

Introduction to Quantum Computation, UPB

Winter 2022, Assignment 1

To be completed by: Friday, October 21

1 Exercises

1. For complex number $c = a + bi$, recall that the *real* and *imaginary* parts of c are denoted $\text{Re}(c) = a$ and $\text{Imag}(c) = b$.

- (a) Prove that $c + c^* = 2 \cdot \text{Re}(c)$.

Proof.

$$c + c^* = (a + bi) + (a - bi) \stackrel{\text{assoc.}, \text{comm.}}{=} (a + a) + (b - b)i = 2a = 2\text{Re}(c) \quad (1)$$

□

- (b) Prove that $cc^* = a^2 + b^2$. How can we therefore rewrite $|c|$ in terms of a and b ?

Proof.

$$cc^* = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - i^2b^2 = a^2 - (-1)b^2 = a^2 + b^2 \quad (2)$$

□

- (c) What is the polar form of $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$? Use the fact that $e^{i\theta} = \cos \theta + i \sin \theta$.

Observation 1.

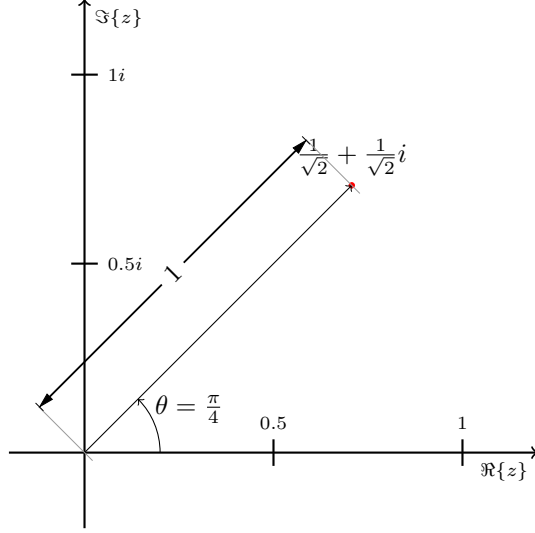
$$c = 1 \cdot e^{\frac{\pi}{4}i} \stackrel{\text{by calculation}}{=} c$$

Where:

Length: 1

Angle: $\frac{\pi}{4}$

- (d) Draw $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ as a vector in the complex plane, ensuring to denote both the length of the vector and its angle with the x axis.



2. Prove that for any normalized vectors $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$,

$$\| |\psi\rangle - |\phi\rangle \|_2 = \sqrt{2 - 2 \cdot \text{Re}(\langle \psi | \phi \rangle)}.$$

Proof.

$$\begin{aligned} \| |\psi\rangle - |\phi\rangle \|_2 &= \sqrt{\langle \psi - \phi | \psi - \phi \rangle} \\ &= \sqrt{\langle \psi | \psi \rangle - \langle \psi | \phi \rangle - \langle \phi | \psi \rangle + \langle \phi | \phi \rangle} \\ &= \sqrt{1 - \langle \psi | \phi \rangle - \langle \phi | \psi \rangle + 1} = \sqrt{2 - \langle \psi | \phi \rangle - \langle \psi | \phi \rangle^*} \\ &= \sqrt{2 - (\langle \psi | \phi \rangle + \langle \psi | \phi \rangle^*)} = \sqrt{2 - 2 \cdot \text{Re}(\langle \psi | \phi \rangle)} \end{aligned} \tag{3}$$

□

Why does it not matter if we replace $\langle \psi | \phi \rangle$ with $\langle \phi | \psi \rangle$ in this equation?

It does not matter, since $\langle \psi | \phi \rangle = \langle \phi | \psi \rangle^*$, and this implies that $\text{Re}(\langle \psi | \phi \rangle) = \text{Re}(\langle \phi | \psi \rangle)$.

3. Define

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

- (a) What is $\text{Tr}(A \cdot |1\rangle\langle 0|)$? (Hint: This can be computed quickly by using the cyclic property of the trace and the outer product representation of A . Do master this trick; it will be used repeatedly in the course and save you much time.)

$$\text{Tr}(A \cdot |1\rangle\langle 0|) = \text{Tr}(\langle 0 | A | 1 \rangle) = A_{0,1} = b. \tag{4}$$

- (b) Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the same tricks as in part A, along with the fact that the trace is linear, to quickly evaluate

$$\text{Tr}(A \cdot |+\rangle\langle +|).$$

$$\begin{aligned}
\text{Tr}(A \cdot |+\rangle\langle +|) &= \text{Tr}(\langle +|A|+\rangle) \\
&= \left(\frac{1}{\sqrt{2}}\right)^2 \cdot \text{Tr}(\langle 0| + \langle 1|)A(|0\rangle + |1\rangle) \\
&= \frac{1}{2} \cdot \text{Tr}(\langle 0|A|0\rangle + \langle 0|A|1\rangle + \langle 1|A|0\rangle + \langle 1|A|1\rangle) \\
&= \frac{1}{2} \cdot (a + c + d + b)
\end{aligned} \tag{5}$$

4. (a) A general property of the outer product is that $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$. Verify that this holds for the case where $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |1\rangle$. (Hint: Write out the full matrix corresponding to $|0\rangle\langle 1|$.)

$$\begin{aligned}
|0\rangle\langle 1| &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
(|0\rangle\langle 1|)^\dagger &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0|
\end{aligned} \tag{6}$$

- (b) Use Part (a) to prove that a normal matrix A satisfies $A = A^\dagger$ if and only if all of A 's eigenvalues are real. (Hint: Since A is normal, you can start by writing A in terms of its spectral decomposition. What does the condition $A = A^\dagger$ enforce in terms of A 's spectral decomposition?)

Let

$$A = \sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i|$$

be the spectral decomposition of A . Where λ_i are the eigenvalues of A and $|\psi_i\rangle$ are the corresponding eigenvectors such that the set of eigenvectors form an orthonormal basis of \mathbb{C}^d .

Direction \Rightarrow : A satisfies $A = A^\dagger$. We show that all eigenvalues are real.

Proof. Since $A = A^\dagger$, we have:

$$A = \sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i| = \left(\sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i| \right)^\dagger = A^\dagger \tag{7}$$

$$\begin{aligned}
\left(\sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i| \right)^\dagger &= \sum_{i=1}^d (\lambda_i |\psi_i\rangle\langle\psi_i|)^\dagger \\
&= \sum_{i=1}^d \lambda_i^\dagger (|\psi_i\rangle\langle\psi_i|)^\dagger \\
&= \sum_{i=1}^d \lambda_i^\dagger |\psi_i\rangle\langle\psi_i|
\end{aligned} \tag{8}$$

All $|\psi_i\rangle\langle\psi_i|$ are linearly independent, which can be easily seen by writing them out in the basis of $|\psi_i\rangle$.

Now:

$$\sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i| = \sum_{i=1}^d \lambda_i^\dagger |\psi_i\rangle\langle\psi_i| \tag{9}$$

Which means that

$$\lambda_i = \lambda_i^\dagger \quad \forall i \tag{10}$$

From which it follows, that all eigenvalues are real. \square

Direction \Leftarrow : All eigenvalues are real. We show that $A = A^\dagger$.

Proof. Since

$$\lambda_i = \lambda_i^\dagger \quad \forall i \quad (11)$$

It follows that:

$$\sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i| = \sum_{i=1}^d \lambda_i^\dagger |\psi_i\rangle\langle\psi_i| \quad (12)$$

By the same argument as above:

$$A = A^\dagger$$

□