LEMMA [Inversion of the Typing Relation]:

1. If $\Gamma \mid \Sigma \vdash x : R$, then $x : R \in \Gamma$
2. If $\Gamma \mid \Sigma \vdash \lambda x : T_1.t_2 : R$, then $R = T_1 \to R_2$ for some R_2, with $\Gamma,\, x : T_1 \vdash t_2 : R_2$.
3. If $\Gamma \mid \Sigma \vdash t_1\ t_2 : R$, then there is some type T_{11} such that $\Gamma \mid \Sigma \vdash t_1 : T_{11} \to R$ and $\Gamma \mid \Sigma \vdash t_2 : T_{11}$.
4. If $\Gamma \mid \Sigma \vdash ref\ t : R$, then $R = Ref\ T_1$ for some T_1 and $\Gamma \mid \Sigma \vdash t_1 : T_1$.
5. If $\Gamma \mid \Sigma \vdash !t : R$, then $\Gamma \mid \Sigma \vdash t : Ref\ R$.
6. If $\Gamma \mid \Sigma \vdash t_1 := t_2 : R$, then $R = Unit$, $\Gamma \mid \Sigma \vdash t_1 : Ref\ T_1$, for some T_1 and $\Gamma \mid \Sigma \vdash t_2 : T_1$.
7. If $\Gamma \mid \Sigma \vdash l : R$, then $R = Ref\ T_1$, for some T_1, and $\Sigma(l) = T_1$.

Proof : Immediate from the definition of the typing relation.

LEMMA [Canonical Forms]:

1. If v is a value of type $T_1 \to T_2$, then $v = \lambda x : T_1.t_2$.
2. If v is a value of type $Unit$, then $v = unit$.
3. If v is a value of type $Ref\ T$, then $v = l$.

Proof : Straightforward.

THEOREM [Progress]: Suppose t is a closed, well-typed term (that is, $\varnothing \mid \Sigma \vdash t : T$ for some T). Then either t is a value or else, for any store μ such that $\varnothing \mid \Sigma \vdash \mu$, there is some term t' with $t \mid \mu \longrightarrow t' \mid \mu'$.
Proof: Straightforward induction on typing derivations.

1. The variable case cannot occur (because t is closed).
2. The abstraction case is immediate, since abstractions are values.
3. For application, where $t = t_1\ t_2$, with $\varnothing \mid \Sigma \vdash t_1 : T_{11} \longrightarrow T_{12}$ and $\varnothing \mid \Sigma \vdash t_2 : T_{11}$, for the inversion lemma. Then, by the induction hypothesis, either t_1 is a value or else it can make a step of evaluation, and likewise t_2. If t_1 can take a step, then rule $EApp1$ applies to t. If t_1 is a value and t_2 can take a step, then rule $EApp2$ applies. Finally, if both t_1 and t_2 are values, then the canonical forms lemma tells us that t_1 has the form $\lambda x : T_{11}.t_{12}$, and so rule $EAppAbs$ applies to t.
4. The *unit* case is immediate, since *unit* is a value.
5. If $t = Ref\ t_1$, then $T = Ref\ T_1$, for some T_1, and $\varnothing \mid \Sigma \vdash t_1 : T_1$. By the induction hypothesis, either t_1 is a value or else it can make a step of evaluation. If t_1 can take a step, then for any store μ such that $\varnothing \mid \Sigma \vdash \mu$, there is some term t_1' with $t_1 \mid \mu \longrightarrow t_1' \mid \mu'$. For this reason, for any store μ such that $\varnothing \mid \Sigma \vdash \mu$, the rule $ERef$ applies to t, which guarantees the existence of t' and μ'. If t_1 is a value ($t_1 = Ref\ v_1$) then the rule $ERefV$ applies to t, obtaining that $t' = l$, with $l \notin dom(\mu)$ and $\mu' = (\mu, l \mapsto v_1)$.

6. If $t = !t_1$, then $\varnothing \mid \Sigma \vdash t_1 : Ref\ T$. By the induction hypothesis, either t_1 is a value or else it can make a step of evaluation. If t_1 can take a step, then for any store μ such that $\varnothing \mid \Sigma \vdash \mu$, there is some term t_1' with $t_1 \mid \mu \longrightarrow t_1' \mid \mu'$. For this reason, for any store μ such that $\varnothing \mid \Sigma \vdash \mu$, the rule $EDeref$ applies to t, which guarantees the existence of t' and μ'. If t_1 is a value ($t_1 = l$, for the Canonical Forms Lemma) then the rule $EDerefLoc$ applies to t (since $\varnothing \mid \Sigma \vdash l : Ref\ T$ and $\varnothing \mid \Sigma \vdash \mu$ then $\mu(l)$ is defined), obtaining that $t' = \mu(l)$, and $\mu' = \mu$.

7. If $t = t_1 := t_2$, then then $T = Unit$, $\varnothing \mid \Sigma \vdash t_1 : Ref\ T_1$, for some T_1 and $\varnothing \mid \Sigma \vdash t_2 : T_1$. By the induction hypothesis, either t_1 is a value or else it can make a step of evaluation, and likewise t_2. If t_1 can take a step, then for any store μ such that $\varnothing \mid \Sigma \vdash \mu$, there is some term t_1' with $t_1 \mid \mu \longrightarrow t_1' \mid \mu'$. For this reason, for any store μ such that $\varnothing \mid \Sigma \vdash \mu$, the rule $EAssing1$ applies to t, which guarantees the existence of t' and μ'. If t_2 can take a step, is the same, but with the rule $EAssing2$. Finally, if both t_1 and t_2 are values, then the canonical forms lemma tells us that t_1 has the form l. Then the rule $EAssing$ applies to t, which guarantees the existence of t' and μ'.

8. If $t = l$, then it is immediate, since locations are values.

LEMMA[Permutation]: If $\Gamma \mid \Sigma \vdash t : T$ and Δ is a permutation of Γ, then $\Delta \mid \Sigma \vdash t : T$.
Proof: Straightforward induction on typing derivations.

LEMMA[Weakening]: If $\Gamma \mid \Sigma \vdash t : T$ and $x \notin dom(\Gamma)$, then $\Gamma, x : S \mid \Sigma \vdash t : T$.
Proof: Straightforward induction on typing derivations.

LEMMA [Preservation of types under substitution]: If $\Gamma, x : S \mid \Sigma \vdash t : T$ and $\Gamma \mid \Sigma \vdash s : S$, then $\Gamma \mid \Sigma \vdash [x \mapsto s]t : T$.
Proof: By induction on a derivation of the statement $\Gamma, x : S \vdash t : T$. For a given derivation, we proceed by cases on the final typing rule used in the proof.

LEMMA [1]: If $\Gamma \mid \Sigma \vdash \mu$, $\Sigma(l) = T$ and $\Gamma \mid \Sigma \vdash v : T$ then $\Gamma \mid \Sigma \vdash [l \mapsto v]\mu$.
Proof: Straightforward induction.

LEMMA [2]: If $\Gamma \mid \Sigma \vdash t : T$ and $\Sigma \subseteq \Sigma'$ then $\Gamma \mid \Sigma' \vdash t : T$.
Proof: Straightforward induction.

LEMMA[Preservation]: If $\Gamma \mid \Sigma \vdash t : T$, $\Gamma \mid \Sigma \vdash \mu$ and $t \mid \mu \longrightarrow t' \mid \mu'$, then for some $\Sigma' \supseteq \Sigma$, $\Gamma \mid \Sigma \vdash t' : T$ and $\Gamma \mid \Sigma' \vdash \mu'$.
Proof:

1. *Case TVar*: $\cdots$

2. *Case TAbs*: $\cdots$

3. *Case TApp*: $\cdots$

4. *Case TLoc*: If the last rule in the derivation is $TLoc$, then we know from the form of this rule that t must be a location l, then t is a value, so it cannot be the case that $t \mid \mu \longrightarrow t' \mid \mu'$ for any t', and the requirements of the theorem are vacuously satisfied.

5. *Case* $TUnit$: If the last rule in the derivation is $TUnit$, then we know from the form of this rule that t must be $unit$, then t is a value, so it cannot be the case that $t \mid \mu \longrightarrow t' \mid \mu'$ for any t', and the requirements of the theorem are vacuously satisfied.

6. *Case* $TRef$: $t = ref\ t_1$, $\Gamma \mid \Sigma \vdash ref\ t_1 : Ref\ T_1$, for some T_1 and $\Gamma \mid \Sigma \vdash t_1 : T_1$. If the last rule in the derivation is $TRef$, then we know from the form of this rule that t must have the form $ref\ t_1$, for some t_1. Now, looking at the evaluation rules with this form on the left-hand side, we find that there are two rules by which $t \mid \mu \longrightarrow t' \mid \mu'$ can be derived: $ERefV$ and $ERef$. We consider each case separately.

   - *Subcase* $ERef$: $t_1 \mid \mu \longrightarrow t_1' \mid \mu'$, $t' = ref\ t_1'$. By the induction hypothesis, $\Gamma \mid \Sigma' \vdash t_1' : T_1$, for some Σ', where $\Gamma \mid \Sigma' \vdash \mu'$. Then we can apply rule $TRef$, to conclude that $\Gamma \mid \Sigma' \vdash ref\ t_1' : Ref\ T_1$, that is $\Gamma \mid \Sigma' \vdash t' : T$, where $\Gamma \mid \Sigma' \vdash \mu'$.
   - *Subcase* $ERefV$: $t = ref\ v_1$, $t' = l$ and $l \notin dom(\mu)$. For $\Sigma' = \Sigma \cup (l, T_1)$, for the rule $TLoc$, is true that $\Gamma \mid \Sigma' \vdash l : Ref\ T_1$, and $\Gamma \mid \Sigma' \vdash (\mu, l \mapsto v_1)$.

7. *Case* $TAssing$: $t = t_1 := t_2$, $T = Unit$, $\Gamma \mid \Sigma \vdash t_1 : Ref\ T_1$, for some T_1, and $\Gamma \mid \Sigma \vdash t_2 : T_1$. If the last rule in the derivation is $TAssing$, then we know from the form of this rule that t must have the form $t_1 :=\ t_2$, for some t_1 and t_2. Now, looking at the evaluation rules with this form on the left-hand side, we find that there are three rules by which $t \mid \mu \longrightarrow t' \mid \mu'$ can be derived: $EAssing1$, $EAssing2$ and $EAssing$. We consider each case separately.

   - *Subcase* $EAssing1$: $t_1 \mid \mu \longrightarrow t_1' \mid \mu'$, $t' = t_1' := t_2$. By the induction hypothesis, $\Gamma \mid \Sigma' \vdash t_1' : Ref\ T_1$, for some Σ', where $\Gamma \mid \Sigma' \vdash \mu'$. Moreover, $\Gamma \mid \Sigma' \vdash t_2 : T_1$, for Lemma[2]. Then we can apply rule $TAssing$, to conclude that $\Gamma \mid \Sigma' \vdash t_1' := t_2 : Unit$, that is $\Gamma \mid \Sigma' \vdash t' : T$, where $\Gamma \mid \Sigma' \vdash \mu'$.
   - *Subcase* $EAssing2$: Similar to the above.
   - *Subcase* $EAssing$: $t = l := v$ and $t' = unit$. Then for Σ, it is satisfied that $\Gamma \mid \Sigma \vdash t' : Unit$, that is $\Gamma \mid \Sigma \vdash t' : T$, and for $\mu' = [l \mapsto v]\mu$ is true that $\Gamma \mid \Sigma \vdash [l \mapsto v]\mu$ for Lemma[1].

8. *Case* $TDeref$: $t = !t_1$ and $\Gamma \mid \Sigma \vdash t_1 : Ref\ T$. If the last rule in the derivation is $TDeref$, then we know from the form of this rule that t must have the form $t = !t_1$, for some t_1. Now, looking at the evaluation rules with this form on the left-hand side, we find that there are two rules by which $t \mid \mu \longrightarrow t' \mid \mu'$ can be derived: $EDeref$ and $EDerefLoc$. We consider each case separately.

   - *Subcase* $EDeref$: $t_1 \mid \mu \longrightarrow t_1' \mid \mu'$, $t' = !t_1'$. By the induction hypothesis, $\Gamma \mid \Sigma' \vdash t_1' : Ref\ T$, for some Σ', where $\Gamma \mid \Sigma' \vdash \mu'$. Then we can apply rule $TDeref$, to conclude that $\Gamma \mid \Sigma' \vdash !t_1' : T$, that is $\Gamma \mid \Sigma' \vdash t' : T$, where $\Gamma \mid \Sigma' \vdash \mu'$.
   - *Subcase* $EDerefLoc$: $t_1 = l$, $\mu(l) = v$, for some v and $t' = v$. We know that $\Gamma \mid \Sigma \vdash l : Ref\ T$, then for the inversion lemma, $\Sigma(l) = T$. We know too, that $\Gamma \mid \Sigma \vdash \mu$, then $\Gamma \mid \Sigma \vdash \mu(l) : \Sigma(l)$, that is $\Gamma \mid \Sigma \vdash v : T$.