Introduction to Quantum Computation, UPB
Winter 2022, Assignment 2

To be completed by: Thursday, October 27

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1 Exercises

1. Use the spectral decompositions of X and Z to prove that $HXH^\dagger = Z$. (Do not simply write out the matrices and multiply!) Why does this immediately also yield that $HZH^\dagger = X$?

$$X = (|+\rangle\langle+| - |-\rangle\langle-|)$$
$$Z = (|0\rangle\langle 0| - |1\rangle\langle 1|)$$

Since for H we have $H = H^\dagger = H^{-1}$, we can write $HXH^\dagger = HXH$.

For the Hadamard gate, we have $H|0\rangle = |+\rangle$ and $H|1\rangle = |-\rangle$, so we can write:

$$H = |+\rangle\langle 0| + |-\rangle\langle 1| \quad \text{(trivial, since Orthonormal-Change-of-Basis)}$$

Additionally, we have $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$, so we can write:

$$H = |0\rangle\langle +| + |1\rangle\langle -| \quad \text{(trivial, since Orthonormal-Change-of-Basis)}$$

Now, we can write:

$$\begin{aligned}
HXH^\dagger &= H(|+\rangle\langle+| - |-\rangle\langle-|)H \\
&= (|0\rangle\langle+| + |1\rangle\langle-|)(|+\rangle\langle+| - |-\rangle\langle-|)(|+\rangle\langle 0| + |-\rangle\langle 1|) \\
&\text{We will not write out the } \langle-|+\rangle \text{ and } \langle+|-\rangle \text{ terms, since they are zero.} \\
&= (|0\rangle\langle+|+\rangle\langle+|+\rangle\langle 0| + |1\rangle\langle-|(-1)|-\rangle\langle-|-\rangle\langle 1|) \\
&= (|0\rangle\langle 0| - |1\rangle\langle 1|) \\
&= Z
\end{aligned}$$

From this it immediately follows, that $HZH^\dagger = X$:

$$\begin{aligned}
& HXH^\dagger = Z && | \cdot H \quad (\text{since } H = H^\dagger = H^{-1}) \\
\iff & HX = ZH && | H^\dagger \cdot \\
\iff & X = HZH^\dagger
\end{aligned}$$

2. (a) Write out the 4-dimensional vector for $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$.

$$\begin{bmatrix} \alpha \cdot \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \\ \beta \cdot \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix} = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

(b) Let $\mathcal{B}_1 = \{|\psi_1\rangle, |\psi_2\rangle\}$ and $\mathcal{B}_2 = \{|\phi_1\rangle, |\phi_2\rangle\}$ be two orthonormal bases for $\mathbb{C}^2$. Prove that

$$\mathcal{B}_3 = \{|\psi_1\rangle \otimes |\phi_1\rangle, |\psi_1\rangle \otimes |\phi_2\rangle, |\psi_2\rangle \otimes |\phi_1\rangle, |\psi_2\rangle \otimes |\phi_2\rangle\}$$

is an orthonormal basis for $\mathbb{C}^4$. In other words, show that for each $|v\rangle \in \mathcal{B}_3$, $\| |v\rangle \|_2 = 1$, and for all pairs of distinct $|v\rangle, |w\rangle \in \mathcal{B}_3$, $\langle v|w\rangle = 0$.

Every $|v\rangle \in \mathcal{B}_3$ is of the form $|\psi_i\rangle \otimes |\phi_j\rangle$, where $i, j \in \{1, 2\}$.
Let $i, j \in \{1, 2\}$. Now let:

$$|\psi_i\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$|\phi_j\rangle = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$$

Since $\mathcal{B}_1, \mathcal{B}_2$ are orthonormal bases, we have $\alpha^2 + \beta^2 = 1$ and $\gamma^2 + \delta^2 = 1$.
Since:

$$|\psi_i\rangle \otimes |\phi_j\rangle = \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix}$$

We need to show that $\alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 = 1$.

$$\begin{aligned} \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 &= \alpha^2 \cdot (\gamma^2 + \delta^2) + \beta^2 \cdot (\gamma^2 + \delta^2) \\ &= \alpha^2 \cdot 1 + \beta^2 \cdot 1 \\ &= 1 \end{aligned}$$

We have shown that, for each $|v\rangle \in \mathcal{B}_3$, $\| |v\rangle \|_2 = 1$.
Now, let $|v\rangle, |w\rangle \in \mathcal{B}_3$ be distinct. Then, $|v\rangle, |w\rangle$ are of the form $|\psi_i\rangle \otimes |\phi_j\rangle$ and $|\psi_k\rangle \otimes |\phi_l\rangle$, respectively, where $i, j, k, l \in \{1, 2\}$. And $(i, j) \neq (k, l)$ or in other words, $i \neq k$ or $j \neq l$.
Now:

$$\begin{aligned} \langle v|w\rangle &= ((\langle\psi_i| \otimes \langle\phi_j|)(|\psi_k\rangle \otimes |\phi_l\rangle)) \\ &= \langle\psi_i|\psi_k\rangle \cdot \langle\phi_j|\phi_l\rangle \\ &\text{Since } i \neq k \text{ or } j \neq l \text{ and the } \psi\text{s and } \phi\text{s are orthogonal,} \\ &= 0 \end{aligned}$$

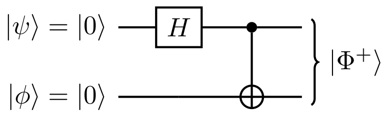
We have succesfully shown, that $\mathcal{B}_3$ is an orthonormal basis for $\mathbb{C}^4$.

3. (a) Prove that $(Z \otimes Y)^\dagger = Z \otimes Y$. Do not write out any matrices explicitly; rather, you must use the properties of the tensor product, dagger, and Y.

$$\begin{aligned} (Z \otimes Y)^\dagger &= Z^\dagger \otimes Y^\dagger && |\dagger \text{ distributes over } \otimes \\ &= Z \otimes Y && |Z,Y \text{ are self-adjoint} \end{aligned}$$

(b) In class, we saw a quantum circuit which, given starting state $|0\rangle \otimes |0\rangle$, prepared the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. In fact, that circuit is a change of basis matrix, mapping the standard basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ to the Bell basis $|\Phi^+\rangle, |\Psi^+\rangle, |\Phi^-\rangle, |\Psi^-\rangle$.

In fact, in class we saw:



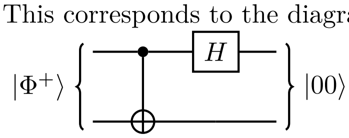
Which is equivalent to:

$$\text{CNOT}(H \otimes I)\,|0\rangle\,|0\rangle = \left|\Phi^+\right\rangle$$

i. Write down a quantum circuit which maps $|\Phi^+\rangle$ to $|0\rangle \otimes |0\rangle$. (Hint: Write out the circuit from class as a sequence of matrix operations. Then, recalling that the inverse of any unitary operation U is given by $U^\dagger$, take the inverse of this entire sequence by taking the dagger.)

$$\begin{aligned}
(\text{CNOT}(H \otimes I))^{-1} &= ((H \otimes I)^{-1}\text{CNOT}^{-1}) \\
&= ((H \otimes I)^{\dagger}\text{CNOT}^{\dagger}) \\
&= ((H^{\dagger} \otimes I^{\dagger})\text{CNOT}) \\
&= (H \otimes I)\text{CNOT}
\end{aligned}$$

This corresponds to the diagram:



ii. Your circuit from 3(b)(i) is actually a change of basis which maps the Bell basis *back* to the standard basis. To verify this, run your circuit from 3(b)(i) on input $|\Psi^-\rangle$ and check that the output is $|1\rangle \otimes |1\rangle$.

$$\begin{aligned}
(H \otimes I)\text{CNOT}\left|\Psi^-\right\rangle &= (H \otimes I)\text{CNOT}\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\
&= (H \otimes I)\frac{1}{\sqrt{2}}\text{CNOT}(|01\rangle - |10\rangle) \\
&= (H \otimes I)\frac{1}{\sqrt{2}}(\text{CNOT}\,|01\rangle - \text{CNOT}\,|10\rangle) \\
&= (H \otimes I)\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \\
&= (H \otimes I)\frac{1}{\sqrt{2}}((|0\rangle - |1\rangle) \otimes |1\rangle) \\
&= (H \otimes I)((\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) \otimes |1\rangle) \\
&= (H \otimes I)(|-\rangle \otimes |1\rangle) \\
&= ((H\,|-\rangle) \otimes (I\,|1\rangle)) \\
&= |1\rangle \otimes |1\rangle \\
&= |11\rangle
\end{aligned}$$