1. Prove that the conjugate gradient algorithm, the preliminary version (Algorithm 5.1, page 108 in [NW]), is equivalent to Algorithm 5.2 (CG) (page 112 in [NW]), i.e., that

$$\alpha_k = \frac{r_k^\top r_k}{p_k^\top A p_k} \text{ and } \beta_{k+1} = \frac{r_{k+1}^\top r_{k+1}}{r_k^\top r_k}. \tag{1}$$

Solution: Recall from Algorithm 5.1, that we defined

$$\alpha_k := -\frac{r_k^\top p_k}{p_k^\top A p_k} \text{ and } \beta_{k+1} := \frac{r_{k+1}^\top A p_k}{p_k^\top A p_k}.$$

We first show the result for α_k; expanding the definition of p_k,

$$\alpha_k := -\frac{r_k^\top p_k}{p_k^\top A p_k} = -\frac{r_k^\top(-r_k + \beta_k p_{k-1})}{p_k^\top A p_k} = \frac{r_k^\top r_k - \beta_k r_k^\top p_{k-1}}{p_k^\top A p_k}$$

but recall that $r_k^\top p_i = 0$ for $i = 0, 1, \ldots, k-1$, and hence $r_k^\top p_{k-1} = 0$. Therefore,

$$\alpha_k = \frac{r_k^\top r_k - \beta_k r_k^\top p_{k-1}}{p_k^\top A p_k} = \frac{r_k^\top r_k}{p_k^\top A p_k}$$

as desired; note that the right-hand side is the definition of α_k in Algorithm 5.2. Now for β_{k+1}, first multiplying by α_k/α_k and using the above result for the α_k in the denominator,

$$\beta_{k+1} := \frac{r_{k+1}^\top A p_k}{p_k^\top A p_k} = \frac{r_{k+1}^\top \alpha_k A p_k}{p_k^\top \alpha_k A p_k} = \frac{r_{k+1}^\top \alpha_k A p_k}{\alpha_k p_k^\top A p_k} = \frac{r_{k+1}^\top \alpha_k A p_k}{r_k^\top r_k}.$$

Finally, recalling the fact that $r_{k+1} = r_k + \alpha_k A p_k$ and hence $\alpha_k A p_k = r_{k+1} - r_k$, and also that $r_{k+1}^\top r_k = 0$, we obtain the desired result

$$\beta_{k+1} = \frac{r_{k+1}^\top \alpha_k A p_k}{r_k^\top r_k} = \frac{r_{k+1}^\top (r_{k+1} - r_k)}{r_k^\top r_k} = \frac{r_{k+1}^\top r_{k+1} - r_{k+1}^\top r_k}{r_k^\top r_k} = \frac{r_{k+1}^\top r_{k+1}}{r_k^\top r_k}$$

which is the definition of β_{k+1} in Algorithm 5.2. Hence, the preliminary and practical versions of the conjugate gradient descent algorithm are equivalent.

2. Let A be an $n \times n$ matrix. A subspace spanned by the columns of an $n \times k$ matrix B is an invariant subspace of A if A maps it into itself, i.e., if $AB \subset \mathsf{span}(B)$. This means that there is a $k \times k$ matrix C such that $AB = BC$. Prove that if a vector $r \in \mathbb{R}^n$ lies in the k-dimensional subspace spanned by the columns of B, i.e., if $r = By$ for some $y \in \mathbb{R}^k$ (r is a linear combination of columns of B with coefficients $y_1, \ldots, y_k$) then the Krylov subspaces generated by r stop expanding at degree $k-1$, i.e,

$$\mathsf{span}\{r, Ar, \ldots, A^p r\} = \mathsf{span}\{r, Ar, \ldots, A^{k-1} r\} \quad \forall p \geq k. \tag{2}$$

Solution: We proceed by induction on p. As a base case, consider $p = k$. In particular, we want to show that

$$\mathsf{span}\{r, Ar, \ldots, A^{k-1}r, A^k r\} = \mathsf{span}\{r, Ar, \ldots, A^{k-1}r\}$$

which is true if and only if $A^k r \in \mathsf{span}\{r, Ar, \ldots, A^{k-1}r\}$. But since $r = By$ and $AB = BC$, this is equivalent to asking whether

$$BC^k y = A^k By = A^k r \in \mathsf{span}\{r, Ar, \ldots, A^{k-1}r\} = \mathsf{span}\{By, BCy, \ldots, BC^{k-1}y\}.$$

In particular, this means we want to know whether there exist constants $d_0, d_1, \ldots, d_{k-1}$ such that

$$BC^k y = d_0 By + d_1 BCy + \cdots + d_{k-1} BC^{k-1} y = B(d_0 I + d_1 C + \cdots + d_{k-1} C^{k-1})y.$$

Hence, the claim is true for the base case if we can write

$$C^k = d_0 I + d_1 C + \cdots + d_{k-1} C^{k-1}$$

and indeed we can. The Cayley-Hamilton theorem states that C is a root of its own characteristic polynomial. Hence, negating the coefficient of the characteristic polynomial allows us to write C^k as a linear combination of $I, C, \ldots, C^{k-1}$. Hence, we have that

$$A^k r = BC^k y \in \mathsf{span}\{By, BCy, \ldots, BC^{k-1}y\} = \mathsf{span}\{r, Ar, \ldots, A^{k-1}r\}$$

so indeed the claim holds when $p = k$.

Now as the inductive hypothesis suppose the claim holds for some $p \geq k$ and consider the case of $p + 1$. We claim that

$$\mathsf{span}\{r, Ar, \ldots, A^p r, A^{p+1} r\} = \mathsf{span}\{r, Ar, \ldots, A^p r\}$$

or equivalently, that $A^{p+1} r \in \mathsf{span}\{r, Ar, \ldots, A^p r\}$. Now observe that $A^{p+1} r = AA^p r$, and by the inductive hypothesis, $A^p r \in \mathsf{span}\{r, Ar, \ldots, A^p r\} = \mathsf{span}\{r, Ar, \ldots, A^{k-1} r\}$, so we can write

$$A^{p+1} r = AA^p r = A(c_0 r + c_1 Ar + \cdots + c_{k-1} A^{k-1} r) = c_0 Ar + c_1 A^2 r + \cdots + c_{k-1} A^k r.$$

Hence, it follows that $A^{p+1} r \in \mathsf{span}\{r, Ar, \ldots, A^k r\} = \mathsf{span}\{r, Ar, \ldots, A^{k-1} r\}$, and so

$$\mathsf{span}\{r, Ar, \ldots, A^p r, A^{p+1} r\} = \mathsf{span}\{r, Ar, \ldots, A^p r\} = \mathsf{span}\{r, Ar, \ldots, A^{k-1} r\},$$

so the claim holds for $p + 1$. By the principle of induction, the claim holds for all $p \geq k$.

3. Prove Theorem 5.5 from [NW], page 115. Here are the steps that you need to work out.

   (a) Construct a polynomial $Q(\lambda)$ of degree $k + 1$ with roots $\lambda_n, \lambda_{n-1}, \ldots, \lambda_{n-k+1}$, and $\frac{1}{2}(\lambda_1 + \lambda_{n-k})$ such that $Q(0) = 1$.

Solution: In order to have $\lambda_n, \lambda_{n-1}, \dots, \lambda_{n-k+1}$ as roots of Q, we will base Q on

$$(\lambda_n - \lambda)(\lambda_{n-1} - \lambda) \dots (\lambda_{n-k+1} - \lambda) = \prod_{j=n-k+1}^{n} (\lambda_j - \lambda).$$

Further, since we also want $\frac{1}{2}(\lambda_1 + \lambda_{n-k})$ as a root, we multiply the above by a factor of $\left(\frac{1}{2}(\lambda_1 + \lambda_{n-k}) - \lambda\right)$ to obtain

$$\left(\frac{1}{2}(\lambda_1 + \lambda_{n-k}) - \lambda\right) \prod_{j=n-k+1}^{n} (\lambda_j - \lambda).$$

It remains to scale the above by a constant factor to ensure $Q(0) = 1$. Substituting $\lambda = 0$ into the above, we find

$$\frac{1}{2}(\lambda_1 + \lambda_{n-k}) \prod_{j=n-k+1}^{n} \lambda_j$$

and hence we define

$$Q(\lambda) := \frac{\left(\frac{1}{2}(\lambda_1 + \lambda_{n-k}) - \lambda\right) \prod_{j=n-k+1}^{n} (\lambda_j - \lambda)}{\frac{1}{2}(\lambda_1 + \lambda_{n-k}) \prod_{j=n-k+1}^{n} \lambda_j}$$
$$= \frac{\frac{1}{2}(\lambda_1 + \lambda_{n-k}) - \lambda}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})} \prod_{j=n-k+1}^{n} \left(1 - \frac{\lambda}{\lambda_j}\right).$$

(b) Argue that $P(\lambda)$ defined as

$$P(\lambda) = \frac{Q(\lambda) - 1}{\lambda} \tag{3}$$

is a polynomial, not a rational function, by referring to the theorem about factoring polynomials. Cite that theorem.

Solution: Recall that we constructed $Q(\lambda)$ such that $Q(0) = 1$, so the polynomial $Q(\lambda) - 1$ has a root at 0. In particular, by the factor theorem this means that we can write

$$Q(\lambda) - 1 = (\lambda - 0)P(\lambda) = \lambda P(\lambda)$$

for some polynomial $P(\lambda)$. Hence,

$$P(\lambda) = \frac{Q(\lambda) - 1}{\lambda}$$

is indeed a polynomial.

(c) Use the ansatz

$$\|x_{k+1} - x^*\|_A^2 \le \min_{P \in \mathcal{P}_k} \max_{1 \le i \le n} [1 + \lambda_i P(\lambda_i)]^2 \|x_0 - x^*\|_A^2. \tag{4}$$

Argue that

$$\|x_{k+1} - x^*\|_A^2 \le \max_{1\le i\le n} Q^2(\lambda_i)\|x_0 - x^*\|_A^2. \tag{5}$$

Solution: Taking as an ansatz,

$$\|x_{k+1} - x^*\|_A^2 \le \min_{P_k\in\mathcal{P}_k} \max_{1\le i\le n} [1 + \lambda_i P_k(\lambda_i)]^2 \|x_0 - x^*\|_A^2$$

then in particular it must be that

$$\|x_{k+1} - x^*\|_A^2 \le \max_{1\le i\le n} [1 + \lambda_i P(\lambda_i)]^2 \|x_0 - x^*\|_A^2$$

since P is a polynomial of degree k by item (b), since Q is a polynomial of degree $k+1$. Further, by definition

$$P(\lambda_i) = \frac{Q(\lambda_i) - 1}{\lambda_i}$$

and hence $1 + \lambda_i P(\lambda_i) = Q(\lambda_i)$, meaning

$$\|x_{k+1} - x^*\|_A^2 \le \max_{1\le i\le n} [1 + \lambda_i P(\lambda_i)]^2 \|x_0 - x^*\|_A^2 = \max_{1\le i\le n} [Q(\lambda_i)]^2 \|x_0 - x^*\|_A^2$$

as desired.

(d) Show that

$$\|x_{k+1} - x^*\|_A^2 \le \max_{\lambda\in[\lambda_1,\lambda_{n-k}]} \left| \frac{\lambda - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})} \right|^2 \|x_0 - x^*\|_A^2. \tag{6}$$

Solution: From item (c), we know that

$$\|x_{k+1} - x^*\|_A^2 \le \max_{1\le i\le n} Q^2(\lambda_i)\|x_0 - x^*\|_A^2$$

hence it suffices to show that

$$\max_{1\le i\le n} Q^2(\lambda_i) \le \max_{\lambda\in[\lambda_1,\lambda_{n-k}]} \left| \frac{\lambda - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})} \right|^2.$$

Toward this end, recall that by construction, $Q(\lambda_i) = 0$ for $n - k + 1 \le i \le n$. Hence,

$$\max_{1\le i\le n} Q^2(\lambda_i) = \max_{1\le i\le n-k} Q^2(\lambda_i).$$

Further, it is certainly true that

$$\max_{1\le i\le n-k} Q^2(\lambda_i) \le \max_{\lambda\in[\lambda_1,\lambda_{n-k}]} Q^2(\lambda),$$

hence we need only show that

$$\max_{\lambda\in[\lambda_1,\lambda_{n-k}]} Q^2(\lambda) \le \max_{\lambda\in[\lambda_1,\lambda_{n-k}]} \left| \frac{\lambda - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})} \right|^2.$$

Recalling our definition of Q, we have that

$$Q^2(\lambda) = \left(\frac{\frac{1}{2}(\lambda_1 + \lambda_{n-k}) - \lambda}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})}\right)^2 \prod_{j=n-k+1}^{n} \left(1 - \frac{\lambda}{\lambda_j}\right)^2$$
$$= \left|\frac{\lambda - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})}\right|^2 \prod_{j=n-k+1}^{n} \left(1 - \frac{\lambda}{\lambda_j}\right)^2$$

and so the result follows provided for all $\lambda \in [\lambda_1, \lambda_{n-k}]$,

$$\prod_{j=n-k+1}^{n} \left(1 - \frac{\lambda}{\lambda_j}\right)^2 \leq 1.$$

To see that this inequality is true, take $j \in \{n-k+1, \ldots, n\}$ and $\lambda \in [\lambda_1, \lambda_{n-k}]$. Then

$$\frac{\lambda}{\lambda_j} \leq \frac{\lambda}{\lambda_{n-k+1}} \leq \frac{\lambda_{n-k}}{\lambda_{n-k+1}} < 1$$

and also $\frac{\lambda}{\lambda_j} > 0$ since A is SPD; hence $\left(1 - \frac{\lambda}{\lambda_j}\right)^2 \leq 1$, so $\prod_{j=n-k+1}^{n} \left(1 - \frac{\lambda}{\lambda_j}\right)^2 \leq 1$. Therefore,

$$\max_{\lambda \in [\lambda_1, \lambda_{n-k}]} Q^2(\lambda) = \max_{\lambda \in [\lambda_1, \lambda_{n-k}]} \left|\frac{\lambda - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})}\right|^2 \prod_{j=n-k+1}^{n} \left(1 - \frac{\lambda}{\lambda_j}\right)^2$$
$$\leq \max_{\lambda \in [\lambda_1, \lambda_{n-k}]} \left|\frac{\lambda - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})}\right|^2$$

and so we obtain the desired result by applying item (c).

(e) Find the maximum of the function in the right-hand side of the last equation in the interval $[\lambda_1, \lambda_{n-k}]$.

Solution: The maximum is attained either at an endpoint of the interval $[\lambda_1, \lambda_{n-k}]$, or at a critical point. Differentiating, we find

$$\frac{d}{d\lambda}\left|\frac{\lambda - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})}\right|^2 = \frac{d}{d\lambda}\left(\frac{\lambda - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})}\right)^2 = 2\left(\frac{\lambda - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})}\right).$$

We see that the above is equal to 0 if and only if $\lambda = \frac{1}{2}(\lambda_1 + \lambda_{n-k})$, giving us a third candidate maximizer. We will now evaluate the right-hand side for each of λ_1, λ_{n-k}, and $\frac{1}{2}(\lambda_1 + \lambda_{n-k})$.

At λ_1, we find

$$\left|\frac{\lambda_1 - \frac{1}{2}(\lambda_1 + \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})}\right|^2 = \left|\frac{\frac{1}{2}(\lambda_1 - \lambda_{n-k})}{\frac{1}{2}(\lambda_1 + \lambda_{n-k})}\right|^2 = \left|\frac{\lambda_1 - \lambda_{n-k}}{\lambda_1 + \lambda_{n-k}}\right|^2$$

which we see is non-negative. The situation is similar for λ_{n-k}, where

$$\left|\frac{\lambda_{n-k}-\frac{1}{2}(\lambda_1+\lambda_{n-k})}{\frac{1}{2}(\lambda_1+\lambda_{n-k})}\right|^2 = \left|\frac{-\frac{1}{2}(\lambda_1-\lambda_{n-k})}{\frac{1}{2}(\lambda_1+\lambda_{n-k})}\right|^2 = \left|\frac{\lambda_1-\lambda_{n-k}}{\lambda_1+\lambda_{n-k}}\right|^2$$

and so λ_1 and λ_{n-k} achieve the same value on the right-hand side. It remains to check $\frac{1}{2}(\lambda_1+\lambda_{n-k})$.

$$\left|\frac{\frac{1}{2}(\lambda_1+\lambda_{n-k})-\frac{1}{2}(\lambda_1+\lambda_{n-k})}{\frac{1}{2}(\lambda_1+\lambda_{n-k})}\right|^2 = 0,$$

and so the maximum is attained at λ_1 and λ_{n-k}, meaning

$$\max_{\lambda\in[\lambda_1,\lambda_{n-k}]}\left|\frac{\lambda-\frac{1}{2}(\lambda_1+\lambda_{n-k})}{\frac{1}{2}(\lambda_1+\lambda_{n-k})}\right|^2 = \left|\frac{\lambda_1-\lambda_{n-k}}{\lambda_1+\lambda_{n-k}}\right|^2.$$

(f) Finish the proof of the theorem.

Solution: From item (d) we have that

$$\|x_{k+1}-x^*\|_A^2 \le \max_{\lambda\in[\lambda_1,\lambda_{n-k}]}\left|\frac{\lambda-\frac{1}{2}(\lambda_1+\lambda_{n-k})}{\frac{1}{2}(\lambda_1+\lambda_{n-k})}\right|^2 \|x_0-x^*\|_A^2$$

and from item (e) we have

$$\max_{\lambda\in[\lambda_1,\lambda_{n-k}]}\left|\frac{\lambda-\frac{1}{2}(\lambda_1+\lambda_{n-k})}{\frac{1}{2}(\lambda_1+\lambda_{n-k})}\right|^2 = \left|\frac{\lambda_1-\lambda_{n-k}}{\lambda_1+\lambda_{n-k}}\right|^2.$$

Taking these facts together we obtain

$$\|x_{k+1}-x^*\|_A^2 \le \left|\frac{\lambda_1-\lambda_{n-k}}{\lambda_1+\lambda_{n-k}}\right|^2 \|x_0-x^*\|_A^2$$

which is the claim of Theorem 5.5.

4. *The coding for this problem can be done in Matlab or Python. If you choose Python, you can export the matrix L_{symm} and the vector b_{symm} to Python and do the rest of the work in Python.* The goal of this problem is to practice the *conjugate gradient algorithm (CG) with and without preconditioning* on a meaningful problem. An explanation for its setup is a bit long, but it is a typical case that a lot of effort is spent on problem setup. I have done the setup for you. Your job will be just to code the CG algorithms.

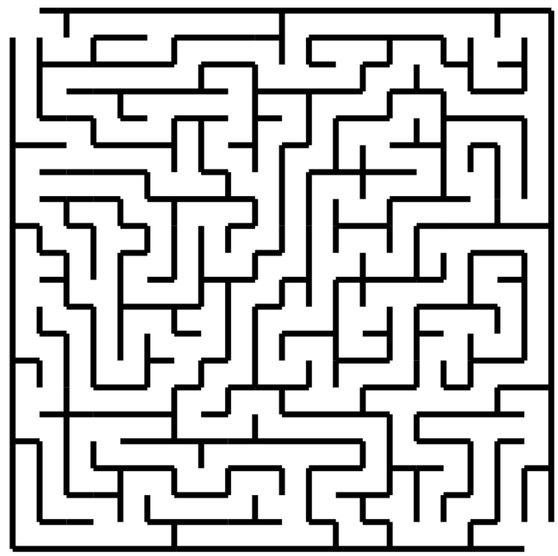


Figure 1: A maze with two exits.

Consider a maze with two exists, A, and B, shown in Fig. 1 taken from this paper by W. Ee and E. Vanden-Eijnden. This maze consists of a 20×20 array of cells, $N = 400$ cells in total. We will number the calls from 1 to 400 column-wise, i.e., the first column contains cells with indices from 1 to 20, the second one from 21 to 40, and so on. Exit A is at cell 1 while Exit B is at cell 400.

Let A be the adjacency matrix for this maze. A is 400×400, $A_{ij} = 1$ if cells i and j are adjacent and there is no wall between them, and $A_{ij} = 0$ otherwise. A random walker makes a step from a cell to any adjacent cell not separated by a wall with equal probability. The stochastic matrix (the other names for it are the transition matrix and the Markov matrix) for this random walk can be found as $P = R^{-1}A$ where R is a diagonal matrix with row sums of A along its diagonal. Row rums of P are all equal to 1, and P_{ij} is the probability that the random walker located at cell i will next move to cell j.

Our goal is to compute the *committor* $x \in \mathbb{R}^N$ for this random walk, i.e., the vector of probabilities whose component x_i is the probability that the walker located at cell i will first arrive at Exit B rather than Exit A. This vector of probabilities satisfies:

$$x_1 = 0, \quad x_N = 1, \quad x_i = \sum_{j=1}^{N} P_{ij} x_j, \quad 2 \le i \le N-1. \tag{7}$$

Equation (7) is obtained from the following reasoning. The probability of exiting via B rather than A from cell i is equal to the sum of products of probabilities to a cell j from i and exit from j via B rather than A. The sum is over all other cells j. Equation (7) can be written in the matrix form as follows. We denote $P - I$ by L. Then

$$L_{2:(N-1),2:(N-1)} x_{2:(N-1)} = b_{2:(N-1)}, \tag{8}$$

where b is obtained by $b = -Le_N$, where e_N is the vector with entries $1, \dots, N-1$ equal to zero and entry N equal to 1. You can check this, or just believe me. Recall that $L = R^{-1}A - I$. It is not a symmetric matrix, but it can be symmetrized as follows. We will mark with tilde all submatrices $(2:(N-1), 2:(N-1))$ and all subvectors with indices in $2:(N-1)$. Then we symmetrize $\tilde{L}$ using the fact that the adjacency matrix A is symmetric, $\tilde{L} = \tilde{R}^{-1}\tilde{A} - \tilde{I}$, and multiplication by $\tilde{R}^{\pm 1/2}$.

$$\tilde{L}\tilde{x} = \tilde{b}, \quad \text{the linear system we need to solve;} \tag{9}$$

$$(\tilde{R}^{-1}\tilde{A} - \tilde{I})\tilde{x} = \tilde{b}, \quad \text{recall what is } \tilde{L}; \tag{10}$$

$$\tilde{R}^{1/2}(\tilde{R}^{-1}\tilde{A} - \tilde{I})\tilde{R}^{-1/2}\tilde{R}^{1/2}\tilde{x} = \tilde{R}^{1/2}\tilde{b}, \quad \text{multiply by } \tilde{R}^{1/2}\text{, insert } \tilde{R}^{-1/2}\tilde{R}^{1/2}; \tag{11}$$

$$(\tilde{R}^{-1/2}\tilde{A}\tilde{R}^{-1/2} - \tilde{I})\tilde{R}^{1/2}\tilde{x} = \tilde{R}^{1/2}\tilde{b}, \quad \text{do some algebra.} \tag{12}$$

The matrix $\tilde{R}^{-1/2}\tilde{A}\tilde{R}^{-1/2} - \tilde{I}$ is symmetric. $\tilde{R}^{1/2}\tilde{x} =: y$ is a new vector of unknowns. $\tilde{R}^{1/2}\tilde{b} =: b_{\mathsf{symm}}$ is the new right-hand side. It is possible to check (just believe me), that the matrix $-L_{\mathsf{symm}}$ is symmetric positive definite.

Thus, here is the linear system with a symmetric positive definite matrix:

$$\boxed{-L_{\mathsf{symm}}y = -b_{\mathsf{symm}}, \quad x_{2:(N-1)} = \tilde{R}^{-1/2}y.} \tag{13}$$

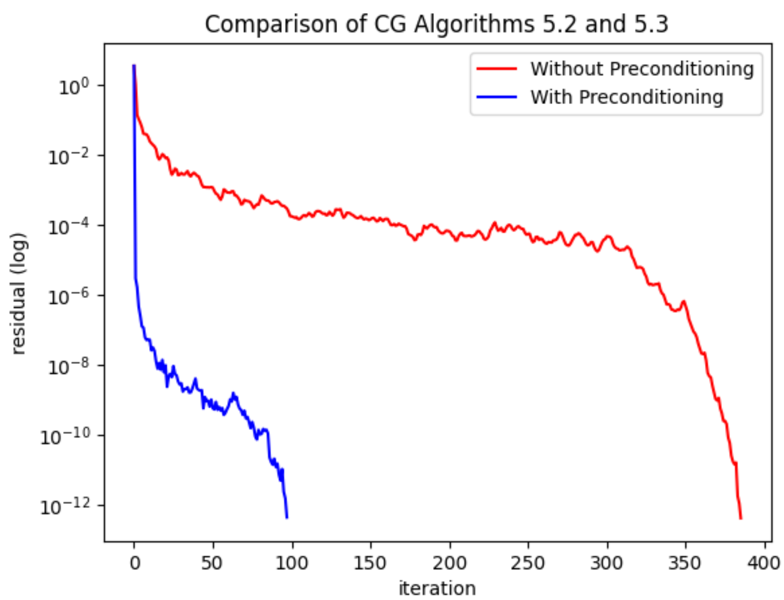
The Matlab code `random_walk_in_maze.m` visualizes the maze, sets up this linear system, solves it using the built-in solver "\", visualizes the solution, and plots the eigenvalues of $-L_{\mathsf{symm}}$. Task. Modify the code to solve Eq. (13) using the conjugate gradient algorithm without and with preconditioning (Algorithms 5.2 and 5.3) in [NW]. Use the incomplete Cholesky preconditioner. The corresponding Matlab command is

```
ichol_fac = ichol(sparse(A));
M = ichol_fac * ichol_fac';
```

Stop iterations when the residual will have a norm less than 10^{-12}. Plot the norm of the residuals after each iteration for the CG algorithm with and without preconditioning in the same figure. Use the logarithmic scale along the y-axis. Visualize the computed solution. Link you code to the pdf file with the homework.

The coding for this problem can be done in Matlab or Python. If you choose Python, you can export the matrix L_{symm} and the vector b_{symm} to Python and do the rest of the work in Python.

Solution: We plot the convergence of the two algorithms below via the code here



We see that the algorithm without conditioning takes about 380 iterations for the residual to not exceed 10^{-12}, while with conditioning, it takes only 100 iterations.