Introduction to Quantum Computation, UPB Winter 2022, Assignment 4

Eli Kogan-Wang

1 Exercises

1. (a) Let $A, B \in \mathcal{L}(\mathbb{C}^d)$ be positive semi-definite matrices. Prove that $A + B$ is positive semi-definite.
Since A, B positive semi-definite: $\forall \left|\psi\right\rangle \in \mathbb{C}^d$ we have $\left\langle\psi\right| A \left|\psi\right\rangle \geq 0$ and $\left\langle\psi\right| B \left|\psi\right\rangle \geq 0$.
Now:
$$\begin{aligned} \left\langle\psi\right| (A+B) \left|\psi\right\rangle &= \left\langle\psi\right| A \left|\psi\right\rangle + \left\langle\psi\right| B \left|\psi\right\rangle \\ &\geq 0 + 0 \\ &= 0 \end{aligned}$$
By linearity of the Vector space $\mathcal{L}(\mathbb{C}^d)$, we have that $A + B$ is positive semi-definite.

   (b) Prove that if ρ and σ density matrices, then so is $p_1\rho + p_2\sigma$ for any $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$.
Let ρ and σ be density matrices. Then ρ and σ are positive semi-definite hermitian matrices. Additionally, $\mathrm{Tr}(\rho) = \mathrm{Tr}(\sigma) = 1$.
Now, let $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$. Then, we have the following:
Positive semi-definiteness:
$$\forall \left|\psi\right\rangle \in \mathbb{C}^d, \left\langle\psi\right| (p_1\rho + p_2\sigma) \left|\psi\right\rangle = p_1 \left\langle\psi\right| \rho \left|\psi\right\rangle + p_2 \left\langle\psi\right| \sigma \left|\psi\right\rangle \geq p_1 0 + p_2 0 = 0$$
Hermitian:
$$\begin{aligned} &\forall \left|\psi\right\rangle \in \mathbb{C}^d, \left\langle\psi\right| (p_1\rho + p_2\sigma) \left|\psi\right\rangle^\dagger \\ &= p_1 \left\langle\psi\right| \rho \left|\psi\right\rangle^\dagger + p_2 \left\langle\psi\right| \sigma \left|\psi\right\rangle^\dagger \\ &= p_1 \left\langle\psi\right| \rho^\dagger \left|\psi\right\rangle + p_2 \left\langle\psi\right| \sigma^\dagger \left|\psi\right\rangle \\ &= p_1 \left\langle\psi\right| \rho \left|\psi\right\rangle + p_2 \left\langle\psi\right| \sigma \left|\psi\right\rangle \\ &= \left\langle\psi\right| (p_1\rho + p_2\sigma) \left|\psi\right\rangle \end{aligned}$$
Trace:
$$\begin{aligned} \mathrm{Tr}(p_1\rho + p_2\sigma) &= p_1\mathrm{Tr}(\rho) + p_2\mathrm{Tr}(\sigma) \\ &= p_1 1 + p_2 1 \\ &= p_1 + p_2 \\ &= 1 \end{aligned}$$

2. Suppose that with probability $1/3$, I give you state $\left|0\right\rangle \in \mathbb{C}^2$, and with probability $2/3$, I give you state $\left|-\right\rangle$. Write down (i.e. as a 2×2 matrix) the density matrix describing the state in your possession.
Let $\left|0\middle\rangle\middle\langle 0\right|$ be the density matrix for $\left|0\right\rangle$ and $\left|-\middle\rangle\middle\langle -\right|$ be the density matrix for $\left|-\right\rangle$.
We will write out $\left|-\middle\rangle\middle\langle -\right|$ in terms of the computational basis $\{\left|0\right\rangle, \left|1\right\rangle\}$:
$$\left|-\middle\rangle\middle\langle -\right| = \frac{1}{2}\left(\left|0\middle\rangle\middle\langle 0\right| - \left|0\middle\rangle\middle\langle 1\right| - \left|1\middle\rangle\middle\langle 0\right| + \left|1\middle\rangle\middle\langle 1\right|\right)$$

Then, the density matrix for the state in my possession is:

$$
\begin{aligned}
\frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|-\rangle\langle -| &= \frac{1}{3}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \frac{2}{3}\cdot\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\\
&= \frac{1}{3}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\\
&= \frac{1}{3}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\\
&= \begin{bmatrix}\frac{2}{3} & \frac{-1}{3}\\ \frac{-1}{3} & \frac{1}{3}\end{bmatrix}
\end{aligned}
$$

3. Define bipartite state $|\psi\rangle = \alpha\,|01\rangle - \beta\,|10\rangle$. Let $\rho = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\psi\rangle\langle\psi|$. Compute $\mathrm{Tr}_B(\rho)$.

   Remember that $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

   And that $\mathrm{Tr}_B(\rho) = \sum_{i=1}^{d_B}(I_A \otimes \langle i|)\rho(I_A \otimes |i\rangle)$.

   Further we will inspect the density operator $|\Phi^+\rangle\langle\Phi^+|$:

$$
\begin{aligned}
|\Phi^+\rangle\langle\Phi^+| &= \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)\\
&= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)\\
&= \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|00\rangle\langle 11| + \frac{1}{2}|11\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|
\end{aligned}
$$

   And additionally, we will inspect the density operator $|\psi\rangle\langle\psi|$:

$$
\begin{aligned}
|\psi\rangle\langle\psi| &= (\alpha\,|01\rangle - \beta\,|10\rangle)(\alpha\,\langle 01| - \beta\,\langle 10|)\\
&= \alpha^2|01\rangle\langle 01| - \alpha\beta|01\rangle\langle 10| - \alpha\beta|10\rangle\langle 01| + \beta^2|10\rangle\langle 10|
\end{aligned}
$$

   We see that:

$$
\begin{aligned}
\mathrm{Tr}_B\rho &= \sum_{i=0}^{1}(I_A \otimes \langle i|)\rho(I_A \otimes |i\rangle) \\
&= (I_A \otimes \langle 0|)\rho(I_A \otimes |0\rangle) + (I_A \otimes \langle 1|)\rho(I_A \otimes |1\rangle) \\
&= (I_A \otimes \langle 0|)\frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\psi\rangle\langle\psi|(I_A \otimes |0\rangle) + (I_A \otimes \langle 1|)\frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\psi\rangle\langle\psi|(I_A \otimes |1\rangle) \\
&= (I_A \otimes \langle 0|)\frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\psi\rangle\langle\psi|(I_A \otimes |0\rangle) \\
&+ (I_A \otimes \langle 1|)\frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\psi\rangle\langle\psi|(I_A \otimes |1\rangle) \\
&= (I_A \otimes \langle 0|) \\
&\frac{1}{2}(\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|00\rangle\langle 11| + \frac{1}{2}|11\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|) \\
&+ \frac{1}{2}(\alpha^2|01\rangle\langle 01| - \alpha\beta|01\rangle\langle 10| - \alpha\beta|10\rangle\langle 01| + \beta^2|10\rangle\langle 10|) \\
&(I_A \otimes |0\rangle) \\
&+ (I_A \otimes \langle 1|) \\
&\frac{1}{2}(\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|00\rangle\langle 11| + \frac{1}{2}|11\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|) \\
&+ \frac{1}{2}(\alpha^2|01\rangle\langle 01| - \alpha\beta|01\rangle\langle 10| - \alpha\beta|10\rangle\langle 01| + \beta^2|10\rangle\langle 10|) \\
&(I_A \otimes |1\rangle) \\
&= (I_A \otimes \langle 0|)\frac{1}{2}(\frac{1}{2}|00\rangle\langle 00|) + \frac{1}{2}(\beta^2|10\rangle\langle 10|)(I_A \otimes |0\rangle) \\
&+ (I_A \otimes \langle 1|)\frac{1}{2}(\frac{1}{2}|11\rangle\langle 11|) + \frac{1}{2}(\alpha^2|01\rangle\langle 01|)(I_A \otimes |1\rangle) \\
&= \frac{1}{4}|0\rangle\langle 0| + \frac{\beta^2}{2}|1\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 1| + \frac{\alpha^2}{2}|0\rangle\langle 0| \\
&= \frac{1}{4}|0\rangle\langle 0| + \frac{\alpha^2}{2}|0\rangle\langle 0| + \frac{\beta^2}{2}|1\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 1| \\
&= \frac{1+2\alpha^2}{4}|0\rangle\langle 0| + \frac{1+2\beta^2}{4}|1\rangle\langle 1|
\end{aligned}
$$

4. Let $|\psi\rangle = \alpha_0 |a_0\rangle |b_0\rangle + \alpha_1 |a_1\rangle |b_1\rangle$ be the Schmidt decomposition of a two-qubit state $|\psi\rangle$. Prove that for any single qubit unitaries U and V, $|\psi\rangle$ is entangled if and only if $|\psi'\rangle = (U \otimes V)|\psi\rangle$ is entangled. (Hint: Prove that the Schmidt rank of $|\psi\rangle$ equals that of $|\psi'\rangle$. Also, you might find Lemma 1 of the Lecture 3 notes useful.)

$$
\begin{aligned}
|\psi'\rangle &= (U \otimes V)|\psi\rangle \\
&= (U \otimes V)(\alpha_0 |a_0\rangle |b_0\rangle + \alpha_1 |a_1\rangle |b_1\rangle) \\
&= (U \otimes V)(\alpha_0 |a_0\rangle |b_0\rangle) + (U \otimes V)(\alpha_1 |a_1\rangle |b_1\rangle) \\
&= \alpha_0(U|a_0\rangle \otimes V|b_0\rangle) + \alpha_1(U|a_1\rangle \otimes V|b_1\rangle)
\end{aligned}
$$

Where $\{U|a_0\rangle, U|a_1\rangle\}$ forms an Orthonormal basis for $\mathbb{C}^{d_A}$. And $\{V|b_0\rangle, V|b_1\rangle\}$ forms an Orthonormal basis for $\mathbb{C}^{d_B}$. We have previously shown that a unitary Operator preserves Orthonomality.

Now we have a Schmidt Decomposition of $|\psi'\rangle$ with coefficients α_0 and α_1.

Now $|\psi\rangle$ is entangled if and only if all the coefficients α_0 and α_1 are non-zero. Now $|\psi'\rangle$ is entangled if and only if all the coefficients α_0 and α_1 are non-zero. Now we have shown that $|\psi\rangle$ is entangled if and only if $|\psi'\rangle$ is entangled.