1. Show that the Pauli matrices X, Y, Z together with the identity matrix I_2 build a complete basis for 2×2 matrices over the set of complex numbers (quaternion space).

Solution: We claim first that the set $\{X, Y, Z, I_2\}$ of Pauli matrices together with the identity matrix spans $\mathbb{C}^{2\times 2}$. Let $A \in \mathbb{C}^{2\times 2}$, in which case we can write

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

where α, β, γ, $\delta \in \mathbb{C}$. Further, we recall that

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We now seek $a_1, a_2, a_3, a_4 \in \mathbb{C}$ such that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = A = a_1 X + a_2 Y + a_3 Z + a_4 I_2 = \begin{bmatrix} a_3 + a_4 & a_1 - a_2 i \\ a_1 + a_2 i & -a_3 + a_4 \end{bmatrix},$$

and so we see that $a_1 = \frac{1}{2}(\beta + \gamma)$, $a_2 = \frac{i}{2}(\beta - \gamma)$, $a_3 = \frac{1}{2}(\alpha - \delta)$ and $a_4 = \frac{1}{2}(\alpha + \delta)$. Hence, we have written our arbitrary $A \in \mathbb{C}^{2\times 2}$ as the linear combination

$$\frac{1}{2}(\beta + \gamma)X + \frac{i}{2}(\beta - \gamma)Y + \frac{1}{2}(\alpha - \delta)Z + \frac{1}{2}(\alpha + \delta)I_2$$

of $\{X, Y, Z, I_2\}$, meaning that this set spans $\mathbb{C}^{2\times 2}$.

It remains to show that $\{X, Y, Z, I_2\}$ is linearly independent. Suppose we have that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = a_1 X + a_2 Y + a_3 Z + a_4 I_2 = \begin{bmatrix} a_3 + a_4 & a_1 - a_2 i \\ a_1 + a_2 i & -a_3 + a_4 \end{bmatrix}.$$

Then since $a_1 + a_2 i = a_1 - a_2 i$, it must be that $a_2 = 0$, and likewise, since $a_3 + a_4 = -a_3 + a_4$, it must be that $a_3 = 0$. It follows that $a_1 = a_4 = 0$ also, and so the equation above has only the trivial solution. Hence, $\{X, Y, Z, I_2\}$ is linearly independent. Since it also spans $\mathbb{C}^{2\times 2}$, it is a basis for $\mathbb{C}^{2\times 2}$.

2. Pauli matrices are often denoted as σ_j ($j = x, y, z$), with $\sigma_x = X$, $\sigma_y = Y$, $\sigma_z = Z$, and combined to a vector of matrices: $\vec{\sigma} = (X, Y, Z)^\top$.

Show that a qubit oriented in an arbitrary direction $\vec{n} = (n_x, n_y, n_z)^\top$ on the Bloch sphere can be represented as $\vec{n} \cdot \vec{\sigma}$, where '$\cdot$' denotes the scalar product (dot-product) of vectors:

$$\vec{a} \cdot \vec{b} = \vec{a}^\top \vec{b} = \sum_i a_i b_i.$$

What are the elements of the resulting matrix?

Solution: By the definition of dot product, we know the elements of the resulting matrix:

$$\vec{n}\cdot\vec{\sigma} = n_xX + n_yY + n_zZ = \begin{bmatrix} 0 & n_x \\ n_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & -in_y \\ in_y & 0 \end{bmatrix} + \begin{bmatrix} n_z & 0 \\ 0 & -n_z \end{bmatrix} = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix}.$$

It now suffices to show that there exists such an $\vec{n}$ any direction on the Bloch sphere; recall the directions on the Bloch sphere are parameterized by angles θ and φ. We can then simply define

$$\vec{n} = (n_x, n_y, n_z)^\top = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))^\top.$$

3. The exponentiation of a square matrix A can be regarded as exponentiation of each matrix element a_{ij} or as short form for the power series (for any $\lambda \in \mathbb{C}$):

$$e^{\lambda A} = \sum_{n=0}^{\infty} \frac{\lambda^n A^n}{n!}.$$

Show that for an involutory $n \times n$ matrix A: $e^{i\varphi A} = I_n\cos(\varphi) + iA\sin(\varphi)$, where I_n is the n-dimensional identity matrix.

Solution: Let $A \in \mathbb{C}^{n\times n}$ be involutory, meaning that $A^2 = I_n$. Then regarding the matrix exponentiation as a power series, we can split the series into even and odd terms

$$e^{i\varphi A} = \sum_{k=0}^{\infty} \frac{(i\varphi)^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{(i\varphi)^{2k} A^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i\varphi)^{2k+1} A^{2k+1}}{(2k+1)!}.$$

But then note that since A is involutory, $A^{2k} = I_n$ and $A^{2k+1} = A$. Hence,

$$e^{i\varphi A} = \sum_{k=0}^{\infty} \frac{(i\varphi)^{2k} I_n}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i\varphi)^{2k+1} A}{(2k+1)!}$$

Further, $(i\varphi)^{2k} = i^{2k}\varphi^{2k} = (-1)^k\varphi^{2k}$ and therefore $(i\varphi)^{2k+1} = (-1)^k\varphi^{2k+1}i$, meaning

$$e^{i\varphi A} = \sum_{k=0}^{\infty} \frac{(-1)^k\varphi^{2k} I_n}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k\varphi^{2k+1} iA}{(2k+1)!} = I_n \sum_{k=0}^{\infty} \frac{(-1)^k\varphi^{2k}}{(2k)!} + Ai\sum_{k=0}^{\infty} \frac{(-1)^k\varphi^{2k+1}}{(2k+1)!}.$$

Finally, recognize that these sums are the Maclaurin series for cos and sin respectively, meaning that

$$e^{i\varphi A} = I_n\cos(\varphi) + iA\sin(\varphi)$$

as desired.

4. A rotation about the x-, y-, or z-axis in 2-state quantum systems (e.g. electron spin or qubit) is realized by $R_j(\theta) = e^{-\frac{i}{2}\sigma_j\theta}$ (with $j = x, y, z$).

(a) Show that: $R_j(\theta) \equiv e^{-\frac{i}{2}\sigma_j\theta} = I_2\cos(\theta/2) - i\sigma_j\sin(\theta/2)$.

Solution: Observe first that

$$\sigma_x^2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\sigma_y^2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\sigma_z^2 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

and hence σ_j is involutory for each $j = x, y, z$. Therefore, we will apply the result of the previous question. In particular,

$$R_j(\theta) \equiv e^{-\frac{i}{2}\sigma_j\theta} = e^{i(-\theta/2)\sigma_j} = I_2\cos(-\theta/2) + i\sigma_j\sin(-\theta/2).$$

Finally, recall that cos is an even function, so $\cos(-\theta/2) = \cos(\theta/2)$, and likewise sin is an odd function, so $\sin(-\theta/2) = -\sin(\theta/2)$. Therefore, we have that

$$R_j(\theta) = I_2\cos(\theta/2) - i\sigma_j\sin(\theta/2)$$

as desired.

(b) Show that a rotation about an arbitrary axis $\vec{n} = (n_x, n_y, n_z)^\top$ can be written as $R_{\vec{n}}(\theta) = e^{-\frac{i}{2}\vec{n}\cdot\vec{\sigma}\theta} = I_2\cos(\theta/2) - i\vec{n}\cdot\vec{\sigma}\sin(\theta/2)$.

Solution: First recall the matrix $\vec{n}\cdot\vec{\sigma}$ from earlier and note that

$$(\vec{n}\cdot\vec{\sigma})^2 = \begin{bmatrix}n_z & n_x - in_y\\ n_x + in_y & -n_z\end{bmatrix}\begin{bmatrix}n_z & n_x - in_y\\ n_x + in_y & -n_z\end{bmatrix} = \begin{bmatrix}n_x^2 + n_y^2 + n_z^2 & 0\\ 0 & n_x^2 + n_y^2 + n_z^2\end{bmatrix}$$

and since $\vec{n} = (n_x, n_y, n_z)^\top$ denotes a direction on the unit sphere, $n_x^2 + n_y^2 + n_z^2 = 1$, meaning $(\vec{n}\cdot\vec{\sigma})^2 = I_2$, so $\vec{n}\cdot\vec{\sigma}$ is involutory. Hence,

$$e^{-\frac{i}{2}\vec{n}\cdot\vec{\sigma}\theta} = e^{i(-\theta/2)\vec{n}\cdot\vec{\sigma}} = I_2\cos(-\theta/2) + i\vec{n}\cdot\vec{\sigma}\sin(-\theta/2).$$

Finally, since cos is an even function and sin is an odd function, we have that

$$e^{-\frac{i}{2}\vec{n}\cdot\vec{\sigma}\theta} = I_2\cos(\theta/2) - i\vec{n}\cdot\vec{\sigma}\sin(\theta/2)$$

as desired.