Bounds on the Hausdorff Measure of Level-N Sierpinski Gaskets

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Abstract

Although a favorite of fractal geometers, the Hausdorff measure of many classical fractals is often difficult to calculate or even bound. We begin by illustrating and extrapolating upon classic proofs for bounds on the Hausdorff measure of the Cantor set. What follows is an extension of a known technique which gives an upper bound to the Hausdorff measure of the Sierpinski gasket, to the class of level-N Sierpinski gaskets.

1 Introduction

Very broadly, a set is fractal if it shows the same pattern at multiple scales. Many prototypical examples of fractal sets can be generated by what are known as iterated function systems—a finite collection of contraction maps on the same space. The classical Cantor set and Sierpinski gasket are examples of fractals that arise from iterated function systems. These fractals will be the center in our discussion.

Fractals have many fascinating properties; among them are dimension and measure. While fractal dimensions have been carefully explored, much is not known about the measure, or size, of fractal sets; see [Fal14], [Hut81]. A common approach to measuring the size of a fractal set is the use of Hausdorff measure, a generalization of the Lebesgue measure. The Hausdorff measure approximates the size of a set with sets of sufficiently small diameters. It also includes a parameter, $s \geq 0$, which grants flexibility in how a set grows as it is scaled. For sets like lines, planes, or cubes, the parameter s is an integer and reflects the topological dimension of the set being measured. For the fractal sets we consider, the parameter s will give a non-integer dimension.

We now define the Hausdorff measure and Hausdorff dimension of a set in $\mathbb{R}^n$.

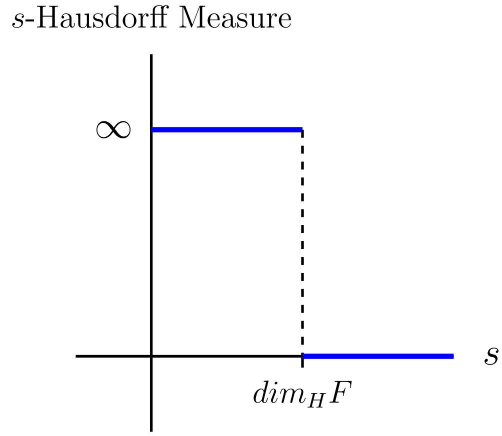


Figure 1: Graph of $\mathcal{H}^s(F)$ against s for a fixed set $F \subseteq \mathbb{R}^n$.

Definition 1.1 (Hausdorff Measure)**.** Suppose that $F \subseteq \mathbb{R}^n$, $s \geq 0$, and $\delta > 0$. Let

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : F \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } |U_i| < \delta\right\} \tag{1}$$

where $|U|$ denotes the diameter of the set U. As δ decreases, the infimum is being taken over a reduced class of permissible covers of F, so $\mathcal{H}^s_\delta$ increases. The limit,

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F),$$

is called the **s-dimensional Hausdorff measure of F**. The **Hausdorff dimension** of F is defined to be

$$\dim_H F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}.$$

We will expand upon the definition of Hausdorff measure. If $r > s$ we have for a δ cover $\{U_j\}_{j \in J}$ of F that

$$\sum_{j \in J} \operatorname{diam}(U_j)^r = \sum_{j \in J} \operatorname{diam}(U_j)^{r-s} \operatorname{diam}(U_j)^s \leq \delta^{r-s} \sum_{j \in J} \operatorname{diam}(U_j)^s$$

and hence $\mathcal{H}^r_\delta(F) \leq \delta^{r-s}\mathcal{H}^s_\delta(F)$.

Letting $\delta \to 0$ we see that if for some s, $\mathcal{H}^s(F) < \infty$ then for all $r > s$, $\mathcal{H}^r(F) = 0$. This means that for most values of s the Hausdorff measure of a fixed set $F \subseteq \mathbb{R}^n$ is either 0 or infinity. The value of s at which the Hausdorff measure of a set switches from being infinite to being 0, gives us a notion of dimension that is better suited for studying fractal sets than the typical topological dimension; see Figure 1.

For fractal sets, even for classical ones, the calculation of Hausdorff measure can be quite involved. The original intent of this work was to investigate and improve upon the calculation of the Hausdorff measure of the Sierpinski gasket (Figure 2). In the process, we investigated a more general class of fractals and we present the results from that work here.

To briefly summarize the advances on the bounds of the Hausdorff measure of the Sierpinski gasket, $\mathcal{S}$, we list the progression below. The reader should note that for $\mathcal{S}$, the Hausdorff dimension is $s = \log_2(3)$.

Upper bound of $\mathcal{H}^s(\mathcal{S})$:

- (1987) $\mathcal{H}^s(\mathcal{S}) \leq 0.9508$, [Mar87]
- (1997) $\mathcal{H}^s(\mathcal{S}) \leq 0.9105$, [Zho97a]
- (1997) $\mathcal{H}^s(\mathcal{S}) \leq 0.8900$, [Zho97b]
- (1999) $\mathcal{H}^s(\mathcal{S}) \leq 0.8180$, [WW99] (In Chinese)
- (2000) $\mathcal{H}^s(\mathcal{S}) \leq 0.8308$, [Fen97]

Lower bound of $\mathcal{H}^s(\mathcal{S})$:

- (2002) $\mathcal{H}^s(\mathcal{S}) \geq 0.5000$, [JZZ02]
- (2004) $\mathcal{H}^s(\mathcal{S}) \geq 0.5631$, [HW04]
- (2006) $\mathcal{H}^s(\mathcal{S}) \geq 0.6704$, [JZZ06]
- (2009) $\mathcal{H}^s(\mathcal{S}) \geq 0.7700$, [Mór09]

Our work, is based on the results of Zhou in 1997, which is not the best method available for calculating an upper bound for the Hausdorff measure of the Sierpinski gasket; however, Zhou's method is readily available for extension to a larger family of level-N Sierpinski gaskets.

The following is a brief summary of this paper's sections:

- Section 2 introduces the classic Cantor set and points out several key observations which will be necessary later on.
- Section 3 illustrates a proof from [Fal14] which bounds the Hausdorff measure of the Cantor set between 1/2 and 1.
- Section 4 presents a novel modification to the proof of Section 3 which pushes the lower bound up slightly.

- Section 5 introduces a class of fractals, the level-N Sierpinski gasket with some guiding figures.
- Section 6 presents the main result of this paper, a novel extension of the method by [Zho97b] in order to find upper bounds on the Hausdorff measure of any level-N Sierpinski gasket.

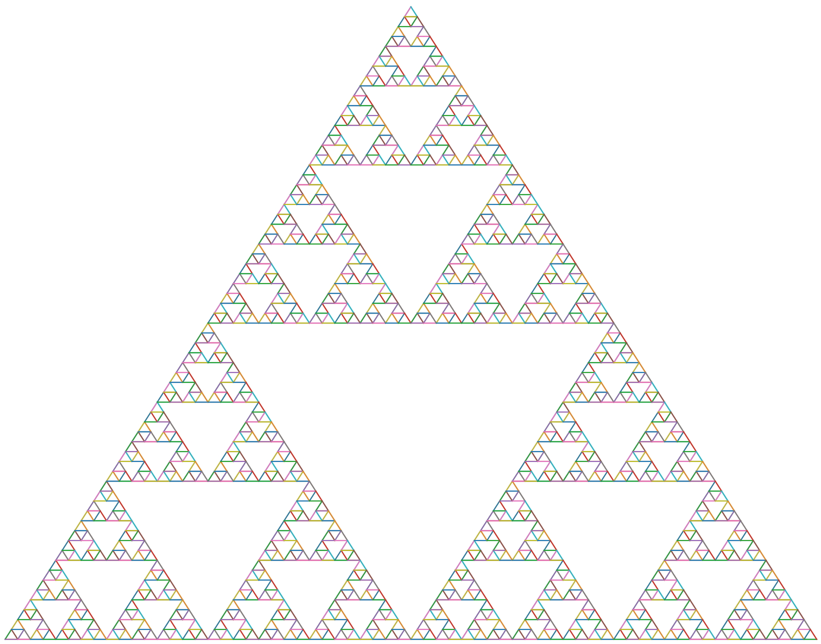


Figure 2: Sierpinski gasket

2 The Cantor Set

The classic Cantor set, denoted $\mathscr{C}$, may be informally constructed in *pre-fractal* stages using the following process:

(Stage 0): Begin with the interval [0,1].

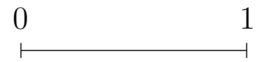
(Stage 1): From Stage 0, remove the open middle third interval $(\frac{1}{3}, \frac{2}{3})$.

(Stage 2): From each interval in Stage 1, remove the open middle third intervals.

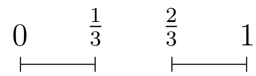
(Stage n): From each of the 2^{n-1} interval in Stage $n-1$, remove the open middle third intervals.

Hence, we have the following intervals at each stage:

(Stage 0): $C_0 = [0, 1]$



(Stage 1): $C_1 = C_0 \backslash \left(\frac{1}{3}, \frac{2}{3}\right)$



(Stage 2): $C_2 = C_1 \backslash \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{4}{9}, \frac{5}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \right)$

(Stage n): $C_n = C_{n-1} \backslash \bigcup_{k=0}^{n} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$

The Cantor set is then defined as the intersection of the sets C_n for all n.

$$\mathscr{C} = \bigcap_{n=0}^{\infty} C_n.$$

One property shared by many fractals is that of self-similarity. One may observe that the Cantor set is made up of pieces that are geometrically similar to the entire set. Using iterated function systems, we may define some fractals in a unified way and also find a very simple method of calculating the Hausdorff dimension of such fractals.

Recall that a *contraction* is a mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$|S(x) - S(y)| \leq r|x - y|$$

with $0 < r < 1$ for all $x, y \in \mathbb{R}^n$. The number r is the *ratio* of the contraction S. If equality holds above, then we say S is a *similarity* which transforms subsets of $\mathbb{R}^n$ into geometrically similar sets. A finite family of contractions $S = \{S_1, S_2, \ldots, S_m\}$, where $S_i : \mathbb{R}^n \to \mathbb{R}^n$, is called an *iterated function system* or IFS. We call a non-empty compact subset F of $\mathbb{R}^n$ an *attractor* or *invariant set* for the IFS if

$$F = \bigcup_{i=1}^{m} S_i(F),$$

that is, if it is made up of its images under the contractions of S. By applying the Banach fixed point theorem [Fal14, p. 135], the fundamental property of an IFS is that it determines a unique attractor. There are a number of classic fractals which can be achieved as the attractor of an IFS.

Furthermore, the attractor of an IFS made up of a collection of similarities is called a *self-similar set*. With the addition of a separation property known as the *open set condition* which requires the existence of a non-empty bounded *open* set V such that

$$V \supset \bigcup_{i=1}^{m} S_i(V)$$

where this union is disjoint, by [Fal14, p. 139], we may calculate the Hausdorff dimension by finding the number s such that

$$\sum_{i=1}^{m} r_i^s = 1.$$

The terms r_i above are the ratios of each similarity S_i of S.

The Cantor set may be defined as the attractor of the following IFS,

$$\begin{cases} S_1(x) = \frac{1}{3}x \\ S_2(x) = \frac{1}{3}x + \frac{2}{3}. \end{cases}$$

Thus, it is easy to calculate that the Hausdorff dimension of $\mathscr{C}$ is $s = \log_3(2)$.

Before continuing to the calculations on the Hausdorff measure of the Cantor set, some essential observations and conventions are made:

1. We call the intervals that make up the sets C_k *level-k intervals* which appear in the construction of $\mathscr{C}$. For example, $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ are level-1 intervals, $[\frac{2}{9}, \frac{1}{3}]$ is a level-2 interval, and so on.
2. C_k consists of 2^k disjoint level-k intervals each of length 3^{-k}.
3. Any pair of closed intervals taken from any two different levels in the construction of $\mathscr{C}$ are either disjoint or one is contained in the other.

3 Bounds on $\mathcal{H}^s(\mathscr{C})$

We will present a result by Falconer [Fal14, p. 52-53] which bounds the Hausdorff measure of the Cantor set.

Proposition 3.1. $\frac{1}{2} \leq \mathcal{H}^s(\mathscr{C}) \leq 1$.

Proof. Since $C_n \subseteq C_m$ for $n \geq m$, by definition, $\mathscr{C} \subset C_k$. Recall that the pre-fractal C_k has 2^k intervals each of length 3^{-k}. Taking the set of these intervals as a 3^{-k}-cover of $\mathscr{C}$, we observe that by (1), $\mathcal{H}^s_{3^{-k}}$ is less than the sum of the lengths of these intervals raised to the Hausdorff dimension of $\mathscr{C}$. In other words,

$$\mathcal{H}^s_{3^{-k}}(\mathscr{C}) \leq 2^k \left(3^{-k}\right)^s = 2^k 2^{-k} = 1$$

since $s = \log_3(2)$. Letting $k \to \infty$ gives $\mathcal{H}^s(\mathscr{C}) \leq 1$.

To prove that $\frac{1}{2} \leq \mathcal{H}^s(\mathscr{C})$, we shall show that $\frac{1}{2}$ is a lower bound of the set described in (1). That is,

$$\frac{1}{2} \leq \sum_{i=1}^{\infty} |U_i|^s \tag{2}$$

for any cover $\{U_i\}$ of $\mathscr{C}$. Note that we shall assume that any member of $\{U_i\}$ has nontrivial diameter. Expanding the intervals to include their end points, and using the compactness of $\mathscr{C}$, we only need to verify (2) if $\{U_i\}$ is a finite collection of closed subintervals of [0,1].

For each U_i, let k_i be the integer such that

$$3^{-(k_i+1)} \leq |U_i| < 3^{-k_i}. \tag{3}$$

Such k_i exists since the union of all such intervals of the form $[3^{-(k_i+1)}, 3^{-k_i})$ is equivalent to the set of positive real numbers.

Thus, U_i can intersect *at most* one level-k interval since the separation of the level-k intervals is at least 3^{-k} in length. For $j \in \mathbb{Z}$, if $j \geq k_i$, then at the level-j pre-fractal, C_j, a level-k interval will have been subdivided into 2^{j-k_i} level-j intervals. Thus, U_i intersects at most 2^{j-k_i} level-j intervals of C_j. Raising the left inequality in (3) to the s-power and multiplying by $2^j 3^s$ results in the following:

$$\begin{aligned} 2^j 3^s 3^{-s(k_i+1)} &\leq 2^j 3^s |U_i|^s \\ 2^j \cdot 2 \cdot 2^{-(k_i+1)} &\leq 2^j 3^s |U_i|^s \\ 2^{j-k_i} &\leq 2^j 3^s |U_i|^s. \end{aligned}$$

Choose P such that $P \geq k_i$ for all U_i. Note that there will be finitely many such k_i since there are finitely many U_i. Raising (3) to the s-power and multiplying by $2^P 3^s$ gives

$$2^{P-k_i} \leq 2^P 3^s |U_i|^s$$

for all U_i. Note also,

$$\sum_{U_i \in \{U_i\}} 2^{P-k_i}$$

describes an upper bound for the number of level$-P$ intervals intersected by $\{U_i\}$. But since $\{U_i\}$ covers $\mathscr{C}$ and hence intersects all 2^P level-P intervals, we have

$$2^P \leq \sum_{U_i \in \{U_i\}} 2^{P-k_i} \leq \sum_{U_i \in \{U_i\}} 2^P 3^s |U_i|^s$$

or after simplification

$$\frac{1}{2} \leq \sum_{U_i \in \{U_i\}} |U_i|^s.$$

Therefore, (2) holds. ■

Example 3.1. As a guiding example for the proof above, consider the set below.

$$\{U_i\} = \left\{ \left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right] \right\}.$$

It is clear $\{U_i\}$ is a cover of $\mathscr{C}$. For each interval, we see that

$$\begin{cases} \operatorname{diam}[0,1/4] & = & 1/4 \Rightarrow & k_{[0,1/4]} = 1 \Rightarrow & \left(\frac{1}{9} \leq \frac{1}{4} < \frac{1}{3}\right) \\ \operatorname{diam}[1/4,1/3] & = & 1/12 \Rightarrow & k_{[1/4,1/3]} = 2 \Rightarrow & \left(\frac{1}{27} \leq \frac{1}{12} < \frac{1}{9}\right) \\ \operatorname{diam}[2/3,1] & = & 1/3 \Rightarrow & k_{[2/3,1]} = 0 \Rightarrow & \left(\frac{1}{3} \leq \frac{1}{3} < 1\right) \end{cases}$$

It is clear that $P \geq 2$ in this case. Choosing $P = 2$ results in the situation given by Figure 3. If we look at the max number of level-P intervals each interval in our cover intersects, then we have

$$\begin{cases} [0,1/4] & \Rightarrow & \text{at most } 2^{2-1} = 2 \text{ level-}P \text{ intervals} \\ [1/4,1/3] & \Rightarrow & \text{at most } 2^{2-2} = 1 \text{ level-}P \text{ intervals} \\ [2/3,1] & \Rightarrow & \text{at most } 2^{2-0} = 4 \text{ level-}P \text{ intervals} \\ & & \text{Total } = 7 \text{ level-}P \text{ intervals} \end{cases}$$

But as we can clearly see, there are only 4 level-P intervals for $P = 2$.

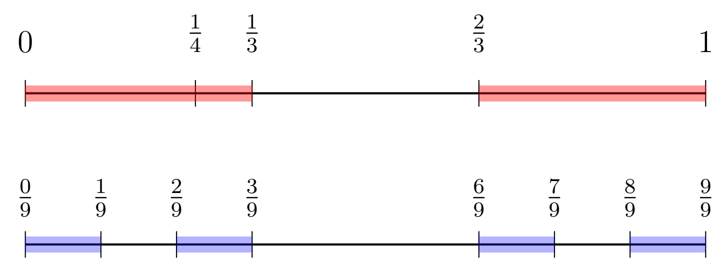


Figure 3: Example of a cover in association with a proper choice of level-P intervals

4 Improved Bounds on $\mathcal{H}^s(\mathscr{C})$

By assuming a weaker bound than that given in (3) on U_i, we allow it to exceed the length previously prescribed. This lets U_i intersect more than a single level-k_i interval, but preserves the number of level-$j \geq k_i$ intervals it intersects due to the size of the gap separating intervals in C_{k_i}. This results in the improved bounds below.

Proposition 4.1. $0.774 \approx 2^{s-1} \leq \mathcal{H}^s(\mathscr{C})$.

Proof. It suffices to show that

$$2^{s-1} \leq \sum_{U_i \in \{U_i\}} |U_i|^s \tag{4}$$

for any cover $\{U_i\}$ of $\mathscr{C}$. We shall assume that every member of $\{U_i\}$ is an interval of nontrivial diameter. Expanding the intervals slightly and using the compactness of $\mathscr{C}$, we only need to verify (4) if $\{U_i\}$ is a finite collection of closed subintervals of [0,1].

For each U_i let k_i be the integer such that

$$2 \cdot 3^{-(k_i+1)} \leq |U_i| < 2 \cdot 3^{-k_i}. \tag{5}$$

Such k_i exists for each U_i since the union of all such intervals $[2 \cdot 3^{-(k_i+1)}, 2 \cdot 3^{-k_i})$ is equal to the set of positive real numbers.

Note that this allows U_i to intersect at most two level-k_i intervals in C_{k_i} (Figure 4). However, the inequality $|U_i| < 2 \cdot 3^{-k_i}$ along with the observation that the gap separating intervals in C_{k_i} is of length at least 3^{-k_i}, gives that U_i will intersect C_{k_i} on a set of length at most 3^{-k_i}.

For $j \in \mathbb{Z}$, if $j \geq k_i$, then by construction of $\mathscr{C}$, U_i intersects at most 2^{j-k_i} level-j intervals of C_j. Raising (5) to the s-power and multiplying by $2^j 3^s$ gives us the following inequality.

$$\begin{aligned} 2^j 3^s 3^{s^2} 3^{-s(k_i+1)} &\leq 2^j 3^s |U_i|^s \\ 2^j 3^s 3^{-s(k_i+1)} &\leq 2^j 3^s 3^{-s^2} |U_i|^s \\ 2^{j-k_i} &\leq 2^j 3^s 2^{-s} |U_i|^s. \end{aligned}$$

If we choose P such that $P \geq k_i$ for all U_i, then

$$3^{-(P+1)} \leq \frac{|U_i|}{2}$$

for all $U_i \in \{U_i\}$, which is a finite set. Raising (5) to the s-power and multiplying by $2^P 3^s$ results in $2^{P-k_i} \leq 2^P 3^s 2^{-s} |U_i|$. Next, note that

$$\sum_{U_i \in \{U_i\}} 2^{P-k_i}$$

describes an upper bound for the number of level-P intervals intersected by $\{U_i\}$. But since $\{U_i\}$ covers $\mathscr{C}$ and hence intersects all 2^P level-P intervals, we have

$$2^P \leq \sum_{U_i \in \{U_i\}} 2^{P-k_i} \leq \sum_{U_i \in \{U_i\}} 2^P 3^s 2^{-s} |U_i|^s.$$

Therefore, by simplification, (4) holds. ■

Now that it has been made clear how the process of finding the Hausdorff measure of even classic fractal sets can be a very involved process that relies heavily upon the geometry of the specific fractal and space that it is in, we'll now proceed to introduce a new family of fractals, the *level-N* Sierpinski gaskets.

5 The Level-N Sierpinski Gasket

We next introduce a class of fractals which in a way generalizes the construction of the Sierpinski gasket. For an integer $N \geq 2$, we informally define the Level-N Sierpinski Gasket, $\mathcal{S}^N$, by the following process; see Figure 6 and Figure 7.

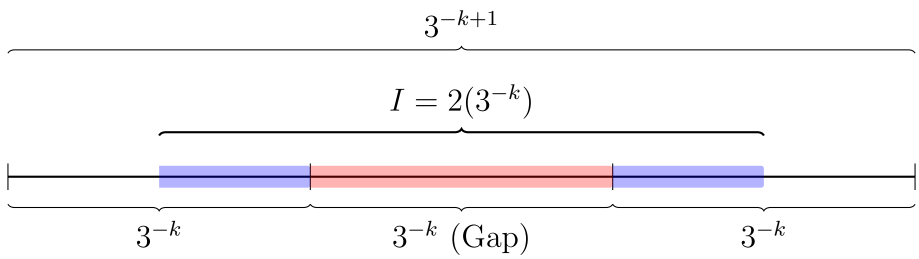


Figure 4: Cumulative Interval of explicit length $2(3^{-k})$ but contributing length of 3^{-k}

1. Begin with a unit triangle with vertices (0,0),(1,0), $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and denote it S_0^N
2. In the next iteration, S_1^N is formed by dividing S_0^N into N^2 equal triangles that are each a scaled copy of S_0^N by a factor of $\frac{1}{N}$.
3. To reach S_{k+1}^N, repeat the division described in step 2 on each upright triangle of S_k^N
4. $\mathcal{S}^N$ is obtained by taking $k \to \infty$ above.

Formally, the level-N Sierpinski gasket is defined as the attractor of the IFS given by the similarities produced by the following algorithm:

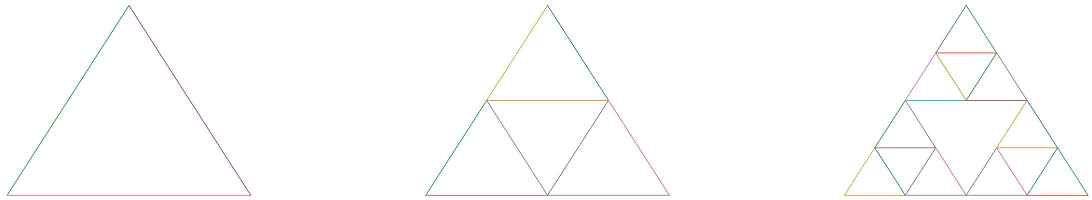
for i from 0 to $N-1$
    for j from 0 to $N-i$
        $f_{i,j}(x) = \frac{1}{N}\left[x + i(1/2, \sqrt{3}/2) + j(1,0)\right]$

For example, when $N = 3$, we have the following IFS:

$$\begin{cases} f_{0,0}(x) = \dfrac{1}{3}x \\ f_{0,1}(x) = \dfrac{1}{3}\left(x + \begin{bmatrix}1\\0\end{bmatrix}\right) \\ f_{0,2}(x) = \dfrac{1}{3}\left(x + \begin{bmatrix}2\\0\end{bmatrix}\right) \end{cases} \qquad \begin{cases} f_{1,0}(x) = \dfrac{1}{3}\left(x + \begin{bmatrix}1/2\\ \sqrt{3}/2\end{bmatrix}\right) \\ f_{1,1}(x) = \dfrac{1}{3}\left(x + \begin{bmatrix}3/2\\ \sqrt{3}/2\end{bmatrix}\right) \\ f_{2,0}(x) = \dfrac{1}{3}\left(x + \begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}\right) \end{cases}$$

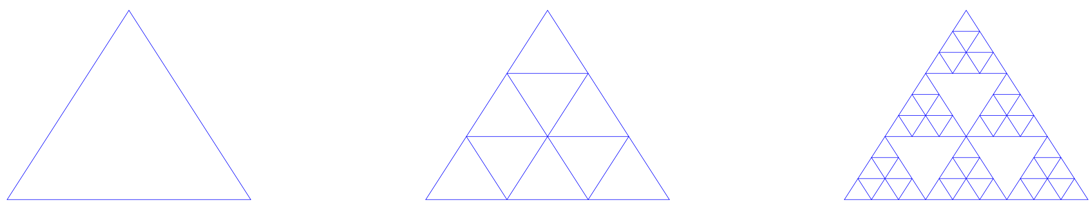
Note that the classic Sierpinski gasket, $\mathcal{S}$, is given by the case $N = 2$; see Figure 5. Also, it is easy to see that there are $\frac{1}{2}N(N+1)$ similarities for the N case. From [Fal14, p. 139], a calculation will yield that the level-N Sierpinski gasket has Hausdorff dimension

$$s = \frac{\log\left(\frac{N(N+1)}{2}\right)}{\log(N)}. \tag{6}$$



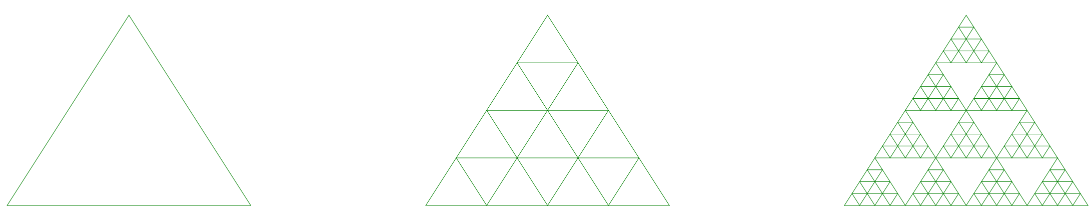
(a) S_0^2 (b) S_1^2 (c) S_2^2

Figure 5: Construction of $\mathcal{S}$



(a) S_0^3 (b) S_1^3 (c) S_2^3

Figure 6: Construction of $\mathcal{S}^3$



(a) S_0^4 (b) S_1^4 (c) S_2^4

Figure 7: Construction of $\mathcal{S}^4$

6 Extension of Zhou's Method

Here we present the main results of this paper. In [Zho97b], Zhou introduces a method which establishes upper bounds for the Hausdorff measure of the classical Sierpinski gasket. We take Zhou's construction and extend it to sets $\mathcal{S}^N$.

Let $f_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a similarity transformation of ratio $\alpha > 0$. If $F \subset \mathbb{R}^2$, then by [Fal14, p. 46], we have the scaling property

$$\mathcal{H}^s(f_\alpha(F)) = \alpha^s \mathcal{H}^s(F). \tag{7}$$

For convenience, we'll define $f_\alpha \circ f_\alpha = f_\alpha^2$ and $f_\alpha^n = f_\alpha \circ f_\alpha^{n-1}$ for $n = 2, 3, 4, \ldots$, with $f_\alpha^1 = f_\alpha$.

Let $\beta = \{U_i : i \in \mathbb{N}\}$ be a δ-cover of $\mathcal{S}^N$ and let $\epsilon_\beta \in \mathbb{R}$ be the error in using β to estimate $\mathcal{H}^s(\mathcal{S}^N)$,

$$\mathcal{H}^s(\mathcal{S}^N) = \epsilon_\beta + \sum_{i=0}^{\infty} |U_i|^s.$$

Let $k \in \mathbb{N}$. Compressing β via $f_{1/N}^k$ results in a δ/N^k-cover of $f_{1/N}^k(\mathcal{S}^N)$. By (6), taking N^{sk} duplicates of $f_{1/N}^k(\beta)$ reconciles a δ/N^k-cover of $\mathcal{S}^N$ by exploiting the set's symmetry. With this in mind, we consider the following propositions:

Proposition 6.1.

$$\mathcal{H}^s(\mathcal{S}^N) = N^{sk} \sum_{i=1}^{\infty} \left(\frac{1}{N^k} |U_i| \right)^s + \epsilon_\beta = \sum_{i=1}^{\infty} |U_i|^s + \epsilon_\beta; \tag{8}$$

that is, β and $f_{1/N}^k(\beta)$ have the same errors.

Proof. If we suppose that the error of using N^{sk} duplicates of $f_{1/N}^k(\beta)$ to estimate $\mathcal{H}^s(\mathcal{S}^N)$ differs from ϵ_β, then factoring constants and simplifying shows them to be equal. ■

Proposition 6.2.

$$\mathcal{H}^s\left(f_{1/N}^k(\mathcal{S}^N)\right) = N^{-sk} \mathcal{H}^s(\mathcal{S}^N) = \sum_{i=1}^{\infty} \left(\frac{1}{N^k} |U_i| \right)^s + \frac{1}{N^{sk}} \epsilon_\beta; \tag{9}$$

that is, the error when using a compression of β scales accordingly.

Proof. The statement is clear using (7) and (8). ■

Proposition 6.3. Let $\delta > 0$. Then

$$\mathcal{H}^s(\mathcal{S}^N) = \mathcal{H}^s_\delta(\mathcal{S}^N). \tag{10}$$

Proof. We first note that $\mathcal{H}^s(\mathcal{S}^N) \geq \mathcal{H}^s_\delta(\mathcal{S}^N)$ is clear by definition. Let $\epsilon > 0$. Since

$$\mathcal{H}^s_\delta(\mathcal{S}^N) = \inf\left\{\sum_{i=1}^{\infty} |A_i|^s : \{A_i\} \text{ is a } \delta\text{-cover of } \mathcal{S}^N\right\},$$

there exists a δ-cover, $\Gamma = \{V_i : i \in \mathbb{N}\}$, such that

$$\sum_{i=1}^{\infty} |V_i| \leq \mathcal{H}^s_\delta(\mathcal{S}^N) + \epsilon.$$

Letting $k \geq 0$ and using N^{sk} duplicates of $f^k_{1/N}(\Gamma)$ as a $\frac{\delta}{N^k}$-cover of $\mathcal{S}^N$, we have

$$\mathcal{H}^s_{\delta/N^k}(\mathcal{S}^N) \leq N^{sk} \sum_{i=1}^{\infty} \left(\frac{1}{N^k}|V_i|\right)^s = \sum_{i=1}^{\infty} |V_i|^s \leq \mathcal{H}^s_\delta(\mathcal{S}^N) + \epsilon.$$

Taking $k \to \infty$ and noting that ϵ is free gives us that

$$\mathcal{H}^s(\mathcal{S}^N) \leq \mathcal{H}^s_\delta(\mathcal{S}^N).$$

■

Our focus is on the level-N Sierpinski gaskets, but one should note that these propositions may be extended to the class of all self-similar fractal sets. Now we move to the main result of the paper.

Theorem 6.4. For the class of level-N Sierpinski gaskets, Zhou's method, [Zho97b], may be extended to show that

$$\mathcal{H}^s(\mathcal{S}^N) \leq \min_{m \in \mathbb{N}} \left(1 + \frac{3}{-5 + \left(\frac{N(1+N)}{2}\right)^m}\right)\left(1 - \frac{1}{N^m - 1}\right)^s. \tag{11}$$

One exception is that for $N = 2$, we minimize over $\mathbb{N}\backslash\{1\}$.

Proof. Our goal is to build a sequence of covers of $\mathcal{S}^N$ whose diameters are decreasing. Beginning from S^N_0, we label the 3 vertices $(0,0) : A$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) : B$, $(1,0) : C$. We iterate $m \in \mathbb{N}$ times to S^N_m and label 6 points,

$$\begin{cases} a^m_1 : \left(\frac{1}{2N^m}, \frac{\sqrt{3}}{2N^m}\right) \\ a^m_2 : \left(\frac{1}{N^m}, 0\right) \\ b^m_1 : \left(\frac{N^m-1}{2N^m}, \frac{\sqrt{3}(N^m-1)}{2N^m}\right) \\ b^m_2 : \left(\frac{N^m+1}{2N^m}, \frac{\sqrt{3}(N^m-1)}{2N^m}\right) \\ c^m_1 : \left(\frac{N^m-1}{N^m}, 0\right) \\ c^m_2 : \left(\frac{2N^m-1}{2N^m}, \frac{\sqrt{3}}{2N^m}\right) \end{cases}$$

which are the pair of vertices from stage m which are nearest to the vertices A, B, and C; see Figure 8a. We form three triangles using A, B, C and their corresponding pairs:

$$\triangle A a_1^m a_2^m, \qquad \triangle B b_1^m b_2^m, \qquad \triangle C c_1^m c_2^m.$$

For convenience, we denote the set of points contained in a single one of these triangles by T_N^m. Thus, by taking three copies of T_N^m, denoted 3-T_N^m, and positioning them correctly, we recover $\triangle A a_1^m a_2^m, \triangle B b_1^m b_2^m$ and $\triangle C c_1^m c_2^m$.

Next, we use these 6 neighbors again to form a hexagon and label the set of points contained in it H_N^m; see Figure 8a.

Note that $|H_N^m| = 1 - \frac{1}{N^m}$. The union of H_N^m with $3(2^0)$ copies of T_N^m forms a cover of $\mathcal{S}^N$, which we label σ_1; see Figure 9a,

$$\sigma_1 = \{H_N^m,\ 3(2^0)\text{-}T_N^m\}.$$

To reach σ_2, we iterate m times again to reach S_{2m}^N. From each of a_1^m, a_2^m, b_1^m, b_2^m, c_1^m, and c_2^m, we mark the pair of nearest neighbors closest to the center of S_{2m}^N; see Figure 8b. Label the triangles formed as T_N^{2m}, of which there are $3(2^1)$, and we shrink our hexagon H_N^m slightly such that these new neighbors lie on its boundary; see Figure 9b.

We label this hexagon H_N^{2m} and note that it has diameter $H_N^{2m} = 1 - \left(\frac{1}{N^m} + \frac{1}{N^{2m}}\right)$. Thus,

$$\sigma_2 = \{H_N^{2m},\ 3(2^0)\text{-}T_N^m,\ 3(2^1)\text{-}T_N^{2m}\}.$$

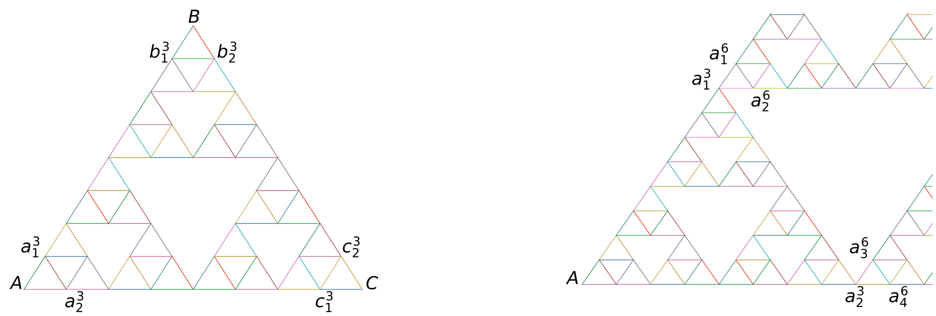
Continuing in this process, we have, at the n^{th}-step,

$$\sigma_n = \{H_N^{nm},\ 3(2^0)\text{-}T_N^m,\ 3(2^1)\text{-}T_N^{2m},\ \ldots,\ 3(2^{n-1})\text{-}T_N^{nm}\}$$

where

$$|H_N^{nm}| = 1 - \sum_{i=1}^{n} \frac{1}{N^{mi}}.$$

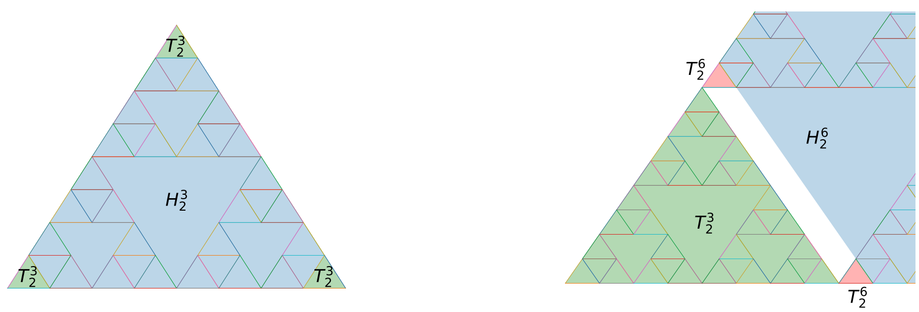
Thus, we have a sequence of coverings of $\mathcal{S}^N$, $\{\sigma_n\}_{n=1}^{\infty}$, however, it is not quite the one that we want.



(a) Points forming σ_1.

(b) Points used in the left corner formation of σ_2

Figure 8: Zhou's Method on $\mathcal{S}$ with $m = 3$ (Points).



(a) Visualization of σ_1.

(b) Visualization of the left corner of σ_2.

Figure 9: Zhou's Method on $\mathcal{S}$ with $m = 3$ (Sets).

Let $n \geq 0$ and let $\beta = \{V_i : i \in \mathbb{N}\}$ be a cover of $\mathcal{S}^N$. For convenience, let us use $f_{1/N}^{nm}(\beta)$ to denote the set of dilations $f_{1/N}^{nm}(V_i)$ for all $i \in \mathbb{N}$. In other words,

$$f_{1/N}^{nm}(\beta) \equiv \{f_{1/N}^{nm}(V_i) : i \in \mathbb{N}\}$$

By definition, $f_{1/N}^{nm}(\beta)$ covers $f_{1/N}^{nm}(\mathcal{S}^N) = f_{1/N}^{nm}(\mathcal{S}^N) \cap T_N^{nm}$. Thus, we may replace T_N^{nm} with $f_{1/N}^{nm}(\beta)$ in σ_n for $n = 1, 2, 3, \ldots$, recovering a new sequence of covers, $\{v_n\}_{n=1}^{\infty}$, where

$$v_n = \{H_N^{nm},\ 3(2^0)\text{-}f_{1/N}^{m}(\beta),\ 3(2^1)\text{-}f_{1/N}^{2m}(\beta),\ \ldots,\ 3(2^{n-1})\text{-}f_{1/N}^{nm}(\beta)\}.$$

By Proposition 6.3, $\mathcal{H}^s(\mathcal{S}^N) = \mathcal{H}_1^s(\mathcal{S}^N)$, for which v_n is one of the covers that the infimum, from $\mathcal{H}_1^s(\mathcal{S}^N)$, is taken over. Thus, taking the sum of diameters of sets in v_n, we have

$$\mathcal{H}^s(\mathcal{S}^N) = \mathcal{H}_1^s(\mathcal{S}^N) \leq \left(1 - \sum_{k=1}^{n} \frac{1}{N^{mk}}\right)^s + 3\left(\sum_{j=1}^{n} \frac{2^{j-1}}{\left(\frac{N(N+1)}{2}\right)^{mj}}\right) \sum_{i=1}^{\infty} |V_i|^s \tag{12}$$

Using the error, ϵ_β, and Proposition 6.2,

$$\left(1 - \sum_{k=1}^{n} \frac{1}{N^{mk}}\right)^s + 3\left(\sum_{j=1}^{n} \frac{2^{j-1}}{\left(\frac{N(N+1)}{2}\right)^{mj}}\right) \left(\mathcal{H}^s(\mathcal{S}^N) - \epsilon_\beta\right). \tag{13}$$

Noting that we may choose β so that $|\epsilon_\beta|$ is sufficiently small, we have that

$$\mathcal{H}^s(\mathcal{S}^N) \leq \left(1 - \sum_{k=1}^{n} \frac{1}{N^{mk}}\right)^s + 3\left(\sum_{j=1}^{n} \frac{2^{j-1}}{\left(\frac{N(N+1)}{2}\right)^{mj}}\right) \mathcal{H}^s(\mathcal{S}^N). \tag{14}$$

Evaluating terms, we have

$$\mathcal{H}^s(\mathcal{S}^N) \leq \left(\frac{N^m - 2 + N^{-nm}}{N^m - 1}\right)^s + \frac{3}{\left(\frac{N(N+1)}{2}\right)^m - 2}\left[1 - \left(\frac{2^{m+1}}{N^m(N+1)^m}\right)^n\right] \mathcal{H}^s(\mathcal{S}^N).$$

Thus,

$$\mathcal{H}^s(\mathcal{S}^N) \leq \frac{\left(\frac{N^m-2}{N^m-1} + \frac{N^{-nm}}{N^m-1}\right)^s}{1 - \frac{3}{\left(\frac{N(N+1)}{2}\right)^m - 2}\left[1 - \left(\frac{2^{m+1}}{N^m(N+1)^m}\right)^n\right]} \tag{15}$$

Using Mathematica and some further analysis, it is possible to show that (15) is a decreasing sequence in n. Please refer to my documentation for a rough justification. Noting that $N \geq 2$, we take the limit as $n \to \infty$. Thus,

$$\begin{aligned}
\mathcal{H}^s(\mathcal{S}^N) &\leq \frac{\left(\frac{N^m-2}{N^m-1}\right)^s}{1 - \frac{3}{\left(\frac{N(N+1)}{2}\right)^m - 2}} \\
&= \left(\frac{1}{\frac{\left(\frac{N(N+1)}{2}\right)^m - 2}{\left(\frac{N(N+1)}{2}\right)^m - 2} - \frac{3}{\left(\frac{N(N+1)}{2}\right)^m - 2}}\right)\left(1 - \frac{1}{N^m - 1}\right)^s \\
&= \left(\frac{\left(\frac{N(N+1)}{2}\right)^m - 2}{\left(\frac{N(N+1)}{2}\right)^m - 5}\right)\left(1 - \frac{1}{N^m - 1}\right)^s \\
&= \left(1 + \frac{3}{-5 + \left(\frac{N(1+N)}{2}\right)^m}\right)\left(1 - \frac{1}{N^m - 1}\right)^s.
\end{aligned}$$

Finally, we simply need to minimize over $m \in \mathbb{N}$, which gives us the desired result in (11). ■

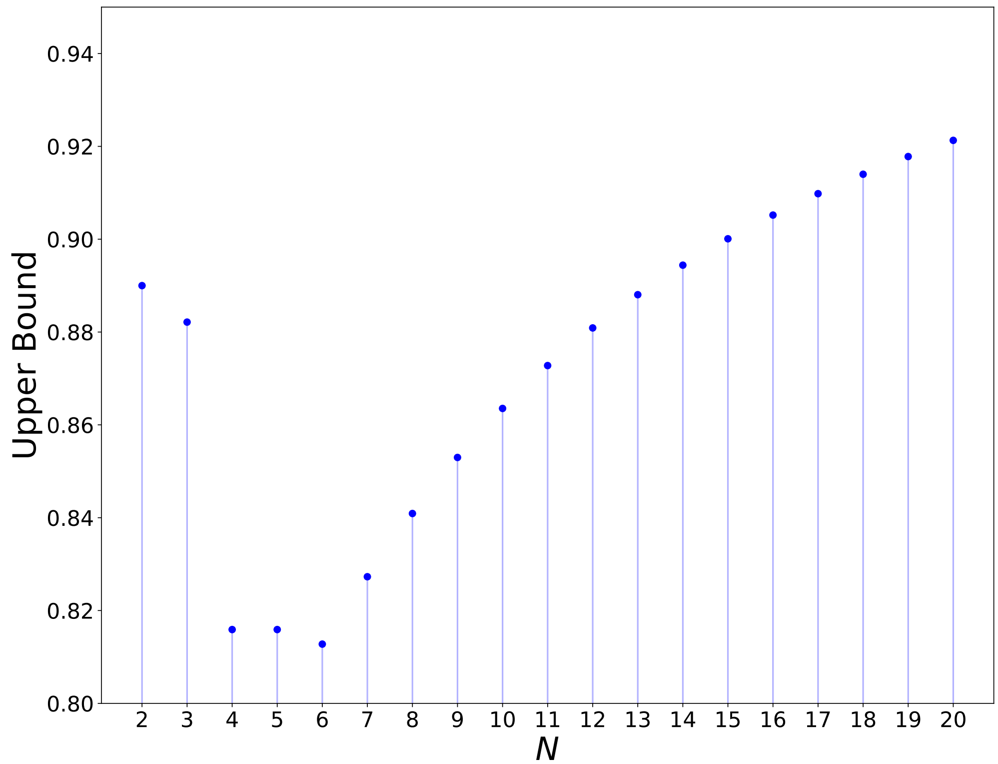


Figure 10: Upper bounds of $\mathcal{H}^s(\mathcal{S}^N)$ vs. N

7 Closing Remark

One notable observation is the following correspondence between N and m:

N	2	3	4	5	6	$\cdots$
$m_{\min}$	3	2	1	1	1	$\cdots$
$\mathcal{H}^s(\mathcal{S}^N) \leq$	0.890039	0.882138	0.815903	0.801163	0.812767	$\cdots$

See Figure 10 below for a graph which plots the upper bound for $\mathcal{H}^s(\mathcal{S}^N)$ for levels 2 through 20.

The upper bounds gradually increase to 1. We speculate that the Hausdorff measure begins to approach a multiple of the 2-dimensional area of the outermost triangle making up $\mathcal{S}^N$. This coincides with the geometric intuition that as N increases, the level-N Sierpinski gasket becomes "denser", converging towards a filled triangle. Another key observation that supports this idea is that the Hausdorff dimension for $\mathcal{S}^N$ given in (6), has limit

$$\lim_{N\to\infty} \frac{\log\left(\frac{N(N+1)}{2}\right)}{\log(N)} = 2.$$

References

[Fal14] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. West Sussex: John Wiley & Sons, 2014.

[Hut81] J. Hutchinson. "Fractals and Self Similarity". In: *Indiana University Mathematics Journal* 30 (1981), pp. 713–747.

[Mar87] J. Marion. "Mesures de Hausdorff D'Ensembles Fractals". In: *Annales mathématiques du Québec* 11.1 (1987), pp. 111–132.

[Zho97a] Z. Zhou. "The Hausdorff measures of the Koch curve and Sierpiński gasket". In: *Progress in Natural Science* 7.4 (1997), p. 401.

[Zho97b] Z. Zhou. "Hausdorff measure of the Sierpinski Gasket". In: *Science in China* 40.10 (1997).

[WW99] H. Wang and X. Wang. "Computer search for the upper estimation of Hausdorff measure of classical fractal sets II". In: *Mathematica Numerica Sinica* 21.4 (1999).

[Fen97] L. Feng. "A new estimate of the Hausdorff measures of the Sierpiński gasket". In: *Progress in Natural Science* 7.4 (1997), p. 401.

[JZZ02] B. Jia, Z. Zhou, and Z. Zhu. "A lower bound for the Hausdorff measure of the Sierpinski gasket". In: *Nonlinearity* 15 (2002), pp. 393–404.

[HW04] R. Houjun and W. Weiyi. "An Approximation Method to Estimate the Hausdorff Measure of the Sierpinski Gasket". In: *Analysis in Theory and Applications* 20.2 (2004), pp. 158–166.

[JZZ06] B. Jia, Z. Zhou, and Z. Zhu. "A New Lower Bound of the Hausdorff Measure of the Sierpinski Gasket". In: *Analysis In Theory And Applications* 22.1 (2006), pp. 8–19.

[Mór09] P. Móra. "Estimate of the Hausdorff Measure of the Sierpinski Triangle". In: *Fractals* 17.2 (2009), pp. 137–148.

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