Introduction to Quantum Computation, UPB
Winter 2022, Assignment 1

To be completed by: Friday, October 21

1 Exercises

1. For complex number $c = a + bi$, recall that the *real* and *imaginary* parts of c are denoted $\mathrm{Re}(c) = a$ and $\mathrm{Imag}(c) = b$.
   (a) Prove that $c + c^* = 2 \cdot \mathrm{Re}(c)$.
   (b) Prove that $cc^* = a^2 + b^2$. How can we therefore rewrite $|c|$ in terms of a and b?
   (c) What is the polar form of $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$? Use the fact that $e^{i\theta} = \cos\theta + i \sin\theta$.
   (d) Draw $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ as a vector in the complex plane, ensuring to denote both the length of the vector and its angle with the x axis.

2. Prove that for any normalized vectors $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$,

$$\| \, |\psi\rangle - |\phi\rangle \, \|_2 = \sqrt{2 - 2 \cdot \mathrm{Re}(\langle\psi|\phi\rangle)}.$$

   Why does it not matter if we replace $\langle\psi|\phi\rangle$ with $\langle\phi|\psi\rangle$ in this equation?

3. Define

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   (a) What is $\mathrm{Tr}(A \cdot |1\rangle\langle 0|)$? (Hint: This can be computed quickly by using the cyclic property of the trace and the outer product representation of A. Do master this trick; it will be used repeatedly in the course and save you much time.)
   (b) Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the same tricks as in part A, along with the fact that the trace is linear, to quickly evaluate

$$\mathrm{Tr}(A \cdot |+\rangle\langle +|).$$

4. (a) A general property of the outer product is that $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$. Verify that this holds for the case where $|\psi\rangle = |0\rangle$ and $|\phi\rangle = |1\rangle$. (Hint: Write out the full matrix corresponding to $|0\rangle\langle 1|$.)
   (b) Use Part (a) to prove that a normal matrix A satisfies $A = A^\dagger$ if and only if all of A's eigenvalues are real. (Hint: Since A is normal, you can start by writing A in terms of its spectral decomposition. What does the condition $A = A^\dagger$ enforce in terms of A's spectral decomposition?)