LEMMA [Inversion of the Typing Relation]:

1. If $\Gamma \mid \Sigma \vdash x : R$ , then $x : R \in \Gamma$
2. If $\Gamma \mid \Sigma \vdash \lambda x : T_1.t_2 : R$, then $R = T_1 \rightarrow R_2$ for some $R_2$, with $\Gamma,\, x : T_1 \vdash t_2 : R_2$.
3. If $\Gamma \mid \Sigma \vdash t_1\ t_2 : R$, then there is some type $T_{11}$ such that $\Gamma \mid \Sigma \vdash t_1 : T_{11} \rightarrow R$ and $\Gamma \mid \Sigma \vdash t_2 : T_{11}$.
4. If $\Gamma \mid \Sigma \vdash ref\ t : R$, then $R = Ref\ T_1$ for some $T_1$ and $\Gamma \mid \Sigma \vdash t_1 : T_1$.
5. If $\Gamma \mid \Sigma \vdash !t : R$, then $\Gamma \mid \Sigma \vdash t : Ref\ R$.
6. If $\Gamma \mid \Sigma \vdash t_1 := t_2 : R$, then $R = Unit$, $\Gamma \mid \Sigma \vdash t_1 : Ref\ T_1$, for some $T_1$ and $\Gamma \mid \Sigma \vdash t_2 : T_1$.
7. If $\Gamma \mid \Sigma \vdash l : R$, then $R = Ref\ T_1$, for some $T_1$, and $\Sigma(l) = T_1$.

*Proof* : Immediate from the definition of the typing relation.

LEMMA [Canonical Forms]:

1. If $v$ is a value of type $T_1 \rightarrow T_2$, then $v = \lambda x : T_1.t_2$.
2. If $v$ is a value of type $Unit$, then $v = unit$.
3. If $v$ is a value of type $Ref\ T$, then $v = l$.

*Proof* : Straightforward.

THEOREM [Progress]: Suppose $t$ is a closed, well-typed term (that is, $\varnothing \mid \Sigma \vdash t : T$ for some $T$). Then either $t$ is a value or else, for any store $\mu$ such that $\varnothing \mid \Sigma \vdash \mu$, there is some term $t'$ with $t \mid \mu \longrightarrow t' \mid \mu'$.
Proof: Straightforward induction on typing derivations.

1. The variable case cannot occur (because $t$ is closed).
2. The abstraction case is immediate, since abstractions are values.
3. For application, where $t = t_1\ t_2$, with $\varnothing \mid \Sigma \vdash t_1 : T_{11} \longrightarrow T_{12}$ and $\varnothing \mid \Sigma \vdash t_2 : T_{11}$, for the inversion lemma. Then, by the induction hypothesis, either $t_1$ is a value or else it can make a step of evaluation, and likewise $t_2$. If $t_1$ can take a step, then rule $EApp1$ applies to $t$. If $t_1$ is a value and $t_2$ can take a step, then rule $EApp2$ applies. Finally, if both $t_1$ and $t_2$ are values, then the canonical forms lemma tells us that $t_1$ has the form $\lambda x : T_{11}.t_{12}$, and so rule *EAppAbs* applies to $t$.
4. The *unit* case is immediate, since *unit* is a value.
5. If $t = Ref\ t_1$, then $T = Ref\ T_1$, for some $T_1$, and $\varnothing \mid \Sigma \vdash t_1 : T_1$. By the induction hypothesis, either $t_1$ is a value or else it can make a step of evaluation. If $t_1$ can take a step, then for any store $\mu$ such that $\varnothing \mid \Sigma \vdash \mu$, there is some term $t_1'$ with $t_1 \mid \mu \longrightarrow t_1' \mid \mu'$. For this reason, for any store $\mu$ such that $\varnothing \mid \Sigma \vdash \mu$, the rule $ERef$ applies to $t$, which guarantees the existence of $t'$ and $\mu'$. If $t_1$ is a value ($t_1 = Ref\ v_1$) then the rule $ERefV$ applies to $t$, obtaining that $t' = l$, with $l \notin dom(\mu)$ and $\mu' = (\mu, l \mapsto v_1)$.

6. If $t = !t_1$, then $\varnothing \mid \Sigma \vdash t_1 : Ref\ T$. By the induction hypothesis, either $t_1$ is a value or else it can make a step of evaluation. If $t_1$ can take a step, then for any store $\mu$ such that $\varnothing \mid \Sigma \vdash \mu$, there is some term $t'_1$ with $t_1 \mid \mu \longrightarrow t'_1 \mid \mu'$. For this reason, for any store $\mu$ such that $\varnothing \mid \Sigma \vdash \mu$, the rule $EDeref$ applies to $t$, which guarantees the existence of $t'$ and $\mu'$. If $t_1$ is a value ($t_1 = l$, for the Canonical Forms Lemma) then the rule $EDerefLoc$ applies to $t$ (since $\varnothing \mid \Sigma \vdash l : Ref\ T$ and $\varnothing \mid \Sigma \vdash \mu$ then $\mu(l)$ is defined), obtaining that $t' = \mu(l)$, and $\mu' = \mu$.

7. If $t = t_1 := t_2$, then then $T = Unit$, $\varnothing \mid \Sigma \vdash t_1 : Ref\ T_1$, for some $T_1$ and $\varnothing \mid \Sigma \vdash t_2 : T_1$. By the induction hypothesis, either $t_1$ is a value or else it can make a step of evaluation, and likewise $t_2$. If $t_1$ can take a step, then for any store $\mu$ such that $\varnothing \mid \Sigma \vdash \mu$, there is some term $t'_1$ with $t_1 \mid \mu \longrightarrow t'_1 \mid \mu'$. For this reason, for any store $\mu$ such that $\varnothing \mid \Sigma \vdash \mu$, the rule $EAssing1$ applies to $t$, which guarantees the existence of $t'$ and $\mu'$. If $t_2$ can take a step, is the same, but with the rule $EAssing2$. Finally, if both $t_1$ and $t_2$ are values, then the canonical forms lemma tells us that $t_1$ has the form $l$. Then the rule $EAssing$ applies to $t$, which guarantees the existence of $t'$ and $\mu'$.

8. If $t = l$, then it is immediate, since locations are values.

LEMMA[Permutation]: If $\Gamma \mid \Sigma \vdash t : T$ and $\Delta$ is a permutation of $\Gamma$, then $\Delta \mid \Sigma \vdash t : T$.
*Proof*: Straightforward induction on typing derivations.

LEMMA[Weakening]: If $\Gamma \mid \Sigma \vdash t : T$ and $x \notin dom(\Gamma)$, then $\Gamma, x : S \mid \Sigma \vdash t : T$.
*Proof*: Straightforward induction on typing derivations.

LEMMA [Preservation of types under substitution]: If $\Gamma, x : S \mid \Sigma \vdash t : T$ and $\Gamma \mid \Sigma \vdash s : S$, then $\Gamma \mid \Sigma \vdash [x \mapsto s]t : T$.
*Proof*: By induction on a derivation of the statement $\Gamma, x : S \vdash t : T$. For a given derivation, we proceed by cases on the final typing rule used in the proof.

LEMMA [1]: If $\Gamma \mid \Sigma \vdash \mu$, $\Sigma(l) = T$ and $\Gamma \mid \Sigma \vdash v : T$ then $\Gamma \mid \Sigma \vdash [l \mapsto v]\mu$.
*Proof*: Straightforward induction.

LEMMA [2]: If $\Gamma \mid \Sigma \vdash t : T$ and $\Sigma \subseteq \Sigma'$ then $\Gamma \mid \Sigma' \vdash t : T$.
*Proof*: Straightforward induction.

LEMMA[Preservation]: If $\Gamma \mid \Sigma \vdash t : T$ and $t \longrightarrow t'$, then $\Gamma \mid \Sigma \vdash t' : T$.
*Proof*: By induction on a derivation of $t : T$. At each step of the induction, we assume that the desired property holds for all subderivations (i.e., that if $s : S$ and $s \longrightarrow s'$, then $s' : S$, whenever s : S is proved by a subderivation of the present one) and proceed by case analysis on the final rule in the derivation.

1. $Case\ TVar$: It cannot be the case that $t \longrightarrow t'$ for any $t'$, and the requirements of the theorem are vacuously satisfied.

2. $Case\ TAbs$: $t = \lambda x : T_2.t_1, with\ T = T_2 \rightarrow T_1$ and $\Gamma, x : T_2 \vdash t_1 : T_1$. If the last rule in the derivation is $TAbs$, then we know from the form of this rule that t must be a function $\lambda x : T_2.t_1$ and $T$ must be $T_2 \rightarrow T_1$, with $\Gamma, x : T_2 \vdash t_1 : T_1$. But then $t$ is a value, so it cannot be the case that $t \longrightarrow t'$ for any $t'$, and the requirements of the theorem are vacuously satisfied.

3. *Case TApp*: $t = t_1\ t_2$, $\vdash\ t_1 : T_2 \to T$ and $\vdash\ t_2 : T_2$. If the last rule in the derivation is *TApp*, then we know from the form of this rule that $t$ must have the form $t_1\ t_2$, for some $t_1$ and $t_2$. We must also have a subderivation with conclusions $\vdash t_1 : T_2 \to T$ and $\vdash t_2 : T_2$. Now, looking at the evaluation rules with this form on the left-hand side, we find that there are three rules by which $t \longrightarrow t'$ can be derived: *EApp*1, *EApp*2 and *EAppAbs*. We consider each case separately.

   - *Subcase EApp*1: $t_1 \longrightarrow t_1',\ t' = t_1'\ t_2$. By the induction hypothesis, $\vdash t_1' : T_2 \to T$, then we can apply rule *TApp*, to conclude that $\vdash t_1'\ t_2 : T$ , that is $\vdash t' : T$.
   - *Subcase EApp*2: $t_2 \longrightarrow t_2',\ t' = t_1\ t_2'$. By the induction hypothesis, $\vdash t_2' : T_2$, then we can apply rule *TApp*, to conclude that $\vdash t_1\ t_2' : T$, that is $\vdash t' : T$.
   - *Subcase EAppAbs*: $t_1 = \lambda x : T_1.t_{12}$, $t_2 = v_2$, $t' = [x \mapsto v_2]t_{12}$ and $\vdash t_{12} : T$ for the inversion lemma. The resulting term $\vdash [x \mapsto v_2]t_{12} : T$, for the Substitution lemma, that is $\vdash t' : T$.