Linear Algebra

Vectors, Matrices and Tensors

- Vectors are denoted by lowercase letters as shown below

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

- Matrices are denoted by uppercase letters as shown below:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

- Tensors are given by a single capital letter

$$A_{i,j,k}$$

Matrices

- Matrix Multiplication
- To multiply matrices A and B, A must have the same amount of columns as the B has rows. In other words, A (has the shape $m \times n$) and B (has the shape $n \times p$) and the product will have the shape $m \times p$
- The element-wise product is given by

$$C_{i,j} = \sum_k A_{i,k} \cdot B_{k,j}$$

- The dot product of two matrices with the same dimensionality is equivalent to the matrix product of $x^T y$
- While matrix products aren't commutative, it is commutative when it is a dot product

$$x^T y = (x^T y)^T = y^T x$$

- Inverse and Identity Matrix
- Identity matrix is filled with all zeros except for the diagonal which has 1's
- Inverse matrix is defined as:

$$AA^{-1} = I_n$$

- For the equation $Ax = b$, both sides can be multiplied by the inverse matrix to solve as show below:

$$A^{-1}Ax = A^{-1}b$$

$$I_n x = A^{-1}b$$

$$x = A^{-1}b$$

- For A^{-1} to exist, it must have exactly one solution for every value of b
- For some values of b, it is possible to have no solutions or infinitely many solutions
-Span and Linear Combinations
- The span is all the possible combinations of a vector and a constant ($c\vec{v}$)
- Colinear vectors are linearly dependent and the span of the vectors is a line
- Non-colinear vectors are linear dependent and the span of the vectors is a plane
- Linear independence is necessary to have a solution for every value of b
- A square matrix is also needed to have an inverse
- A square matrix with linear dependent columns is called a singular

Norms

- The norm measures the size of a vector and is given by:

$$||x||_p = (\sum_i |x_i|^p)^{1/p}$$

- The most common norm is the L^2 norm which is known as the Euclidean norm
- The squared Euclidean norm can be calculated by $x^T x$
- The L^1 norm is used when the difference between zero and non-zero elements is very important
- The L^∞ also arises and it is the absolute value of the largest magnitude in the vector
- Measuring the size of the matrix requires the Frobenius norm given by:

$$||A||_F = \sqrt{\sum_{i,j} A_{i,j}^2}$$

- The Frobenius norm is analogous to the L^2 norm
- The dot product of two vectors can be rewritten as norms as shown below:

$$x^T y = ||x||_2 ||y||_2 \cos\theta$$