Real Analysis Reference Sheet

σ-Algebras

- **Def.** (Algebra) An algebra of set on $X(\neq \varnothing)$ is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements.

- **Def.** (σ-algebra) A σ-algebra is an algebra that is closed under countable unions.

- **Def.** (Borel σ-algebra) If X is a metric or topological space, then the σ-algebra generated by the family of open sets in X is called the Borel σ-algebra on X and is denoted $\mathcal{B}_X$.

- **Def.** (product σ-algebra) Let $\{X_\alpha\}_{\alpha\in A}$ be an indexed collection of nonempty sets. Let $X = \prod_{\alpha\in A} X_\alpha$ and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each α, the product σ-algebra on X is the σ-algebra generated by the set

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$$

Measures

- **Def.** (Measure) Let X be a nonempty set equipped with σ-algebra, $\mathcal{M}$. A measure on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

  (i) $\mu(\varnothing) = 0$

  (ii) If $(E_j)_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$. (Countable additivity).

- **Thm 1.8** (Properties of measures).

  Let $(X, \mathcal{M}, \mu)$ be a measure space.

  (a) (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

  (b) (Subadditivity) If $(E_j)_1^\infty \subset \mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$.

  (c) (Continuity from below) If $(E_j)_1^\infty$ is an increasing sequence in $\mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

  (d) (Continuity from above) If $(E_j)_1^\infty$ is a decreasing sequence in $\mathcal{M}$ and $\mu(E_1) < \infty$, then $\mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

- (Types of measures) If $\mu(X) < \infty$ then μ is called a finite measures. If there exists a sequence $(E_j)_1^\infty \subset \mathcal{M}$ such that $X = \bigcup_1^\infty E_j$ and $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$, then μ is called a σ-finite measure. If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, then μ is called a semifinite measure.

- **Def.** (Complete measure) A measure, μ, whose domain (the σ-alg.) contains all subsets of null-sets is called complete. Null-sets are sets, $N \in \mathcal{M}$ such that $\mu(N) = 0$.

- **Thm 1.9** Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}$ and $F \subset N$ for some $N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$. The measure $\overline{\mu}$ is called the completion of μ and $\overline{\mathcal{M}}$ is called the completion of $\mathcal{M}$ w.r.t. μ.

- **Def.** (Outer measure) An outer measure on $X(\neq \varnothing)$ is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

  (i) $\mu^*(\varnothing) = 0$

  (ii) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$

  (iii) $\mu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$.

- **Def.** (μ^*-measurable sets) A set $A \subseteq X$ is called μ^*-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subseteq X$$

- **Thm 1.11** (**Caratheodory's Theorem**) If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*- measurable sets forms a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

- **Def.** (premeasure) If $\mathcal{A}$ is an algebra on X, then a function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a premeasure if $\mu_0(\varnothing) = 0$ and μ_0 is countably additive on disjoint sets.

- (Outer measure induced by premeasure, 1.12)

$$\mu^*(E) = \inf\{\textstyle\sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subseteq \bigcup_1^\infty A_j\}$$

- **Prop 1.13** If μ_0 is a permeasure on $\mathcal{A}$ and μ^* is an induced outer measure, then $\mu^*|_{\mathcal{A}} = \mu_0$ and every set in $\mathcal{A}$ is μ^*-measurable.

- (Caratheodory's contruction of measures) Start with a premeasure μ_0 on an algebra $\mathcal{A}$, use μ_0 to induce an outer measure μ^*, and then extend μ_0 to a complete measure $\mu = \mu^*|_{\mathcal{M}}$ defined on the σ-algebra, $\mathcal{M}$, of μ^*-measurable sets.

- **Def.** (Lebesgue-Stieltjes measure) Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, right continuous function. Then there is a unique measure on $\mathcal{B}_{\mathbb{R}}$ (Borel σ-alg. on $\mathbb{R}$) such that the measure of any interval (a, b) is simply its length $b - a$ for all $a, b \in \mathbb{R}$. Caratheodory's contruction may then be applied to extend this measure to a complete measure, denoted μ_F, whose domain, $\mathcal{M}_\mu$, is strictly larger than $\mathcal{B}_{\mathbb{R}}$. This complete measure is called the Lebesgue-Stieltjes measure associated to F and

$$\mu_F(E) = \inf \left\{ \sum_1^\infty [F(b_j) - F(a_j)] : E \subseteq \bigcup_1^\infty (a_j, b_j) \right\}$$

- **Prop 1.20** If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$.

- **Def.** (Lebesgue measure) The Lebesgue measure is the complete measure μ_F associated to the function $F(x) = x$. We denote this measure by $m : \mathcal{L} \to [0, \infty]$ where $\mathcal{L}$ denotes the set of Lebesgue measurable sets (m-measurable). Note $\mathcal{B}_{\mathbb{R}} \subset \mathcal{L}$ strictly.

  The most significant properties of the Lebesgue measure are its invariance under translations and simple behavior under dilation.

Measurable Functions

- **Def.** (Measurable functions) Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measure spaces. Then a mapping $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-measurable or just measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$. This is similar to the definition of continuous mappings between topological spaces. If $\mathcal{N}$ is a σ-algebra generated by some set $\mathcal{E}$, then we may simply show $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

  For complex-valued functions on X, we say they are measurable if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable. Such functions have nice closure properties. If $f, g : X \to \mathbb{C}$ are measurable, then so are $f + g$, fg, $\max\{f, g\}$ and $\min\{f, g\}$.

- **Def.** (simple function) A simple function on X is a finite linear combination of characteristic functions of sets in $\mathcal{M}$ with complex coefficients.

$$f = \sum_1^n z_j \chi_{E_j}$$

  where $E_j = f^{-1}(\{z_j\})$ and range$(f) = \{z_j : 1 \leq j \leq n\}$. This is called the standard representation.

- **Thm 2.10** If $f : X \to \mathbb{C}$ is measurable, there is a sequence $(\phi_n)_1^\infty$ of simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

- **Prop 2.11** The following are true iff μ is complete:

  (a) If f is measurable and $f = g$ μ-a.e., then g is measurable.

  (b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Integration

- (Integration of nonnegative functions) Define the space $L^+(X)$ to the set of measurable nonnegative functions on X. If ϕ is a simple function in $L^+(X)$ with standard representation $\phi = \sum_1^n a_j \chi_{E_j}$, then define the integral of ϕ w.r.t. μ by

$$\int_X \phi \; d\mu = \sum_1^n a_j \mu(E_j).$$

  and for $A \in \mathcal{M}$, $\int_A \phi \; d\mu = \int_X \phi \chi_A \; d\mu$. Some general properties:

(a) If $c \geq 0$, $\int c\phi = c \int \phi$.
(b) $\int(\phi + \psi) = \int \phi + \int \psi$
(c) If $\phi \leq \psi$, then $\int \phi \leq \int \psi$.
(d) The map $A \mapsto \int_A d\mu$ is a measure on $\mathcal{M}$.

Now, for any $f \in L^+(X)$, we define its integral by

$$\int f \ d\mu = \sup\left\{\int \phi \ d\mu : 0 \leq \phi \leq f, \phi \text{ is simple}\right\}$$

- (Integration of complex-valued functions) For a real-valued function, f, if f^+, f^- are its positive and negative parts and at least one of $\int f^+$ and $\int f^-$ is finite, then we define $\int f = \int f^+ - \int f^-$. If both $\int f^+, \int f^-$ are finite, then we say f is integrable. Note $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

  Next, for a complex-valued function, f, we say that f is integrable on a set E if $\int_E |f| < \infty$ and define

  $$\int f = \int \operatorname{Re} f + i \int \operatorname{Im} f$$

  Note that the space of complex-valued integrable functions is a complex vector space and the integral is a linear functional on it. The space if integrable complex-valued functions on X is denoted $L^1(X)$ or $L^1(\mu)$. Two functions f, g are equivalent in $L^1(X)$ is $f = g$ μ-a.e. $L^1(X)$ is also a metric space with distance $\int |f - g| d\mu$.

- **Thm 2.26** If $f \in L^1(\mu)$ and $\epsilon > 0$, there is an integrable simple function $\phi = \sum_1^n a_j \chi_{E_j}$ such that $\int |f - \phi| d\mu < \epsilon$. That is, the integrable simple functions are dense in L^1 in its metric.

- **Cor 3.6** If $f \in L^1(\mu)$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_E f d\mu| < \epsilon$ whenever $\mu(E) < \delta$.

- **Thm 2.14** (**The Monotone Convergence Theorem**) If $(f_n)_1^\infty \subset L^+$ such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n\to\infty}(= \sup_n f_n)$, then $\int f = \lim_{n\to\infty} \int f_n$

- **Prop 2.16** If $f \in L^+$ then $\int f = 0$ iff $f = 0$ a.e.

- **Lemma 2.18** (**Fatou's Lemma**) If $(f_n)_1^\infty$ is any sequence in L^+, then

  $$\int (\liminf f_n) \leq \liminf \int f_n.$$

- **Thm 2.24** (**Dominated Convergence Theorem**) Let $(f_n) \subseteq L^1(X)$ such that

  (a) $f_n \to f$ μ-a.e.
  (b) There exists $g \in L^1$, $g \geq 0$ such that $|f_n| \leq g$ μ-a.e. for all n

  Then $f \in L^1$ and $\int_X f = \lim_{n\to\infty} \int_X f_n$.

- **Thm 2.28**. (Relation between the Lebesgue and Riemann integrals) Let f be a bounded real-valued function on $[a, b]$.

  (a) If f is Riemann integrable, then f is Lebesgue measurable (and hence integrable on $[a, b]$ since it is bounded), and
  $$\int_a^b f(x)dx = \int_{[a,b]} f \ dm.$$
  (b) f is Riemann integrable iff the set of points $x \in [a, b]$ such that f is discontinuous at x has Lebsegue measure zero.

- **Thm 2.26** If $f \in L^1(m)$ then there is a continuous function g that vanishes outside a bounded interval such that $\|f - g\|_1 < \epsilon$

- **Thm 2.27** (Differentiation under the integral sign) Suppose that $f : X \times [a, b] \to \mathbb{C}$ and that $f(\cdot, t) : X \to \mathbb{C}$ is integrable for each $t \in [a, b]$. Let $F(t) = \int_X f(x, t) d\mu(x)$.

  (a) Suppose that there exists $g \in L^1(\mu)$ such that $|f(x, t)| \leq g(x)$ for all x, t. If $\lim_{t\to t_0} f(x, t) = f(x, t_0)$ for every x, then $\lim_{t\to t_0} F(t) = F(t_0)$; in particular, if $f(x, \cdot)$ is continuous for each x, then F is continuous.
  (b) Suppose that $\partial f/\partial t$ exists and there is a $g \in L^1(\mu)$ such that $|(\partial f/\partial t)(x, t)| \leq g(x)$ for all x, t. Then F is differentiable and $F'(x) = \int (\partial f/\partial t)(x, t) d\mu(x)$.

Modes of Convergence

- **Def.** (pointwise convergence) If $(f_n)_1^\infty$ is a sequence of measurable complex-valued functions then $f_n \to f$ pointwise if $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. We may also define pointwise μ-a.e. convergence similarly.

- **Def.** (uniform convergence) $(f_n)_1^\infty$ converges to f uniformly if $\|f_n - f\|_\infty = \sup_{x\in X} |f_n(x) - f(x)| \to 0$.

- **Def.** (Convergence in measure) $(f_n)_1^\infty$ converges to f in measure if for every $\epsilon > 0$

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$$

as $n \to \infty$.

- **Def.** (Convergence in L^p space) $(f_n)_1^\infty$ converges to f if $\|f_n - f\|_p = (\int |f_n - f|^p)^{1/p} \to 0$ as $n \to \infty$.

- (Relationships between modes of convergence)
  1. Uniform conv. $\implies$ Pointwise conv. $\implies$ μ-a.e. conv.
  2. If $f_n \to f$ in L^1 then $f_n \to f$ in measure
  3. If $f_n \to f$ in L^1 then there is a subsequence of f_n that converges to f μ-a.e.

- **Thm 2.33** (**Egoroff's Theorem**) Suppose that $\mu(X) < \infty$ and $(f_n)_1^\infty$ and f are all measurable complex-valued functions on X such that $f_n \to f$ μ-a.e. Then for every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

- **Exc 2.44** (**Lusin's Theorem**) If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subseteq [a, b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous.

Product Measures

- **Thm 2.37** (**The Fubini-Tonelli Theorem**) Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

  (a) (Tonelli) If $f \in L^+(X\times Y)$, then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and

$$\begin{aligned}\int f d(\mu \times \nu) &= \int \left[\int f(x, y) d\nu(y)\right] d\mu(x) \\ &= \int \left[\int f(x, y) d\mu(x)\right] d\nu(y)\end{aligned}$$

  (b) (Fubini) If $f \in L^1(\mu\times\nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively and the integral equality of Tonelli's holds as well.

- **Def.** (Lebesgue measure on $\mathbb{R}^n$) The Lebesgue measure on $\mathbb{R}^n$ denoted m^n is the product of Lebesgue measure on $\mathbb{R}$ with itself n times on the n times product space of $\mathcal{B}_\mathbb{R}$ or $\mathcal{L}$.

Differentiation of Measures

- **Def.** (signed measure) A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that $\nu(\varnothing) = 0$, ν can only map to either $+\infty$ or $-\infty$ but not both, and if $(E_j)_1^\infty \subset \mathcal{M}$ is disjoint, then $\nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$ where this sum converges absolutely if $\nu(\bigcup_1^\infty) < \infty$.

  Every signed measure ν can either be represented as the difference between two positive measures $\mu_1 - \mu_2$ or if μ is a measure on $\mathcal{M}$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite, then defining $\nu(E) = \int_E f d\mu$ also produces a signed measure.

- **Thm 3.3** (**The Hahn Decomposition Theorem**) If ν is a signed measure on $(X, \mathcal{M})$, there exists a positive set P and a negative set N for ν such that $P \cup N = X$, $P \cap N = \varnothing$. If another such pair P', N' exists, then $P \triangle P'$ and $N \triangle N'$ are null for ν.

- **Def.** (mutually singular measures) Two signed measures μ, ν on $(X, \mathcal{M})$ are mutually singular if there exists $E, F \in \mathcal{M}$ such that $E \cap F = \varnothing$, $E \cap F = X$ and E is null for μ and F is null for ν. We denote this by $\mu \perp \nu$.

- **Thm 3.4** (**The Jordan Decomposition Theorem**) If ν is a signed measure, there exists unique positive measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

  Def. (total variation) The total variation of a signed measure ν is $|\nu| = \nu^+ + \nu^-$.

- **Def.** (absoluteley continuous measures) Suppose ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. We say that ν is absolutely continuous w.r.t. μ if $\nu(E) = 0$ whenever $\mu(E) = 0$. We denote this by $\nu \ll \mu$.

- **Thm 3.8** (**The Lebesgue-Radon-Nikodym Theorem**) Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There exists a unique σ-finite signed measure λ, ρ on $(X, \mathcal{M})$ such that

  $$\lambda \perp \mu, \ \ \rho \ll \mu, \ \ \nu = \lambda + \rho.$$

  Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that

  $$d\rho = f d\mu \Leftrightarrow \rho(E) = \int_E f d\mu, \ \ \forall E \in \mathcal{M}$$

  and any two such functions are equal μ-a.e. The decomposition $\nu = \lambda + \rho$ is called the Lebesgue decomposition of ν w.r.t. μ. When $\nu \ll \mu$, we have that $d\nu = f d\mu$ for some f and this f is called the Radon-Nikodym derivatve of ν w.r.t. μ. and is denoted $f = d\nu/d\mu$.

- **Def.** (Hardy-Littlewood maximal function). Let $f \in L^1_{\text{loc}}$, i.e. that f is integrable on any bounded measurable subset of $\mathbb{R}^n$, then

  $$H(f)(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| dy$$

- **Def.** (Lebesgue set) For $f \in L^1_{\text{loc}}$, the Lebesgue set, L_f is defined to be the following:

  $$\left\{ x : \lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dy = 0 \right\}$$

- **Def.** (**The Lebesgue Differentiation Theorem**) Suppose $f \in L^1_{\text{loc}}$. For every x the Lebesgue set of f, in particular, for almost every x, we have

  $$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0$$

  and

  $$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x)$$

  for every family $\{E_r\}_{r>0}$ that shrinks nicely to x. $\{E_r\}$ shrinks nicely to x if $E_r \subseteq B_r(x)$ for each $r > 0$ and there is some constant α independent of r such that $m(E_r) > \alpha m(B_r(x))$.

- **Thm 3.22** Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$, and let $d\nu = d\lambda + f dm$ be its Lebesgue-Radon-Nikodym representation. Then for m-a.e. $x \in \mathbb{R}^n$,

  $$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

  for every family $\{E_r\}_{r>0}$ that shrinks nicely to x. It is particularly useful in application to use balls centered around x.

Differentiation of functions on $\mathbb{R}$

- **Def.** (regular measure) A Borel measure ν on $\mathbb{R}$ will be called regular if $\nu(K) < \infty$ for every compact set K and $\nu(E) = \inf\{\nu(U) : U \text{ open}, E \subseteq U\}$ for every $E \in \mathcal{B}_{\mathbb{R}}$. A signed or complex measure will be called regular if its total variation is regular.

- **Def.** (total variation of a function) Let $F : \mathbb{R} \to \mathbb{C}$. The total variation of F on $[a, b]$ is defined as

  $$T_F([a,b]) = \sup \left\{ \textstyle\sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N} \right.$$
  $$\left. a = x_0 < \cdots < x_n = b \right\}$$

- **Def.** (bounded variation) If $T_F([a, b]) < \infty$ then F is of bounded variation and we denote $F \in BV([a, b])$.

- **Def.** (absolutely continuous function) A function $F : \mathbb{R} \to \mathbb{C}$ is called absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any finite set of disjoint intervals $\{(a_j, b_j)\}_1^N$,

  $$\sum_1^N (b_j - a_j) < \delta \implies \sum_1^N |F(b_j) - F(a_j)| < \epsilon$$

- **Thm 3.35** (The Fundamental Theorem of Calculus for Lebesgue Integrals) If $-\infty < a < b < \infty$ and $F : [a,b] \to \mathbb{C}$, the following are equivalent:

  (a) F is absolutely continuous on $[a,b]$.

  (b) $F(x) - F(a) = \int_a^x f(t)dt$ for some $f \in L^1([a,b],m)$,

  (c) F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b],m)$, and $F(x)-F(a) = \int_a^x F'(t)dt$.

Point Set Topology

- **Def.** (topology) A topology on X is a family $\mathcal{T}$ of subsets of X that contains $\varnothing$ and X and is closed under arbitrary unions and finite intersections.

- **Def.** (neighborhood base) If $\mathcal{T}$ is a topology on X, a neighborhood base for $\mathcal{T}$ at $x \in X$ is a family $\mathcal{N} \subseteq \mathcal{T}$ such that

  (i) $x \in V$ for all $V \in \mathcal{N}$

  (ii) If $U \in \mathcal{T}$ and $x \in U$, there exists $V \in \mathcal{N}$ such that $x \in V$ and $V \subseteq U$.

  A base for $\mathcal{T}$ is a family $\mathcal{B} \subseteq \mathcal{T}$ that contains a neighborhood base for $\mathcal{T}$ at each $x \in X$.

- **Def.** (first and second countable) A topological space $(X, \mathcal{T})$ is first countable if there is a countable neighborhood base for $\mathcal{T}$ at every point of X. The space is second countable if $\mathcal{T}$ has a countable base.

- **Def.** (separable space) $(X, \mathcal{T})$ is separable if X has a countable dense subset. Every second countable space is separable.

  Def. (Hausdorff space) A space is called Hausdorff if for all $x, y \in X$, $x \neq y$, there are disjoint open sets U, V with $x \in U$ and $y \in V$.

- **Def.** (weak topology) The weak topology of a topological space $(X, \mathcal{T})$ is the weakest topology (the one with the least open sets) under which every element of X^* is continuous on X.

- **Def.** (weak* topology) The weak* topology is the weakest topology on X^* such that the maps, $T_x(\phi) = \phi(x)$ is continuous on X^* for any $x \in X$. Convergence in the weak* topology is essentially pointwise convergence. That is $f_n \to f$ iff $f_n(x) \to f(x)$ for all $x \in X$.

- **Def.** (nets) To develop a generalization of sequences that work well in arbitrary topological spaces, begin by defining a type of indexed set called a directed set, which is a set $A(\neq \varnothing)$ equipped with a binary relation $\lesssim$ such that

  (i) $\alpha \lesssim \alpha$ for all $\alpha \in A$.

  (ii) if $\alpha \lesssim \beta$ and $\beta \lesssim \gamma$ then $\alpha \lesssim \gamma$.

  (iii) for any $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha \lesssim \gamma$ and $\beta \lesssim \gamma$.

  A net in a set X is a mapping $\alpha \mapsto x_\alpha$ from a directed set A into X. Denote such a mapping by $\langle x_\alpha \rangle_{\alpha \in A}$. Let X be a topological space and E a subset of X. A net $\langle x_\alpha \rangle_{\alpha \in A}$ is eventually in E if there exists $\alpha_0 \in A$ such that $x_\alpha \in E$ for $\alpha \gtrsim \alpha_0$. A point $x \in X$ is a limit of $\langle x_\alpha \rangle$ if for every neighborhood U of x, $\langle x_\alpha \rangle$ is eventually in U.

- **Def.** (local compactness) A topological space is locally compact if every $x \in X$ has a neighborhood whose closure is compact.

- **Def.** We call locally compact Hausdorff spaces LCH spaces for short.

- **Def.** The support of a complex-valued function $f : X \to \mathbb{C}$ is defined as

  $$\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$$

  then define the following spaces:

  1. $C(X) = \{f : X \to \mathbb{C} \text{ is continuous}\}$
  2. $BC(X) = \{f \in C(X) : f \text{ bounded}\}$
  3. $C_c(X) = \{f \in C(X) : \text{supp}(f) \text{ compact}\}$
  4. $C_0(X) = \{f \in C(X) : f \text{ vanishes at } \infty\}$

  It may be shown that

  $$C_c(X) \subset C_0(X) \subset BC(X) \subset C(X)$$

- **Lemma 4.32** (**Urysohn's Lemma**) If X is an LCH space and $K \subseteq U \subseteq X$ where K is compact and U is open, there exists $f \in C(X, [0,1])$ such that $f = 1$ on K and $f = 0$ outside a compact subset of U.

- **Prop 4.35** If X is an LCH space, then $C_0(X) = \overline{C_c(X)}$ in $\| \cdot \|_\infty$.

Elements of Functional Analysis

- **Def.** (Banach space) A normed vector space that is complete w.r.t. the norm metric is called a Banach space.

- **Def.** (bounded linear map) A linear map $T : X \to Y$ between two normed vector spaces is called bounded if there exists $C \geq 0$ such that $\|T(x)\|_Y \leq C\|x\|_X$ for all $x \in X$. If T is linear then continuity on X and boundedness on X are equivalent.

- **Def.** (operator norm) Let $L(X, Y)$ be the space of bounded linear maps from $X \to Y$. Then $L(X, Y)$ is a vector space and the function $T \mapsto \|T\|$ is defined by

$$\|T\| = \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|T(x)\|_Y$$

- **Def.** (isometry) If $T \in L(X, Y)$, T is called an isometry if $\|T(x)\|_Y = \|x\|_X$. An isometry is injective but not necessarily surjective. It is, however an isomorphism onto its range (i.e. bijective and T^{-1} is bounded).

- **Def.** (dual space) If X is a vector space over $\mathbb{C}$, then a linear map from $X \to \mathbb{C}$ is called a linear functional. If X is a normed vector space then the space $L(X, \mathbb{C})$ of bounded linear functionals on X is called the dual space of X and is denoted by X^*. X^* is a Banach space with its operator norm.

- **Thm 5.6** (**The Hahn-Banach Theorem**) Let X be a real vector space, ρ a sublinear functional on X, $\mathcal{M}$ a subspace of X, and f a complex linear functional on $\mathcal{M}$ such that $|f(x)| \leq \rho(x)$ for all $x \in \mathcal{M}$. Then there exists a complex linear functional F on X such that $|F(x)| \leq \rho(x)$ for all $x \in X$ and $F_{\mathcal{M}} = f$.

- **Thm 5.8** (Consequences of the Hahn-Banach Thm) Let X be a normed vector space.

  (a) If $\mathcal{M}$ is a closed subspace of X and $x \in X \backslash \mathcal{M}$, there exists $f \in X^*$ such that $f(x) \neq 0$ and $f|_{\mathcal{M}} = 0$.

  (b) If $x \neq 0 \in X$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

  (c) The bounded linear functions on X separate points.

  (d) If $x \in X$, define $\hat{x} : X^* \to \mathbb{C}$ be $\hat{x}(f) = f(x)$. Then the map $x \mapsto \hat{x}$ is a linear isometry from X into X^{**} (the dual of X^*).

- **Thm 5.9** (**The Baire Category Theorem**) Let X be a complete metric space.

  (a) If $(U_n)_1^\infty$ is a sequence of open dense subsets of X, then $\bigcap_1^\infty U_n$ is dense in X

  (b) X is not a countable union of nowhere dense sets, i.e. not meager.

- **Def.** (meager set) If X is a topological space, a set $E \subseteq X$ is called meager if E is a countable union of nowhere dense sets. A set is called nowhere dense if its closure has empty interior (i.e. no point in it can be contained in an open ball that's contained in the set). Otherwise, E is called residual. Intuitively, nowhere dense sets are naturally very small, so a meager set still has a sense of smallness, but has nicer properties than nowhere dense sets. (σ-ideal).

- **Thm 5.10** (**The Open Mapping Theorem**) Let X, Y both be Banach spaces. If $T \in L(X, Y)$ is surjective, then T is open, i.e. that $T(U)$ is open in Y whenever U is open in X.

- **Thm 5.12** (**The Closed Graph Theorem**) If X, Y are normed vector spaces and T is a linear map from $X \to Y$, define the graph of T to be $\Gamma(T) = \{(x, y) \in X \times Y : y = T(x)\}$. Then T is closed if $\Gamma(T)$ is a closed subspace of $X \times Y$.

  If X, Y are Banach spaces and $T : X \to Y$ is a closed linear map, then T is bounded.

- **Thm 5.13** (**The Uniform Boundedness Principle**) Suppose that X, Y are normed vector spaces and A is a subset of $L(X, Y)$.

  (a) If $\sup_{T \in A} \|T(x)\|_Y < \infty$ for all x in some nonmeager subset of X, then $\sup_{T \in A} \|T\| < \infty$

  (b) If X is a Banach space and $\sup_{T \in A} \|T(x)\|_Y$ is finite for all $x \in X$, then $\sup_{T \in A} \|T\| < \infty$.

- **Def.** (weak convergence) Let X be a normed vector space. A net $\langle x_\alpha \rangle_{\alpha \in A}$ is said to converge weakly to $x \in X$ iff $f(x_\alpha) \to f(x)$ for all $f \in X^*$.

Hilbert Spaces

- **Def.** (Hilbert Space) Let $\mathcal{H}$ be a complex vector space. An inner product on $\mathcal{H}$ is a map $(x, y) \mapsto \langle x, y \rangle$ from $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that

  (i) $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$ for all $x, y, z \in \mathcal{H}$, and $a, b \in \mathbb{C}$.

  (ii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$

  (iii) $\langle x, x \rangle \in (0, \infty)$ for all nonzero $x \in \mathcal{H}$.

  $\langle \cdot, \cdot \rangle$ induces a norm $\|x\| = \sqrt{\langle x, x \rangle}$ on $\mathcal{H}$ and if $\mathcal{H}$ is complete w.r.t $\| \cdot \|$ then we say $\mathcal{H}$ is a Hilbert space, a special kind of Banach space which generalizes finite Euclidean spaces. Structurally, every Hilbert space looks like some ℓ^2 space (prop 5.30).

- **Thm 5.19** (**The Schwarz Inequality**)

  $| \langle x, y \rangle | \leq \|x\| \|y\|$ for all $x, y \in \mathcal{H}$ with equality iff x, y are linearly independent.

- **Thm 5.22** (The Parallelogram Law) For all $x, y \in \mathcal{H}$, $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

- **Thm 5.23** (The Pythagorean Theorem)

  If $(x_j)_1^n \subset \mathcal{H}$ and $x_j \perp x_k$ for $j \neq k$, then

$$\| \sum_1^n x_j \|^2 = \sum_1^n \|x_j\|^2$$

- $L^2(X, \mu)$ is a Hilbert space with inner product $\langle f, g \rangle = \int f \overline{g} d\mu$. An important special case of this is obtained by taking μ to be counting measure on $(X, \mathcal{P}(X))$. Here we denote $L^2(X, \mu)$ be $\ell^2(X, \mu)$ the set of functions $f : X \to \mathbb{C}$ such that $\sum_{x \in X} |f(x)|^2 < \infty$.

- **Thm 5.24** If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$; that is , each $x \in \mathcal{H}$ can be uniquely expressed as $x = y + z$ where $y \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$. Moreover, y, z are the unique elements of $\mathcal{M}, \mathcal{M}^\perp$ whose distance to x is minimal. Note $\mathcal{M}^\perp$ is called the orthogonal complement of $\mathcal{M}$.

- **Thm 5.25** (**Riesz Representation Theorem for Hilbert Spaces**) If $f \in \mathcal{H}^*$, there is a unique $y \in \mathcal{H}$ such that $f(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$.

- **Thm 5.26** (Bessel's Inequality) If $\{u_\alpha\}_{\alpha \in A}$ is an orthonormal set in $\mathcal{H}$, then for any $x \in \mathcal{H}$, $\sum_{\alpha \in A} | \langle x, u_\alpha \rangle |^2 \leq \|x\|^2$. In particular, the set $\{\alpha : | \langle x, u_\alpha \rangle |^2 \neq 0\}$ is countable.

- **Thm 5.27** (**Parseval's Identity**) If $\{u_\alpha\}_{\alpha \in A}$ is an orthonormal set in $\mathcal{H}$, the following are equivalent:

  (a) If $\langle x, u_\alpha \rangle = 0$ for all $\alpha \in A$, then $x = 0$.

  (b) (Parseval's) $\|x\|^2 = \sum_{\alpha \in A} | \langle x, u_\alpha \rangle |^2$ for all $x \in \mathcal{H}$

  (c) For each $x \in \mathcal{H}$, $x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$, which converges.

- (Some bounded linear operators) Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces.

  1. A unitary map is an invertible (inverse is bounded) map $U : \mathcal{H}_1 \to \mathcal{H}_2$ that preserves inner product. Unitary maps are the true isomorphisms in the category of Hilbert spaces.
  2. Let $\mathcal{H}$ be a Hilbert space and $T \in L(\mathcal{H}, \mathcal{H})$. Then there is a unique $T^* \in L(\mathcal{H}, \mathcal{H})$ called the adjoint of T, such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in \mathcal{H}$. Note T is unitary iff T is invertible and $T^{-1} = T^*$.
  3. Let $\mathcal{M} \subseteq \mathcal{H}$ be a closed subspace of $\mathcal{H}$ and for $x \in \mathcal{H}$, define $P(x)$ to be the element of $\mathcal{M}$ such that $x - P(x) \in M^\perp$. If defined so, $P \in L(\mathcal{H}, \mathcal{H})$ and $P^* = P$, $P^2 = P$, range$(P) = \mathcal{M}$ and ker$(P) = \mathcal{M}^\perp$. P is called the orthogonal projection onto $\mathcal{M}$.

L^p Spaces

- **Def.** (L^p space) We define L^p space by the set of measurable functions $f : X \to \mathbb{C}$ such that $\|f\|_p < \infty$ where

$$\|f\|_p = \left[\int_X |f|^p d\mu \right]^{1/p}$$

If our measure is the counting measure on X then we usually denote L^p space by ℓ^p.

- Two real numbers $p > 1$ and $q > 1$ are called conjugate exponents if $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 1$. then we generally say $q = \infty$ (for norms).

- (**Young's Inequality**) If a, b are nonnegative real numbres and if p, q are conjugate exponents, then
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$
where equality holds iff $a^p = b^q$.

- **Thm 6.2** (**Holder's Inequality**) Suppose p, q are conjugate exponents. If f, g are measurable functions on X, then
$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$
In particular, if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and in this case equality holds above iff $\alpha|f|^p = \beta|g|^q$ a.e. for some α, β not both zero.

- **Thm 6.5** (**Minkowski's Inequality**) If $1 \leq p < \infty$ and $f, g \in L^p$, then
$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

- **Thm 6.6** For every finite p, L^p is a Banach space.

- **Prop 6.7** For finite p, the set of simple functions $f = \sum_1^n a_j \chi_{E_j}$, where $\mu(E_j) < \infty$ for all j, is dense in L^p.

- **Thm 6.8cde**
  - (c) $\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.
  - (d) L^∞ is a Banach space.
  - (e) The simple function are dense in L^∞.

- **Prop 6.10** (Interpolation) If $0 < p < q < r \leq \infty$, then
$$L^p \cap L^r \subseteq L^q \text{ and } \|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-r}$$
where $\lambda \in (0, 1)$ is defined by
$$\lambda = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}$$

- **Prop 6.12** (Relationship between L^p spaces) If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p(\mu) \supseteq L^q(\mu)$ and
$$\|f\|_p \leq \|f\|_q \mu(X)^{(1/p)-(1/q)}$$

- The most important L^p spaces are L^1 for integrability, L^2 because it is a Hilbert space, and L^∞ because its topology is closely related to that of uniform convergence.

- **Thm 6.15** (**Representation of** $(L^p)^*$) Let p, q be conjugate exponents. If $1 < p < \infty$, then for each $\phi \in (L^p)^*$ there exists $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$, and hence L^q is isometrically isomorphic to $(L^p)^*$. The same conclusion holds for $p = 1$ provided μ is σ-finite.

Radon Measures

- **Def.** (regular measure) If μ is a Borel measure on X and E a Borel subset of X. The measure μ is called outer regular on E if
$$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}$$
and inner regular on E if
$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}$$
μ is called regular if μ is both outer and inner regular.

- If $f \in C_c(X)$ with $0 \leq f \leq 1$ for all $x \in X$, we write
  1. $K \prec f$ if $f(x) = 1$ for all $x \in K$ where K is compact.
  2. $f \prec V$ if $\text{supp}(f) \subseteq V$ where V is open.

- **Def.** A Borel measure on X is called a Radon measure if
  - (i) $\mu(K) < \infty$ for K compact.
  - (ii) μ is outer regular for all Borel sets E.
  - (iii) μ is inner regular for all open sets E or σ-finite E.
  - (iv) μ is complete.

- **Thm 7.2** (**The Riesz Representation Theorem for positive linear functions**) If I is a positive linear functional on $C_c(X)$, there is a unique Radon measure μ on X such that $I(f) = \int f d\mu$ for all $f \in C_c(X)$. Thus, there is a 1-1 correspondence between the set of positive linear functions on $C_c(X)$ and the set of Radon measures on X.

- **Prop 7.9** If μ is a Radon measure on X, $C_c(X)$ is dense in $L^p(\mu)$ for $1 \le p < \infty$.

- Due to the representation theorem, we have 2 ways to determine any Radon measure μ on X:

  1. Either normally by $\mu(E) = \int \chi_E d\mu$, for $E \in \mathcal{M}$
  2. or $\mu(E) = \int_X f d\mu$ for the correct $f \in C_c(X)$.

  The reason is that one can approximate χ_E by $f \in C_c(X)$ when E is nice.

- **Lemma 7.15** If $I \in (C_0(X, \mathbb{R}))^*$, there exists positive functions $I^\pm \in (C_0(X, \mathbb{R}))^*$ such that $I = I^+ - I^-$. This is a "Jordan decomposition" for real linear functionals on $C_0(X, \mathbb{R})$.

- **Thm 7.17** (**The Riesz Representation Theorem for** $(C_0(X))^*$) Let X be a LCH space, and for $\mu \in M(X)$ the space of complex Radon measures on X, and $f \in C_0(X)$ let $I_\mu(f) = \int f d\mu$. Then the map $\mu \mapsto I_\mu$ is an isometric isomorphism from $M(X) \to (C_0(X))^*$.

- **Cor 7.18** If X is a compact Hausdorff space, then $(C(X))^*$ is isometrically isomorphic to $M(X)$.

Elements of Fourier Analysis

- $C^\infty(\mathbb{R}^n)$ is the set of infinitely continuously differentiable functions on $\mathbb{R}^n$.

- **Def.** (multi-index notation) We first abbreviate partial derivatives by $\partial_j := \frac{\partial}{\partial x_j}$ in $\mathbb{R}^n$. Now for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, we set

$$|\alpha| = \sum_1^n \alpha_j,$$

  and

$$X^\alpha \partial^\beta = \left(\prod_1^n \alpha_j \right) \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$$

- One useful C^∞ space is C_c^∞, the space of compactly supported C^∞ functions. One nontrivial example in this space is

$$\psi(x) = \begin{cases} e^{\frac{1}{|x|^2-1}}, & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}$$

- **Def.** (locally convex space and Frechet space) Recall a seminorm is a norm that isn't positive definite (i.e. $\rho(x) = 0$ iff $x = 0$). A family of seminorms $\{\rho_\alpha\}_{\alpha \in A}$ is said to separate points if $\rho_\alpha(x) = 0$ for all $\alpha \in A$ iff $x = 0$.

  A locally convex space is a vector space X with a family of seminorms that separate points. The natural topology on such a space is the weakest topology in which all ρ_α and addition are continuous. This topology may be generated by the set of all open balls w.r.t to each seminorm.

  A locally convect space that is defined by a countable family of seminorms and is complete is called a Frechet space.

- **Def.** (Schwartz space) Schwartz space, $\mathcal{S}$, consists of C^∞ functions which, together with their derivatives, vanish at infinity faster that any power of $|x|$. That is, for any $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ we define

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

  then

$$\mathcal{S} = \{f \in C^\infty : \|f\|_{(N,\alpha)} < \infty \text{ for all } N, \alpha\}$$

  It is important to note that if $f \in \mathcal{S}$, then $\partial^\alpha f \in L^p$ for all α and all $p \in [1, \infty]$.

- **Prop 8.3** If $f \in C^\infty$, then $f \in \mathcal{S}$ iff $x^\beta \partial^\alpha f$ is bounded for all multi-indices $\alpha, \beta \in \mathbb{R}^n$ iff $\partial^\alpha (x^\beta f)$ is bounded for all multi-indices $\alpha, \beta \in \mathbb{R}^n$. This is a very useful alternative definition for Schwartz functions.

- **Prop 8.2** $\mathcal{S}$ is a Frechet space with the topology defined by the seminorms $\| \cdot \|_{(N,\alpha)}$

- **Def.** (convolution) Let f, g be measurable functions on $\mathbb{R}^n$. The convolution of f and g is the function $f * g$ defined by

$$(f * g)(x) = \int f(x - y)g(y)dy$$

for all x such that the integral exists.

- **Prop 8.6** Assuming that all integrals in question exist, we have
  (a) $f * g = g * f$
  (b) $(f * g) * h = f * (g * h)$
  (c) For $z \in \mathbb{R}^n$, $\tau_z(f*g) = (\tau_z f)*g = f*(\tau_z g)$ where $\tau_z(f) = f(x - z)$ for all $x \in \mathbb{R}^n$.
  (d) If A is the closure of $\{x+y : x \in \text{supp}(f), y \in \text{supp}(g)\}$, then $\text{supp}(f * g) \subseteq A$.

- **Prop 8.9** (**Young's Inequality**) Suppose $1 \leq p, q, r \leq \infty$ and $p^{-1} + q^{-1} = r^{-1} + 1$. Then if $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and $\|f * g\|_r \leq \|f\|_p \|g\|_q$.

- **Thm 8.15** (Approximate identities) For a function ϕ on $\mathbb{R}^n$ and $t > 0$ we define

$$\phi_t(x) = t^{-n}\phi(t^{-n}x)$$

If $\phi \in L^1$ and $\int \phi(x)dx = a$ then
  (a) If $f \in L^p$ $(1 \leq p < \infty)$, then $f * \phi_t \to af$ in the L^p norm as $t \to 0$.
  (b) If f is bounded and uniformly continuous, then $f * \phi_t \to af$ uniformly as $t \to 0$.
  (c) If $f \in L^\infty$ and f is continuous on an open set U, then $f*\phi_t \to af$ uniformly on compact subsets of U as $t \to 0$.

- **Prop 8.17** C_c^∞ (and hence also $\mathcal{S}$) is dense in L^p $(1 \leq p < \infty)$ and in C_0.

- **Thm 8.20** Let $E_k(x) = e^{2\pi ikx}$, then $\{E_k : k \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$. It is also dense in $C(\mathbb{T}^n)$ which is dense in $L^2(\mathbb{T}^n)$

- **Def.** (Fourier transform on $L^2(\mathbb{T}^n)$) If $f \in L^2(\mathbb{T}^n)$, we define its Fourier transform $\hat{f}$, a function on $\mathbb{Z}^n$, by

$$\mathcal{F}(f)(k) = \hat{f}(k) = \langle f, E_k \rangle = \int f(x)e^{-2\pi ikx}dx$$

and we call the series

$$\sum_{k \in \mathbb{Z}^n} \hat{f}(k)E_k$$

the Fourier series of f. The Fourier transform maps $L^2(\mathbb{T}^n)$ onto $\ell^2(\mathbb{Z}^n)$ with $\|\hat{f}\|_2 = \|f\|_2$, and that the Fourier series of f converges to f in the L^2 norm.

- **Thm 8.21** (**The Hausdorff-Young Inequality**) Suppose that $1 \leq p \leq 2$ and q is the conjugate exponent of p. If $f \in L^p(\mathbb{T}^n)$, then $\hat{f} \in \ell^q(\mathbb{Z}^n)$ and $\|\hat{f}\|_q \leq \|f\|_p$.

- **Def.** (Fourier transform on $L^1(\mathbb{R}^n)$) Let $f \in L^1$. Then

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int f(x)e^{-2\pi i\xi x}dx$$

- **Thm 8.22** (Elementary properties of the Fourier transform) Suppose $f, g \in L^1(\mathbb{R}^n)$.
  (a) $\widehat{(\tau_y f)}(t) = e^{-2\pi ity}\hat{f}(t)$ and $\tau_y(\hat{f}) = \hat{h}$ where $h(x) = e^{2\pi iyx}f(x)$.
  (b) If T is an invertible linear tranformation of $\mathbb{R}^n$ and $S = (T^*)^{-1}$ is its inverse transpose, then $\widehat{(f \circ T)} = \hat{f} \circ T$; and if $T(x) = y^{-1}x$, $(y > 0)$, then $\widehat{(f \circ T)}(t) = y^n\hat{f}(yt)$, so that $\widehat{(f_y)}(t) = \hat{f}(yt)$ where $f_y(t) = y^{-n}f(y^{-1}t)$
  (c) $\widehat{(f * g)} = \hat{f}\hat{g}$.
  (d) If $x^\alpha f \in L^1$ for $|\alpha| \leq k$ then $\hat{f} \in C^k$ and $\partial^\alpha \hat{f} = \widehat{[(-2\pi ix)^\alpha f]}$.
  (e) If $f \in C^k$, $\partial^\alpha f \in L^1$ for $|\alpha| \leq k$, and $\delta^\alpha f \in C_0$ for $|\alpha| \leq k-1$, then $\widehat{(\partial^\alpha f)}(t) = (2\pi it)^\alpha \hat{f}(t)$
  (f) (**The Riemann-Lebesgue Lemma**) $\mathcal{F}(L^1(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$.

- **Thm 8.26** (**The Fourier Inversion Theorem**) If $f \in L^1$, we define

$$\mathcal{F}^{-1}(f)(x) = \hat{f}(-x) = \int f(\xi)e^{2\pi i\xi x}d\xi.$$

if $\hat{f} \in L^1$ as well, then f agrees almost everywhere with a continuous function f_0, and $\mathcal{F}^{-1}(\hat{f}) = \hat{(}\mathcal{F}^{-1}(f)) = f_0$.

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