Algorithmische Geometrie H13

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Aufgabe 24 (5 Punkte):

Let $S = \{p_1, p_2, \ldots, p_n\}$ be a set of points in the plane which lie on a circle. We shall prove that the smallest angle of any triangulation of the convex hull of S is the same.

Take i, such that $\|p_i - p_{i+1}\| = \min_{1 \leq j \leq n} \|p_j - p_{j+1}\|$, where $p_{n+1} = p_1$. This means that the distance between p_i and p_{i+1} is the smallest among all distances between consecutive points.

It is also known, that any triangulation of a set of points includes all the edges of the convex hull of the set. This means that p_i and p_{i+1} are connected by an edge in any triangulation of the convex hull of S.

Since a triangulation consists of triangles, there exists a vertex p_j of a triangle, such that p_i, p_{i+1} and p_j are vertices of the same triangle. Since p_j is a vertex on the same circle as p_i and p_{i+1}, the angle $\angle p_i p_j p_{i+1}$ is constant for all j. This is because the angle subtended by a chord on a circle is constant, and the angle $\angle p_i p_j p_{i+1}$ is half of the angle subtended by the arc $p_i p_{i+1}$. See https://en.wikipedia.org/wiki/Inscribed_angle for more information, this was proven ca. 300 BC by Euclid as Proposition 20 of Book 3 of his Elements.

Any angles in the triangulation are such angles subtended by the arcs of the circle, and are therefore only dependent on the distance between the points.

Therefore, the smallest angle of any triangulation of the convex hull of S is the same, since it only depends on the distance between the points.