Introduction to Quantum Computation, UPB
Winter 2022, Assignment 5

Eli Kogan-Wang

1 Exercises

1. Define $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$, and consider projective measurement $M = \{|\psi\rangle\langle\psi|, I - |\psi\rangle\langle\psi|\}$ with labels corresponding to outcomes $S = \{1, -1\}$, respectively. Suppose state $|0\rangle \in \mathbb{C}^2$ is measured via M. What is the expected value for the measurement?

   Assume that $||\psi\rangle| = 1$ since otherwise the measurement is not well-defined.

   Consider

   $$|\psi\rangle\langle\psi|0\rangle = |\psi\rangle(\alpha\langle 0| + \beta\langle 1|)|0\rangle = \alpha|\psi\rangle$$

   Now $\Pr(\text{Outcome } |\psi\rangle\langle\psi|) = \alpha\alpha^\dagger = |\alpha|^2$ and $\Pr(\text{Outcome } I - |\psi\rangle\langle\psi|) = 1 - \alpha\alpha^\dagger = \beta\beta^\dagger = |\beta|^2$.

   The expected value is therefore $|\alpha|^2 - |\beta|^2 = |\alpha|^2 - (1 - |\alpha|^2) = 2|\alpha|^2 - 1$.

2. In this question, we consider how well the CHSH game strategy from class fares if we use a *less* entangled state as a shared resource between Alice and Bob. Specifically, imagine we use the same observables as before, but now we replace $|\Phi^+\rangle$ as a shared state with $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$. Intuitively, as α gets closer to 1, this state becomes less entangled, and for $\alpha = 1$, it becomes a product state (i.e. non-entangled).

   (a) What is the probability of winning the CHSH game with shared state $|\psi\rangle$? (Hint: Recall from Lecture 5 that for *any* $|\psi\rangle$, the quantity $\mathrm{Tr}(A \otimes B|\psi\rangle\langle\psi|)$ equals Pr(output same bits) − Pr(output different bits), i.e. the interpretation of this quantity does not depend on our choice of $|\psi\rangle$.)

   We know that $\mathrm{Tr}(A_i \otimes B_j|\psi\rangle\langle\psi|) = \Pr(\text{output same bits}) - \Pr(\text{output different bits})$ for the inputs $i, j \in \{0, 1\}$.

   Additionally, we want to output the same bits if and only if i, j are not $1, 1$.

   So we can calculate:

   $$\begin{aligned}\Pr(\text{win}) - \Pr(\text{lose}) = &\, Pr(q_A q_B = 00)\mathrm{Tr}(A_0 \otimes B_0|\psi\rangle\langle\psi|)\\ &+ Pr(q_A q_B = 01)\mathrm{Tr}(A_0 \otimes B_1|\psi\rangle\langle\psi|)\\ &+ Pr(q_A q_B = 10)\mathrm{Tr}(A_1 \otimes B_0|\psi\rangle\langle\psi|)\\ &- Pr(q_A q_B = 11)\mathrm{Tr}(A_1 \otimes B_1|\psi\rangle\langle\psi|)\end{aligned}$$

   Recall that $A_0 = Z$ and $A_1 = X$, and $B_0 = H$ and $B_1 = ZHZ$.

   Now for 00:

$$
\begin{aligned}
\mathrm{Tr}(A_0 \otimes B_0 |\psi\rangle\langle\psi|) &= \mathrm{Tr}(Z \otimes H |\psi\rangle\langle\psi|) \\
&= \mathrm{Tr}(Z \otimes H(\alpha^\dagger |0\rangle |0\rangle + \beta |1\rangle |1\rangle)(\alpha \langle 0| \langle 0| + \beta \langle 1| \langle 1|)) \\
&= \alpha^\dagger \mathrm{Tr}(Z \otimes H(|0\rangle \langle 0|)(\alpha \langle 0| \langle 0| + \beta \langle 1| \langle 1|)) \\
&+ \beta^\dagger \mathrm{Tr}(Z \otimes H(|1\rangle |1\rangle)(\alpha \langle 0| \langle 0| + \beta \langle 1| \langle 1|)) \\
&= |\alpha|^2 \mathrm{Tr}(Z \otimes H(|0\rangle |0\rangle)(\langle 0| \langle 0|)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}(Z \otimes H(|0\rangle |0\rangle)(\langle 1| \langle 1|)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}(Z \otimes H(|1\rangle |1\rangle)(\langle 0| \langle 0|)) \\
&+ |\beta|^2 \mathrm{Tr}(Z \otimes H(|1\rangle |1\rangle)(\langle 1| \langle 1|)) \\
&= |\alpha|^2 \mathrm{Tr}((|0\rangle |+\rangle)(\langle 0| \langle 0|)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}((|0\rangle |+\rangle)(\langle 1| \langle 1|)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}((- |1\rangle |-\rangle)(\langle 0| \langle 0|)) \\
&+ |\beta|^2 \mathrm{Tr}((- |1\rangle |-\rangle)(\langle 1| \langle 1|)) \\
&= |\alpha|^2 \mathrm{Tr}((\langle 0| \langle 0|)(|0\rangle |+\rangle)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}((\langle 1| \langle 1|)(|0\rangle |+\rangle)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}((\langle 0| \langle 0|)(- |1\rangle |-\rangle)) \\
&+ |\beta|^2 \mathrm{Tr}((\langle 1| \langle 1|)(- |1\rangle |-\rangle)) \\
&= |\alpha|^2 \langle 0|0\rangle \langle 0|+\rangle \\
&+ \alpha^\dagger \beta \langle 1|0\rangle \dots \\
&+ \beta^\dagger \alpha \langle 0|1\rangle \dots \\
&+ |\beta|^2 (-1) \langle 1|1\rangle \langle 1|-\rangle \\
&= |\alpha|^2 \langle 0|+\rangle \\
&- |\beta|^2 \langle 1|-\rangle \\
&= \frac{|\alpha|^2}{\sqrt{2}} + \frac{|\beta|^2}{\sqrt{2}} \\
&= \frac{|\alpha|^2 + |\beta|^2}{\sqrt{2}}
\end{aligned}
$$

And for 01:

Consider $ZHZ|0\rangle = ZH|0\rangle = Z|+\rangle = |-\rangle$ and $ZHZ|1\rangle = -ZH|1\rangle = -Z|-\rangle = -|+\rangle = (-1)|+\rangle$.

$$
\begin{aligned}
\mathrm{Tr}(A_0 \otimes B_1|\psi\rangle\langle\psi|) &= \mathrm{Tr}(Z \otimes ZHZ|\psi\rangle\langle\psi|) \\
&= \mathrm{Tr}(Z \otimes ZHZ(\alpha^\dagger \left|0\right\rangle \left|0\right\rangle + \beta \left|1\right\rangle \left|1\right\rangle)(\alpha \left\langle 0\right| \left\langle 0\right| + \beta \left\langle 1\right| \left\langle 1\right|)) \\
&= \alpha^\dagger \mathrm{Tr}(Z \otimes ZHZ(\left|0\right\rangle \left\langle 0\right|)(\alpha \left\langle 0\right| \left\langle 0\right| + \beta \left\langle 1\right| \left\langle 1\right|)) \\
&+ \beta^\dagger \mathrm{Tr}(Z \otimes ZHZ(\left|1\right\rangle \left|1\right\rangle)(\alpha \left\langle 0\right| \left\langle 0\right| + \beta \left\langle 1\right| \left\langle 1\right|)) \\
&= \left|\alpha\right|^2 \mathrm{Tr}(Z \otimes ZHZ(\left|0\right\rangle \left|0\right\rangle)(\left\langle 0\right| \left\langle 0\right|)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}(Z \otimes ZHZ(\left|0\right\rangle \left|0\right\rangle)(\left\langle 1\right| \left\langle 1\right|)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}(Z \otimes ZHZ(\left|1\right\rangle \left|1\right\rangle)(\left\langle 0\right| \left\langle 0\right|)) \\
&+ \left|\beta\right|^2 \mathrm{Tr}(Z \otimes ZHZ(\left|1\right\rangle \left|1\right\rangle)(\left\langle 1\right| \left\langle 1\right|)) \\
&= \left|\alpha\right|^2 \mathrm{Tr}((\left|0\right\rangle (-1) \left|+\right\rangle)(\left\langle 0\right| \left\langle 0\right|)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}((\left|0\right\rangle (-1) \left|+\right\rangle)(\left\langle 1\right| \left\langle 1\right|)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}((-\left|1\right\rangle \left|-\right\rangle)(\left\langle 0\right| \left\langle 0\right|)) \\
&+ \left|\beta\right|^2 \mathrm{Tr}((-\left|1\right\rangle \left|-\right\rangle)(\left\langle 1\right| \left\langle 1\right|)) \\
&= \left|\alpha\right|^2 \mathrm{Tr}((\left\langle 0\right| \left\langle 0\right|)(\left|0\right\rangle (-1) \left|+\right\rangle)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}((\left\langle 1\right| \left\langle 1\right|)(\left|0\right\rangle (-1) \left|+\right\rangle)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}((\left\langle 0\right| \left\langle 0\right|)(-\left|1\right\rangle \left|-\right\rangle)) \\
&+ \left|\beta\right|^2 \mathrm{Tr}((\left\langle 1\right| \left\langle 1\right|)(-\left|1\right\rangle \left|-\right\rangle)) \\
&= \left|\alpha\right|^2 \left\langle 0|0\right\rangle (-1) \left\langle 0|+\right\rangle \\
&+ \alpha^\dagger \beta \left\langle 1|0\right\rangle \dots \\
&+ \beta^\dagger \alpha \left\langle 0|1\right\rangle \dots \\
&+ \left|\beta\right|^2 (-1) \left\langle 1|1\right\rangle \left\langle 1|-\right\rangle \\
&= -\left|\alpha\right|^2 \left\langle 0|+\right\rangle \\
&- \left|\beta\right|^2 \left\langle 1|-\right\rangle \\
&= -\frac{\left|\alpha\right|^2}{\sqrt{2}} + \frac{\left|\beta\right|^2}{\sqrt{2}} \\
&= \frac{-\left|\alpha\right|^2 + \left|\beta\right|^2}{\sqrt{2}}
\end{aligned}
$$

And for 10:

Consider $X\left|0\right\rangle = \left|1\right\rangle$ and $X\left|1\right\rangle = \left|0\right\rangle$.

$$
\begin{aligned}
\mathrm{Tr}(A_1 \otimes B_0|\psi\rangle\langle\psi|) &= \mathrm{Tr}(X \otimes H|\psi\rangle\langle\psi|) \\
&= \mathrm{Tr}(X \otimes H(\alpha^\dagger |0\rangle |0\rangle + \beta |1\rangle |1\rangle)(\alpha \langle 0| \langle 0| + \beta \langle 1| \langle 1|)) \\
&= \alpha^\dagger \mathrm{Tr}(X \otimes H(|0\rangle \langle 0|)(\alpha \langle 0| \langle 0| + \beta \langle 1| \langle 1|)) \\
&+ \beta^\dagger \mathrm{Tr}(X \otimes H(|1\rangle |1\rangle)(\alpha \langle 0| \langle 0| + \beta \langle 1| \langle 1|)) \\
&= |\alpha|^2 \mathrm{Tr}(X \otimes H(|0\rangle |0\rangle)(\langle 0| \langle 0|)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}(X \otimes H(|0\rangle |0\rangle)(\langle 1| \langle 1|)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}(X \otimes H(|1\rangle |1\rangle)(\langle 0| \langle 0|)) \\
&+ |\beta|^2 \mathrm{Tr}(X \otimes H(|1\rangle |1\rangle)(\langle 1| \langle 1|)) \\
&= |\alpha|^2 \mathrm{Tr}((|1\rangle |+\rangle)(\langle 0| \langle 0|)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}((|1\rangle |+\rangle)(\langle 1| \langle 1|)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}((|0\rangle |-\rangle)(\langle 0| \langle 0|)) \\
&+ |\beta|^2 \mathrm{Tr}((|0\rangle |-\rangle)(\langle 1| \langle 1|)) \\
&= |\alpha|^2 \mathrm{Tr}((\langle 0| \langle 0|)(|1\rangle |+\rangle)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}((\langle 1| \langle 1|)(|1\rangle |+\rangle)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}((\langle 0| \langle 0|)(|0\rangle |-\rangle)) \\
&+ |\beta|^2 \mathrm{Tr}((\langle 1| \langle 1|)(|0\rangle |-\rangle)) \\
&= |\alpha|^2 \langle 0|1\rangle \dots \\
&+ \alpha^\dagger \beta \langle 1|1\rangle \langle 1|+\rangle \\
&+ \beta^\dagger \alpha \langle 0|0\rangle \langle 0|-\rangle \\
&+ |\beta|^2 (-1) \langle 1|0\rangle \dots \\
&= \alpha^\dagger \beta \langle 1|+\rangle \\
&+ \beta^\dagger \alpha \langle 0|-\rangle \\
&= \frac{\alpha^\dagger \beta}{\sqrt{2}} + \frac{\beta^\dagger \alpha}{\sqrt{2}} \\
&= \frac{2\Re(\alpha^\dagger \beta)}{\sqrt{2}} = \sqrt{2}\Re(\alpha^\dagger \beta)
\end{aligned}
$$

And for 11:

$$
\begin{aligned}
\mathrm{Tr}(A_1 \otimes B_1|\psi\rangle\langle\psi|) &= \mathrm{Tr}(X \otimes ZHZ|\psi\rangle\langle\psi|) \\
&= \mathrm{Tr}(X \otimes ZHZ(\alpha^\dagger |0\rangle |0\rangle + \beta |1\rangle |1\rangle)(\alpha \langle 0| \langle 0| + \beta \langle 1| \langle 1|)) \\
&= \alpha^\dagger \mathrm{Tr}(X \otimes ZHZ(|0\rangle \langle 0|)(\alpha \langle 0| \langle 0| + \beta \langle 1| \langle 1|)) \\
&+ \beta^\dagger \mathrm{Tr}(X \otimes ZHZ(|1\rangle |1\rangle)(\alpha \langle 0| \langle 0| + \beta \langle 1| \langle 1|)) \\
&= |\alpha|^2 \,\mathrm{Tr}(X \otimes ZHZ(|0\rangle |0\rangle)(\langle 0| \langle 0|)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}(X \otimes ZHZ(|0\rangle |0\rangle)(\langle 1| \langle 1|)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}(X \otimes ZHZ(|1\rangle |1\rangle)(\langle 0| \langle 0|)) \\
&+ |\beta|^2 \,\mathrm{Tr}(X \otimes ZHZ(|1\rangle |1\rangle)(\langle 1| \langle 1|)) \\
&= |\alpha|^2 \,\mathrm{Tr}((|1\rangle (-1) |+\rangle)(\langle 0| \langle 0|)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}((|1\rangle (-1) |+\rangle)(\langle 1| \langle 1|)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}((|0\rangle |-\rangle)(\langle 0| \langle 0|)) \\
&+ |\beta|^2 \,\mathrm{Tr}((|0\rangle |-\rangle)(\langle 1| \langle 1|)) \\
&= |\alpha|^2 \,\mathrm{Tr}((\langle 0| \langle 0|)(|1\rangle (-1) |+\rangle)) \\
&+ \alpha^\dagger \beta \mathrm{Tr}((\langle 1| \langle 1|)(|1\rangle (-1) |+\rangle)) \\
&+ \beta^\dagger \alpha \mathrm{Tr}((\langle 0| \langle 0|)(|0\rangle |-\rangle)) \\
&+ |\beta|^2 \,\mathrm{Tr}((\langle 1| \langle 1|)(|0\rangle |-\rangle)) \\
&= |\alpha|^2 \langle 0|1\rangle \ldots \\
&- \alpha^\dagger \beta \langle 1|1\rangle \langle 1|+\rangle \\
&+ \beta^\dagger \alpha \langle 0|0\rangle \langle 0|-\rangle \\
&+ |\beta|^2 (-1) \langle 1|0\rangle \ldots \\
&= -\alpha^\dagger \beta \langle 1|+\rangle \\
&+ \beta^\dagger \alpha \langle 0|-\rangle \\
&= -\frac{\alpha^\dagger \beta}{\sqrt{2}} + \frac{\beta^\dagger \alpha}{\sqrt{2}} \\
&= -\frac{2\Im(\alpha^\dagger \beta) i}{\sqrt{2}} = -\sqrt{2}\Im(\alpha^\dagger \beta) i
\end{aligned}
$$

Calculating Pr(win) − Pr(lose) from the formulas above, we get:

$$\begin{aligned}
\Pr(\text{win}) - \Pr(\text{lose}) &= \frac{1}{4}\left(\frac{|\alpha|^2 + |\beta|^2}{\sqrt{2}} + \frac{-|\alpha|^2 + |\beta|^2}{\sqrt{2}} + \sqrt{2}\Re(\alpha^\dagger\beta) - (-\sqrt{2}\Im(\alpha^\dagger\beta)i)\right) \\
&= \frac{1}{4}\left(\frac{2|\beta|^2}{\sqrt{2}} + \sqrt{2}\Re(\alpha^\dagger\beta) + \sqrt{2}\Im(\alpha^\dagger\beta)i\right) \\
&= \frac{\sqrt{2}}{4}\left(|\beta|^2 + \Re(\alpha^\dagger\beta) + \Im(\alpha^\dagger\beta)i\right) \\
&= \frac{\sqrt{2}}{4}\left(|\beta|^2 + \Re(\alpha^\dagger\beta) + \Im(\alpha^\dagger\beta)i\right) \\
&= \frac{\sqrt{2}}{4}\left(|\beta|^2 + \alpha^\dagger\beta\right) \\
&= \frac{\sqrt{2}}{4}\left((1 - |\alpha|^2) + \alpha^\dagger\beta\right)
\end{aligned}$$

And since $1 = \Pr(\text{win}) + \Pr(\text{lose}) \iff 1 - 2(\Pr(\text{lose})) = \Pr(\text{win}) - \Pr(\text{lose})$, we get:
Now $\Pr(\text{lose}) = \frac{\Pr(\text{win}) - \Pr(\text{lose}) - 1}{-2}$
And $\Pr(\text{win}) = 1 - \Pr(\text{lose})$

$$\begin{aligned}
\Pr(\text{win}) &= 1 - \Pr(\text{lose}) \\
&= 1 - \frac{\Pr(\text{win}) - \Pr(\text{lose}) - 1}{-2} \\
&= 1 - \frac{\frac{\sqrt{2}}{4}\left((1 - |\alpha|^2) + \alpha^\dagger\beta\right) - 1}{-2} \\
&= 1 + \frac{\frac{\sqrt{2}}{4}\left((1 - |\alpha|^2) + \alpha^\dagger\beta\right) - 1}{2} \\
&= \frac{2 + \frac{\sqrt{2}}{4}\left((1 - |\alpha|^2) + \alpha^\dagger\beta\right) - 1}{2} \\
&= \frac{1 + \frac{\sqrt{2}}{4}\left((1 - |\alpha|^2) + \alpha^\dagger\beta\right)}{2}
\end{aligned}$$

And we're done! We've found the probability of winning the game! It's $\frac{1 + \frac{\sqrt{2}}{4}\left((1 - |\alpha|^2) + \alpha^\dagger\beta\right)}{2}$, simple as that!

(b) Based on your answer above, what is the probability of Alice and Bob winning with this strategy if $\alpha = 1$, i.e. $|\psi\rangle$ is unentangled?

Now since $\alpha = 1$: $\beta = 0$.

And $\Pr(\text{win}) = \frac{1 + \frac{\sqrt{2}}{4}\left((1 - |\alpha|^2) + \alpha^\dagger\beta\right)}{2}$

$$
\begin{aligned}
\Pr(\text{win}) &= \frac{1+\frac{\sqrt{2}}{4}\left((1-|\alpha|^2)+\alpha^\dagger\beta\right)}{2} \\
&= \frac{1+\frac{\sqrt{2}}{4}\left((1-1)+1^\dagger 0\right)}{2} \\
&= \frac{1+\frac{\sqrt{2}}{4}\left(0+1^\dagger 0\right)}{2} \\
&= \frac{1+\frac{\sqrt{2}}{4}\left(0+0\right)}{2} \\
&= \frac{1+\frac{\sqrt{2}}{4}\left(0\right)}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

We win with probability $\frac{1}{2}$! This is even worse than the classical strategy!

An interesting question is: For which α does the quantum strategy win with probability $\frac{3}{4}$?

3. This question studies a 3-player non-local game called the *GHZ game*. There are now three players, Alice, Bob and Charlie, each of which receives a question q_a, q_b, or q_c, respectively, such that $q_a q_b q_c \in \{000, 011, 101, 110\}$. The players each return a bit $r_a, r_b, r_c \in \{0, 1\}$, respectively, and win if

$$q_a \vee q_b \vee q_c = r_a \oplus r_b \oplus r_c,$$

where $\vee$ denotes the binary OR operation.

An analysis similar to the CHSH game shows that the optimal winning classical strategy yields success probability 3/4. Your task in this question is to analyze an optimal quantum strategy.

The 3-qubit state the players share is

$$|\psi\rangle = \frac{1}{2}\left(|000\rangle - |011\rangle - |101\rangle - |110\rangle\right) \in \mathbb{C}^8.$$

Each player will use the same measuring strategy: Given input bit 0, they will apply local unitary $U_0 = I$, and if they get input 1, they apply local unitary $U_1 = H$. They then measure their qubit in the standard basis, and return the answer (0 or 1). As for CHSH, we assume the labels for the measurement outcomes are $+1, -1$ for measurement outcomes $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, respectively.

(a) Suppose Alice gets input 0. What is her observable? What if she gets input 1?

In general, Alice's observable is $I - H$.

(b) Suppose the questions are $q_A q_B q_C = 000$. What is the probability the players win?

All players apply $U_0 = I$.

Now we get with probability $\frac{1}{4}$ each the following results:

$|000\rangle$: Alice measures $|0\rangle\langle 0|$, Bob measures $|0\rangle\langle 0|$, Charlie measures $|0\rangle\langle 0|$ and they win.

$|011\rangle$: Alice measures $|0\rangle\langle 0|$, Bob measures $|1\rangle\langle 1|$, Charlie measures $|1\rangle\langle 1|$ and they win.

$|101\rangle$: Alice measures $|1\rangle\langle 1|$, Bob measures $|0\rangle\langle 0|$, Charlie measures $|1\rangle\langle 1|$ and they win.

$|110\rangle$: Alice measures $|1\rangle\langle 1|$, Bob measures $|1\rangle\langle 1|$, Charlie measures $|0\rangle\langle 0|$ and they win.

They win with probability $\frac{4}{4} = 1$.

(c) Suppose the questions are $q_A q_B q_C = 011$. What is the probability the players win?

Alice applies $U_0 = I$, Bob applies $U_1 = H$ and Charlie applies $U_1 = H$.

We get with probability $\frac{1}{4}$ each the following results:

$|0++\rangle$: Alice measures $|0\rangle\langle 0|$, Bob measures $|+\rangle\langle +|$, Charlie measures $|+\rangle\langle +|$.
We get results $|000\rangle$, $|001\rangle$, $|010\rangle$, $|011\rangle$ with probability $\frac{1}{4}$ each.
Only $|001\rangle$ and $|010\rangle$ win, so this contributes $\frac{2}{4} \cdot \frac{1}{4} = \frac{1}{8}$ to the probability of winning.
$|0--\rangle$: Alice measures $|0\rangle\langle 0|$, Bob measures $|-\rangle\langle -|$, Charlie measures $|-\rangle\langle -|$.
We get results $|000\rangle$, $|001\rangle$, $|010\rangle$, $|011\rangle$ with probability $\frac{1}{4}$ each.
Only $|001\rangle$ and $|010\rangle$ win, so this contributes $\frac{2}{4} \cdot \frac{1}{4} = \frac{1}{8}$ to the probability of winning.
$|1-+\rangle$: Alice measures $|1\rangle\langle 1|$, Bob measures $|-\rangle\langle -|$, Charlie measures $|+\rangle\langle +|$.
We get results $|100\rangle$, $|101\rangle$, $|110\rangle$, $|111\rangle$ with probability $\frac{1}{4}$ each.
Only $|100\rangle$ and $|111\rangle$ win, so this contributes $\frac{2}{4} \cdot \frac{1}{4} = \frac{1}{8}$ to the probability of winning.
$|1+-\rangle$: Alice measures $|1\rangle\langle 1|$, Bob measures $|+\rangle\langle +|$, Charlie measures $|-\rangle\langle -|$.
We get results $|100\rangle$, $|101\rangle$, $|110\rangle$, $|111\rangle$ with probability $\frac{1}{4}$ each.
Only $|100\rangle$ and $|111\rangle$ win, so this contributes $\frac{2}{4} \cdot \frac{1}{4} = \frac{1}{8}$ to the probability of winning.
They win with probability $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$.
Probably. I haven't applied observables.