In Zhou's, his proof of the upper bound of the SG produces a sequence of upper limits of the Hausdorff measure of SG. He then shows that this sequence is decreasing by analyzing the analogous function be replacing the discrete variable "n" with the continuous variable "x". Here is the calculation of that derivative

In[∘]:= $\mathbf{F1[x\_] := \frac{25}{22+3\,(2/27)\,\hat{}\,x}\left(\frac{6}{7}+\frac{1}{7}\left(\frac{1}{8}\right)^{x}\right)^{Log[2,3]};}$

In[∘]:= `D[F1[x], x]`

Out[∘]= $$-\left(\left(25\left(\frac{6}{7}+\frac{8^{-x}}{7}\right)^{\frac{Log[3]}{Log[2]}}\left(2^{x}\times 3^{1-3x}\,Log[2]-2^{x}\times 3^{2-3x}\,Log[3]\right)\right)\Big/\left(22+2^{x}\times 3^{1-3x}\right)^{2}\right)-\frac{25\times 8^{-x}\left(\frac{6}{7}+\frac{8^{-x}}{7}\right)^{-1+\frac{Log[3]}{Log[2]}}Log[3]\,Log[8]}{7\left(22+2^{x}\times 3^{1-3x}\right)Log[2]}$$

In[∘]:= `Simplify[D[F1[x], x] < 0]`

Out[∘]= $$27^{x}\left(6+8^{-x}\right)^{\frac{Log[3]}{Log[2]}}\left(3\times 2^{x}+22\times 27^{x}\right)^{2}\left(Log[2]+2^{1+3x}\,Log[8]\right)\left(3\times 2^{x}\left(1+3\times 2^{1+3x}\right)Log[2]^{2}-9\times 2^{x}\left(1+3\times 2^{1+3x}\right)Log[2]\,Log[3]+Log[8]\left(22\times 27^{x}\,Log[3]+2^{x}\,Log[27]\right)\right)>0$$

Simplifying the significant sign ambiguous part, we have

In[∘]:= $$\mathbf{FullSimplify\left[\left(3\times 2^{x}\left(1+3\times 2^{1+3x}\right)Log[2]^{2}-9\times 2^{x}\left(1+3\times 2^{1+3x}\right)Log[2]\,Log[3]+Log[8]\left(22\times 27^{x}\,Log[3]+2^{x}\,Log[27]\right)\right)>0\right]}$$

Out[∘]= $$2^{x}\,Log[2]+22\times 27^{x}\,Log[3]>3\times 2^{1+4x}\,Log\left[\frac{27}{2}\right]$$

which is clear.

Continuing, we show that the increment choice of m=3 is the best one for the classic Sierpinski Gasket. In a similar process we generalize the function above and proceed to show it is decreasing.

In[∘]:= $$\mathbf{f2[x\_] := \frac{\left(\frac{2\hat{}m-2}{2\hat{}m-1}+\frac{1}{2\hat{}m-1}\left(\frac{1}{2\hat{}m}\right)^{x}\right)^{Log[2,3]}}{\frac{3\hat{}m-5}{3\hat{}m-2}+\frac{3}{3\hat{}m-2}\left(\frac{2}{3\hat{}m}\right)^{x}};}$$

Check for simplification of the function itself.

In[∘]:=
```
FullSimplify[FunctionExpand[f2[x]]]
```

Out[∘]=
$$\frac{\left(-2+3^{m}\right)\left(\frac{-2+2^{m}+\left(2^{-m}\right)^{x}}{-1+2^{m}}\right)^{\frac{\mathrm{Log}[3]}{\mathrm{Log}[2]}}}{-5+3^{m}+3\times 2^{x}\left(3^{-m}\right)^{x}}$$

Doesn’t give anything too useful, so we take the derivative and proceed.

In[∘]:=
```
D[f2[x], x]
```

Out[∘]=
$$\frac{\left(2^{-m}\right)^{x}\left(\frac{\left(2^{-m}\right)^{x}}{-1+2^{m}}+\frac{-2+2^{m}}{-1+2^{m}}\right)^{-1+\frac{\mathrm{Log}[3]}{\mathrm{Log}[2]}}\mathrm{Log}[3]\,\mathrm{Log}\left[2^{-m}\right]}{\left(-1+2^{m}\right)\left(\frac{3\times 2^{x}\left(3^{-m}\right)^{x}}{-2+3^{m}}+\frac{-5+3^{m}}{-2+3^{m}}\right)\mathrm{Log}[2]}-$$
$$\left(\left(\frac{\left(2^{-m}\right)^{x}}{-1+2^{m}}+\frac{-2+2^{m}}{-1+2^{m}}\right)^{\frac{\mathrm{Log}[3]}{\mathrm{Log}[2]}}\left(\frac{3\times 2^{x}\left(3^{-m}\right)^{x}\mathrm{Log}[2]}{-2+3^{m}}+\frac{3\times 2^{x}\left(3^{-m}\right)^{x}\mathrm{Log}\left[3^{-m}\right]}{-2+3^{m}}\right)\right)\Big/$$
$$\left(\frac{3\times 2^{x}\left(3^{-m}\right)^{x}}{-2+3^{m}}+\frac{-5+3^{m}}{-2+3^{m}}\right)^{2}$$

Assuming m>=1, Simplify the derivative set less than 0

In[∘]:=
```
Assuming[{m ≥ 1}, Simplify[D[f2[x], x] < 0]]
```

Out[∘]=
$$\left(\frac{3}{2}\right)^{m\,x}\left(\frac{-2+2^{m}+2^{-m\,x}}{-1+2^{m}}\right)^{\frac{\mathrm{Log}\left[\frac{3}{2}\right]}{\mathrm{Log}[2]}}\left(3\times 2^{x}-5\times 3^{m\,x}+3^{m+m\,x}\right)^{2}\left(3\times 2^{x}\left(1-2^{1+m\,x}+2^{m+m\,x}\right)\mathrm{Log}[2]^{2}+\right.$$
$$\left.m\,\mathrm{Log}[3]\left(3^{m+m\,x}\mathrm{Log}[2]-2^{(1+m)\,x}\left(-2+2^{m}\right)\mathrm{Log}[8]-3^{m\,x}\mathrm{Log}[32]\right)\right)>0$$

From the result

In[∘]:=
```
Assuming[{m ≥ 1, x ≥ 1}, FullSimplify[(3 × 2^x (1 - 2^(1+m*x) + 2^(m+m*x)) Log[2]^2 +
    m Log[3] (3^(m+m*x) Log[2] - 2^((1+m)*x) (-2 + 2^m) Log[8] - 3^(m*x) Log[32])) > 0]]
```

Out[∘]=
$$3^{m\,x}\left(-5+3^{m}\right)m\,\mathrm{Log}[3]+2^{x}\left(\mathrm{Log}[8]+2^{m\,x}\left(-2+2^{m}\right)\left(\mathrm{Log}[8]-m\,\mathrm{Log}[27]\right)\right)>0$$

From here, the simplification repeats, so we must proceed by hand.

In[∘]:=
```
Assuming[{m ≥ 1, x ≥ 1}, FullSimplify[FunctionExpand[
   3^(m*x) (-5 + 3^m) m * Log[3] + 3 * 2^x (Log[2] + 2^(m*x) (-2 + 2^m) (Log[2] - m * Log[3])) > 0]]]
```

Out[∘]=
$$3^{m\,x}\left(-5+3^{m}\right)m\,\mathrm{Log}[3]+2^{x}\left(\mathrm{Log}[8]+2^{m\,x}\left(-2+2^{m}\right)\left(\mathrm{Log}[8]-m\,\mathrm{Log}[27]\right)\right)>0$$

Note that it is simple to show that 3^m-5 > 3(2^m-2) for m>=3, so we make the following substitution

$$m * \text{Log}[3] \left[3^{m x} \left(3^m - 5\right) - 3 * 2^x 2^{m x} \left(2^m - 2\right)\right] > 0$$

$$3^{m x} \left(3^m - 5\right) = \left(3/2\right)^{m x} 2^{m x} \left(3^m - 5\right) \geq \left(27/8\right)^x 2^{m x} \left(3^m - 5\right) \geq \left(27/8\right)^x 2^{m x} \left(3^m - 5\right) > 3 * \left(27/8\right)^x 2^{m x} \left(2^m - 2\right)$$

Having proven that the expression above is indeed positive for m>=3, we can now take the limit and minimize in m to verify the best choice.

In[∘]:= $\text{lf2}[m\_] := \dfrac{\left(\frac{2\text{^}m-2}{2\text{^}m-1}\right)^{\text{Log}[2,3]}}{\frac{3\text{^}m-5}{3\text{^}m-2}};$

In[∘]:= `NMinimize[{lf2[m], m ≥ 3}, m]`

Out[∘]= `{0.890039, {m → 3.}}`

This indicates that the true min must occur at either m=2 (since we never checked m=2)or m=3 By examination, it is at m=3. Note if we minimized over m>=2, we’d get a non-integer minimum for m, but by the geometry of our method, we must restrict to integers.

In[∘]:= `lf2[2] // N`

Out[∘]= `0.920322`

In[∘]:= `lf2[3] // N`

Out[∘]= `0.890039`

Repeat for level-3 Sierpinski Gasket

In[∘]:= $f3[x\_] := \dfrac{\left(\frac{3\text{^}m-2}{3\text{^}m-1} + \frac{1}{3\text{^}m-1}\left(\frac{1}{3\text{^}m}\right)^x\right)^{\text{Log}[3,6]}}{\frac{6\text{^}m-5}{6\text{^}m-2} + \frac{3}{6\text{^}m-2}\left(\frac{2}{6\text{^}m}\right)^x};$

Assuming m>=1, Simplify the derivative set less than 0

In[∘]:= `Assuming[{m ≥ 1}, Simplify[D[f3[x], x] < 0]]`

Out[∘]= $$6^{m x} \left(\frac{-2 + 3^m + 3^{-m x}}{-1 + 3^m}\right)^{\frac{\text{Log}[6]}{\text{Log}[3]}} \left(1 - 2 \times 3^{m x} + 3^{m+m x}\right) \left(3 \times 2^x - 5 \times 6^{m x} + 6^{m+m x}\right)^2 \left(\left(3 \times 2^x - 5 \times 6^{m x} + 6^{m+m x}\right) m \,\text{Log}[6] + 3 \times 2^x \left(1 - 2 \times 3^{m x} + 3^{m+m x}\right) \text{Log}\left[2^{1-m} \times 3^{-m}\right]\right) > 0$$

Continue simplifying the sign ambiguous portion

In[∘]:=
$$\text{Assuming}\left[\{m \geq 1\}, \text{FullSimplify}\left[\text{FunctionExpand}\left[\left(\left(3\times 2^x - 5\times 6^{m x} + 6^{m+m x}\right) m \,\text{Log}[6] + 3\times 2^x \left(1 - 2\times 3^{m x} + 3^{m+m x}\right) \text{Log}\left[2^{1-m}\times 3^{-m}\right]\right) > 0\right]\right]\right]$$

Out[∘]=
$$\left(3\times 2^x + 6^{m x}\left(-5 + 6^m\right)\right) m \,\text{Log}[6] + 3\times 2^x \left(1 + 3^{m x}\left(-2 + 3^m\right)\right)\left(\text{Log}[2] - m\,\text{Log}[6]\right) > 0$$

In[∘]:=
$$\text{Assuming}\left[\{m \geq 1, x \geq 1\}, \text{FullSimplify}\left[\text{FunctionExpand}\left[\left(3\times 2^x + 6^{m x}\left(-5 + 6^m\right)\right) m \,\text{Log}[6] + 3\times 2^x \left(1 + 3^{m x}\left(-2 + 3^m\right)\right)\left(\text{Log}[2] - m\,\text{Log}[6]\right) > 0\right]\right]\right]$$

Out[∘]=
$$\left(3\times 2^x + 6^{m x}\left(-5 + 6^m\right)\right) m \,\text{Log}[6] + 3\times 2^x \left(1 + 3^{m x}\left(-2 + 3^m\right)\right)\left(\text{Log}[2] - m\,\text{Log}[6]\right) > 0$$

Output has repeated, so we proceed by hand. Note that it is easy to show that 6^m-5 >3(3^m-2) for m>=2, so we have that

$$m * \text{Log}[6]\left[6^{m x}\left(6^m - 5\right) - 3 * 2^x * 3^{m x}\left(3^m - 2\right)\right] > 0$$

$$6^{m x}\left(6^m - 5\right) \geq 3 * 2^{m x}\, 3^{m x}\left(3^m - 2\right) \geq 3 * 4^x\, 3^{m x}\left(3^m - 2\right) \geq 3 * 2^x * 3^{m x}\left(3^m - 2\right)$$

Thus, we can minimize the limit (in x) for m>=2.

In[∘]:=
$$\text{lf3}[\text{m\_}] := \frac{\left(\frac{3\text{^}m-2}{3\text{^}m-1}\right)^{\text{Log}[3,6]}}{\frac{6\text{^}m-5}{6\text{^}m-2}};$$

In[∘]:= `NMinimize[{lf3[m], m ≥ 2}, m]`

Out[∘]= `{0.882138, {m → 2.}}`

Checking m=1 just in case (since we never checked it)

In[∘]:= `lf3[1] // N`

Out[∘]= `1.29152`

Generalize fully and from the cases of A=2, and A=3 above, we notice that Mathematica’s full extent of simplification results in a similar form of equation.

In[∘]:=
$$\text{fA}[\text{x\_}, \text{A\_}] := \frac{\left(\frac{A\text{^}m-2}{A\text{^}m-1} + \frac{1}{A\text{^}m-1}\left(\frac{1}{A\text{^}m}\right)^x\right)^{\text{Log}\left[A, \frac{A (A+1)}{2}\right]}}{\frac{(A*(A+1)/2)\text{^}m-5}{(A*(A+1)/2)\text{^}m-2} + \frac{3}{(A*(A+1)/2)\text{^}m-2}\left(\frac{2}{(A*(A+1)/2)\text{^}m}\right)^x};$$

In[∘]:= **Assuming[{A ≥ 2, m ≥ 1, x ≥ 1}, Simplify[D[fA[x, A], x] < 0]]**

Out[∘]=
$$(-1+A^m)\left(\frac{-2+A^m+A^{-m x}}{-1+A^m}\right)^{\frac{\mathrm{Log}\left[\frac{1+A}{2}\right]}{\mathrm{Log}[A]}}\left(2^{1+m}-(A(1+A))^m\right)\left(3\times 2^{m+x+m x}-5\times 2^m (A(1+A))^{m x}+(A(1+A))^{m(1+x)}\right)^2\left(\left(A^{m+3 m x}(1+A)^{m+2 m x}+3\times 2^{m+x+m x}(A^2(1+A))^{m x}-5\times 2^m\left(A^3(1+A)^2\right)^{m x}\right) m\,\mathrm{Log}\left[\frac{1}{2}A(1+A)\right]+3\left(2^{m+x+m x}(A^2(1+A))^{m x}+2^{m+x+m x}A^{m+m x}(A^2(1+A))^{m x}-2^{(1+m)(1+x)}(A^3(1+A))^{m x}\right)\mathrm{Log}\left[2^{1+m}(A(1+A))^{-m}\right]\right)<0$$

In[∘]:= **Assuming[{A ≥ 4, m ≥ 1, x ≥ 1}, FullSimplify[FunctionExpand[**
$$\left(\left(A^{m+3 m x}(1+A)^{m+2 m x}+3\times 2^{m+x+m x}(A^2(1+A))^{m x}-5\times 2^m\left(A^3(1+A)^2\right)^{m x}\right) m\,\mathrm{Log}\left[\frac{1}{2}A(1+A)\right]+3\left(2^{m+x+m x}(A^2(1+A))^{m x}+2^{m+x+m x}A^{m+m x}(A^2(1+A))^{m x}-2^{(1+m)(1+x)}(A^3(1+A))^{m x}\right)\mathrm{Log}\left[2^{1+m}(A(1+A))^{-m}\right]\right)>0]]$$

Out[∘]=
$$\left(A^{m+3 m x}(1+A)^{m+2 m x}-5\times 2^m\left(A^3(1+A)^2\right)^{m x}\right) m\,\mathrm{Log}\left[\frac{1}{2}A(1+A)\right]+2^{m+x+m x}\left(-2(A^3(1+A))^{m x}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)+(A^2(1+A))^{m x}\left(\mathrm{Log}[8]+A^{m(1+x)}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)\right)\right)>0$$

In[∘]:= **Assuming[{A ≥ 2, m ≥ 1, x ≥ 1},**
FullSimplify[
$$\left(\left(A^{m+3 m x}(1+A)^{m+2 m x}-5\times 2^m\left(A^3(1+A)^2\right)^{m x}\right) m\,\mathrm{Log}\left[\frac{1}{2}A(1+A)\right]+2^{m+x+m x}\left(-2(A^3(1+A))^{m x}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)+(A^2(1+A))^{m x}\left(\mathrm{Log}[8]+A^{m(1+x)}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)\right)\right)\right)]]$$

Out[∘]=
$$\left(A^{m+3 m x}(1+A)^{m+2 m x}-5\times 2^m\left(A^3(1+A)^2\right)^{m x}\right) m\,\mathrm{Log}\left[\frac{1}{2}A(1+A)\right]+2^{m+x+m x}\left(-2(A^3(1+A))^{m x}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)+(A^2(1+A))^{m x}\left(\mathrm{Log}[8]+A^{m(1+x)}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)\right)\right)$$

In[∘]:= **Assuming[{A ≥ 4, m ≥ 1, x ≥ 1},**
FullSimplify[
$$\left(A^{m+3 m x}(1+A)^{m+2 m x}-5\times 2^m\left(A^3(1+A)^2\right)^{m x}\right) m\,\mathrm{Log}\left[\frac{1}{2}A(1+A)\right]+2^{m+x+m x}\left(-2(A^3(1+A))^{m x}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)+(A^2(1+A))^{m x}\left(\mathrm{Log}[8]+A^{m(1+x)}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)\right)\right)]]$$

Out[∘]=
$$\left(A^{m+3 m x}(1+A)^{m+2 m x}-5\times 2^m\left(A^3(1+A)^2\right)^{m x}\right) m\,\mathrm{Log}\left[\frac{1}{2}A(1+A)\right]+2^{m+x+m x}\left(-2(A^3(1+A))^{m x}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)+(A^2(1+A))^{m x}\left(\mathrm{Log}[8]+A^{m(1+x)}\left((1+m)\mathrm{Log}[8]-3m\,\mathrm{Log}[A(1+A)]\right)\right)\right)$$

Simplification repeats so we proceed by hand.

Note that (1+m)log(8) - 3mlog(A(1+A)) = log8 - 3m log(A(A+1)/2), so

$$2^{m+x+m\,x}\left(-2\left(A^3(1+A)\right)^{m\,x}\left((1+m)\,\text{Log}[8]-3\,m\,\text{Log}[A(1+A)]\right)+\left(A^2(1+A)\right)^{m\,x}\left(\text{Log}[8]+A^{m(1+x)}\left((1+m)\,\text{Log}[8]-3\,m\,\text{Log}[A(1+A)]\right)\right)\right)==$$
$$2^{m+x+m\,x}\left(\left(A^m A^{3\,m\,x}(1+A)^{m\,x}-2\,A^{3\,m\,x}(1+A)^{m\,x}\right)\left(3\,\text{Log}[2]-3\,m\,\text{Log}[A(1+A)/2]\right)+3*A^{2\,m\,x}(1+A)^{m\,x}\,\text{Log}[2]\right)$$

Collect all portions with a factor of mLog[A(1+A)/2].

$$m\,\text{Log}[A(A+1)/2]\left(A^{m+3\,m\,x}(1+A)^{m+2\,m\,x}-5\times2^m\left(A^3(1+A)^2\right)^{m\,x}+3\times2^{m+x+m\,x+1}A^{3\,m\,x}(A+1)^{m\,x}-3\times2^{m+x+m\,x}A^{m(1+x)}A^{2\,m\,x}(1+A)^{m\,x}\right)$$

Simplify it

In[∘]:=
```
Assuming[{A ≥ 4, m ≥ 1, x ≥ 1},
 FullSimplify[m Log[A (A + 1) / 2] (A^(m+3 m x) (1 + A)^(m+2 m x) - 5 × 2^m (A^3 (1 + A)^2)^(m x) +
   3 × 2^(m +x+m x+1) A^(3 m x) (A + 1)^(m x) - 3 × 2^(m + x + m x) A^(m (1+x)) A^(2 m x) (1 + A)^(m x))]]
```

Out[∘]=
$$\left(-5\times2^m\left(A^3(1+A)^2\right)^{m\,x}+A^{3\,m\,x}(1+A)^{m\,x}\left(A^m(1+A)^{m+m\,x}-3\times2^{m+x+m\,x}(-2+A^m)\right)\right)m\,\text{Log}\left[\frac{1}{2}A(1+A)\right]$$

Our goal is to show this term is positive for A>=4,m>=1,x>=1

In[∘]:=
```
Assuming[{A ≥ 4, m ≥ 1, x ≥ 1},
 FullSimplify[-5 × 2^m (A^3 (1 + A)^2)^(m x) + A^(3 m x) (1 + A)^(m x) (A^m (1 + A)^(m+m x) - 3 × 2^(m+x+m x) (-2 + A^m)) > 0]]
```

Out[∘]=
$$A^{3\,m\,x}(1+A)^{m\,x}\left(A^m(1+A)^{m+m\,x}-3\times2^{m+x+m\,x}(-2+A^m)\right)>5\times2^m\left(A^3(1+A)^2\right)^{m\,x}$$

Simplification repeats, proceed by hand. Note each term has a factor of A^(3mx)(1+A)^(mx) so we may cancel it out. Thus,

In[∘]:=
```
Assuming[{A ≥ 4, m ≥ 1, x ≥ 1},
 FullSimplify[(A^m (1 + A)^(m+m x) - 3 × 2^(m+x+m x) (-2 + A^m)) > 5 × 2^m (1 + A)^(m x)]]
```

Out[∘]=
$$(1+A)^{m\,x}\left(-5\times2^m+(A(1+A))^m\right)>3\times2^{m+x+m\,x}(-2+A^m)$$

$$(1+A)^{m\,x}2^m\left(-5+\left(\frac{A(1+A)}{2}\right)^m\right)>2^{(1+m)\,x}*3\times2^m(-2+A^m)$$

It is clear that $(1+A)^{m x} > 2^{(1+m) x}$, and both sides has a term of 2^m. Thus, we simply need to show that

$$-5+\left(\frac{A(1+A)}{2}\right)^m > 3\left(-2+A^m\right)$$

However, we notice that the above holds only for A>=5 (m>=1) does not hold for A=4 (m>=1). For A=4 the above holds for m>=2, so we may simply minimize over m>=2 and then check m=1 manually. So we are done!

Taking the limit now we have that the Hausdorff Measure of any Level-N Gasket is bounded above by

In[ ]:= $\text{Fmin}[A\_, m\_] := \dfrac{\left(2-\left(A\,(1+A)/2\right)^m\right)\left(1+\frac{1}{1-A^m}\right)^{\frac{\text{Log}\left[\frac{1}{2} A\,(1+A)\right]}{\text{Log}[A]}}}{5-\left(A\,(1+A)/2\right)^m};$