1. Suppose you need to evaluate the derivative of a function $f(x)$ by forward difference, i.e.,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}. \tag{1}$$

The function is not available analytically but can be evaluated at any point x with a relative error ϵ such that $|\epsilon| \le 10^{-14}$. Suppose the function and its second derivative are of the order of 1. Give a rough estimate of the optimal value for h that minimizes the error in $f'(x)$.

Solution: Since evaluating f introduces some error, our approximation will be given by

$$\frac{f(x+h)(1+\epsilon_1) - f(x)(1+\epsilon_2)}{h}$$

where $|\epsilon_1|, |\epsilon_2| \le 10^{-14}$. Further, by Taylor expansion we have that

$$f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n = f(x) + f'(x)h + f''(x)\frac{h^2}{2} + O(h^3)$$

and so our approximation yields

$$\begin{aligned} &\frac{\left(f(x) + f'(x)h + f''(x)\frac{h^2}{2} + O(h^3)\right)(1+\epsilon_1) - f(x)(1+\epsilon_2)}{h} \\ &= \frac{\left(f'(x)h + f''(x)\frac{h^2}{2} + O(h^3)\right)(1+\epsilon_1) + f(x)(\epsilon_1 - \epsilon_2)}{h} \\ &= \left(f'(x) + f''(x)\frac{h}{2} + O(h^2)\right)(1+\epsilon_1) + \frac{f(x)(\epsilon_1 - \epsilon_2)}{h} \end{aligned}$$

Then ignoring the $O(h^2)$ terms and approximating $f(x)$ and $f''(x)$ by 1 (since it was specified they are of this order), our approximation of $f'(x)$ roughly yields

$$\left(f'(x) + \frac{h}{2}\right)(1+\epsilon_1) + \frac{(\epsilon_1 - \epsilon_2)}{h}$$

and so by subtracting $f'(x)$, we find that the error is roughly given by

$$\left(f'(x) + \frac{h}{2}\right)(1+\epsilon_1) + \frac{(\epsilon_1 - \epsilon_2)}{h} - f'(x) = f'(x)\epsilon_1 + \frac{h}{2}(1+\epsilon_1) + \frac{\epsilon_1 - \epsilon_2}{h}.$$

In order to minimize this error, we differentiate this expression with respect to h:

$$\frac{d}{dh}\left[f'(x)\epsilon_1 + \frac{h}{2}(1+\epsilon_1) + \frac{\epsilon_1 - \epsilon_2}{h}\right] = \frac{1+\epsilon_1}{2} - \frac{\epsilon_1 - \epsilon_2}{h^2}$$

Then setting this expression equal to 0 and solving for h, we find

$$h^2 = \frac{2(\epsilon_1 - \epsilon_2)}{1+\epsilon_1} \le \frac{4 \cdot 10^{-14}}{1 - 10^{-14}} \approx 4 \cdot 10^{-14}$$

suggesting the optimal value for h to minimize the error in $f'(x)$ is

$$h \approx \sqrt{4 \cdot 10^{-14}} = 2 \cdot 10^{-7}.$$

2. Consider the polynomial space $\mathcal{P}_n(x)$, $x \in [-1, 1]$. Let T_k, $k = 0, 1, \ldots, n$, be the Chebyshev basis in it. The Chebyshev polynomials are defined via

$$T_k = \cos(k \arccos x).$$

(a) Use the trigonometric formula

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

to derive the three-term recurrence relationship for the Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad k = 1, 2, \ldots \tag{2}$$

Solution: We will first prove the base cases of the recurrence. By definition of the Chebyshev polynomials,

$$T_0 = \cos(0 \arccos x) = \cos(0) = 1$$

and likewise

$$T_1 = \cos(1 \arccos x) = \cos(\arccos x) = x$$

since $x \in [-1, 1]$ where arccos is defined. It remains to show that

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$$

for $k \in \mathbb{N}$ such that $k \geq 1$. By the definition of Chebyshev polynomials, the above equation is true if and only if

$$\cos((k+1)\arccos x) = 2x\cos(k \arccos x) - \cos((k-1)\arccos x).$$

Then, adding $\cos((k-1)\arccos x)$ to both sides yields

$$\cos((k+1)\arccos x) + \cos((k-1)\arccos x) = 2x\cos(k \arccos x)$$

which by the given trigonometric identity is true if and only if

$$2\cos\left(\frac{2k \arccos x}{2}\right)\cos\left(\frac{2\arccos x}{2}\right) = 2x\cos(k \arccos x)$$

or equivalently,

$$2\cos(k \arccos x)\cos(\arccos x) = 2x\cos(k \arccos x).$$

Finally, note that $\cos(\arccos x) = x$ for $x \in [-1, 1]$, and hence the original equation is true if and only if

$$2x\cos(k \arccos x) = 2x\cos(k \arccos x),$$

which is clearly true.

(b) Consider the differentiation map

$$\frac{d}{dx} : \mathcal{P}_n \to \mathcal{P}_{n-1}.$$

Write the matrix of the differentiation map with respect to the Chebyshev bases in $\mathcal{P}_n$ and $\mathcal{P}_{n-1}$ for $n = 7$. *Hint: you might find helpful properties of Chebyshev polynomials presented in Section 3.3.1 of Gil, Segura, Temme, "Numerical Methods For Special Functions".*

Solution: From part (a), we know that $T_0(x) = 1$ and $T_1(x) = x$. We first compute the Chebyshev polynomials up to T_7 using the recurrence proven in part (a):

- $T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$
- $T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 3x$
- $T_4(x) = 2xT_3(x) - T_2(x) = 8x^4 - 8x^2 + 1$
- $T_5(x) = 2xT_4(x) - T_3(x) = 16x^5 - 20x^3 + 5x$
- $T_6(x) = 2xT_5(x) - T_4(x) = 32x^6 - 48x^4 + 18x^2 - 1$
- $T_7(x) = 2xT_6(x) - T_5(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

We then differentiate each of the Chebyshev polynomials found above and write them in terms of the Chebyshev basis:

- $T_0'(x) = 0$
- $T_1'(x) = 1 = T_0(x)$
- $T_2'(x) = 4x = 4T_1(x)$
- $T_3'(x) = 12x^2 - 3 = 6T_2(x) + 3T_0(x)$
- $T_4'(x) = 32x^3 - 16x = 8T_3(x) + 8T_1(x)$
- $T_5'(x) = 80x^4 - 60x^2 + 5 = 10T_4(x) + 10T_2(x) + 5T_0(x)$
- $T_6'(x) = 192x^5 - 192x^3 + 36x = 12T_5(x) + 12T_3(x) + 12T_1(x)$
- $T_7'(x) = 448x^6 - 560x^4 + 168x^2 - 7 = 14T_6(x) + 14T_4(x) + 14T_2(x) + 7T_0(x)$

and hence the matrix of the differentiation map is

$$\begin{bmatrix} 0 & 1 & 0 & 3 & 0 & 5 & 0 & 7 \\ 0 & 0 & 4 & 0 & 8 & 0 & 12 & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & 0 & 14 \\ 0 & 0 & 0 & 0 & 8 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \end{bmatrix}$$

with respect to the Chebyshev bases $\mathcal{P}_7$ and $\mathcal{P}_6$.

3. Let $A = (a_{ij})$ be an $m \times n$ matrix.

   (a) Prove that the l_1-norm of A is

$$\|A\|_1 = \max_j \sum_i |a_{ij}|,$$

i.e., the maximal column sum of absolute values. Find the maximizing vector.

Solution: We will first show that $\|A\|_1 \le \max_j \sum_i |a_{ij}|$. By the definition of matrix norms and the l_1-norm for vectors,

$$\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{x \neq 0} \frac{\sum_{i=1}^m |(Ax)_i|}{\sum_{j=1}^n |x_j|}.$$

But then note that by the definition of matrix multiplication,

$$(Ax)_i = \sum_{k=1}^n a_{ik} x_k$$

and hence

$$\|A\|_1 = \max_{x \neq 0} \frac{\sum_{i=1}^m |\sum_{k=1}^n a_{ik} x_k|}{\sum_{j=1}^n |x_j|}.$$

Further, note that by the triangle inequality and multiplicativity of absolute value,

$$\left| \sum_{k=1}^n a_{ik} x_k \right| \le \sum_{k=1}^n |a_{ik} x_k| = \sum_{k=1}^n |a_{ik}| |x_k|,$$

so we can upper bound

$$\begin{aligned} \|A\|_1 &= \max_{x \neq 0} \frac{\sum_{i=1}^m |\sum_{k=1}^n a_{ik} x_k|}{\sum_{j=1}^n |x_j|} \\ &\le \max_{x \neq 0} \frac{\sum_{i=1}^m \sum_{k=1}^n |a_{ik}| |x_k|}{\sum_{j=1}^n |x_j|} = \max_{x \neq 0} \frac{\sum_{k=1}^n \sum_{i=1}^m |a_{ik}| |x_k|}{\sum_{j=1}^n |x_j|}. \end{aligned}$$

Further, note that for $1 \le k \le n$,

$$\sum_{i=1}^m |a_{ik}| \le \max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}|,$$

and so we can eliminate dependence on x, since

$$\begin{aligned} \|A\|_1 &\le \max_{x \neq 0} \frac{\sum_{k=1}^n \sum_{i=1}^m |a_{ik}| |x_k|}{\sum_{j=1}^n |x_j|} \le \max_{x \neq 0} \frac{\sum_{k=1}^n \max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}| |x_k|}{\sum_{j=1}^n |x_j|} \\ &= \max_{x \neq 0} \frac{\max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}| \sum_{k=1}^n |x_k|}{\sum_{j=1}^n |x_j|} = \max_{x \neq 0} \max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}| = \max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}|. \end{aligned}$$

Hence, we have that $\|A\|_1 \leq \max_{1\leq j\leq n} \sum_{i=1}^m |a_{ij}|$. Now towards showing the other direction, let j' be the maximizer of

$$\max_{1\leq j\leq n} \sum_{i=1}^m |a_{ij}|$$

and further, let v be the vector with 1 as the j'th entry, and all zeros otherwise,

$$v = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}^\top.$$

Then note that

$$\|v\|_1 = |0| + \cdots + |0| + |1| + |0| + \cdots + |0| = 1$$

and also that

$$\|Av\|_1 = \sum_{i=1}^m |(Av)_i| = \sum_{i=1}^m \left| \sum_{j=1}^n a_{ij} v_j \right| = \sum_{i=1}^m |a_{ij'}| = \max_{1\leq j\leq n} \sum_{i=1}^m |a_{ij}|$$

since $v_j = 0$ when $j \neq j'$. Hence,

$$\|A\|_1 = \max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} \geq \frac{\|Av\|_1}{\|v\|_1} = \|Av\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^m |a_{ij}|$$

and therefore both directions hold, so

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^m |a_{ij}|$$

as desired.

(b) Prove that the max-norm or l_∞-norm of A is

$$\|A\|_{\max} = \max_i \sum_j |a_{ij}|,$$

i.e., the maximal row sum of absolute values. Find the maximizing vector.

Solution: We will first show that $\|A\|_{\max} \leq \max_i \sum_j |a_{ij}|$. By the definition of matrix norms and the l_∞-norm for vectors,

$$\|A\|_{\max} = \max_{x\neq 0} \frac{\|Ax\|_{\max}}{\|x\|_{\max}} = \max_{x\neq 0} \frac{\max_{1\leq i\leq m} |(Ax)_i|}{\max_{1\leq k\leq n} |x_k|}.$$

But then note that by the definition of matrix multiplication,

$$(Ax)_i = \sum_{j=1}^n a_{ij} x_j$$

and hence

$$\|A\|_{\max} = \max_{x \neq 0} \frac{\max_{1 \leq i \leq m} \left| \sum_{j=1}^n a_{ij} x_j \right|}{\max_{1 \leq k \leq n} |x_k|}.$$

Further, note that by the triangle inequality and the multiplicativity of absolute value,

$$\left| \sum_{j=1}^n a_{ij} x_j \right| \leq \sum_{j=1}^n |a_{ij} x_j| = \sum_{j=1}^n |a_{ij}||x_j| \leq \sum_{j=1}^n |a_{ij}| \max_{1 \leq k \leq n} |x_k| = \max_{1 \leq k \leq n} |x_k| \sum_{j=1}^n |a_{ij}|.$$

Therefore, we can upper bound $\|A\|_{\max}$ and eliminate dependence on x since

$$\begin{aligned} \|A\|_{\max} = \max_{x \neq 0} \frac{\max_{1 \leq i \leq m} \left| \sum_{j=1}^n a_{ij} x_j \right|}{\max_{1 \leq k \leq n} |x_k|} &\leq \max_{x \neq 0} \frac{\max_{1 \leq i \leq m} \max_{1 \leq k \leq n} |x_k| \sum_{j=1}^n |a_{ij}|}{\max_{1 \leq k \leq n} |x_k|} \\ &= \max_{x \neq 0} \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}| = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|, \end{aligned}$$

and so we have that $\|A\|_{\max} \leq \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$. Now towards showing the other direction, let i' denote the row of A with the maximal row sum and define

$$v = \begin{bmatrix} \mathsf{sign}(a_{i'1}) & \mathsf{sign}(a_{i'2}) & \dots & \mathsf{sign}(a_{i'n}) \end{bmatrix}^\top.$$

Then note that

$$\|v\|_{\max} = \max_{1 \leq j \leq n} |\mathsf{sign}(a_{i'j})| = \max_{1 \leq j \leq n} 1 = 1$$

and also that

$$\begin{aligned} \|Av\|_{\max} = \max_{1 \leq i \leq m} |(Av)_i| = \max_{1 \leq i \leq m} \left| \sum_{j=1}^n a_{ij} v_j \right| &= \left| \sum_{j=1}^n a_{i'j} v_j \right| \\ &= \left| \sum_{j=1}^n a_{i'j} \mathsf{sign}(a_{i'j}) \right| = \left| \sum_{j=1}^n |a_{i'j}| \right| = \sum_{j=1}^n |a_{i'j}| = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|. \end{aligned}$$

Hence,

$$\|A\|_{\max} = \max_{x \neq 0} \frac{\|Ax\|_{\max}}{\|x\|_{\max}} \geq \frac{\|Av\|_{\max}}{\|v\|_{\max}} = \|Av\|_{\max} = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$$

and therefore both directions hold, so

$$\|A\|_{\max} = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$$

as desired.