Introduction to Quantum Computation, UPB
Winter 2022, Assignment 3

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1 Exercises

1. (a) Let $A \in \mathcal{L}(\mathbb{C}^d)$ be Hermitian. Prove that if for all $|\psi\rangle \in \mathbb{C}^d$, $\langle\psi| A |\psi\rangle \geq 0$, then A has only non-negative eigenvalues. (Hint: Start by taking the spectral decomposition of A, and then make clever choices for $|\psi\rangle$.)
   Since A is Hermitian, we have that $A = A^\dagger$ (A is self-adjoint) and we have proven that all since A is normal, A is diagonizable/there exists a spectral decomposition of A.

$$A = \sum_{i=1}^{d} \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

   Where λ_i are the eigenvalues of A and $|\lambda_i\rangle$ are the corresponding eigenvectors such that $\{|\lambda_i\rangle\}$ is an orthonormal basis of $\mathbb{C}^d$.
   Additionally, since A is self-adjoint, we have that all eigenvalues λ_i are real.
   Given $\langle\psi| A |\psi\rangle \geq 0$, for all $|\psi\rangle \in \mathbb{C}^d$, we can choose $|\psi\rangle = |\lambda_i\rangle$ for all i and we have that:

$$\langle\lambda_i| A |\lambda_i\rangle \overset{\text{by calculation with spectr. decomp}}{=} \lambda_i \geq 0$$

   Which is what we wanted to prove.

   (b) Let $A \in \mathcal{L}(\mathbb{C}^d)$ be Hermitian. Prove that if A has only non-negative eigenvalues, then for all $|\psi\rangle \in \mathbb{C}^d$, $\langle\psi| A |\psi\rangle \geq 0$. (Hint: Write $|\psi\rangle$ with respect to the eigenbasis of A.)
   Similiarly, let:

$$A = \sum_{i=1}^{d} \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

   Where λ_i are the eigenvalues of A and $|\lambda_i\rangle$ are the corresponding eigenvectors such that $\{|\lambda_i\rangle\}$ is an orthonormal basis of $\mathbb{C}^d$.
   Now, since $\{|\lambda_i\rangle\}$ is an orthonormal basis of $\mathbb{C}^d$, we have that:

$$|\psi\rangle = \sum_{i=1}^{d} \mu_i |\lambda_i\rangle$$

   For some $\mu_i \in \mathbb{C}$.
   And now:

$$
\begin{aligned}
\langle\psi| A |\psi\rangle &= \sum_{i=1}^{d} \mu_i^\dagger \langle\lambda_i| A \sum_{j=1}^{d} \mu_j |\lambda_j\rangle \\
&= \sum_{i=1}^{d}\sum_{j=1}^{d} \mu_i^\dagger \mu_j \langle\lambda_i| A |\lambda_j\rangle \\
&= \sum_{i=1}^{d}\sum_{j=1}^{d} \mu_i^\dagger \mu_j \lambda_i \delta_{i,j} \\
&= \sum_{i=1}^{d} \mu_i^\dagger \mu_i \lambda_i \\
&= \sum_{i=1}^{d} |\mu_i|^2 \lambda_i
\end{aligned}
$$

And since $|\mu_i|^2 \geq 0$ for all i, we have that $\langle\psi| A |\psi\rangle \geq 0$, for all $|\psi\rangle \in \mathbb{C}^d$ as desired.

2. Let $|\psi\rangle = |-\rangle \in \mathbb{C}^2$. Suppose we measure in the Z basis $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. What are the probabilities for each possible measurement outcome, and the corresponding post-measurement states?

$$
|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle
$$

$$
\begin{aligned}
Pr(\text{outcome } \ |0\rangle : |\psi\rangle) &= \mathrm{Tr}(|0\rangle\langle 0||-\rangle\langle -||0\rangle\langle 0|) \\
&\overset{cyclic}{=} \mathrm{Tr}(|0\rangle\langle 0||0\rangle\langle 0||-\rangle\langle -|) \\
&= \mathrm{Tr}(|0\rangle\langle 0||-\rangle\langle -|) \\
&= \mathrm{Tr}(\langle 0|-\rangle \langle -|0\rangle) \\
&= \langle 0|-\rangle \langle -|0\rangle \\
&= \langle 0|-\rangle \langle 0|-\rangle^\dagger \\
&= |\langle 0|-\rangle|^2 \\
&= \left| \frac{1}{\sqrt{2}} \langle 0|0\rangle - \frac{1}{\sqrt{2}} \langle 0|1\rangle \right|^2 \\
&= \frac{1}{2}
\end{aligned}
$$

Post measurement state if we measure $|0\rangle$:

$$
\begin{aligned}
|\psi'\rangle &= \frac{|0\rangle \langle 0|-\rangle}{\sqrt{\frac{1}{2}}} \\
&= \frac{|0\rangle \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \\
&= |0\rangle
\end{aligned}
$$

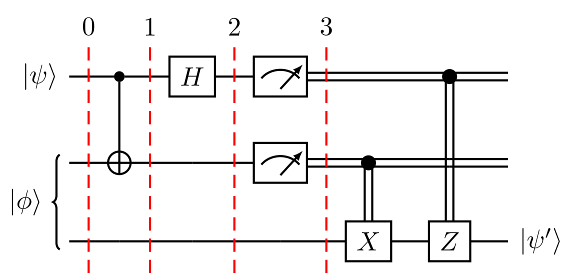
$$
\begin{aligned}
Pr(\text{outcome } \ |1\rangle : |\psi\rangle) &= \mathrm{Tr}(|1\rangle\!\langle 1||-\rangle\!\langle -||1\rangle\!\langle 1|) \\
&\overset{cyclic}{=} \mathrm{Tr}(|1\rangle\!\langle 1||1\rangle\!\langle 1||-\rangle\!\langle -|) \\
&= \mathrm{Tr}(|1\rangle\!\langle 1||-\rangle\!\langle -|) \\
&= \mathrm{Tr}(\langle 1|-\rangle \langle -|1\rangle) \\
&= \langle 1|-\rangle \langle -|1\rangle \\
&= \langle 1|-\rangle \langle 1|-\rangle^{\dagger} \\
&= |\langle 1|-\rangle|^{2} \\
&= \left| \frac{1}{\sqrt{2}} \langle 1|0\rangle - \frac{1}{\sqrt{2}} \langle 1|1\rangle \right|^{2} \\
&= \frac{1}{2}
\end{aligned}
$$

Post measurement state if we measure $|1\rangle$:

$$
\begin{aligned}
|\psi'\rangle &= \frac{|1\rangle \langle 1|-\rangle}{\sqrt{\frac{1}{2}}} \\
&= \frac{-|1\rangle \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \\
&= -|1\rangle
\end{aligned}
$$

3. Consider the teleportation protocol we saw in class. Does it still work if we replace the use of the entangled Bell state $|\phi^+\rangle$ with the unentangled state $|00\rangle$ (i.e. Alice and Bob share the state $|00\rangle$)? How about if we use $\sqrt{2/5}\,|00\rangle + \sqrt{3/5}\,|11\rangle$ instead of $|\phi^+\rangle$?

   In class we saw for $|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ the following circuit.



   For $|\phi\rangle = |\phi^+\rangle = \frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle$ we have analized the circuit and determined that $|\psi'\rangle = |\psi\rangle$.

   Now consider $|\phi\rangle = |00\rangle$, then we have

$$
\begin{aligned}
|\phi_0\rangle &= |\psi\rangle \otimes |00\rangle \\
&= (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |00\rangle \\
&= \alpha\,|0\rangle \otimes |00\rangle + \beta\,|1\rangle \otimes |00\rangle \\
&= \alpha\,|000\rangle + \beta\,|100\rangle \\
&= (\alpha\,|00\rangle + \beta\,|10\rangle) \otimes |0\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\phi_1\rangle &= (\text{CNOT} \otimes I)\,|\phi_0\rangle \\
&= \alpha\,|00\rangle \otimes |0\rangle + \beta\,|11\rangle \otimes |0\rangle \\
&= \alpha\,|000\rangle + \beta\,|110\rangle \\
&= \alpha\,|0\rangle\,|00\rangle + \beta\,|1\rangle\,|10\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\phi_2\rangle &= (H \otimes I \otimes I)\,|\phi_1\rangle \\
&= \alpha\,|+\rangle\,|00\rangle + \beta\,|-\rangle\,|10\rangle \\
&= \alpha\,|000\rangle + \alpha\,|100\rangle + \beta\,|010\rangle - \beta\,|110\rangle
\end{aligned}
$$

Alice measures $B = \{|00\rangle\langle 00| \otimes I, |01\rangle\langle 01| \otimes I, |10\rangle\langle 10| \otimes I, |11\rangle\langle 11| \otimes I\}$

For outcome $|00\rangle$ we have $\Pr = |\alpha|^2$ and $|\psi'\rangle = \frac{\alpha}{|\alpha|}\,|0\rangle$.

For outcome $|01\rangle$ we have $\Pr = |\beta|^2$ and $|\psi'\rangle = \frac{\beta}{|\beta|}\,|0\rangle$.

For outcome $|10\rangle$ we have $\Pr = |\alpha|^2$ and $|\psi'\rangle = \frac{\alpha}{|\alpha|}\,|0\rangle$.

For outcome $|11\rangle$ we have $\Pr = |\beta|^2$ and $|\psi'\rangle = \frac{-\beta}{|\beta|}\,|0\rangle$.

We have transmitted no information and have destroyed our entangled pair.

Now consider $|\phi\rangle = \sqrt{2/5}\,|00\rangle + \sqrt{3/5}\,|11\rangle$, then we have

$$
\begin{aligned}
|\phi_0\rangle &= |\psi\rangle \otimes |\phi\rangle \\
&= (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes (\sqrt{2/5}\,|00\rangle + \sqrt{3/5}\,|11\rangle) \\
&= (\alpha\sqrt{2/5}\,|000\rangle + \alpha\sqrt{3/5}\,|011\rangle + \beta\sqrt{2/5}\,|100\rangle + \beta\sqrt{3/5}\,|111\rangle)
\end{aligned}
$$

$$
\begin{aligned}
|\phi_1\rangle &= (\text{CNOT} \otimes I)\,|\phi_0\rangle \\
&= \alpha\sqrt{2/5}\,|000\rangle + \alpha\sqrt{3/5}\,|111\rangle + \beta\sqrt{2/5}\,|110\rangle + \beta\sqrt{3/5}\,|101\rangle
\end{aligned}
$$

$$
\begin{aligned}
|\phi_2\rangle &= (H \otimes I \otimes I)\,|\phi_1\rangle \\
&= \alpha\sqrt{2/5}\,|+\rangle\,|00\rangle + \alpha\sqrt{3/5}\,|+\rangle\,|11\rangle + \beta\sqrt{2/5}\,|-\rangle\,|10\rangle + \beta\sqrt{3/5}\,|-\rangle\,|01\rangle \\
&= \frac{1}{\sqrt{2}} \\
&\cdot (\alpha\sqrt{2/5}(|0\rangle + |1\rangle)\,|00\rangle \\
&+ \alpha\sqrt{3/5}(|0\rangle + |1\rangle)\,|11\rangle \\
&+ \beta\sqrt{2/5}(|0\rangle - |1\rangle)\,|10\rangle \\
&+ \beta\sqrt{3/5}(|0\rangle - |1\rangle)\,|01\rangle) \\
\\
&= \frac{1}{\sqrt{2}} \\
&\cdot (|00\rangle\,(\alpha\sqrt{2/5}\,|0\rangle + \beta\sqrt{3/5}\,|1\rangle) \\
&+ |01\rangle\,(\alpha\sqrt{3/5}\,|1\rangle + \beta\sqrt{2/5}\,|0\rangle) \\
&+ |10\rangle\,(\alpha\sqrt{2/5}\,|0\rangle - \beta\sqrt{3/5}\,|1\rangle) \\
&+ |11\rangle\,(\alpha\sqrt{3/5}\,|1\rangle - \beta\sqrt{2/5}\,|0\rangle)) \\
&= \frac{1}{\sqrt{2}} \\
&\cdot (\alpha\sqrt{2/5}\,|000\rangle + \beta\sqrt{3/5}\,|001\rangle \\
&+ \alpha\sqrt{3/5}\,|011\rangle + \beta\sqrt{2/5}\,|010\rangle \\
&- \alpha\sqrt{2/5}\,|100\rangle - \beta\sqrt{3/5}\,|101\rangle \\
&- \alpha\sqrt{3/5}\,|111\rangle - \beta\sqrt{2/5}\,|110\rangle)
\end{aligned}
$$

We shall consider the measurement outcomes:

For $|00\rangle$ we have

$$
\begin{aligned}
\|\,(|00\rangle\langle 00| \otimes I)\,|\phi_2\rangle\,\|_2^2 &= \left\| \frac{1}{\sqrt{2}} \cdot \alpha \cdot \sqrt{2/5}\,|000\rangle + \frac{1}{\sqrt{2}} \cdot \beta \cdot \sqrt{3/5}\,|001\rangle \right\|_2^2 \\
&= \frac{1}{2} \cdot |\alpha|^2 \cdot \frac{2}{5} + \frac{1}{2} \cdot |\beta|^2 \cdot \frac{3}{5}
\end{aligned}
$$

And the new state is

$$
\begin{aligned}
&\frac{\frac{1}{\sqrt{2}} \cdot \alpha \cdot \sqrt{2/5}\,|000\rangle + \frac{1}{\sqrt{2}} \cdot \beta \cdot \sqrt{3/5}\,|001\rangle}{\sqrt{\frac{1}{2} \cdot |\alpha|^2 \cdot \frac{2}{5} + \frac{1}{2} \cdot |\beta|^2 \cdot \frac{3}{5}}} \\
=&\frac{1}{\sqrt{2}} \cdot \frac{\alpha \cdot \sqrt{2/5}\,|000\rangle + \beta \cdot \sqrt{3/5}\,|001\rangle}{\sqrt{\frac{1}{2} \cdot |\alpha|^2 \cdot \frac{2}{5} + \frac{1}{2} \cdot |\beta|^2 \cdot \frac{3}{5}}} \\
=&\frac{\alpha \cdot \sqrt{2/5}\,|000\rangle + \beta \cdot \sqrt{3/5}\,|001\rangle}{\sqrt{|\alpha|^2 \cdot \frac{2}{5} + |\beta|^2 \cdot \frac{3}{5}}}
\end{aligned}
$$

The other measurement outcomes behave similarly. We will write them out in their un-normalized form:

As before, we may apply the X-gate, controlled by the measurement of the second qubit, to the third qubit.

The new state for the third qubit is

Outcome $|00\rangle$: $\alpha\sqrt{2/5}\,|0\rangle + \beta\sqrt{3/5}\,|1\rangle$

Outcome $|01\rangle$: $\alpha\sqrt{3/5}\,|0\rangle + \beta\sqrt{2/5}\,|1\rangle$

Outcome $|10\rangle$: $\alpha\sqrt{2/5}\,|0\rangle - \beta\sqrt{3/5}\,|1\rangle$

Outcome $|11\rangle$: $\alpha\sqrt{3/5}\,|0\rangle - \beta\sqrt{2/5}\,|1\rangle$

As before, we now apply the Z-gate, controlled by the measurement of the first qubit, to the third qubit.

The new state for the third qubit is

Outcome $|00\rangle$: $\alpha\sqrt{2/5}\,|0\rangle + \beta\sqrt{3/5}\,|1\rangle$

Outcome $|01\rangle$: $\alpha\sqrt{3/5}\,|0\rangle + \beta\sqrt{2/5}\,|1\rangle$

Outcome $|10\rangle$: $\alpha\sqrt{2/5}\,|0\rangle + \beta\sqrt{3/5}\,|1\rangle$

Outcome $|11\rangle$: $\alpha\sqrt{3/5}\,|0\rangle + \beta\sqrt{2/5}\,|1\rangle$

Our result is a state, that is strongly correlated with the original state $|\psi\rangle$.

Since the outcome vectors are normalized, I don't believe that the result is a linear tranformation of the original state. It may or may not be difficult to recover the original state from the result.