1. *Additional reading for this problem: J. Demmel, Applied Numerical Linear Algebra, available online via the UMCP library.*

The goal of this exercise is to understand how one can compute a QR decomposition using *Householder reflections*.

(a) Let u be a unit vector in $\mathbb{R}^n$, i.e., $\|u\|_2 = 1$. Let $P = I - 2uu^\top$. This matrix performs reflection with respect to the hyperplane orthogonal to the vector u. Show that $P = P^\top$ and $P^2 = I$.

Solution: By the definition of P and the linearity of transpose,

$$P^\top = (I - 2uu^\top)^\top = I^\top - 2(uu^\top)^\top.$$

But then since $I^\top = I$ and $(uu^\top)^\top = (u^\top)^\top u^\top = uu^\top$,

$$P^\top = I^\top - 2(uu^\top)^\top = I - 2uu^\top = P$$

and hence $P = P^\top$ as desired.

Now observe that since $I^2 = I$ and $I(uu^\top) = (uu^\top)I = uu^\top$,

$$P^2 = (I - 2uu^\top)(I - 2uu^\top) = I - 4uu^\top + 4(uu^\top)^2.$$

But then because $\|u\|_2 = 1$,

$$(uu^\top)^2 = u(u^\top u)u^\top = u\|u\|_2 u^\top = uu^\top$$

and hence

$$P^2 = I - 4uu^\top + 4(uu^\top)^2 = I - 4uu^\top + 4uu^\top = I.$$

(b) Let $x \in \mathbb{R}^n$ be any vector, $x = \left[x_1, \dots, x_n\right]^\top$. Let u be defined as follows:

$$\tilde{u} := \begin{bmatrix} x_1 + \mathsf{sign}(x_1)\|x\|_2 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \equiv x + \mathsf{sign}(x_1)\|x\|_2 e_1, \quad u = \frac{\tilde{u}}{\|\tilde{u}\|_2}, \tag{1}$$

where $e_1 = \left[1, 0, \dots, 0\right]^\top$. The matrix with the vector u constructed according to (1) will be denoted $\mathsf{House}(x)$:

$$P = I - 2uu^\top \equiv I - 2\frac{\tilde{u}\tilde{u}^\top}{\tilde{u}^\top\tilde{u}} \equiv \mathsf{House}(x).$$

Calculate Px.

Solution: We claim first that $2\tilde{u}^\top x = \tilde{u}^\top\tilde{u}$. To see this, note that

$$\begin{aligned} 2\tilde{u}^\top x &= 2(x + \mathsf{sign}(x_1)\|x\|_2 e_1)^\top x = 2(x^\top + \mathsf{sign}(x_1)\|x\|_2 e_1^\top)x \\ &= 2x^\top x + 2\mathsf{sign}(x_1)\|x\|_2 e_1^\top x = 2x^\top x + 2\mathsf{sign}(x_1)\|x\|_2 x_1 = 2x^\top x + 2|x_1|\|x\|_2 \end{aligned}$$

and also that since $x^\top e_1 = e_1^\top x = x_1$ and $e_1^\top e_1 = 1$ and $\mathsf{sign}(x_1)^2 = 1$,

$$\begin{aligned}\tilde{u}^\top \tilde{u} &= (x + \mathsf{sign}(x_1)\|x\|_2 e_1)^\top (x + \mathsf{sign}(x_1)\|x\|_2 e_1)\\ &= (x^\top + \mathsf{sign}(x_1)\|x\|_2 e_1^\top)(x + \mathsf{sign}(x_1)\|x\|_2 e_1) = x^\top x + 2\mathsf{sign}(x_1)\|x\|_2 x_1 + \|x\|_2^2\\ &= x^\top x + 2|x_1|\|x\|_2 + x^\top x = 2x^\top x + 2|x_1|\|x\|_2\end{aligned}$$

Hence, we have that $2\tilde{u}^\top x = \tilde{u}^\top \tilde{u}$ (notably a scalar quantity). Then it follows that

$$2\frac{\tilde{u}\tilde{u}^\top}{\tilde{u}^\top \tilde{u}} x = \tilde{u} \cdot \frac{2\tilde{u}^\top x}{\tilde{u}^\top \tilde{u}} = \tilde{u}$$

and so

$$Px = \left(I - 2\frac{\tilde{u}\tilde{u}^\top}{\tilde{u}^\top \tilde{u}}\right) x = x - 2\frac{\tilde{u}\tilde{u}^\top}{\tilde{u}^\top \tilde{u}} x = x - \tilde{u} = -\mathsf{sign}(x_1)\|x\|_2 e_1.$$

(c) Let A be an $m \times n$ matrix, $m \geq n$, with columns a_j, $j = 1, \dots, n$. Let $A_0 = A$. Let $P_1 = \mathsf{House}(a_1)$. Then $A_1 := P_1 A_0$ has the first column with the first entry nonzero and the other entries being zero. Next, we define P_2 as

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$$

where the matrix $\tilde{P}_2 = \mathsf{House}(A_1(2:m, 2))$. The notation $A_1(2:m,2)$ is Matlab's syntax indicating this is the vector formed by entries 2 through m of the 2nd column of A_1. Then we set $A_2 = P_2 A_1$. And so on.

This algorithm can be described as follows. Let $A_0 = A$. Then for $j = 1, 2, \dots, n$ we set

$$P_j = \begin{bmatrix} I_{(j-1)\times(j-1)} & 0 \\ 0 & \tilde{P}_j \end{bmatrix}; \quad \tilde{P}_j = \mathsf{House}(A_{j-1}(j:m, j)), \quad A_j = P_j A_{j-1}.$$

Check that the resulting matrix A_n is upper triangular, its entries $(A_n)_{ij}$ are all zeros for $i > j$. Propose an `if`-statement in this algorithm that will guarantee that A_n has positive entries $(A_n)_{jj}$, $1 \leq j \leq n$.

Solution: We proceed by induction on k to show that for $0 \leq k \leq n$, A_k is upper triangular in columns $1 \leq j \leq k$. The base case of $k = 0$ is trivially true, since the hypothesis asserts nothing about A_0; there are no columns $1 \leq j \leq 0$.

Now for some $0 \leq k \leq n-1$, suppose A_k is upper triangular in columns $1 \leq j \leq k$. Further, note that the entry

$$\begin{aligned}(A_{k+1})_{ij} = (P_{k+1}A_k)_{ij} &= \sum_{\ell=1}^{m} (P_{k+1})_{i\ell}(A_k)_{\ell j}\\ &= \sum_{\ell=1}^{k} (P_{k+1})_{i\ell}(A_k)_{\ell j} + \sum_{\ell=k+1}^{m} (P_{k+1})_{i\ell}(A_k)_{\ell j}.\end{aligned}$$

Further, recall that by definition,

$$P_{k+1} = \begin{bmatrix} I_{k\times k} & 0 \\ 0 & \tilde{P}_{k+1} \end{bmatrix}$$

and so it follows that

$$(P_{k+1})_{i\ell} = \begin{cases} (\tilde{P}_{k+1})_{i-k,\ell-k} & i,\ell > k \\ 1 & i = \ell \le k \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Now focusing first on the left summation, note that because A_k is upper triangular in columns $1 \le j \le k$, and the first case of (2) is not possible since $\ell \le k$, we have that

$$\sum_{\ell=1}^{k} (P_{k+1})_{i\ell}(A_k)_{\ell j} = \sum_{\ell=1}^{j} (P_{k+1})_{i\ell}(A_k)_{\ell j} = \sum_{\ell=1}^{j} (A_k)_{\ell j} \begin{cases} 1 & i = \ell \\ 0 & i \ne \ell \end{cases} = \begin{cases} (A_k)_{ij} & i \le j \\ 0 & i > j \end{cases}.$$

We now focus on the right summation, in which the second case of (2) is not possible since $\ell > k$. Then

$$\begin{aligned} \sum_{\ell=k+1}^{m} (P_{k+1})_{i\ell}(A_k)_{\ell j} &= \sum_{\ell=k+1}^{m} (A_k)_{\ell j} \begin{cases} 0 & i \le k \\ (\tilde{P}_{k+1})_{i-k,\ell-k} & i > k \end{cases} \\ &= \begin{cases} 0 & i \le k \\ \sum_{\ell=k+1}^{m} (\tilde{P}_{k+1})_{i-k,\ell-k}(A_k)_{\ell j} & i > k \end{cases} \end{aligned}$$

and hence

$$(A_{k+1})_{ij} = \begin{cases} (A_k)_{ij} & i \le j \\ 0 & i > j \end{cases} + \begin{cases} 0 & i \le k \\ \sum_{\ell=k+1}^{m} (\tilde{P}_{k+1})_{i-k,\ell-k}(A_k)_{\ell j} & i > k \end{cases}.$$

From the above, note that for $1 \le j \le k$, if $i > j$ then both summands will be 0 (the right will be 0 because $\ell > k \ge j$ so $(A_k)_{\ell j} = 0$ since $j \le k$). Hence, it suffices to show that A_{k+1} is upper triangular in column $j = k+1$;

The left summand cannot contribute any nonzero values if $i > j$, so we need only consider the right summand. Re-indexing to write it as a matrix multiplication using the Matlab notation and taking $j = k+1$,

$$\begin{aligned} \sum_{\ell=k+1}^{m} (\tilde{P}_{k+1})_{i-k,\ell-k}(A_k)_{\ell j} &= \sum_{\ell=1}^{m-k} (\tilde{P}_{k+1})_{i-k,\ell}(A_k)_{k+\ell,j} \\ &= (\tilde{P}_{k+1}A_k(k+1:m,j))_{i-k} = (\tilde{P}_{k+1}A_k(k+1:m,k+1))_{i-k}. \end{aligned}$$

But then by item (b),

$$\begin{aligned} (\tilde{P}_{k+1}A_k(k+1:m,k+1))_{i-k} &= (-\mathsf{sign}((A_k)_{k+1,k+1})\|A_k(k+1:m,k+1)\|_2 e_1)_{i-k} \\ &= -\mathsf{sign}((A_k)_{k+1,k+1})\|A_k(k+1:m,k+1)\|_2 (e_1)_{i-k} \end{aligned}$$

which is 0 if $i > j = k+1$ since then $i - k \geq 2$ so $(e_1)_{i-k} = 0$. Hence, we have that A_{k+1} is upper triangular in columns $1 \leq j \leq k+1$, so by the principle of induction, for any $0 \leq k \leq n$, A_k is upper triangular in columns $1 \leq j \leq k$. Letting $k = n$, we obtain the desired result: A_n is upper triangular in columns $1 \leq j \leq n$.

If we want to ensure that the diagonal entries $(A_n)_{jj} \geq 0$ for $1 \leq j \leq n$, we could alter the definition of $\tilde{u}$ such that

$$\tilde{u} \equiv x + \frac{|x_1|}{x_1}\|x\|_2 e_1,$$

that is, if $A_j j < 0$ then use $\tilde{u} = x - \mathsf{sign}(x_1)\|x\|_2 e_1$.

(d) Extract the QR decomposition of A given the matrices P_j, $1 \leq j \leq n$, and A_n.

Solution: From (c) we know that A_n is upper triangular. Hence, it suffices to construct an orthogonal matrix Q from the matrices P_j, $1 \leq j \leq n$ such that $A = QA_n$.

Towards this end, note that

$$A_n = P_n A_{n-1} = P_n P_{n-1} A_{n-2} = \cdots = P_n P_{n-1} \dots P_1 A_0 = P_n P_{n-1} \dots P_1 A$$

so Q should be defined such that

$$A = Q P_n P_{n-1} \dots P_1 A$$

and so

$$Q = P_1^{-1} P_2^{-1} \dots P_n^{-1}.$$

But finally, from (a) we know that each P_j is orthogonal ($P_j P_j^\top = P_j^2 = I$) and hence $P_j^{-1} = P_j$. Hence, the QR decomposition of A is given by

$$Q = P_1 P_2 \dots P_n, \quad R = A_n$$

where the orthogonality of Q is guaranteed as the product of orthogonal matrices.

2. Prove items (1)−(6) of Theorem 3 on page 14 of `LinearAlgebra.pdf`.

Solution:

(1) Let A be symmetric and $A = U\Lambda U^\top$ is an eigendecomposition of A. Then since A is symmetric and $U^\top U = I$,

$$U\Lambda^2 U^\top = (U\Lambda U^\top)(U\Lambda U^\top) = A^2 = AA^\top = (U\Sigma V^\top)(V\Sigma^\top U^\top) = U\Sigma^2 U^\top$$

and so $\Sigma^2 = \Lambda^2$; in order to guarantee $\sigma_i \geq 0$, it must be that $\sigma_i = |\lambda_i|$. Now

$$U\Lambda U^\top = A = U\Sigma V^\top = U|\Lambda|V^\top$$

and so left multiplying by $\mathsf{sign}(\Lambda)\Lambda^{-1}U^\top$, we obtain

$$\mathsf{sign}(\Lambda)U^\top = \mathsf{sign}(\Lambda)\Lambda^{-1}|\Lambda|V^\top = \mathsf{sign}(\Lambda)\mathsf{sign}(\Lambda)V^\top = V^\top.$$

Finally, taking the transpose,

$$V = (\mathsf{sign}(\Lambda)U^\top)^\top = U\mathsf{sign}(\Lambda)^\top = U\mathsf{sign}(\Lambda),$$

so $v_i = u_i \mathsf{sign}(\lambda_i)$.

(2) Since U and V are orthogonal (so in particular $U^\top U = I$ and $V^\top = V^{-1}$) and Σ is diagonal (so in particular $\Sigma^\top \Sigma = \Sigma^2$), we have that

$$A^\top A = (U\Sigma V^\top)^\top (U\Sigma V^\top) = V\Sigma^\top U^\top U\Sigma V^\top = V\Sigma^\top \Sigma V^\top = V\Sigma^2 V^{-1}.$$

Hence, $V\Sigma^2 V^{-1}$ is the eigendecomposition of $A^\top A$, so the eigenvalues of $A^\top A$ are the diagonal entries of Σ^2, those being σ_i^2. Further, the eigenvectors of $A^\top A$ are the columns of V, the right singular vectors.

(3) Since U and V are orthogonal (so in particular $U^\top = U^{-1}$ and $V^\top V = I$) and Σ is diagonal (so in particular $\Sigma\Sigma^\top = \Sigma^2$), we have that

$$AA^\top = (U\Sigma V^\top)(U\Sigma V^\top)^\top = U\Sigma V^\top V\Sigma^\top U^\top = U\Sigma\Sigma^\top U^\top = U\Sigma^2 U^\top.$$

Note that U and Σ are $m \times n$ and $n \times n$ respectively; the eigendecomposition of $AA^\top$ is obtained by padding U and Σ with zeros to make each $m \times m$. Hence, the eigenvalues of $AA^\top$ are the σ_i^2 from original Σ^2 and the $m - n$ padded zeros.

Further, the eigenvectors of $AA^\top$ corresponding to the σ_i^2 are the columns of U, the left singular vectors, and the eigenvectors corresponding to 0 can be any $m-n$ vectors orthogonal to each and the columns of U.

(4) By the non-negativity of norms,

$$\min_x \|Ax - b\| \geq 0.$$

Now take $x = V\Sigma^{-1}U^\top b$, in which case

$$\min_x \|Ax - b\| \leq \|(U\Sigma V^\top)(V\Sigma^{-1}U^\top b) - b\| = \|b - b\| = \|0\| = 0$$

and hence $\min_x \|Ax - b\|$ attains its minimum value at $x = V\Sigma^{-1}U^\top b$. Further, since A is full rank, this solution is unique.

(5) Recall we have seen a property relating the 2-norm of A to $\rho(A^\top A)$, the largest eigenvalue of $A^\top A$:

$$\|A\|_2 = \sqrt{\rho(A^\top A)}.$$

Further, by item (2), we know that the eigenvalues of $A^\top A$ are σ_i^2, the largest of which is σ_1^2. Hence,

$$\|A\|_2 = \sqrt{\rho(A^\top A)} = \sqrt{\sigma_1^2} = \sigma_1.$$

Now suppose A is square and nonsingular; then $A^{-1} = V\Sigma^{-1}U^\top$, since

$$AA^{-1} = U\Sigma V^\top V\Sigma^{-1}U^\top = U\Sigma\Sigma^{-1}U^\top = UU^\top = I$$

and hence $V\Sigma^{-1}U^\top$ is a singular value decomposition of A^{-1}. Note that the diagonal entries of Σ^{-1} are $1/\sigma_i$, and so the largest singular value of A^{-1} is the reciprocal of the smallest singular value of A, σ_n. Hence, we have that

$$\|A^{-1}\|_2 = \frac{1}{\sigma_n}.$$

(6) Note that for $1 \leq k \leq n$,

$$Av_k = U\Sigma V^\top v_k = U\Sigma e_k = U\sigma_k = \sigma_k u_k$$

and $\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \ldots \sigma_n = 0$ by assumption. In particular, this means that $Av_k \neq 0$ for $1 \leq k \leq r$ and $Av_k = 0$ for $r+1 \leq k \leq n$, and so we have both that

$$\text{span}\left(\frac{1}{\sigma_1}Av_1, \ldots, \frac{1}{\sigma_r}Av_r\right) = \text{span}(u_1, \ldots, u_r) \subseteq \text{range(A)}$$

and

$$\text{span}(v_{r+1}, \ldots, v_n) \subseteq \text{null}(A)$$

so since $v_1, \ldots, v_n$ are orthogonal (and hence linearly independent) then $\text{rank}(A) \geq r$ and $\text{nullity}(A) \geq n - r$. Now towards a contradiction suppose $\text{rank}(A) > r$. Then

$$\text{rank}(A) + \text{nullity}(A) \geq \text{rank}(A) + n - r > r + n - r = n$$

which contradicts the rank-nullity theorem. Hence, it must be that $\text{rank}(A) = r$, so

$$\text{null}(A) = \text{span}(v_{r+1}, \ldots, v_n)$$

and

$$\text{range(A)} = \text{span}(u_1, \ldots, u_r)$$

as desired.

3. Let A be an $m \times n$ matrix where $m < n$ and rows of A are linearly independent. Then the system of linear equations $Ax = b$ is underdetermined, i.e., infinitely many solutions. Among them, we want to find the one that has the minimum 2-norm. Check that the minimum 2-norm solution is given by

$$x^* = A^\top(AA^\top)^{-1}b.$$

Hint: One way to solve this problem is the following. Check that x^* is a solution to $Ax = b$. Show that if $x^* + y$ is also a solution of $Ax = b$ then $Ay = 0$. Then check that the 2-norm of $x^* + y$ is minimal if $y = 0$.

Solution: We observe first that x^* as defined is indeed a solution to $Ax = b$ since

$$Ax^* = A(A^\top(AA^\top)^{-1}b) = (AA^\top)(AA^\top)^{-1}b = b.$$

Now suppose that $x^* + y$ is also a solution to $Ax = b$ (noting that every solution to $Ax = b$

can be written in this manner). Then it follows that

$$b = A(x^* + y) = Ax^* + Ay = b + Ay$$

and hence $Ay = 0$.

Finally, we claim that if $x^* + y$ is a solution to $Ax = b$, then $\|x^* + y\|_2^2 = \|x^*\|_2^2 + \|y\|_2^2$. To see this, by definition of the 2-norm and the fact that vector addition is element-wise,

$$\|x^* + y\|_2^2 = \sum_{i=1}^{n} ((x^* + y)_i)^2 = \sum_{i=1}^{n} (x_i^* + y_i)^2 = \sum_{i=1}^{n} (x_i^*)^2 + 2(x_i^*)(y_i) + (y_i)^2.$$

But then splitting the summation, we find

$$\|x^* + y\|_2^2 = \sum_{i=1}^{n} (x_i^*)^2 + \sum_{i=1}^{n} 2(x_i^*)(y_i) + \sum_{i=1}^{n} (y_i)^2 = \|x^*\|_2^2 + 2\sum_{i=1}^{n} (x_i^*)(y_i) + \|y\|_2^2$$

and further, the remaining sum is simply $(x^*)^\top y$, meaning

$$\|x^* + y\|_2^2 = \|x^*\|_2^2 + 2(x^*)^\top y + \|y\|_2^2.$$

But now note that

$$(x^*)^\top y = (A^\top (AA^\top)^{-1} b)^\top y = b^\top ((AA^\top)^{-1})^\top Ay = b^\top ((AA^\top)^\top)^{-1} Ay = b^\top (AA^\top)^{-1} Ay$$

and recall that earlier we showed $Ay = 0$, hence $(x^*)^\top y = 0$ and so

$$\|x^* + y\|_2^2 = \|x^*\|_2^2 + \|y\|_2^2 \tag{3}$$

when $x^* + y$ is a solution to $Ax = b$. Therefore, by the non-negativity of norms,

$$\|x^* + y\|_2^2 = \|x^*\|_2^2 + \|y\|_2^2 \geq \|x^*\|_2^2$$

and so taking the square root of both sides yields $\|x^* + y\|_2 \geq \|x^*\|_2$. Further, note that equality is attained if and only if $\|y\|_2 = 0$, or equivalently, $y = 0$.

Hence, we have shown that $\|x^* + y\|_2$ is minimal exactly when $y = 0$, and since every solution to $Ax = b$ can be written in this form, this completes the proof.

4. Let A be a 3×3 matrix, and let T be its Schur form, i.e., there is a unitary matrix Q (i.e., $Q^*Q = QQ^* = I$ where Q^* denotes the transpose and complex conjugate of Q) such that

$$A = QTQ^*, \text{ where } T = \begin{bmatrix} \lambda_1 & t_{12} & t_{13} \\ 0 & \lambda_2 & t_{23} \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Assume that λ_j, $j = 1, 2, 3$ are all distinct.

(a) Show that if v is an eigenvector of T then Qv is the eigenvector of A corresponding to the same eigenvalue.

Solution: Let v be an eigenvector of T with corresponding eigenvalue λ, such that $Tv = \lambda v$. Then since $Q^*Q = I$, we have that

$$A(Qv) = (QTQ^*)Qv = QT(Q^*Q)v = QTv = Q\lambda v = \lambda(Qv),$$

and so Qv is an eigenvector of A also corresponding to λ.

(b) Find the eigenvectors of T. *Hint: Check that* $v_1 = [1, 0, 0]^\top$. *Look for* v_2 *of the form* $v_2 = [a, 1, 0]^\top$, *and then for* v_3 *of the form* $v_3 = [b, c, 1]^\top$, *where* a, b, c *are to be expressed via the entries of the matrix* T.

Solution: Since T is (upper) triangular, its eigenvalues are its diagonal entries λ_1, λ_2, and λ_3. Hence, we want to find vectors v_1, v_2, and v_3 such that

$$Tv_1 = \lambda_1 v_1, \quad Tv_2 = \lambda_2 v_2, \quad Tv_3 = \lambda_3 v_3.$$

That is, for each $1 \leq i \leq 3$, we want to find a_i, b_i, and c_i such that

$$\begin{aligned}\begin{bmatrix}\lambda_1 & t_{12} & t_{13}\\ 0 & \lambda_2 & t_{23}\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}a_i\\ b_i\\ c_i\end{bmatrix} - \lambda_i\begin{bmatrix}a_i\\ b_i\\ c_i\end{bmatrix} &= \begin{bmatrix}\lambda_1 a_i + t_{12}b_i + t_{13}c_i\\ \lambda_2 b_i + t_{23}c_i\\ \lambda_3 c_i\end{bmatrix} - \begin{bmatrix}\lambda_i a_i\\ \lambda_i b_i\\ \lambda_i c_i\end{bmatrix}\\ &= \begin{bmatrix}(\lambda_1 - \lambda_i)a_i + t_{12}b_i + t_{13}c_i\\ (\lambda_2 - \lambda_i)b_i + t_{23}c_i\\ (\lambda_3 - \lambda_i)c_i\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.\end{aligned}$$

Then for $i = 1$, $c_1 = 0$ since $(\lambda_3 - \lambda_1)c_1 = 0$. Further, then $(\lambda_2 - \lambda_1)b_1 + t_{23}c_1 = 0$, it must be that $b_1 = 0$. Finally, we see that a_1 is free, and hence an eigenvector corresponding to λ_1 is

$$v_1 = [1, 0, 0]^\top.$$

Now repeating the same procedure for $i = 2$, again we find that $c_2 = 0$, but now b_2 is free since any choice will satisfy $(\lambda_2 - \lambda_2)b_2 + t_{23}c_2 = 0$. Hence, from the first equation it follows that

$$a_2 = \frac{t_{12}b_2}{\lambda_2 - \lambda_1}$$

and so taking $b_2 = 1$ yields that an eigenvector corresponding to λ_2 is

$$v_2 = \begin{bmatrix}\frac{t_{12}}{\lambda_2 - \lambda_1} & 1 & 0\end{bmatrix}^\top.$$

Finally, for $i = 3$, we see that c_3 is free since any choice will satisfy the third equation $(\lambda_3 - \lambda_3)c_3 = 0$. Taking $c_3 = 1$, from the second equation we have that

$$b_3 = \frac{t_{23}}{\lambda_3 - \lambda_2}$$

and hence from the first equation, we have that

$$a_3 = \frac{t_{12}b_3 + t_{13}}{\lambda_3 - \lambda_1} = \frac{t_{12}t_{23}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} + \frac{t_{13}}{\lambda_3 - \lambda_1}$$

and hence an eigenvector corresponding to λ_3 is

$$v_3 = \begin{bmatrix} \frac{t_{12}t_{23}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} + \frac{t_{13}}{\lambda_3-\lambda_1} & \frac{t_{23}}{\lambda_3-\lambda_2} & 1 \end{bmatrix}^\top .$$

(c) Write out the eigenvectors of A in terms of the found eigenvectors of T and the columns of Q: $Q = [q_1, q_2, q_3]$.

Solution: By item (a), we know that if v is an eigenvector of T then Qv is an eigenvector of A corresponding to the same eigenvalue. Hence, using the eigenvectors of T found in item (b),

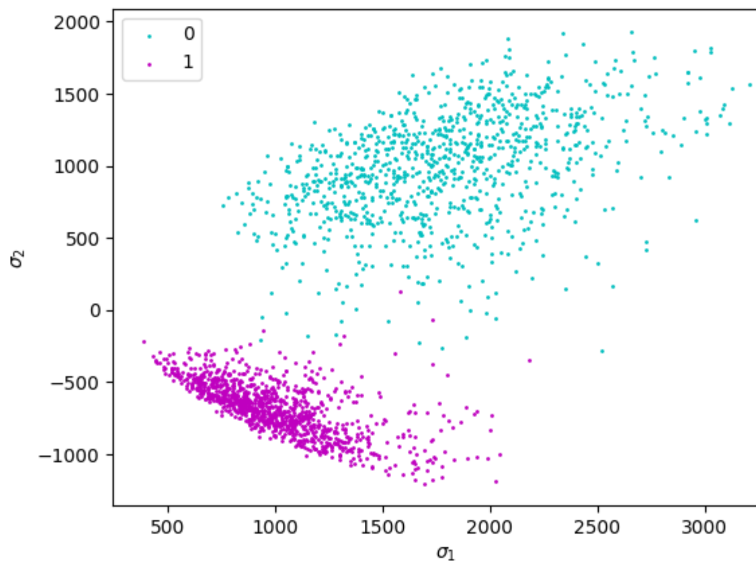
$$\begin{aligned} v_1' &= Qv_1 = q_1 \\ v_2' &= Qv_2 = \left(\frac{t_{12}}{\lambda_2-\lambda_1}\right) q_1 + q_2 \\ v_3' &= Qv_3 = \left(\frac{t_{12}t_{23}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} + \frac{t_{13}}{\lambda_3-\lambda_1}\right) q_1 + \left(\frac{t_{23}}{\lambda_3-\lambda_2}\right) q_2 + q_3 \end{aligned}$$

are eigenvectors of A corresponding to eigenvalues λ_1, λ_2, and λ_3 respectively.

5. Download the MNIST dataset. For your convenience, I prepared it as the `mnist.mat` file. This file contains 60,000 training images and 10,000 test images of handwritten digits from 0 to 9, and labels for the training and test images. Each image is 20-by-20 pixels because I stripped off paddings with zeros. You can use Matlab or Python.

   (a) Convert the set of test images into a matrix A of size $10^4 \times 400$. Compute an SVD of this matrix. Project the data onto the space spanned by the first two right singular vectors v_1 and v_2. Display only points corresponding to digits 0 and 1 in this 2D space and color the points corresponding to 1 and 0 in different colors. Check if 0s and 1s cluster in this 2D space.

Solution: The points corresponding to 0s and 1s indeed appear to cluster in this 2D space; there is a clear decision boundary between them.



(b) Do this task for $k = 10$, 20, and 50. Compute $A_k = U_k \Sigma_k V_k^\top$, where U_k (V_k) is comprised of the first k left (right) singular vectors, and $\Sigma_k = \mathsf{diag}\{\sigma_1, \ldots, \sigma_k\}$. Then take the first four rows of A and A_k, reshape each of these rows back to 20×20 images, and display them.

Solution: This diagram and the one above were produced by this code.