n edges, with one marked point per vertex. An example of such a graph in $\overline{\mathcal{M}}_{1,3}$ is given in figure 6.

6. RESULTS

Theorem 6.1. *Let S' be the fixed set of f. Then, for $n \geq 2$ we have $\#S' = \frac{(n-1)!}{2}$.*

Proof. Let $n \geq 2$ and let $G \in S'$. Every graph G has n vertices with one marked point on each. Fix a marked point p_1 on any vertex, say v_1. Then there are $(n-1)!$ remaining distributions of marked points along the vertices. However, reflecting about v_1 pairs graphs via isomorphism. Hence, there are $\frac{(n-1)!}{2}$ unique graphs. □

Theorem 6.2. *Let $n \geq 3$. Then we have*

$$\mathcal{X}_{1,n}(-1) = (-1)^n \frac{(n-1)!}{2}.$$

Proof. Recall, the k^{th} coefficient of $\mathcal{X}_{1,n}(t)$ counts the number graphs with k edges. So, when $t = -1$, the sign of each term is given by $(-1)^k$. Our sign-reversing involution on S ensures we may pair (non-fixed) graphs whose degree varies by ± 1 with each other. So $\mathcal{X}_{1,n}(-1)$ only counts graphs belonging to our fixed set. These are exclusively the graphs with n edges, and so sign of this term is $(-1)^n$. Then, applying theorem 6.1 gives the desired count. □

7. DEFINITIONS AND REMARKS

Some of the definitions listed here are found in many places, and restated here in language best suited for this paper. Others, such as span, are constructed.

Definition 7.1. Edge Contraction: Let $G = (V, E)$ and suppose v and w are distinct, incident vertices. Then *contracting* is an operation that removes e from G and merges v and w.

By merging v and w we essentially create a new vertex u, who belongs to all the same edge sets as both v and w. Moreover, in our case, u also has all of the marked points on v and w. Thus, if v and w are stable vertices, then u a stable vertex.

Definition 7.2. Pushing out an Edge: Let $G = (V, E)$ and suppose $v \in V$, possibly with marked points $p_1, \ldots, p_k$. Then pushing out an edge from v is the operation that replaces v with a new genus-zero vertex v' and adds a new edge $e = \{v, v'\}$. Note, this construction gives v' all the original information of v, marked points and edges.

A nice visual intuition for this definition is seen in figures 7 and 8. One may notice however that the two operations seem different. Indeed, in figure 8 we do not include the loop edges in this operation. Formally, one could instead define the operation in terms of the span (see definition below) of v and its non-loop edges. With this condition, pushing out always maintains stability.

We may also think of pushing out (as we use it) as an operation on v enforcing that v be as minimally stable as possible. That is, we are taking some vertex v, and if it is genus-one, we change its valence to 1, the minimum amount for it to be stable. Similarly, we take some genus-zero v and give it valence 3.

Definition 7.3. Let $G = (V, E)$ be a graph and let $v \in e$. We say the *span* of v with respect to e, denoted $\mathrm{span}(v; e)$ is the subgraph $H = (V', E')$ such that if we take e to be an outward edge from v, then

(1) $w \in V' \iff$ there exists a path $p : v \to w$

(2) $xy = e \in E' \iff$ there exists a path $p : v \to x \to y$ or $p : v \to y \to x$

where *span*$(v; e_1, e_2 ... e_k)$ is defined similarly.

In simpler terms, we want to just isolate a certain part of the graph.

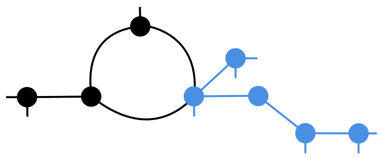

FIGURE 9. Span of v ($\mathrm{span}(v; e_1, e_2)$ where v is the blue-vertex along the loop and e_1, e_2 are the non-loop edges it belongs to.

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