

Natural Language Processing with Deep Learning

Lecture 2 — Mathematical foundations of deep learning

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Natural Language Processing Group
Paderborn University

We focus on Trustworthy Human Language Technologies



www.trusthlt.org

Motivation

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Problem 1: Minimize functions

Problem 2: Minimize multivariate functions

Problem 3: When functions become heavily nested

Efficient computation of gradient

1. Who of you knows stochastic gradient descent and backpropagation?

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2. Who of you ever implemented it from scratch?

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How does this lecture relate to the previous one and to entire course?

How deep will we go?

We won't cover

- Set theory: The assembler of mathematics
 - Sets $A = \{a, b, c\}$, $a \in A$, no ordering
 - Ordered tuples $(a, b) \neq (b, a)$

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- Number theory
 - Set of natural numbers $\mathbb{N}_0 = \{0, 1, \dots\}$
 - Set of real numbers \mathbb{R} , infinity

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- Number theory
 - Set of natural numbers $\mathbb{N}_0 = \{0, 1, \dots\}$
 - Set of real numbers \mathbb{R} , infinity
- Sequences and limits

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ — tuples of reals, e.g., $(1.3, -44.67)$, also a two-dimensional vector

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Why is it important?

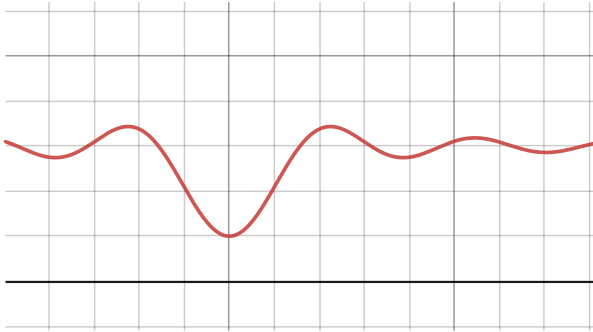
In supervised machine learning ...

Why is it important?

In supervised machine learning ...

- We have some training data (e.g., for classification)
- We have a learning algorithm
- We want to minimize some kind of error (e.g., misclassification) of the learning algorithm on training data

Problem: Find minimum of any function



- For "easy" functions, closed-form solution (high school math)
- For complicated functions not trivial and cumbersome

Function of single variable

We typically use Euler's notation with arbitrary but somehow standard naming conventions (x, y, f)

$$y = f(x) \quad f : \mathbb{R} \rightarrow \mathbb{R}$$

$f : A \rightarrow B$ where A is domain, B is co-domain

Function composition

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad g : \mathbb{R} \rightarrow \mathbb{R}$$

$$h = g \circ f$$

$$h(x) = g(f(x)) \text{ or } (g \circ f)(x) = g(f(x))$$

Lines in a Cartesian plane are characterized by linear equations.

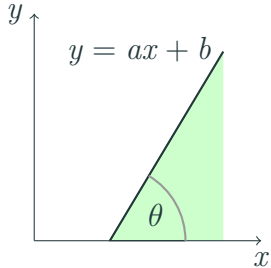
Every line L (including vertical lines) is the set of all points whose coordinates (x, y) satisfy a linear equation:

$$L = \{(x, y) \mid w_1x + w_2y = w_3\}$$

where w_1 , w_2 and w_3 are fixed real numbers (called coefficients) such that w_1 and w_2 are not both zero.

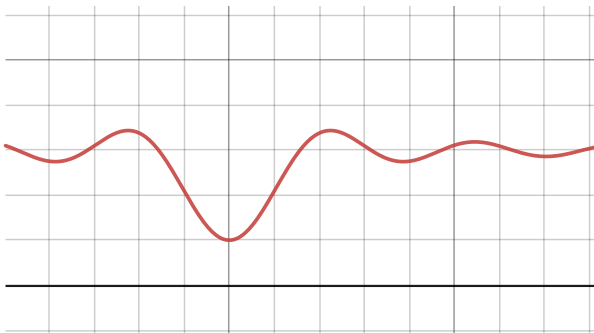
Linear function in two dimensions

Usually we use **slope-intercept** form $y = ax + b$



$$\theta = \arctan(a) \quad a = \tan(\theta)$$

Approximate function by a line at point



"Steepness" at c ?

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

The derivative of f at c

Derivative-computing function

We want a function D which, when given a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ as input, produces another function $g : \mathbb{R} \rightarrow \mathbb{R}$ output, such that $g(c) = f'(c)$ for every c .

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This derivative-computing function D is often written as

$$\frac{d}{dx}$$

but this causes inconsistent notation like

$$\frac{d}{dx}(f), \quad \frac{df}{dx}, \quad \frac{dy}{dx}$$

and forces one to choose a variable name x or y

Variant 1 (Lagrange's notation)

Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two functions which have derivatives.
Then the derivative of $g(f(x))$ is $g'(f(x)) \cdot f'(x)$

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Variant 3 (Leibnitz's notation)

Call $h(x) = g(f(x))$. Then using $\frac{dh}{dx}$ for the derivative of h ,
the chain rule for this would be $\frac{dh}{dx} = \frac{dh}{df} \frac{df}{dx}$

Chain rule example

Consider $y = e^{\sin(x^2)}$. Composite of three functions:

$$y = f(u) = e^u$$

$$u = g(v) = \sin v = \sin(x^2)$$

$$v = h(x) = x^2$$

Chain rule example

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$$y = f(u) = e^u$$

$$u = g(v) = \sin v = \sin(x^2)$$

$$v = h(x) = x^2$$

Their derivatives are

$$\frac{dy}{du} = f'(u) = e^u = e^{\sin(x^2)}$$

$$\frac{du}{dv} = g'(v) = \cos v = \cos(x^2)$$

$$\frac{dv}{dx} = h'(x) = 2x$$

Chain rule example (cont.)

Consider $y = e^{\sin(x^2)}$. Composite of three functions:

$$y = f(u) = e^u, u = g(v) = \sin v = \sin(x^2), v = h(x) = x^2$$

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Their derivatives are

$$\frac{dy}{du} = e^{\sin(x^2)}, \frac{du}{dv} = \cos(x^2), \frac{dv}{dx} = 2x$$

Derivative of their composite at the point $x = a$ is (in Leibniz notation)

$$\frac{dy}{dx} = \left. \frac{dy}{du} \right|_{u=g(h(a))} \cdot \left. \frac{du}{dv} \right|_{v=h(a)} \cdot \left. \frac{dv}{dx} \right|_{x=a}$$

Gradient-based optimization: Find minimum of a function

We want $\hat{x} = \operatorname{argmin}_x f(x)$

Pre-requisites:

- We can evaluate $y = f(x)$ for any x
- We can evaluate its derivative $f'(c)$ (or $\frac{dy}{dx}(c)$) for any c

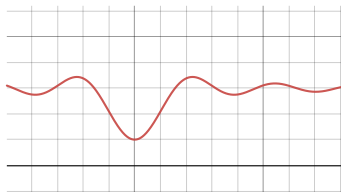
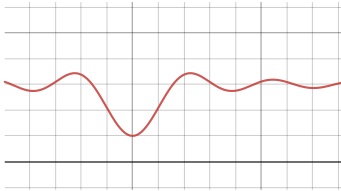


Figure 1: $3 - \frac{\sin(2x)}{x}$

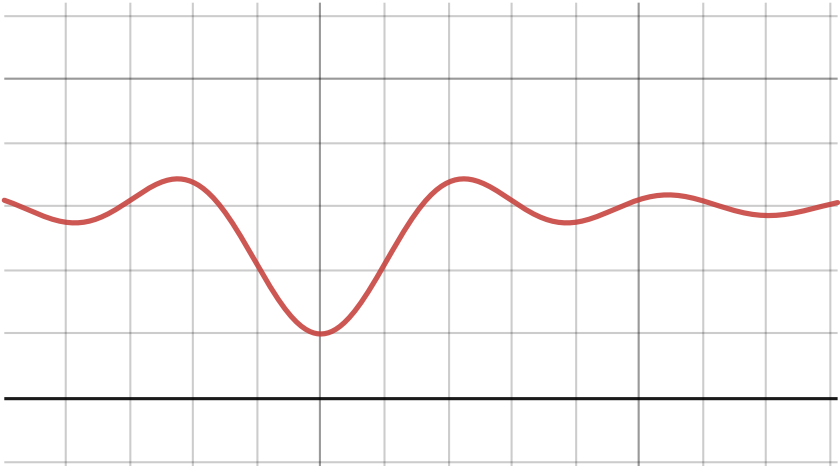
Gradient-based optimization: Find minimum of a function



1. Start with initial random value x_i
2. $u = f'(x_i)$ — direction and strength of change at x_i
3. Next value $x_{i+1} \leftarrow x_i - \eta \cdot u$
4. With small enough η (eta), $f(x_{i+1}) < f(x_i)$

Repeating 2 + 3 (with properly decreasing values of η) will find minimum point x_i

Gradient-based optimization: Workout example



Problem 2: Minimize multivariate functions

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Problem 3: When functions become heavily nested

Efficient computation of gradient

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

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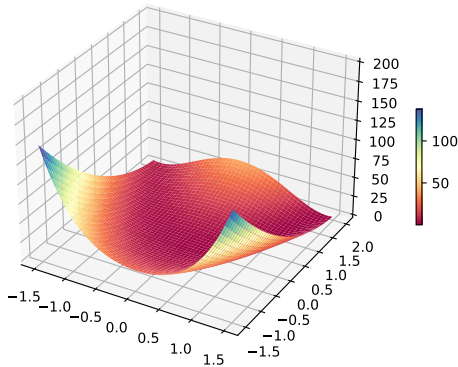


Figure 2: $f(x, y) = (a - x)^2 + b(y - x^2)^2$, $a = 1$, $b = 100$

Partial derivative: the directional derivative wrt. a single variable

$\frac{\partial f}{\partial x_2}$ — "the partial derivative of f with respect to x_2 "

Example: $f(x_1, x_2, x_3) = (x_1)^2 x_2 + \cos(x_3)$

$$\frac{\partial f}{\partial x_1} = 2x_2 x_1 \quad \frac{\partial f}{\partial x_2} = (x_1)^2 \quad \frac{\partial f}{\partial x_3} = -\sin(x_3)$$

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The resulting total derivative matrix Df is called the **gradient** of f , denoted ∇f

Example: $f(x_1, x_2, x_3) = (x_1)^2 x_2 + \cos(x_3)$

$$\nabla f = \left(\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \frac{\partial f}{\partial x_3} \right) = (2x_2 x_1 \quad (x_1)^2 \quad -\sin(x_3))$$

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For every differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and every point $\mathbf{x} \in \mathbb{R}^n$, the gradient $\nabla f(\mathbf{x})$ points in the direction of steepest ascent of f at \mathbf{x} .

Warning!

Sometimes we call gradient the function for computing concrete values for a given input (as above), sometimes the vector of concrete numbers computed for the given input

J. Kun (2020). *A Programmer's Introduction to Mathematics*. 2nd ed., p. 252

Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we want to find

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} f(\mathbf{x})$$

1. Start at some random position with a random value vector $\mathbf{x}_i = (x_1, \dots, x_n)$
2. Compute the gradient and update the position

$$\mathbf{x}_{i+1} \leftarrow \mathbf{x}_i - \eta \cdot \nabla f(\mathbf{x}_i)$$

3. After enough iterations or some stopping criterion we have $\hat{\mathbf{x}}$

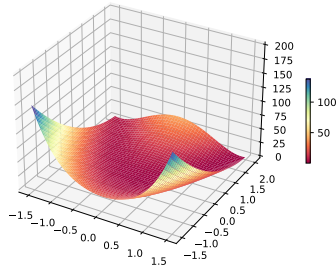
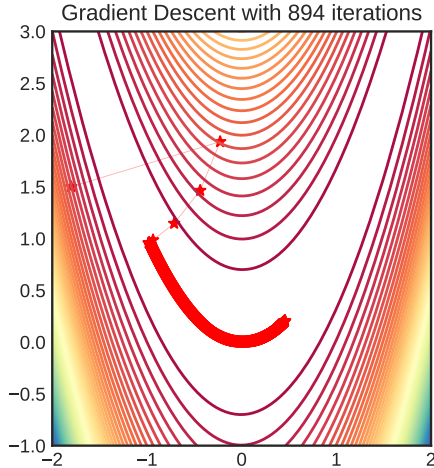
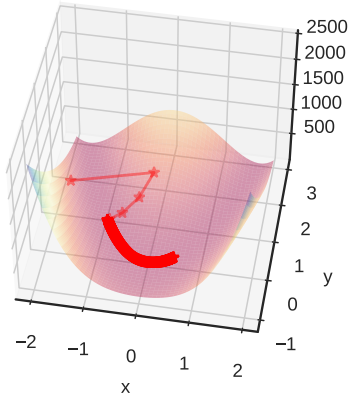


Figure 3: $f(x, y) = (a - x)^2 + b(y - x^2)^2$, $a = 1$, $b = 100$

$$\nabla f = (-400xy + 400x^3 + 2x - 2; \quad 200y - 200x^2)$$

Gradient for minimizing multivariate functions



Random starting point $(-1.8; 1.5)$, minimum at $(1; 1)$

Problem 3: When functions become heavily nested

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Efficient computation of gradient

In reality we work with deeply composed functions

Example

Minimize function e wrt. w_0, w_1, \dots, w_K

$$e = -\frac{1}{N} \sum_{i=1}^N y_{[i]} \log \left(\frac{1}{1 + \exp \left(w_0 + \sum_{j=1}^K w_k \cdot \boldsymbol{x}_{[i][k]} \right)} \right)$$

Where $\boldsymbol{x}_{[1]}, \dots, \boldsymbol{x}_{[N]}$, and $y_{[1]}, \dots, y_{[N]}$ are constants

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$$\nabla f = \left(\frac{\partial e}{\partial w_0}; \frac{\partial e}{\partial w_1}; \dots; \frac{\partial e}{\partial w_K} \right)$$

$$\frac{\partial e}{\partial w_1} = \dots$$

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$$\nabla f = \left(\frac{\partial e}{\partial w_0}; \frac{\partial e}{\partial w_1}; \dots; \frac{\partial e}{\partial w_K} \right)$$

$\frac{\partial e}{\partial w_1} = \dots$ Good luck!

Chain Rule for Multivariable Functions

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y . Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y) .

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Be ready for possible notation madness

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial g} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial h} \frac{\partial h}{\partial t}$$

Chain rule for multivariable functions (two independent variables)

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v , and $z = f(x, y)$ is a differentiable function of x and y . Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v , and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

Problem 3: When functions become heavily nested

Efficient computation of gradient

Working example

$$e = (a + b)(b + 1)$$

Compute gradient wrt. a and b

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Compute gradient wrt. a and b

This one is easy by hand, but that's not the point

$$e = (a + b)(b + 1) = ab + a + b^2 + b$$

$$\frac{\partial e}{\partial a} = b + 1 \quad \frac{\partial e}{\partial b} = a + 2b + 1$$

Add some intermediate variables and function names

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

$$s = f_3(b) := b + 1$$

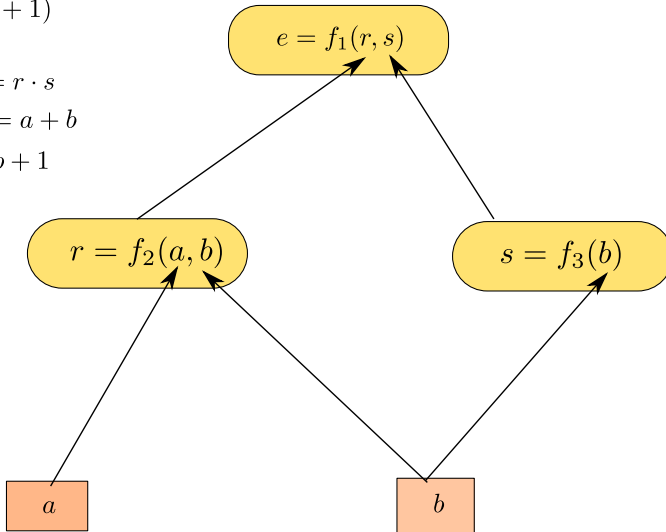
Build computational graph and evaluate (forward step)

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

$$s = f_3(b) := b + 1$$



Important: a, b will be some concrete real numbers, therefore r, s, e will be concrete real numbers too!

- DAG — direct acyclic graph (not necessarily a tree!)
- Each node — a differentiable function with arguments
- Leaves — variables (e.g., a , b) or constants
- Arrows — Function composition

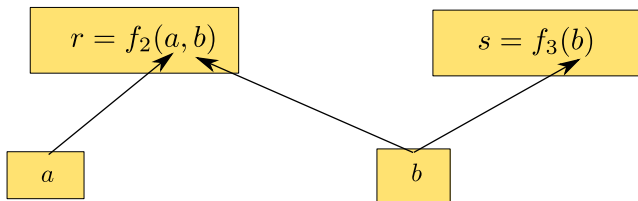


Figure 4: r , s are parents of b ; a , b are children (arguments) of r

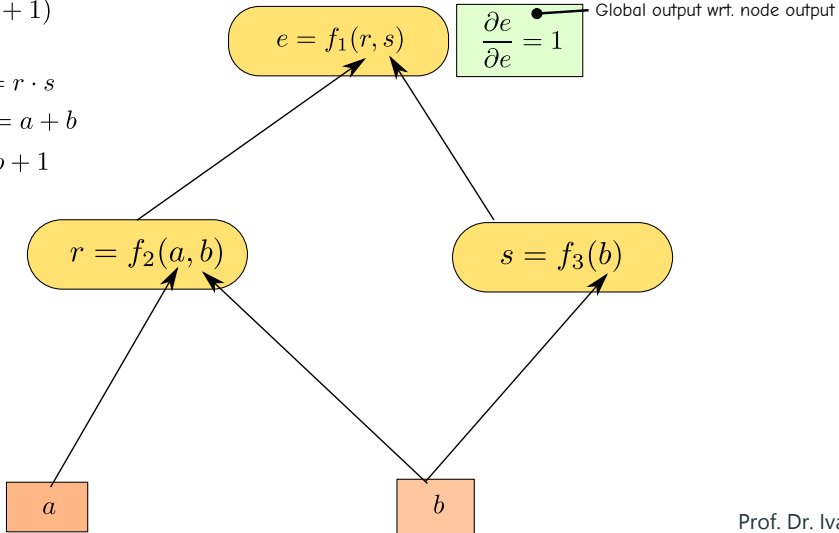
Goal: $\frac{\partial e}{\partial a}$ and $\frac{\partial e}{\partial b}$ (gradient), but let's do $\frac{\partial e}{\partial \star}$ for every node

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

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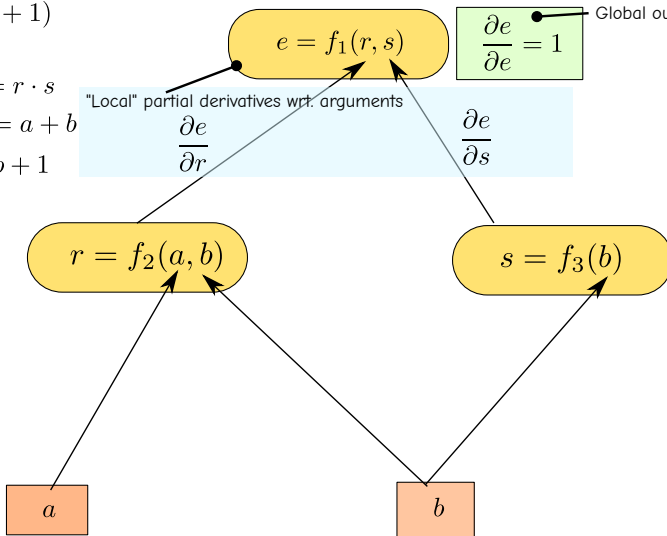
Since $e = r \cdot s$, par. derivatives are easy: $\frac{\partial e}{\partial r} = s$ and $\frac{\partial e}{\partial s} = r$

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

$$s = f_3(b) := b + 1$$



Sanity check: r, s are some concrete real numbers, therefore $\frac{\partial e}{\partial r}$ and $\frac{\partial e}{\partial s}$ will be concrete real numbers too!

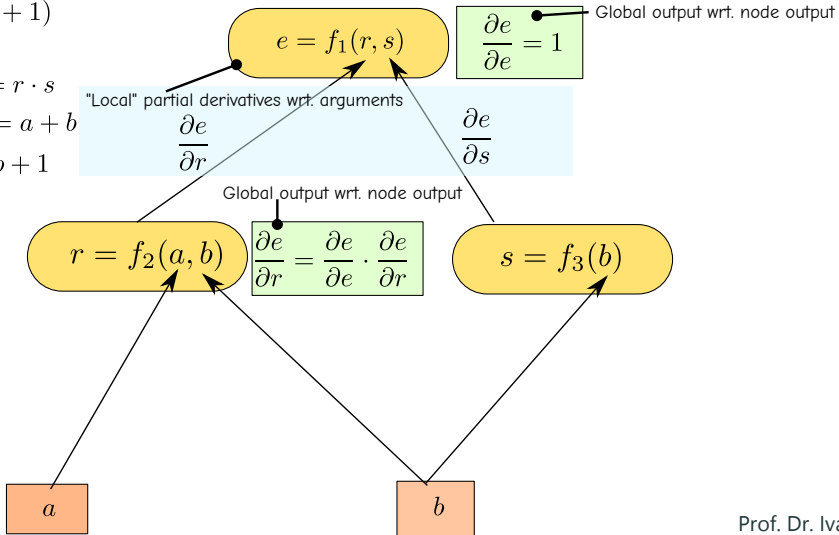
Proceed to next child r and compute $\frac{\partial e}{\partial r}$ – use chain rule!

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

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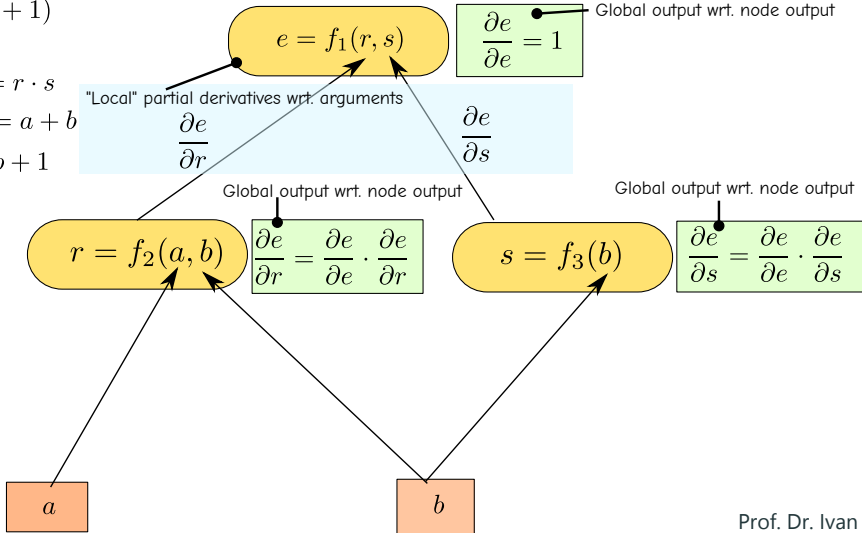
Proceed to next child s and compute $\frac{\partial e}{\partial s}$ – use chain rule!

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

$$s = f_3(b) := b + 1$$



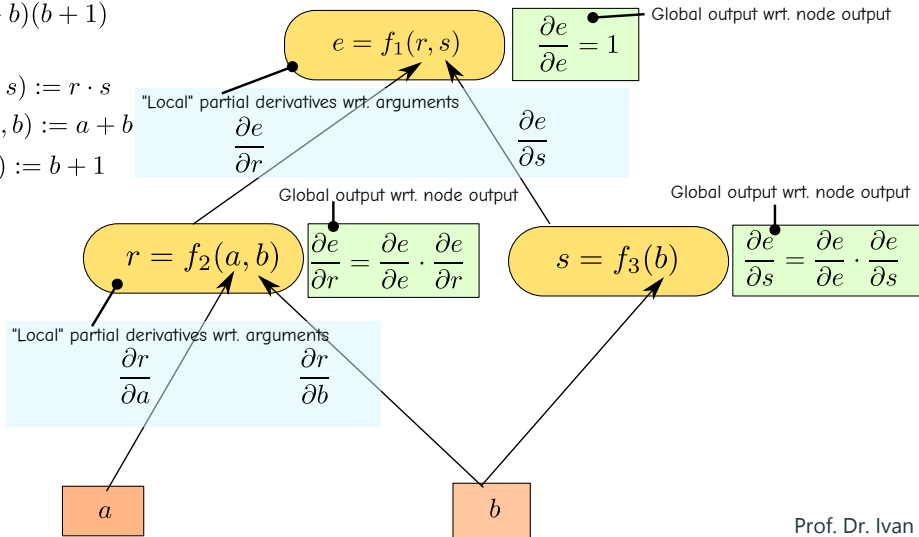
Since $r = a + b$, par. derivatives are easy: $\frac{\partial r}{\partial a} = 1$ and $\frac{\partial r}{\partial b} = 1$ 

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$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

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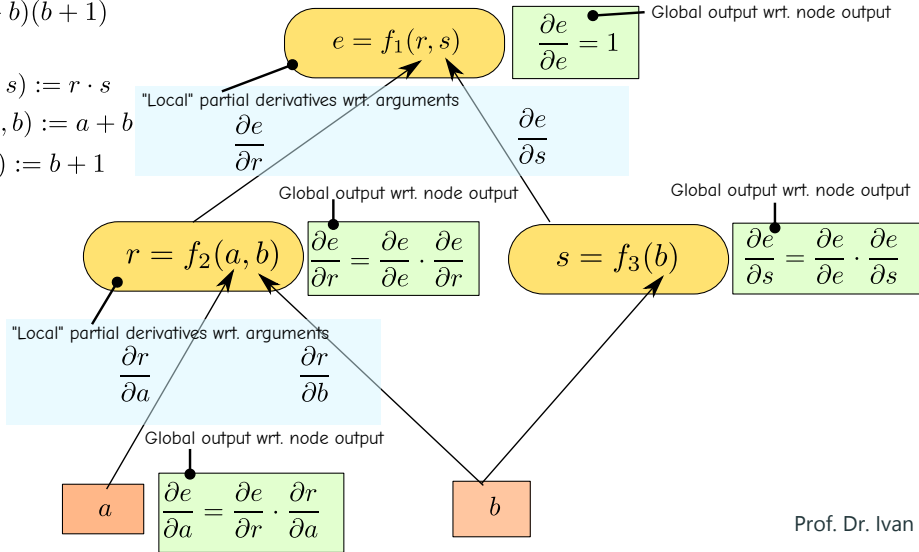
Proceed to next child a and compute $\frac{\partial e}{\partial a}$ – use chain rule!

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

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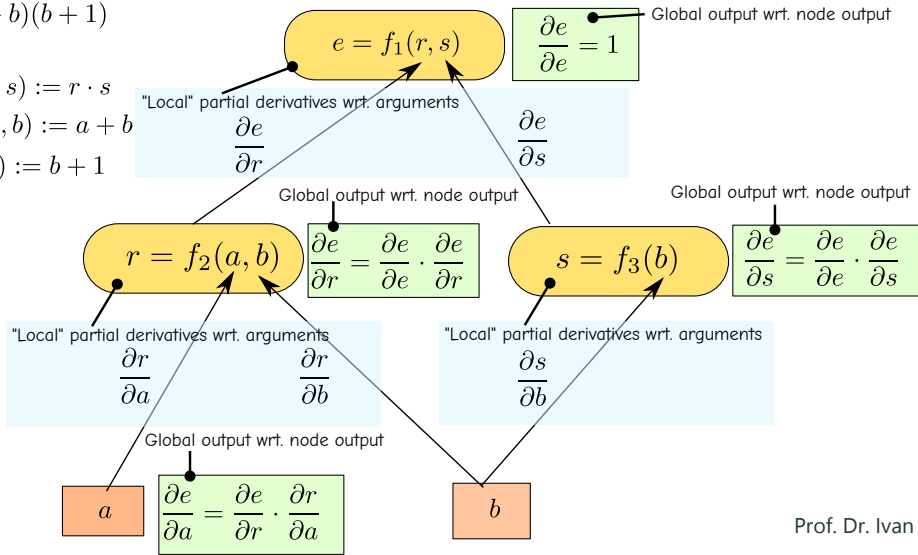
Since $s = b + 1$, par. derivatives are easy: $\frac{\partial s}{\partial b} = 1$

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

$$s = f_3(b) := b + 1$$



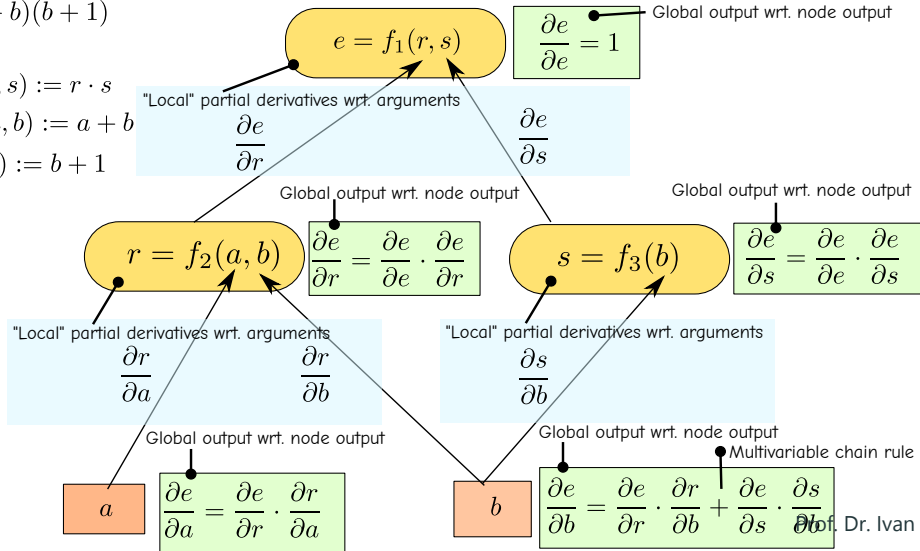
Proceed to b and compute $\frac{\partial e}{\partial b}$ – use multivariate chain rule

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

$$s = f_3(b) := b + 1$$



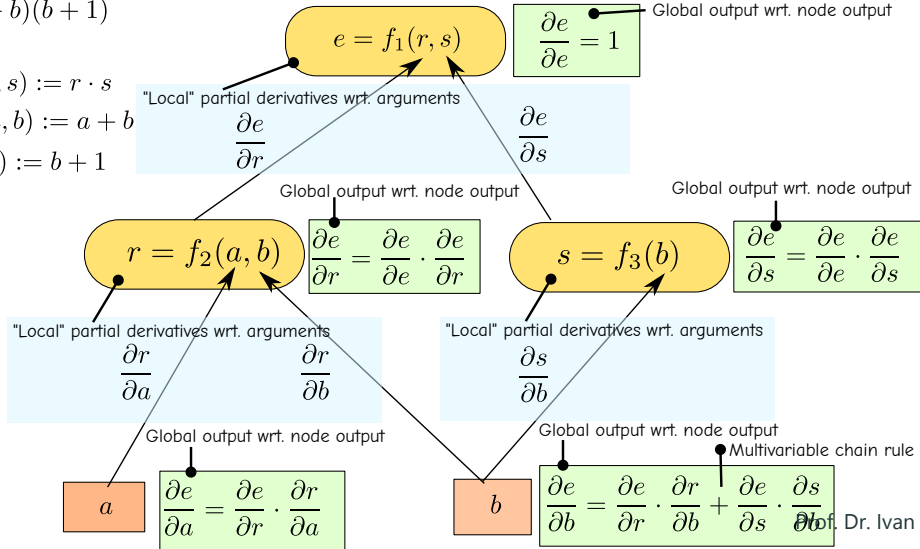
Goal: $\nabla e = \left(\frac{\partial e}{\partial a}; \frac{\partial e}{\partial b} \right)$ — we computed it for concrete a and b

$$e = (a + b)(b + 1)$$

$$e = f_1(r, s) := r \cdot s$$

$$r = f_2(a, b) := a + b$$

$$s = f_3(b) := b + 1$$



Generic node in a computational graph

Adapted from J. Kun (2020). **A Programmer's Introduction to Mathematics**. 2nd ed., p. 265

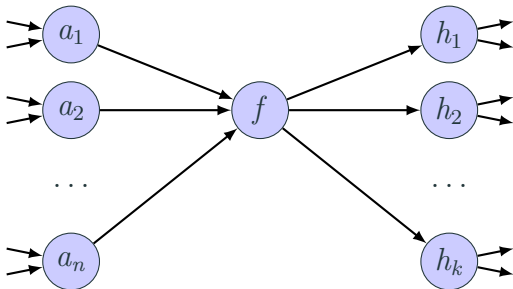
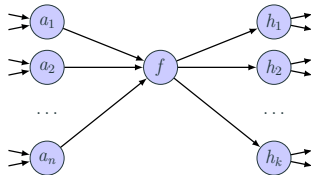


Figure 5: A generic node of a computation graph. Node f has many inputs, its output feeds into many nodes, and each of its inputs and outputs may also have many inputs and outputs.

Generic node in a computational graph $f(a_1, \dots, a_n)$



Assuming the graph is a function $e = g(\dots)$, we compute

$$\frac{\partial e}{\partial f} = \sum_{i=1}^k \frac{\partial e}{\partial h_i} \cdot \frac{\partial h_i}{\partial f}$$

and

$$\frac{\partial f}{\partial a_i} \quad \text{for } a_i, \dots, a_n$$

What each node must implement?

For example a function $s = f(a, b, c, d)$

- How to compute the output value s (given the parameters a, b, c, d)
- How to compute partial derivatives wrt. the parameters, i.e. $\frac{\partial s}{\partial a}, \frac{\partial s}{\partial b}, \frac{\partial s}{\partial c}, \frac{\partial s}{\partial d}$

- Forward computation: Compute all nodes' output (and cache it)
- Backward computation (Backprop): Compute the overall function's partial derivative with respect to each node

Ordering of the computations? Recursively or build a graph's topology upfront and iterate

- We can express any arbitrarily complicated function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ as a computational graph
- For computing the gradient ∇f at a concrete point (x_1, x_2, \dots, x_n) we run the forward pass and backprop
- When caching each node's intermediate output and partial derivatives, we avoid repeating computations \rightarrow efficient algorithm

Recap

Motivation

Problem 1: Minimize functions

Problem 2: Minimize multivariate functions

Problem 3: When functions become heavily nested

Efficient computation of gradient

- We can quite efficiently find a minimum of any differentiable nested multivariate function
 - Iterative gradient descent takes the most promising direction
 - Backpropagation utilizes computational graphs and caching → computes gradients efficiently
- We have not touched neural networks yet at all!

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Credits

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