IMPOSSIBILITY OF MAGIC SQUARE OF NINE SQUARES

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Abstract. Here we attempt to prove the impossibility of a 3×3 magic square consisting of 9 distinct squares.

1. General Formula of Order 3 Magic Squares

[Ste97] [Boy05] [Rab02] [Gar98]
Mark Underwood and Tim Roberts (quadratic residues)
Morgenstern (redux of Rabern, http://multimagie.com/MorgensternMssProperties.pdf)

2. Elementary Number Theory

We will need several results of modular arithmetic in the section that follow. We prove those results here.

Proposition 2.1. Addition TODO

Proposition 2.2. Congruence of powers TODO

Proposition 2.3. For all $a \in \mathbb{Z}$, either $a^2 \equiv 0 \mod 4$ or $a^2 \equiv 1 \mod 4$.

Proof. If a is even, we can write it as $a = 2k$ for some $k \in \mathbb{Z}$, in which case $a^2 = 4k^2$, so $a^2 \equiv 0 \mod 4$. Conversely, if a is odd, we can write it as $a = 2k + 1$ for some $k \in \mathbb{Z}$, in which case $a^2 = 4(k^2 + k) + 1$, so $a^2 \equiv 1 \mod 4$. □

Proposition 2.4. Coprime divisibility TODO

3. Properties of Magic Squares of Squares

We would like to be able to impose restrictions on a, b, and c, both in order to accelerate algorithms searching for magic squares of squares, as well as to potentially derive a contradiction implying their impossibility. Towards this end, we claim the following.

Lemma 3.1. Both $a \equiv 0 \mod 3$ and $b \equiv 0 \mod 3$.

Proof. Since c is a square, either $c \equiv 0 \mod 3$ or $c \equiv 1 \mod 3$. If the former, meaning $c \equiv 0 \mod 3$, then

(1) $a \equiv 1 \mod 3$ implies $c - a \equiv 2 \mod 3$, hence $c - a$ is not a square.
(2) $a \equiv 2 \mod 3$ implies $c + a \equiv 2 \mod 3$, hence $c + a$ is not a square.

Each case above implies an entry is non-square, so it must be that $a \equiv 0 \mod 3$. Otherwise, $c \equiv 1 \mod 3$, in which case

(1) $a \equiv 1 \mod 3$ implies $c + a \equiv 2 \mod 3$, hence $c + a$ is not a square.
(2) $a \equiv 2 \mod 3$ implies $c - a \equiv 2 \mod 3$, hence $c - a$ is not a square.

In either case, $a \equiv 0 \mod 3$. The argument for b is analogous. □

Lemma 3.2. Both $a \equiv 0 \mod 4$ and $b \equiv 0 \mod 4$.

Proof. Since c is a square, either $c \equiv 0 \mod 4$ or $c \equiv 1 \mod 4$. If the former, meaning $c \equiv 0 \mod 4$, then

(1) $a \equiv 1 \mod 4$ implies $c - a \equiv 3 \mod 4$, hence $c - a$ is not a square.
(2) $a \equiv 2 \mod 4$ implies $c + a \equiv 2 \mod 4$, hence $c + a$ is not a square.
(3) $a \equiv 3 \mod 4$ implies $c + a \equiv 3 \mod 4$, hence $c + a$ is not a square.

Each case above implies an entry is non-square, so it must be that $a \equiv 0 \mod 4$. Otherwise, $c \equiv 1 \mod 4$, in which case

(1) $a \equiv 1 \mod 4$ implies $c + a \equiv 2 \mod 4$, hence $c + a$ is not a square.
(2) $a \equiv 2 \mod 4$ implies $c + a \equiv 3 \mod 4$, hence $c + a$ is not a square.
(3) $a \equiv 3 \mod 4$ implies $c - a \equiv 2 \mod 4$, hence $c - a$ is not a square.

In either case, $a \equiv 0 \mod 4$. The argument for b is analogous. □

Theorem 3.3. Both $a \equiv 0 \mod 12$ and $b \equiv 0 \mod 12$.

Proof. Since $\gcd(3, 4) = 1$, this is immediate from the lemmata above. □

Corollary 3.4. All entries of a given magic square of squares are congruent to $c \mod 12$.

Proof. TODO □

Corollary 3.5. All entries of a given magic square of squares are of the same parity. That is, either all entries are even or all entries are odd.

Proof. This follows trivially from the previous corollary. □

We will use this result later to show that the special class of so-called *primitive* magic squares of squares have all odd entries.

4. Primitive Magic Squares of Squares

We will now show that the impossibility of a magic square of squares is implied the impossibility of a special class of so-called *primitive* magic squares of squares and impose even more restrictions on a, b, and c.

Definition 4.1. We say that a magic square is **primitive** if the gcd of all elements is one.

Theorem 4.2. If there exists a 3×3 magic square of squares, then there exists a 3×3 magic squares of squares which is primitive.

Proof. TODO □

Note that by the contrapositive of the theorem above, the impossibility of primitive 3×3 magic squares of squares implies the impossibility of 3×3 magic squares of squares in general.

By considering only primitive magic squares of squares, we are able to impose even more restrictions on a, b, and c.

Theorem 4.3. All entries are congruent to $1 \mod 24$.

Proof. TODO This result is due to Zimmerman. $\square$

Corollary 4.4. $a \equiv 0 \mod 24$, $b \equiv 0 \mod 24$, and $c \equiv 1 \mod 24$.

References

[Boy05] Christian Boyer. Some notes on the magic squares of squares problem. *The Mathematical Intelligencer*, 27(2):52–64, March 2005.

[Gar98] Martin Gardner. Some new discoveries about 3×3 magic squares. *Math Horizons*, 5(3):11–13, 1998.

[Rab02] Landon Rabern. Properties of magic squares of squares. *Rose-Hulman Undergraduate Mathematics Journal*, 4(1), July 2002.

[Ste97] Ian Stewart. Alphamagic squares. *Scientific American*, 276(1):106–109, 1997.