Introduction to Quantum Computation, UPB
Winter 2022, Assignment 1

Solution by Eli Kogan-Wang

1 Exercises

1. For complex number $c = a + bi$, recall that the *real* and *imaginary* parts of c are denoted $\mathrm{Re}(c) = a$ and $\mathrm{Imag}(c) = b$.

   (a) Prove that $c + c^* = 2 \cdot \mathrm{Re}(c)$.

   Proof.

   $$c + c^* = (a + bi) + (a - bi) \overset{\text{assoc., comm.}}{=} (a + a) + (b - b)i = 2a = 2Re(c) \tag{1}$$

   □

   (b) Prove that $cc^* = a^2 + b^2$. How can we therefore rewrite $|c|$ in terms of a and b?

   Proof.

   $$cc^* = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - i^2b^2 = a^2 - (-1)b^2 = a^2 + b^2 \tag{2}$$

   □

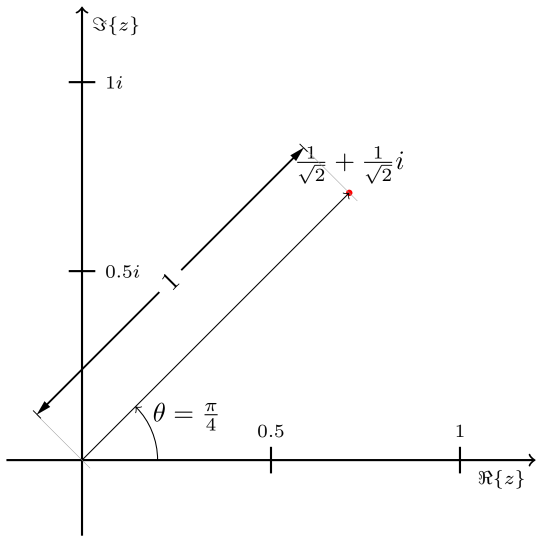
   (c) What is the polar form of $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$? Use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$.

   Observation 1.

   $$c = 1 \cdot e^{\frac{\pi}{4}i} \overset{by\ calculation}{=} c$$

   Where:
   Length: 1
   Angle: $\frac{\pi}{4}$

   (d) Draw $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ as a vector in the complex plane, ensuring to denote both the length of the vector and its angle with the x axis.



2. Prove that for any normalized vectors $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$,

$$\| \, |\psi\rangle - |\phi\rangle \, \|_2 = \sqrt{2 - 2 \cdot \mathrm{Re}(\langle\psi|\phi\rangle)}.$$

Proof.

$$\begin{aligned}
\| \, |\psi\rangle - |\phi\rangle \, \|_2 &= \sqrt{\langle\psi - \phi|\psi - \phi\rangle} \\
&= \sqrt{\langle\psi|\psi\rangle - \langle\psi|\phi\rangle - \langle\phi|\psi\rangle + \langle\phi|\phi\rangle} \\
&= \sqrt{1 - \langle\psi|\phi\rangle - \langle\phi|\psi\rangle + 1} = \sqrt{2 - \langle\psi|\phi\rangle - \langle\psi|\phi\rangle^*} \\
&= \sqrt{2 - (\langle\psi|\phi\rangle + \langle\psi|\phi\rangle^*)} = \sqrt{2 - 2 \cdot \mathrm{Re}(\langle\psi|\phi\rangle)}
\end{aligned} \tag{3}$$

□

Why does it not matter if we replace $\langle\psi|\phi\rangle$ with $\langle\phi|\psi\rangle$ in this equation?

It does not matter, since $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$, and this implies that $\mathrm{Re}(\langle\psi|\phi\rangle) = \mathrm{Re}(\langle\phi|\psi\rangle)$.

3. Define

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(a) What is $\mathrm{Tr}(A \cdot |1\rangle\langle 0|)$? (Hint: This can be computed quickly by using the cyclic property of the trace and the outer product representation of A. Do master this trick; it will be used repeatedly in the course and save you much time.)

$$\mathrm{Tr}(A \cdot |1\rangle\langle 0|) = \mathrm{Tr}(\langle 0|A|1\rangle) = A_{0,1} = b. \tag{4}$$

(b) Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the same tricks as in part A, along with the fact that the trace is linear, to quickly evaluate

$$\mathrm{Tr}(A \cdot |+\rangle\langle +|).$$

$$
\begin{aligned}
\mathrm{Tr}(A \cdot |+\rangle\langle+|) = & \qquad \mathrm{Tr}(\langle+|A|+\rangle) \\
= \left(\frac{1}{\sqrt{2}}\right)^2 \cdot & \qquad \mathrm{Tr}((\langle 0| + \langle 1|)A(|0\rangle + |1\rangle)) \\
= \frac{1}{2}\cdot & \qquad \mathrm{Tr}(\langle 0|A|0\rangle + \langle 0|A|1\rangle + \langle 1|A|0\rangle + \langle 1|A|1\rangle) \\
= \frac{1}{2}\cdot & \qquad\quad (a + c + d + b)
\end{aligned} \tag{5}
$$

4. (a) A general property of the outer product is that $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$. Verify that this holds for the case where $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |1\rangle$. (Hint: Write out the full matrix corresponding to $|0\rangle\langle 1|$.)

$$
\begin{aligned}
|0\rangle\langle 1| &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
(|0\rangle\langle 1|)^\dagger &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0|
\end{aligned} \tag{6}
$$

(b) Use Part (a) to prove that a normal matrix A satisfies $A = A^\dagger$ if and only if all of A's eigenvalues are real. (Hint: Since A is normal, you can start by writing A in terms of its spectral decomposition. What does the condition $A = A^\dagger$ enforce in terms of A's spectral decomposition?)

Let

$$
A = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|
$$

be the spectral decomposition of A. Where λ_i are the eigenvalues of A and $|\psi_i\rangle$ are the corresponding eigenvectors such that the set of eigenvectors form an orthonormal basis of $\mathbb{C}^d$.

Direction $\Rightarrow$: A satisfies $A = A^\dagger$. We show that all eigenvalues are real.

Proof. Since $A = A^\dagger$, we have:

$$
A = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i| = \left(\sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|\right)^\dagger = A^\dagger \tag{7}
$$

$$
\begin{aligned}
\left(\sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|\right)^\dagger &= \sum_{i=1}^{d} (\lambda_j |\psi_i\rangle\langle\psi_i|)^\dagger \\
&= \sum_{i=1}^{d} \lambda_j^\dagger (|\psi_i\rangle\langle\psi_i|)^\dagger \\
&= \sum_{i=1}^{d} \lambda_j^\dagger |\psi_i\rangle\langle\psi_i|
\end{aligned} \tag{8}
$$

All $|\psi_i\rangle\langle\psi_i|$ are linearly independent, which can be easily seen by writing them out in the basis of $|\psi_i\rangle$.

Now:

$$
\sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i| = \sum_{i=1}^{d} \lambda_j^\dagger |\psi_i\rangle\langle\psi_i| \tag{9}
$$

Which means that

$$
\lambda_i = \lambda_i^\dagger \quad \forall i \tag{10}
$$

From which it follows, that all eigenvalues are real. $\square$

Direction $\Leftarrow$: All eigenvalues are real. We show that $A = A^\dagger$.

Proof. Since

$$\lambda_i = \lambda_i^\dagger \quad \forall i \tag{11}$$

It follows that:

$$\sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i| = \sum_{i=1}^{d} \lambda_j^\dagger |\psi_i\rangle\langle\psi_i| \tag{12}$$

By the same argument as above:

$$A = A^\dagger$$

□