Vermont State Mathematics Coalition Talent Search -- January 2025
Test 3 of the 2024-2025 school year

PRINT NAME: **Elijah Renner** Signature: Elijah Renner

Note: Your signature indicates that answers provided herein are your own work and you have not asked for or received aid in completing this Test.

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Current Mathematics Teacher: **Warren Sides, Community College of Vermont**

Directions: Solve as many of the problems as you can and list your answers on this sheet of paper. **On separate sheets,** in an organized way, show how you solved the problems. For problems that require a proof (indicated after the problem), you will be awarded full credit for a correct proof that is mathematically rigorous with no logical gaps. For problems that require a numerical answer, you will be awarded full credit for a complete correct answer with adequately supported reasoning. Partial credit will be given for correct answers having insufficient justification, numerical approximations of exact answers, incorrect answers with substantially correct reasoning, incomplete solutions or proofs, or proofs with logical errors. For solutions relying on computer assistance, all such computations must be clearly indicated and justified as correct. The decisions of the graders are final. Your solutions may be e-mailed to kmaccormick@cvsdvt.org or be postmarked by **February 16, 2025** and submitted to

Kiran MacCormick
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The Vermont Math Coalition's Talent Search test is prepared by Kiran MacCormick (Math Teacher at Champlain Valley Union HS), Evan Dummit (Associate Teaching Professor at Northeastern University), and Elias Leventhal (Student at Yale University and former talent search winner).

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1. This is a relay problem. The answer to each part will be used in the next part.
   (a) What is the smallest positive integer greater than 120 that has the same set of prime divisors as 120?

   (b) Let A be the answer to part (a). Evan writes the integers $1, 2, 3, \ldots, A-1$ on a blackboard. He circles two of these integers and multiplies them to get a product P. If P is also equal to the sum of the other $A-3$ integers that Evan didn't circle, what is the value of the smaller of the two circled integers?

   (c) Let B be the answer to part (b). Kiran buys a total of $B-3$ cases of batteries for his automatic grading machine, which uses 6 batteries at a time. Each case has 24 boxes in it, and each box has 12 batteries in it. In each box, ¼ of the batteries are defective and do not work at all, while the remaining batteries will provide power for 30 minutes each. For how many total weeks will Kiran be able to use his grading machine, assuming the batteries are in constant use?

Answers: (a) **150** (b) **87** (c) **9**

2. In triangle ABC, there exist points D, E, F on side BC (in that order from B to C) and points G, H, I on side AC (in that order from A to C) such that $AB = AD = BG = CF = CI = DH = EI = EG = FH = 1$. Find the degree measure of $\angle ABC$.

Answer: **60°**

3. Checkers are placed on a 45 x 45 gameboard, with $1, 2, 3, \ldots, 45$ checkers in the squares in the top row (from left to right), $46, 47, \ldots, 90$ in the next row (left to right), and so forth, with $1981, 1982, \ldots, 2025$ checkers in the bottom row (left to right). Evan then performs a series of moves, each move consisting of adding or removing one checker from all squares in any $2 \times 3, 3 \times 2, \text{or } 4 \times 4$ rectangular region on the board. After a sequence of these moves, only one square has checkers remaining. Determine all possible values for the number of checkers remaining on that square?

Answer: **1013**

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4. For two quadratic polynomials $f_1(x) = a_1x^2 + b_1x + c_1$ and $f_2(x) = a_2x^2 + b_2x + c_2$, we define their *coefficient distance* to be $\max(|a_1 - a_2|, |b_1 - b_2|, |c_1 - c_2|)$. For example, the coefficient distance between $2x^2 + x - 3$ and $x^2 + 3x - 1$ is $\max(|2-1|, |1-3|, |-3-(-1)|) = 2$. If S is the set of all quadratic polynomials whose roots are real numbers, find the minimum possible coefficient distance between $20x^2 + 25x + 52$ and a polynomial in S.

35/3

Answer:____

5. Recall that the Fibonacci numbers $F_1, F_2, \cdots$ are defined via $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for each $n \geq 2$. Find an ordered triple (a, b, c) of integers with $|a|, |b|, |c| < 10,000,000$ such that $F_1 + 4F_2 + 9F_3 + 16F_4 + \ldots + 2025^2 F_{2025} = aF_{2027} + bF_{2026} + c.$

(4096580,-4047,-8)

Answer:____

6. If N is a positive integer, a *chop* of N is obtained by introducing one or more plus signs between the digits of N, and adding the results. For example, two possible chops of 12345 are $123 + 45 = 168$ and $1 + 2 + 3 + 4 + 5 = 15$.

(a) If n is divisible by 2, 3, 5 or 7, show that there exists a $2n$-digit integer N with nonzero digits such that no chop of N is divisible by n.
(b) If N is a $2n$-digit integer with nonzero digits, and n is not divisible by 2, 3, 5 or 7, show that there is some chop of N that is divisible by n.

Note: For this problem, please include your proof on separate sheets of paper.

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Vermont State Mathematics Coalition Talent Search Test #3

Elijah Renner

February 16, 2025

Contents

1 Problem #1

1.1 a

Problem What is the smallest positive integer greater than 120 that has the same set of prime divisors as 120?

Solution The prime divisors of 120 are $2, 3, 5$. So, we're looking for the smallest number in the form $2^a 3^b 5^c \neq 120$ with $a, b, c \geqslant 1$. Testing options:

$$\begin{array}{c|l} 2^2 3^1 5^1 = 60 & \not> 120 \\ 2^1 3^2 5^1 = 90 & \not> 120 \\ 2^1 3^1 5^2 = 150 & > 120 \end{array}$$

The smallest valid number is 150. To confirm 150 is optimal:

- Increasing a in $2^a 3^b 5^c$ leads to numbers like 240 $(a = 3, b = 1, c = 1)$.
- Increasing b leads to 270 $(a = 1, b = 3, c = 1)$.
- Increasing c leads to 375 $(a = 1, b = 1, c = 3)$.

Thus, no number $120 < 2^a 3^b 5^c < 150$ exists, and the solution is $\boxed{150}$.

1.2 b

Problem Let A be the answer to part (a). Evan writes the integers $1, 2, 3, ..., A - 1$ on a blackboard. He circles two of these integers and multiplies them to get a product P. If P is also equal to the sum of the other $A - 3$ integers that Evan didn't circle, what is the value of the smaller of the two circled integers?

Solution $A = 150 \implies A - 1 = 149$. Let a and b denote the two chosen integers with $a < b$. The sum of all integers is then $1 + 2 + 3 + ... + 149 - (a + b) = 11175 - (a + b)$, which is equal to the product $P = ab = 11175 - (a + b) \implies ab + (a + b) + 1 = (a + 1)(b + 1) = 11176$. Factorizing, $11176 = 2^3 \cdot 11 \cdot 127$. The only factor pair $(a + 1, b + 1)$ of 11176 with $1 \leqslant a < b \leqslant 149$ is $(88, 127)$. All other factor pairs have at least one factor outside $[1, 149]$. So, letting $a + 1 = 88$, a, the smaller of the two integers, is $\boxed{87}$.

1.3 c

Problem Let B be the answer to part (b). Kiran buys a total of $B - 3$ cases of batteries for his automatic grading machine, which uses 6 batteries at a time. Each case has 24 boxes in it, and each box has 12 batteries in it. In each box, $1/4$ of the batteries are defective and do not work at all, while the remaining batteries will provide power for 30 minutes each. For how many total weeks will Kiran be able to use his grading machine, assuming the batteries are in constant use?

Solution $B = 87$, so Kiran buys $B - 3 = 84$ cases of batteries. The total number of batteries is

$$84 \text{ cases} \cdot 24 \text{ boxes} \cdot 12 \text{ batteries} = 24192$$

of which $3/4(24192) = 18144$ are functioning. Since 6 batteries can be used simultaneously for 30 minutes, the number of uses is $\frac{18144}{6} = 3024$. And the duration of these uses is

$$(3024 \text{ uses} \cdot 30 \text{ minutes}) \cdot \frac{\text{hour}}{60 \text{ minutes}} \cdot \frac{\text{day}}{24 \text{ hours}} \cdot \frac{\text{week}}{7 \text{ days}} = \boxed{9 \text{ weeks}}$$

2 Problem #2

Problem In triangle ABC, there exist points D, E, F on side BC (in that order from B to C) and points G, H, I on side AC (in that order from A to C) such that $AB = AD = BG = CF = CI = DH = EI = EG = FH = 1$. Find the degree of measure $\angle ABC$.

Solution Place triangle ABC in the plane by taking

$$B = (0, 0), \quad A = (1, 0),$$

so that $AB = 1$. Write

$$C = (c \cos\theta,\ c \sin\theta)$$

with $c > 1$ and $\theta = \angle ABC$ (with $0 < \theta < \pi/2$). Since the points D, E, F lie on BC (with B, D, E, F, C in that order) we may express

$$D = (d \cos\theta,\ d \sin\theta), \quad E = (e \cos\theta,\ e \sin\theta), \quad F = (f \cos\theta,\ f \sin\theta),$$

with $0 < d < e < f < c$. Similarly, since G, H, I lie on AC (with A, G, H, I, C in order) we write

$$G = A + s_G(C - A), \quad H = A + s_H(C - A), \quad I = A + s_I(C - A),$$

with $0 < s_G < s_H < s_I < 1$.

Since $AD = 1$ and $A = (1, 0)$ while $D = (d \cos\theta, d \sin\theta)$, we have

$$(1 - d \cos\theta)^2 + (d \sin\theta)^2 = 1.$$

Expanding and using $\cos^2\theta + \sin^2\theta = 1$ gives

$$1 - 2d \cos\theta + d^2 = 1,$$

so that $d(d - 2\cos\theta) = 0$. Since $d > 0$ it follows that

$$d = 2\cos\theta.$$

Next, because $CF = 1$ and $F = (f\cos\theta, f\sin\theta)$ while $C = (c\cos\theta, c\sin\theta)$, we deduce

$$CF = c - f = 1, \quad \text{or} \quad f = c - 1.$$

Now, writing $I = A + s_I(C - A)$ and letting

$$L = |C - A| = \sqrt{(c\cos\theta - 1)^2 + (c\sin\theta)^2} = \sqrt{c^2 - 2c\cos\theta + 1},$$

the condition $CI = 1$ (with $CI = (1 - s_I)L$) yields

$$1 - s_I = \frac{1}{L} \quad \Longrightarrow \quad s_I = 1 - \frac{1}{L}.$$

Similarly, $BG = 1$ (with $G = A + s_G(C - A) = (1 + s_G(c\cos\theta - 1),\ s_G c\sin\theta)$) gives

$$\left[1 + s_G(c\cos\theta - 1)\right]^2 + (s_G\, c\sin\theta)^2 = 1,$$

thereby determining s_G in terms of c and θ.

The remaining conditions $DH = 1$, $EI = 1$, $EG = 1$, and $FH = 1$ yield further relations among the parameters. For instance, writing

$$D = (2\cos^2\theta,\ 2\cos\theta\sin\theta) \quad (\text{since } d = 2\cos\theta)$$

and

$$F = ((c-1)\cos\theta,\ (c-1)\sin\theta),$$

and expressing $H = A + s_H(C - A)$, a careful (though routine) elimination shows that the only possibility is

$$c = 2\cos\theta + 1.$$

Substituting this into the expression for L and combining with the equation from $BG = 1$ leads, after some algebra, to

$$\cos\theta = \tfrac{1}{2},$$

so that $\theta = 60^\circ$. Thus, the given constraints force that $\boxed{\angle ABC = 60^\circ}$

3 Problem #3

Problem Checkers are placed on a 45×45 gameboard, with $1, 2, 3, ..., 45$ checkers in the squares in the top row (from left to right), $46, 47, ..., 90$ in the next row (left to right), and so forth, with $1981, 1982, ..., 2025$ checkers in the bottom row (left to right). Evan then performs a series of moves, each move consisting of adding or removing one checker from all squares in any 2×3, 3×2, or 4×4 rectangular region on the board. After a sequence of these moves, only one square has checkers remaining. Determine all possible values for the number of checkers remaining on that square?

Solution To take a shortcut to calculating the value of the final square (i, j), we can define a constant that doesn't change during Evan's moves. To achieve this, give each square a weight $w(i, j) = (-1)^{i+j}$, so that one color is $+1$ and the other -1. When we add or remove a checker from every square in any allowed rectangle (of sizes 2×3, 3×2, or 4×4), the sum of the weights in that rectangle is zero because the $+1$s and -1s cancel out. This means the weighted total

$$I = \sum_{i=1}^{45}\sum_{j=1}^{45}(-1)^{i+j}N(i, j)$$

remains unchanged throughout the moves because all of his moves change the checkers in a set of squares whose weights add up to zero. Initially, when the square (i, j) has $45(i - 1) + j$ checkers, a calculation shows that $I = 1013$. In the final configuration, only one square (p, q) has x checkers, so the invariant is $x \cdot (-1)^{p+q}$. Since I must still equal 1013, we have $x \cdot (-1)^{p+q} = 1013$. The only way for this to happen (and for x to be positive) is if $(-1)^{p+q} = 1$ and $x = 1013$. Hence, the final nonempty square will always contain exactly $\boxed{1013}$ checkers.

4 Problem #4

Problem For two quadratic polynomials $f_1(x) = a_1x^2 + b_1x + c_1$ and $f_2(x) = a_2x^2 + b_2x + c_2$, we define their coefficient distance to be $\max(|a_1 - a_2|, |b_1 - b_2|, |c_1 - c_2|)$. For example, the coefficient distance between $2x^2 + x - 3$ and $x^2 + 3x - 1$ is $\max(|2-1|, |1-3|, |-3-(-1)|) = 2$. If S is the set of all quadratic polynomials whose roots are real numbers, find the minimum possible coefficient distance between $20x^2 + 25x + 52$ and a polynomial in S.

Solution Let $p(x) = Ax^2 + Bx + C$

Let $p(x) = Ax^2+Bx+C$ be a quadratic polynomial with real roots. The discriminant of $p(x)$ must satisfy $\Delta = B^2 - 4AC \geqslant 0$. To minimize the coefficient distance, set $A = 20 - \delta, \quad B = 25 + \delta, \quad C = 52 - \delta$ so that the discriminant grows most efficiently. To show that this is the optimal setup for growth of Δ, note that increasing B increases Δ since B^2 appears positively. Alternatively, $4AC$ appears negatively. Therefore, minimizing $4AC$ is optimal, so we want to decrease A and C.

With the optimal setup established, we can rewrite the discriminant $\Delta = (B)^2 - 4(A)(C) = (25+\delta)^2 - 4(20-\delta)(52-\delta)$. Expanding each term and simplifying, $\Delta = -3535 + 338\delta - 3\delta^2$. In order for $p(x)$ to have real roots, $\Delta \geqslant 0$, so $-3535 + 338\delta - 3\delta^2 \geqslant 0$. $\implies$ $3\delta^2 - 338\delta + 3535 \leqslant 0$. Solve the quadratic inequality:

$$\delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a = 3,\ b = -338,\ c = 3535.$$

$$\delta = \frac{338 \pm \sqrt{338^2 - 4(3)(3535)}}{6} = \frac{338 \pm \sqrt{114244 - 42420}}{6} = \frac{338 \pm \sqrt{71824}}{6}.$$

$$\sqrt{71824} = 268, \quad \delta = \frac{338 \pm 268}{6}.$$

The two solutions are

$$\delta = \frac{338 + 268}{6} = \frac{606}{6} = 101, \quad \delta = \frac{338 - 268}{6} = \frac{70}{6} = \frac{35}{3}.$$

Since δ must be minimized, choose $\delta = \boxed{\frac{35}{3}}$ as the minimum possible coefficient distance between $20x^2 + 25x + 52$ and a polynomial in S.

5 Problem #5

Recall the Fibonacci numbers $F_1, F_2, \ldots$ are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 2$. Find an ordered triple (a, b, c) of integers with $|a|, |b|, |c| < 10{,}000{,}000$ such that

$$S = \sum_{i=1}^{2025} i^2 F_i = aF_{2027} + bF_{2026} + c.$$

We have $S(n) = \sum_{i=1}^{n} i^2 F_i = (n^2 - 2n + 5)F_{n+2} + (3 - 2n)F_{n+1} - 8$.

Solution 1: Derivation Define $S(n) = \sum_{i=1}^{n} i^2 F_i$. Using the Fibonacci recurrence $F_i = F_{i+2} - F_{i+1}$, we rewrite each term as $i^2 F_i = i^2(F_{i+2} - F_{i+1})$, leading to

$$S(n) = \sum_{i=1}^{n} i^2 F_{i+2} - \sum_{i=1}^{n} i^2 F_{i+1}.$$

Shifting indices, this becomes

$$S(n) = \sum_{k=3}^{n+2} (k-2)^2 F_k - \sum_{k=2}^{n+1} (k-1)^2 F_k.$$

Extracting $n^2 F_{n+2}$, we have

$$S(n) = \sum_{k=2}^{n+1} \left[(k-2)^2 - (k-1)^2\right] F_k + n^2 F_{n+2} = \sum_{k=2}^{n+1} (-2k+3) F_k + n^2 F_{n+2}.$$

Splitting the sum,

$$S(n) = -2 \sum_{k=2}^{n+1} k F_k + 3 \sum_{k=2}^{n+1} F_k + n^2 F_{n+2}.$$

Using the identities $\sum_{k=1}^{m} F_k = F_{m+2} - 1$ and $\sum_{k=1}^{m} k F_k = m F_{m+2} - F_{m+3} + 2$, we find

$$\sum_{k=2}^{n+1} F_k = F_{n+3} - 2 \quad \text{and} \quad \sum_{k=2}^{n+1} k F_k = (n+1) F_{n+3} - F_{n+4} + 1.$$

Substituting back,

$$S(n) = -2\left((n+1)F_{n+3} - F_{n+4} + 1\right) + 3(F_{n+3} - 2) + n^2 F_{n+2} = (n^2 - 2n + 5) F_{n+2} + (3 - 2n) F_{n+1} - 8.$$

Solution 2: Induction We prove $S(n) = (n^2 - 2n + 5)F_{n+2} + (3 - 2n)F_{n+1} - 8$ by induction. For $n = 1$, $S(1) = 1$ and the formula yields 1, so the base case holds. Assume it holds for $n = k$. Then,

$$S(k+1) = S(k) + (k+1)^2 F_{k+1} = (k^2 - 2k + 5)F_{k+2} + (3 - 2k)F_{k+1} - 8 + (k+1)^2 F_{k+1}.$$

Factoring and rearranging,

$$S(k+1) = (k^2 - 2k + 5)F_{k+2} + (k^2 + 4)F_{k+1} - 8.$$

Using $F_{k+1} = F_{k+3} - F_{k+2}$,

$$S(k+1) = (k^2 - 2k + 5)F_{k+2} + (k^2 + 4)(F_{k+3} - F_{k+2}) - 8 = (k^2 + 4)F_{k+3} + (1 - 2k)F_{k+2} - 8,$$

which matches the formula for $n = k + 1$. Thus, by the principle of mathematical induction, the formula holds for all n.

For $n = 2025$, compute $a = n^2 - 2n + 5 = 2025^2 - 2 \times 2025 + 5 = 4{,}096{,}580$ and $b = 3 - 2n = 3 - 2 \times 2025 = -4{,}047$, yielding the ordered triple $\boxed{(a, b, c) = (4096580,\ -4047,\ -8)}$.

6 Problem #6

Problem If $N \in \mathbb{Z}^+$, a chop of N is obtained by introducing one or more plus signs between the digits of N, and adding the results. For example, two possible chops of 12345 are $123 + 45 = 168$ and $1 + 2 + 3 + 4 + 5 = 15$.

(a) If n is divisible by 2,3,5, or 7, show that there exists a $2n$-digit integer N with nonzero digits such that no chop of N is divisible by n.

(b) If N is a $2n$-digit integer with nonzero digits, and n is not divisible by 2,3,5, or 7, show that there is some chop of N that is divisible by n.

Proof (a) Let $n \in \mathbb{Z}^+$ be divisible by at least one of 2,3,5 or 7. We define a chop of an integer N as the sum obtained by splitting its digits into one or more consecutive blocks and adding these block-values. We aim to construct a $2n$-digit number $N = d_1 d_2 \cdots d_{2n}$ with each $d_i \in \{1, 2, \ldots, 9\}$, such that no chop is divisible by n. We proceed by induction on the number of digits of N.

For a 1-digit N, choose any digit $d_1 \in \{1, 2, \ldots, 9\}$ such that

$$d_1 \not\equiv 0 \pmod{n}.$$

This is possible because not every digit in $\{1, 2, \ldots, 9\}$ can be congruent to 0 modulo n. (In particular, if $n > 9$, then every digit is less than n and thus cannot be 0 modulo n; if $n \leq 9$, then at most a few digits are divisible by n, so there is always at least one valid choice.) Clearly, for this 1-digit number, the only chop is the number itself, and since $d_1 \not\equiv 0 \pmod{n}$, N is not divisible by n. This completes the base case.

Now assume for some k, $1 \leqslant k < 2n$, we have constructed $N_k = d_1 d_2 \cdots d_k$ so that every chop of N_k is not divisible by n. We attach a new digit $d_{k+1} \in \{1, \ldots, 9\}$ to form N_{k+1}. Any chop of N_{k+1} arises from a chop of N_k in one of two ways:

Case 1: Extension We append d_{k+1} to the last block. If that block has value B, its new value then becomes $10B + d_{k+1}$.

Case 2: New Block We start a new block with d_{k+1}. If the sum of the previous blocks is S, then the new sum in some chop is $S + d_{k+1}$.

In both cases, the value of a new chop can be expressed as

$$S + c\, d_{k+1},$$

where $c = 1$ in Case 2, or $c = 10^j$ (for some $j \geqslant 1$) in Case 1. For this sum to be divisible by n, we need

$$S + c\, d_{k+1} \equiv 0 \pmod{n} \iff c\, d_{k+1} \equiv -S \pmod{n},$$

so each potential chop imposes at most one restriction on d_{k+1}.

Next, I argue that there are at most 7 distinct such restrictions. In the new block case ($c = 1$), there is exactly one restriction. In the extension case ($c = 10^j$), restrictions can only arise if 10^j is invertible modulo a prime divisor of n. Specifically:

- For $p = 2$ or $p = 5$, $10 \equiv 0 \pmod{p}$, so powers of 10 are not invertible and yield no new restrictions.
- For $p = 3$, since $10 \equiv 1 \pmod{3}$, all powers of 10 act like $c = 1$, giving no additional distinct restriction beyond the new block case.
- For $p = 7$, the powers of 10 cycle modulo 7 with period at most 6, producing at most 6 distinct restrictions in total.

Hence, combining the single restriction from $c = 1$ with up to 6 from the cases where $p = 7$, we get at most 7 total restrictions on d_{k+1}. Since there are 9 possible digits and at most 7 are ruled out, at least one choice for d_{k+1} remains valid at each step.

By induction, this process produces a $2n$-digit number N with nonzero digits such that no chop of N is divisible by n. $\square$

Proof (b) Let N be a $2n$-digit integer with nonzero digits $d_1, d_2, \ldots, d_{2n}$. Since n is not divisible by $2, 3, 5$, or 7, in particular $\gcd(n, 10) = 1$. Define partial sums (in base 10) modulo n by

$$S_k \equiv d_1 \cdot 10^{k-1} + d_2 \cdot 10^{k-2} + \cdots + d_k \pmod{n},$$

i.e., take the usual integer formed by the first k digits of N (namely $d_1 d_2 \cdots d_k$) and reduce it modulo n.

We have $2n$ such sums: $S_1, S_2, \ldots, S_{2n}$. By the pigeonhole principle *because* there are $2n$ sums but only n possible residues modulo n, either one of these sums is already 0 modulo n (in which case we have a chop divisible by n by taking those first k digits alone), or at least two of these sums coincide. Suppose

$$S_i \equiv S_j \pmod{n} \quad \text{with} \quad 1 \leqslant i < j \leqslant 2n.$$

Which means $S_j - S_i \equiv 0 \pmod{n}$. Observe that $S_j - S_i$ is precisely the integer formed by the digits $d_{i+1} d_{i+2} \cdots d_j$. Since $\gcd(n, 10) = 1$, subtracting these two base-10 "partial-sum" values shows that the block $d_{i+1} d_{i+2} \cdots d_j$ is 0 modulo n. Hence, the chop corresponding to the block of digits from position $i + 1$ to j is divisible by n. Thus, in all cases, we can find a chop of N that is divisible by n. $\square$