If $\text{val}(v) > 3$, push out an edge. If $\text{val}(v) = 3$, then it must contain an edge not belonging to the cycle (since we are disregarding on single leg vertices), so contract that edge. If $\text{val}(v) = 3$, then after contracting an edge, $\text{val}(v) > 3$ so we would push out an edge. We are guaranteed to choose the same v by our marked point of smallest index condition. Thus the function is an involution on such graphs.

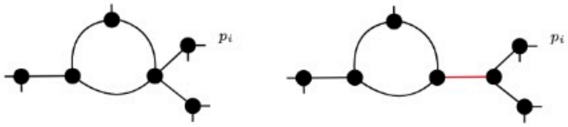

FIGURE 8. Involution on graph, contracted/pushed out edge in red

Our sign-function $\sigma : S \to \{\pm 1\}$ is defined by $\sigma(G) = (-1)^{e(G)}$. Our involution adds or subtracts an edge on every graph not belonging to the fixed set so, $\sigma(f(G)) = -\sigma(G)$ and thus it is a sign-reversing involution. Finally, our function f fixes all graphs with no genus-one component such that every vertex along the loop is single-leg, but this can only happen if our graph has n edges, with one marked point per vertex. An example of such a graph in $\overline{\mathcal{M}}_{1,3}$ is given in figure 6.

6. Results

Theorem 6.1. *Let S' be the fixed set of f. Then, for $n \geq 2$ we have $\#S' = \frac{(n-1)!}{2}$.*

Proof. Let $n \geq 2$ and let $G \in S'$. Every graph G has n vertices with one marked point on each. Fix a marked point p_1 on any vertex, say v_1. Then there are $(n-1)!$ remaining distributions of marked points along the vertices. However, reflecting about v_1 pairs graphs via isomorphism. Hence, there are $\frac{(n-1)!}{2}$ unique graphs. □

Theorem 6.2. *Let $n \geq 3$. Then we have*

$$\mathcal{X}_{1,n}(-1) = (-1)^n \frac{(n-1)!}{2}.$$

Proof. Recall, the k^{th} coefficient of $\mathcal{X}_{1,n}(t)$ counts the number graphs with k edges. So, when $t = -1$, the sign of each term is given by $(-1)^k$. Our sign-reversing involution on S ensures we may pair (non-fixed) graphs whose degree varies by ± 1 with each other. So $\mathcal{X}_{1,n}(-1)$ only counts graphs belonging to our fixed set. These are exclusively the graphs with n edges, and so sign of this term is $(-1)^n$. Then, applying theorem 6.1 gives the desired count. □

7. Definitions and Remarks

Some of the definitions listed here are found in many places, and restated here in language best suited for this paper. Others, such as span, are constructed.

Definition 7.1. Edge Contraction: Let $G = (V, E)$ and suppose v and w are distinct, incident vertices. Then *contracting* is an operation that removes e from G and merges v and w.

By merging v and w we essentially create a new vertex u, who belongs to all the same edge sets as both v and w. Moreover, in our case, u also has all of the marked points on v and w. Thus, if v and w are stable vertices, then u a stable vertex.

Definition 7.2. Pushing out an Edge: Let $G = (V, E)$ and suppose $v \in V$, possibly with marked points $p_1, \ldots, p_k$. Then pushing out an edge from v is the operation that replaces v with a new genus-zero vertex v' and adds a new edge $e = \{v, v'\}$. Note, this construction gives v' all the original information of v, marked points and edges.