Advanced Analytics of Finance

Homework 3

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1) a) $Y|X = \beta_0 + \beta_1 X + \varepsilon^{\sim N(0,\sigma^2)}$ — ε indep of X & across observations

$P(Y/X = x_i : \beta_0, \beta_1, \sigma^2) \to$ likelihood

$$\prod_{i=1}^{n} p(y_i / x_i, \beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i - (\beta_0 + \beta_1 x_i))^2}{2\sigma^2}}$$

y_i are independent

$$L(\beta_0, \beta_1, \sigma^2) = -\frac{n}{2}\log(2\pi) - n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - (\beta_0 + \beta_1 x_i))^2$$

$$\frac{\partial}{\partial\beta_0} = \frac{1}{2\sigma^2} 2 \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i)) = $$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i)) = 0 \quad (1)$$

$$\frac{\partial}{\partial\beta_1} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i)) x_i = 0 \quad (2)$$

From (1) we have

$$\sum_{i=1}^{n} y_i - \underbrace{\sum_{i=1}^{n} \beta_0}_{n\beta_0} + \beta_1 \sum_{i=1}^{n} x_i = 0$$

$$\beta_0 = \bar{y} - \beta_1 \bar{x}$$

From (2) $\rightarrow$ $\sum_{i=1}^{n} y_i x_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$

(1) & (2)

$$\sum_{i=1}^{n} y_i x_i - [\bar{y} - \beta_1 \bar{x}] \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$\sum_{i=1}^{n} y_i x_i - \bar{y} \sum_{i=1}^{n} x_i + \beta_1 \left(\bar{x} \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i^2 \right) = 0$$

$$\beta_1 = \frac{\sum_{i=1}^{n} (x_i y_i - \bar{y} x_i)}{\sum_{i=1}^{n} (x_i^2 - \bar{x} x_i)}$$

since $\sum_{i=1}^{n} (\bar{x}^2 - x_i \bar{x}) = n\bar{x}^2 - \bar{x} \sum x_i = \frac{1}{n} \sum x_i^2 - \frac{1}{n} \sum x_i^2 = 0$

& $\sum_{i=1}^{n} (\bar{x}\bar{y} - y_i \bar{x}) = n\bar{x}\bar{y} - \bar{x} \sum y_i = \frac{1}{n} \sum x_i y_i - \frac{1}{n} \sum x_i y_i = 0$

we can write

$$\beta_1 = \frac{\sum (x_i y_i - \bar{y} x_i) + \sum (\bar{y}\bar{x} - y_i \bar{x})}{\sum (x_i^2 - x_i \bar{x}) + \sum (\bar{x}^2 - x_i \bar{x})} =$$

$$= \frac{1/n \sum (x_i - \bar{x})(y_i - \bar{y})}{1/n \sum (x_i - \bar{x})^2} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \beta_1$$

MAP estimator

let w denote our coefficients $\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}$ and hence

$$Y = w^T x + \varepsilon^{\sim N(0,\sigma^2)}$$

assume that $w \sim N(0, \lambda^{-1} I)$

log posterior ~~d~~istribution

$$\log P(Y|X,w) = N\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_k - x^T w)^2$$

$$+ \log \lambda^{\frac{N}{2}} - \log(2\pi)^{\frac{N}{2}} - \frac{\lambda}{2} w^T w$$

$$\hat{w} = \underset{w}{\operatorname{argmax}}\left(-\frac{1}{2\sigma^2}\sum_{k=1}^{N}(y_k - x^T w)^2 - \frac{\lambda}{2} w^T w\right)$$

$$= \underset{w}{\operatorname{argmin}}\left(\frac{1}{2\sigma^2}\sum_{k=1}^{N}(y_k - x^T w)^2 - \frac{\lambda}{2} w^T w\right)$$

$$\frac{\partial}{\partial w} = \frac{1}{\sigma^2}\sum x(y_k - x^T w) - \lambda w = 0$$

for $\sigma = 1$ we have

$$\sum_{k=1}^{N} x y_k - \sum_{k=1}^{N} x x^T w - \lambda w = 0$$

$$X^T Y - X^T X W - \lambda W = 0$$

$$W(X^T X + \lambda) = X^T Y$$

$$W = (X^T X + \lambda)^{-1} X^T Y$$

b) Rolls critique states that the true market portfolio includes a large range of investment opportunities including international securities, real estate, precious metals etc. Hence, the true market portfolio is empirically impossible. When we use a proxy for the market index (like S&P 500) which is different from the true market return, our beta estimates suffer from an attenuation bias, which means that the estimated risk premia are biased towards zero.

c) $y \sim a + bx$ & $x \sim c + dy$

$$\hat{b} = \frac{cov(y,x)}{var(x)} \qquad \hat{d} = \frac{cov(x,y)}{var(y)}$$

$$\hat{b}\hat{d} = \frac{cov(x,y)}{\sigma_x \sigma_x} \times \frac{cov(x,y)}{\sigma_y \sigma_y} = \frac{cov(x,y)}{\sigma_x \sigma_y} \times \frac{cov(x,y)}{\sigma_x \sigma_y}$$

$$= (\rho_{x,y})^2$$

since $\rho_{xy} \in [0,1]$

$\Rightarrow \rho_{xy}^2 \in [0,1]$

$\Rightarrow 0 \leq \hat{b}\hat{d} \leq 1$

(2) N portfolio managers $\quad t = 1, \ldots, T$ periods of time

a) ~~$H_0: \frac{1}{T}\sum_{t=1}^{T}\mu_t^1 = \frac{1}{T}\sum_{t=1}^{T}\mu_t^2 = \ldots = \frac{1}{T}\sum_{t=1}^{T}\mu_t^N$~~

$$H_0 : \mu_1 = \mu_2 \ldots = \mu_N$$

$$\hat{\mu} = \begin{pmatrix} \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_N \end{pmatrix}_{N \times 1}$$

X_t^i iid for $i = 1 \ldots N$

for $(X_t^1, \ldots X_t^N) \neq 0$

$$\hat{\mu}_1 = \frac{1}{T}\sum_{t=1}^{T} X_t^1 \quad \text{for all } N$$

$$E(X_t^i) = \mu_i$$

b) $$f(x_t, \mu) = \hat{\mu} - \mu = \begin{pmatrix} \frac{1}{T}\sum_{t=1}^{T} X_t^1 - \mu_1 \\ \vdots \\ \frac{1}{T}\sum_{t=1}^{T} X_t^N - \mu_N \end{pmatrix}_{N \times 1}$$

- the asymptotic distribution of GMM

$$\hat{\theta}_{GMM} \overset{a}{\sim} N\Big(\theta, \frac{1}{N}(\underset{N\times N}{\hat{d}'}\,\underset{N\times N}{\hat{S}^{-1}}\,\underset{N\times N}{\hat{d}})^{-1}\Big) = N(\mu, \frac{1}{n}\Omega)$$

where $\hat{d} = \left.\frac{\partial \hat{E}(f(x_i, \theta))}{\partial \theta'}\right|_{\hat{\theta}}$ & $\hat{S} = \hat{E}[f(x_i, \hat{\theta}) f(x_i, \hat{\theta})']$

$$\hat{d} = \left.\frac{\partial E(f(x_t, \mu))}{\partial \mu}\right|_{\hat{\mu}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & \cdots \\ 0 & \cdots & -1 \end{pmatrix}_{N \times N} = -I_N$$

Hence $\Omega = (\hat{d}' \hat{S}^{-1} \hat{d})^{-1} = \hat{S}$

where $\hat{S}^*$ is the covariance matrix

$$\hat{S} = E[(\hat{\mu} - E(\hat{\mu}))(\hat{\mu} - E(\hat{\mu}))^T]$$

lets take the ij

$$\hat{S}_{ij} = E\left[\left(\frac{1}{T}\sum_{t=1}^{T} x_t^{(i)} - E\left[\frac{1}{T}\sum_{t=1}^{T} x_t^{(i)}\right]\right)\right.$$

$$\left.\left(\frac{1}{T}\sum_{t=1}^{T} x_t^{(j)} - E\left[\frac{1}{T}\sum_{t=1}^{T} x_t^{(j)}\right]\right)\right] =$$

$$= \frac{1}{T^2} E\left[\left(\sum_{t=1}^{T} (x_t^i - E(x_t^i)\right)\left(\sum_{t=1}^{T} (x_t^j - E(x_t^j)\right)\right]$$

c) We have that $E[\hat{\mu}_{N\times 1}] = E[\frac{1}{T}\sum_{t=1}^{T} x_t]$

$Var(\hat{\mu}) = \hat{S} = \Omega$ as shown above $= E[\frac{1}{T} \times T\mu_t] = \mu$

$$= \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}$$

hence with GMM o

we have

$$\hat{\mu} \overset{a}{\sim} N(\mu, \frac{1}{N}\Omega) -$$

multivariate normal

d) $\hat{\delta}_k = \hat{\mu}_k - \hat{\mu}_1 \quad k = 2 \dots N$

$$\hat{\delta} = \begin{pmatrix} \hat{\delta}_2 \\ \vdots \\ \hat{\delta}_N \end{pmatrix}_{(N-1)\times 1} = \begin{pmatrix} \hat{\mu}_2 - \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_N - \hat{\mu}_1 \end{pmatrix}$$

~~under~~ under $H_0 : \mu_1 = \dots \mu_N$

hence $E(\hat{\delta}) = E\begin{pmatrix} \hat{\mu}_2 - \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_N - \hat{\mu}_1 \end{pmatrix} = \begin{pmatrix} E(\hat{\mu}_2 - \hat{\mu}_1) \\ \vdots \\ E(\hat{\mu}_N - \hat{\mu}_1) \end{pmatrix} =$

0 we have that $\hat{\delta} \overset{a}{\sim} N(0, \frac{1}{N-1} V)$

where V is the var-covar matrix

$V = E[(\hat{\mu}_k - \hat{\mu}_1 - E[\hat{\mu}_k - \hat{\mu}_1]] \dots$

$$V_{ij} = E\left[\left(\frac{1}{T}\sum x_t^i - \frac{1}{T}\sum x_t^j\right) - E\left(\frac{1}{T}\sum(x_t^i - x_t^j)\right)\right]$$

$$= E\left[\left(\frac{1}{T}\sum(x_t^i - x_t^j) - \frac{1}{T}\sum E(x_t^i - x_t^j)\right]\right.$$

$$= \left[\frac{1}{T}\sum_{t=1}^{T}(x_t^j - x_t^1) - \frac{1}{T}\sum_{t=1}^{T} E(x_t^j - x_t^i)\right]$$

e) $W = \hat{\delta}' V^{-1} \hat{\delta}$ 	V - is positive semi-definite matrix → invertible

if $Z \sim N(0,1)$

$Q = \sum_{i=1}^{K} Z_i \Rightarrow Q \sim \chi^2(K)$

if we normalize by dividing by σ

we would have $\sim \frac{\chi^2}{\sigma^2}$

since we have $N-1$ $\hat{\delta}$s

$dF = N-1$ 	$\hat{\delta}' V^{-1} \hat{\delta} \sim \chi^2(N-1)$

f) reject if $W_{stat} > \chi^2(N-1)_{0,95}$ with size $\alpha = 0,05$

g) we would have var-covar matrix.

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \cdots \\ 0 & \ddots & \\ \vdots & & \sigma_n^2 \end{bmatrix} \Rightarrow W \text{ statistic}$$

since instead of covariances we would have 0s hence Σ will be lower, which would lead the W-stat to be higher, hence more prone to reject, making our test statistic biased towards rejecting.

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Problem Set 3
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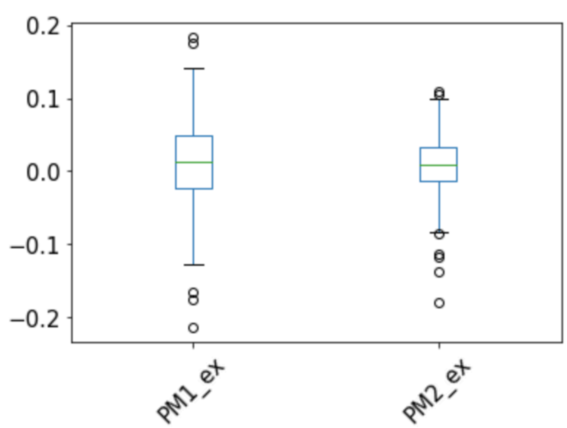
Question 3

Part A

Below are the scatterplots of managers' monthly excess returns against the market excess returns:

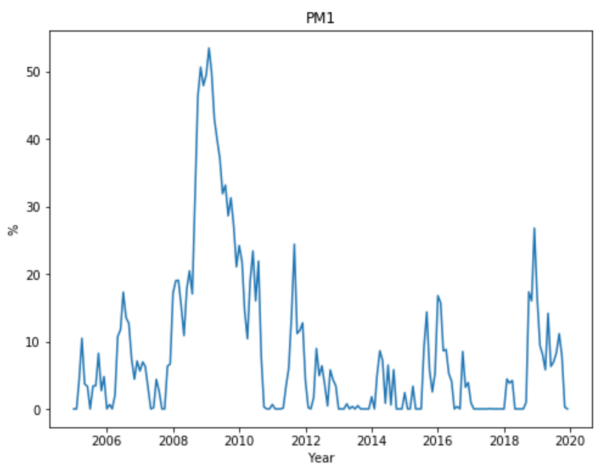
As we can see, both managers' excess returns are positively correlated with market excess returns, however for the second manager, the excess monthly returns are more concentrated towards being higher compared with the first manager.

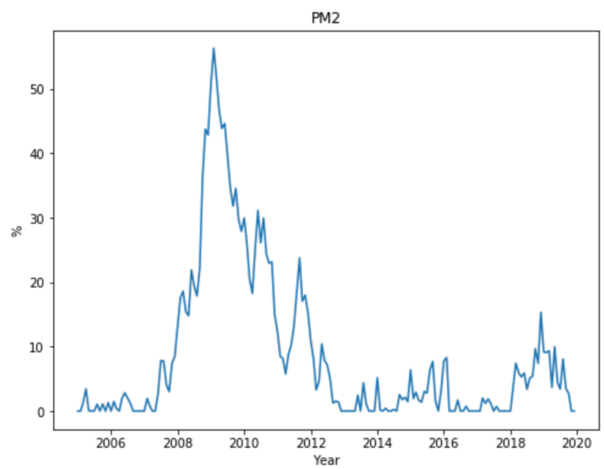
To explore even further, we plot the boxplots of managers' monthly returns:



We can observe, that the median returns of both managers are very close to each other and there are some outlier excess returns for both of them, however for the second manager the outliers seem to be more on the negative side. In general, we can deal with these outliers by winsorizing them, in other words those excess returns in that are higher in magnitude than median ± 3.5*Interquartile Range would be change to equal that amount.

	Mean, %	St. Dev, %	Sharpe Ratio	Information Ratio	Maximum Drawdown
Manager 1	12.309	21.088	0.584	0.377	53.489
Manager 2	8.553	15.398	0.555	-0.006	56.322





The table above reports annualized mean, standard deviation Sharpe Ratio and Information Ratio of both managers, as well as the maximum drawdown during the period from 2005 to 2019. As we can observe, the first manager has both higher average excess returns and higher volatility, however the volatility is not as high compared to the excess returns hence the Sharpe ratio of the first manger is higher as well. In addition, the information ratio is higher for the first manager and maximum drawdown is lower, meaning that the first manager outperforms the market for the given amount of tracking error, however the second manager underperforms the market.

Part B

In order to understand if any manager has the ability to outperform the market and generate alpha that is significant for our sample time period we perform regressions for each of the managers excess returns:

$$R^e_{i,t} = \alpha_i + \beta_i R^e_{m,t} + \gamma_i R_{hml,t} + \delta_i R_{smb,t} + \epsilon_{i,t}$$

The result of this regression for the first manager is reported below. As can be seen, the alpha is negative but close to zero and not significant, whereas all the market and Fama French factors are significant at a 1% level. The coefficients on the market and SMB factor are positive, whereas it is negative for the HML factor, suggesting that HML is not part of this managers strategy.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                 PM1_ex   R-squared:                       0.992
Model:                            OLS   Adj. R-squared:                  0.992
Method:                 Least Squares   F-statistic:                     6983.
Date:                Sun, 15 Mar 2020   Prob (F-statistic):         1.11e-182
Time:                        16:11:01   Log-Likelihood:                 679.79
No. Observations:                 180   AIC:                            -1352.
Df Residuals:                     176   BIC:                            -1339.
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const      -7.841e-06      0.000     -0.018      0.985      -0.001       0.001
Mkt-RF         1.2961      0.011    115.753      0.000       1.274       1.318
SMB            0.8003      0.020     40.441      0.000       0.761       0.839
HML           -0.6925      0.017    -41.443      0.000      -0.725      -0.660
==============================================================================
Omnibus:                        0.320   Durbin-Watson:                   1.986
Prob(Omnibus):                  0.852   Jarque-Bera (JB):                0.180
Skew:                          -0.074   Prob(JB):                        0.914
Kurtosis:                       3.044   Cond. No.                         48.6
==============================================================================
```

For the second manager we have the following result:

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                 PM2_ex   R-squared:                       0.963
Model:                            OLS   Adj. R-squared:                  0.962
Method:                 Least Squares   F-statistic:                     1511.
Date:                Sun, 15 Mar 2020   Prob (F-statistic):         2.59e-125
Time:                        16:11:01   Log-Likelihood:                 601.29
No. Observations:                 180   AIC:                            -1195.
Df Residuals:                     176   BIC:                            -1182.
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0010      0.001      1.468      0.144      -0.000       0.002
Mkt-RF         0.9518      0.017     54.962      0.000       0.918       0.986
SMB           -0.1182      0.031     -3.861      0.000      -0.179      -0.058
HML            0.5333      0.026     20.634      0.000       0.482       0.584
==============================================================================
Omnibus:                        0.180   Durbin-Watson:                   2.008
Prob(Omnibus):                  0.914   Jarque-Bera (JB):                0.029
Skew:                          -0.006   Prob(JB):                        0.986
Kurtosis:                       3.061   Cond. No.                         48.6
==============================================================================
```

Here we can see that again the alpha is very small and insignificant, suggesting that market excess returns and the factors explain the excess returns produced by this manager. We can also observe that the R-squared is very high 96.3% and was 99.2% for the previous manager, suggesting that Fama-French model can almost completely explain the excess returns produces by these managers. In this case, we can see that the loading on the SMB factor is negative, which means that this manager does not include SMB in his/her strategy.

Part C

To understand which PM is better at beating the market, we perform a difference in means test, to understand if their expected excess returns are significantly different from each other or not.

$$H_0: \mu_1 = \mu_2$$
$$H_1: \mu_1 \neq \mu_2$$

In this case, the variance of the mean excess returns is taken to be the unbiased variance of the excess returns. Using the central limit theorem, we would have that the test statistic for this hypothesis is normally distributed which we calculate by the following formula

$$Test\ Statistic = \frac{\overline{\mu}_1 - \overline{\mu}_2}{\sqrt{\frac{\sigma_1^2}{T} + \frac{\sigma_2^2}{T}}}$$

We obtain that our test statistic is 0.557 which is less than 1.96. Hence, we would fail to reject at 5% significance level that the two mean excess returns are not equal. This result is expected since we saw from the above regressions that alphas for both managers were very close to zero and insignificant.

In order to understand, whether the Sharpe ratios of both managers are different from each other and whether one is higher than the other we have the following hypothesis:

$$H_0: SR_1 = SR_2$$
$$H_1: SR_1 \neq SR_2$$

We define

$$h(\theta) = \frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}$$

To be the difference in Sharpe Ratio estimates for both managers, hence according to the standard GMM estimation, the asymptotic variance-covariance matrix of $h(\hat{\theta})$ would be $\widehat{\Omega} = \frac{\hat{V}}{T}$ where

$$\hat{V} = (\hat{d}'\hat{S}^{-1}\hat{d})^{-1}$$

And $\hat{d}$ is

$$\frac{\partial \hat{E}(h(\theta))}{\partial \theta'} = \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -2\widehat{\sigma_1} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2\widehat{\sigma_2} \end{matrix}$$

We have that

$$\hat{S} = E(h(\theta)h(\theta)')$$

From the above we obtain that

$$\frac{h(\widehat{\theta}) - h(\theta_0)}{\sqrt{\widehat{\operatorname{Var}}\left[h(\widehat{\theta})\right]}} \sim \mathcal{N}(0,1)$$

Hence, our test statistic with these calculations is 0.011 and we fail to reject that the Sharpe Ratios of these managers are different from each other.

We further perform a test to determine whether the Information Ratios are the same for both managers:

$$H_0: IR_1 = IR_2$$
$$H_1: IR_1 \neq IR_2$$

We perform the same two sample t-test as we did above for the means, using the fact that the variance of the information ratios can be used as the standard error of the alphas from the regressions, which is due to the fact that we can treat the standard errors of the residuals as constant. We have the following test statistic

$$Test\ Statistic = \frac{\widehat{\mathrm{IR}}_1 - \widehat{\mathrm{IR}}_2}{\sqrt{\frac{\sigma_1^2}{T} + \frac{\sigma_2^2}{T}}}$$

And obtain that it is equal to -14.10, hence we can reject the null hypothesis with 1% significance level that the information ratios of both managers are not equal to each other.