A SIGN-REVERSING INVOLUTION ON GENUS-ONE BOUNDARY STRATA

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Abstract. We give an elementary calculation of the Euler characteristic of the boundary complex of the moduli space of genus-one stable curves. Our result agrees with the work of Chan–Galatius–Payne, who proved that this boundary complex is homotopy equivalent to a particular wedge of spheres, but the proof technique in our case is purely combinatorial: we construct a sign-reversing involution on genus-one boundary strata modeled on a similar calculation in genus zero due to Blankers–Gillespie–Levinson.

1. Introduction

Within the Deligne–Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed, genus-g stable curves, the boundary strata are the closures of the loci of curves with a fixed topological type and distribution of marked points across components. These strata play a key role in the geometric and combinatorial study of the moduli space; for example, their cohomology classes are among the generators of the tautological subring $R^*(\overline{\mathcal{M}}_{g,n}) \subseteq H^*(\overline{\mathcal{M}}_{g,n})$, which is a widely-studied and more tractable substitute for the full cohomology ring.

Boundary strata can be specified by the combinatorial information of a dual graph, in which the size of the edge set $E(G)$ corresponds to the codimension in $\overline{\mathcal{M}}_{g,n}$ of the boundary stratum associated to the dual graph G. We consider the quantity

$$\chi_{g,n} := \sum_{\substack{\text{dual graphs} \\ G \text{ for } \overline{\mathcal{M}}_{g,n}}} (-1)^{|E(G)|}. \tag{1}$$

As we explain in Section 2, this is essentially equivalent to the Euler characteristic of the boundary complex $\Delta_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$, a Δ-complex that encodes the containment relations between boundary strata.

In genus zero, the topology of the boundary complex is well-understood: $\Delta_{0,n}$ is homotopy equivalent to a wedge of $(n-2)!$ spheres of dimension $n-4$ by [3], from which the formula

$$\chi_{0,n} = (-1)^{n-3}(n-2)! \tag{2}$$

readily follows. Recent work of Blankers–Gillespie–Levinson gave a short, elementary proof of (2) by constructing a sign-reversing involution on the set of boundary strata for $\overline{\mathcal{M}}_{0,n}$ with precisely $(n-2)!$ fixed points, each of which is an $(n-3)$-edge graph.

A similar story holds in genus one, in which case work of Chan–Galatius–Payne [2] shows that, whenever $n \geq 3$, the boundary complex $\Delta_{1,n}$ is homotopy equivalent to a wedge of $(n-1)!/2$ spheres of dimension $n-1$. This similarly yields a formula for $\chi_{1,n}$, and in this note, we give a purely combinatorial proof of this formula by way of a sign-reversing involution inspired by [1]. The resulting formula is the following:

Theorem 1. *For any $n \geq 3$, one has*

$$\chi_{1,n} = (-1)^n \frac{(n-1)!}{2}.$$

In Section 2, we develop the requisite background on boundary strata and their associated dual graphs, and in Section 3, we describe the sign-resulting involution that proves Theorem 1.

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2. Background

The points of the moduli space $\overline{\mathcal{M}}_{g,n}$ are in bijection with tuples $(C; x_1, \ldots, x_n)$, where C is an algebraic curve over $\mathbb{C}$ with arithmetic genus g and at worst nodal singularities, $x_1, \ldots, x_n \in C$ are distinct smooth points referred to as the **marked points**, and the tuple is **stable** in the sense that it has finitely many automorphisms. Equivalently, the condition of stability requires that every irreducible component of C of arithmetic genus zero has at least three special points (marked points or half-nodes) and every irreducible component of arithmetic genus one has at least one special point.

Each element $(C; x_1, \ldots, x_n)$ of $\overline{\mathcal{M}}_{g,n}$ has an associated **dual graph**, which consists of the following data:

- a vertex v for each irreducible component of C, labeled with the geometric genus $g(v)$ of the irreducible component;
- an edge between (not necessarily distinct) vertices for each node joining the corresponding irreducible components;
- a half-edge (or "leg") for each marked point, labeled by the index in $\{1, 2, \ldots, n\}$ of the marked point.

For instance, the dual graphs for $\overline{\mathcal{M}}_{1,2}$ are shown in Figure 1. We refer to both the halves of the edges and the legs as **half-edges**, and we use the term **valence of** v, denoted $\mathrm{val}(v)$, for the total number of incident half-edges to a vertex v.

For any dual graph G, we denote by S_G the closure of the locus in $\overline{\mathcal{M}}_{g,n}$ of elements with dual graph G. Such loci are referred to as **boundary strata**, and it is a standard fact that

$$\mathrm{codim}_{\overline{\mathcal{M}}_{g,n}}(S_G) = |E(G)|,$$

where $E(G)$ denotes the set of edges of G. In particular, the maximum number of edges of a dual graph G for $\overline{\mathcal{M}}_{g,n}$ is equal to $\dim(\overline{\mathcal{M}}_{g,n}) = 3g - 3 + n$.

Inclusions among boundary strata can be described in terms of edge-contractions on the associated dual graph. To do so, for a dual graph G and an edge e between distinct vertices v and w, define the **contraction** of G along e to be the dual graph obtained by replacing v and w by a single vertex of genus $g(v) + g(w)$ containing all of the incident half-edges from both v and w. Similarly, if e is a self-edge at a vertex v, define the contraction of G along e to be the dual graph obtained by removing e and increasing $g(v)$ to $g(v) + 1$. Geometrically,

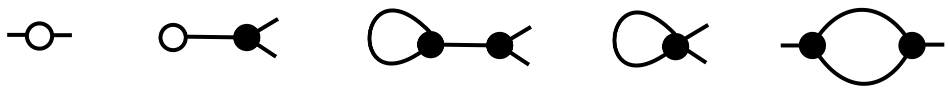


FIGURE 1. The dual graphs for $\overline{\mathcal{M}}_{1,2}$. We omit the labels on legs, and we indicate genus-zero vertices by a filled circle and genus-one vertices by an open circle.

these operations correspond to smoothing nodes, and $S_G \subseteq S_H$ if and only if H can be obtained from G by contracting a subset of edges.

This idea allows one to combinatorially encode the inclusions of boundary strata into a Δ-complex $\Delta_{g,n}$ known as the **boundary complex**. The precise definition of $\Delta_{g,n}$ is not relevant for this work (see, for example, [2, Section 3]), but roughly, it can be viewed as the result of taking a k-simplex for each codimension-$(k+1)$ boundary stratum and gluing the boundaries of these simplices according to edge contraction. Alternatively, $\Delta_{g,n}$ can be defined as the subspace of the moduli space $M_{g,n}^{\mathrm{trop}}$ parameterizing tropical curves of volume 1, or in other words, as the link of $M_{g,n}^{\mathrm{trop}}$ at its cone point.

The Euler characteristic of $\Delta_{g,n}$ is computed as the signed sum of the number of simplices in each dimension:

$$\chi(\Delta_{g,n}) = \sum_{i=0}^{3g-4+n} (-1)^i \cdot |\{(i+1)\text{-edge dual graphs for } \overline{\mathcal{M}}_{g,n}\}|.$$

From here, one readily sees that $\chi(\Delta_{g,n})$ is related to the quantity $\chi_{g,n}$ of (1) by

$$\chi_{g,n} = -\chi(\Delta_{g,n}) + 1.$$

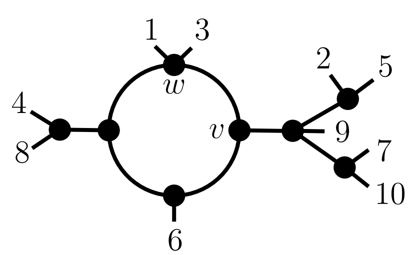
We now turn to computing this quantity in the case when $g = 1$.

3. The sign-reversing involution

Fix $n \geq 3$, and let $\mathcal{S}_{1,n}$ denote the set of dual graphs for $\overline{\mathcal{M}}_{1,n}$. Note that, for $G \in \mathcal{S}_{1,n}$, it must either be the case that G has a unique genus-one vertex, or that G has a unique cycle C as a subgraph. In the latter case, for each vertex v of C, we denote by

$$L_v \subseteq \{1, 2, \ldots, n\}$$

the labels of the vertices that map to v under the contraction of all edges in $E(G) \setminus E(C)$. For example, let G be the following graph:



Then we have

$$L_v = \{2, 5, 7, 9, 10\} \quad \text{and} \quad L_w = \{1, 3\}.$$

Equipped with this notation, we define an involution

$$i : \mathcal{S}_{1,n} \to \mathcal{S}_{1,n} \tag{3}$$

as follows, where the five cases are illustrated in Figure 2:

(1) If $G \in \mathcal{S}_{1,n}$ has a genus-one vertex v and $\mathrm{val}(v) = 1$, then v has a unique incident edge e, and we set $i(G)$ to be the contraction of G along e.
(2) If $G \in \mathcal{S}_{1,n}$ has a genus-one vertex v and $\mathrm{val}(v) > 1$, set $i(G)$ to be the graph obtained from G by changing the genus of v to zero and attaching a new genus-one vertex (with no marked points) to v.
(3) If $G \in \mathcal{S}_{1,n}$ has a cycle C as a subgraph and $|L_v| = 1$ for all v, set $i(G) = G$.
(4) If $G \in \mathcal{S}_{1,n}$ has a cycle C as a subgraph and $|L_v| > 1$ for at least one v, then among all v with $|L_v| > 1$, choose the one for which the minimum element of L_v is smallest. If $\mathrm{val}(v) = 1$, then the fact that $|L_v| > 1$ implies that the unique incident half-edge to v must come from an edge e, and we set $i(G)$ to be the contraction of G along e.
(5) If, in the situation of the previous item, $\mathrm{val}(v) > 1$, set $i(G)$ to be the graph obtained from G by replacing v by a pair of genus-zero vertices v_1 and v_2 joined by an edge e, where v_1 is incident only to e and the two adjacent edges of C, and v_2 inherits all other half-edges of v.

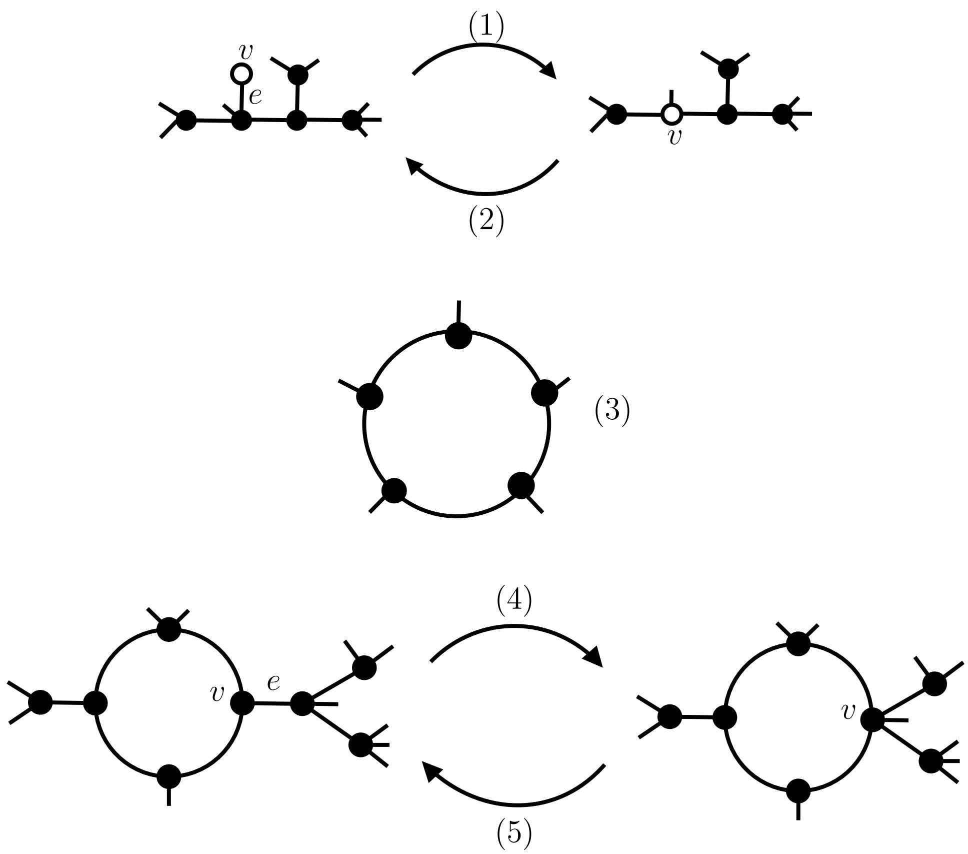


FIGURE 2. An illustration of cases (1)–(5) in the definition of the involution i, with genus-zero vertices indicated by a filled circle and genus-one vertices by an open circle, and with the labels on half-edges omitted.

Using this definition of i, the proof of Theorem 1 is nearly immediate.

Proof of Theorem 1. It is straightforward to see that i is an involution: if G satisfies case (1) in the definition of i, then $i(G)$ satisfies case (2) and vise versa, whereas if G satisfies case (4), then $i(G)$ satisfies case (5) and vise versa.

Moreover, i is sign-reversing in the sense that, if

$$\sigma(G) := (-1)^{|E(G)|},$$

then for any G not fixed by i, we have

$$\sigma(i(G)) = -\sigma(G),$$

since $i(G)$ either adds or removes a single edge from G. Thus, all terms in the sum

$$\chi_{1,n} = \sum_G \sigma(G)$$

cancel aside from those corresponding to graphs fixed by i, which are graphs with a cycle in which every vertex v has $|L_v| = 1$. In other words, such graphs have $G = C$, as shown in the middle picture of Figure 2. The number of such graphs is the number of cyclic orderings of $\{1, 2, \ldots, n\}$, which is $(n-1)!/2$. Since each has exactly n edges, each contributes $(-1)^n$ to $\chi_{1,n}$, and the formula of Theorem 1 follows. □

References

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