Problem Set #2

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Question 1

Interview Questions

a) We have that $\mu_x = 20\%, \mu_M = 8\%, \sigma_x = 50\%, \sigma_M = 20\%$ and $\rho_{X,M} = 50\%$

From our regression model $X = \alpha + \beta * M$, which implies that

$$\beta = \frac{Cov(X,M)}{Var(M)} = \frac{\rho_{X,M}\sigma_x\sigma_M}{\sigma_x^2}$$
$$= \frac{0.5 * 0.5 * 0.2}{0.04} = 1.25$$

b) False Two random variables X and Y are independent if their joint distribution is the product of their marginal distributions, which implies that $E(XY) = E(X)E(Y)$, this in turn would imply that covariance between X and Y, $Cov(X,Y) = E(XY) - E(X)E(Y)$ is zero, however the it does not hold the other way. For example, define X to be random variables that can be $-1, 0, 1$ with equal probability 1/3. and define Y such that

$$Y = \begin{cases} 1 & \text{if } X = 0 \\ 0 & \text{o.w.} \end{cases}$$

This implies that variable XY is 0 in all cases, hence $E(XY) = 0$. We also have that $E(X) = 0andE(Y) = 1/3$, hence

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0$$

However, Y clearly depends on the realizations of X.

c) For beta estimations of stocks we are faced with a trade-off between information and noise. In one hand, shorter intervals used will have more observations and also have more noise. The noise is created by stocks not trading and this would bias betas towards 1. However, given that beta is an unstable measure of systematic risk, to estimate the historical beta, we would prefer to have long horizons of daily returns since using weekly or monthly returns would lead to more information loss than reduction of noise.

Question 2

R^2 and correlation

a) We have the linear regression of the form $y_i = \beta_0 + \beta_1 x_i$, which implies that our predicted dependent variable is

$$\hat{y_i} = \hat{\beta}_0 + \hat{\beta}_1 x_i \tag{1}$$

where

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \tag{2}$$

and

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \tag{3}$$

The formula for R^2 is

$$R^2 = \frac{\sum_{i=1}^n (\hat{y_i} - \bar{y})^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \tag{4}$$

Substituting (1) into (4) we get

$$R^2 = \frac{\sum_{i=1}^n (\hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y})^2}{\sum_{i=1}^n (y_i - \bar{y})^2} \tag{5}$$

Plugging in (2) into (5) we get the numerator equals

$$\begin{aligned}
&= \sum_{i=1}^n (\bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x_i - \bar{y})^2 \\
&= \sum_{i=1}^n (\hat{\beta}_1 x_i - \hat{\beta}_1 \bar{x})^2 \\
&= \hat{\beta}_1{}^2 \sum_{i=1}^n (x_i - \bar{x})^2
\end{aligned} \tag{6}$$

Using (3) and (5) we obtain

$$\frac{(\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}))^2 \sum_{i=1}^n (x_i - \bar{x})^2}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2} = \frac{(\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}))^2}{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2} \tag{7}$$

Hence,

$$R^2 = \left[\frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}} \right]^2 = \hat{cor}(x, y)^2 \tag{8}$$

Where

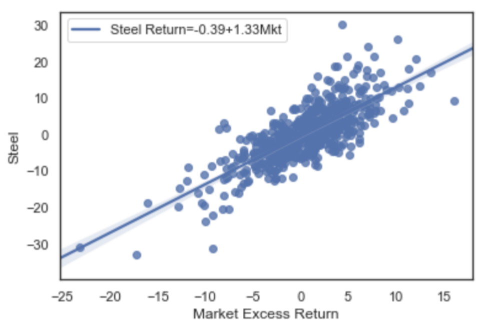
$$\hat{cor}(x, y) = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}} \tag{9}$$

Question 3

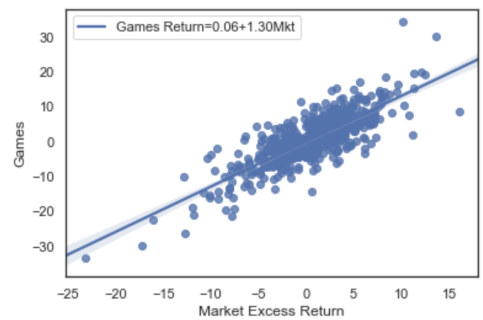
Market timing through cyclical and defensive sector index

a) We obtained the data from Ken French's website and defined the full sample to be from 01/1971 to 12/2019.

b) We use the market model to estimate industry alphas and betas $R^e_{i,t} = \alpha + \beta R^e_{m,t} + \epsilon_{i,t}$, where i stands for each industry. Below are the plots and regression lines for industries with top 5 betas which are "Steel Works Etc", "Recreation","Business Equipment", "Personal and Business Services" and "Fabricated Products and Machinery".



(a)



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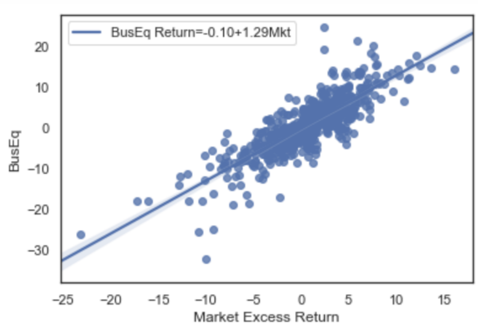
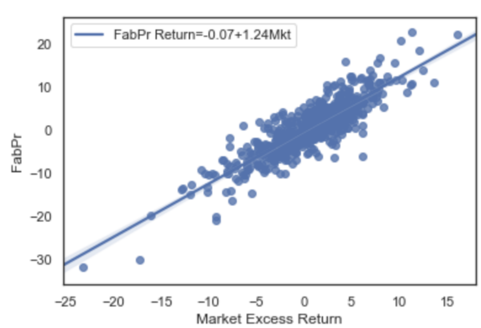


(c)

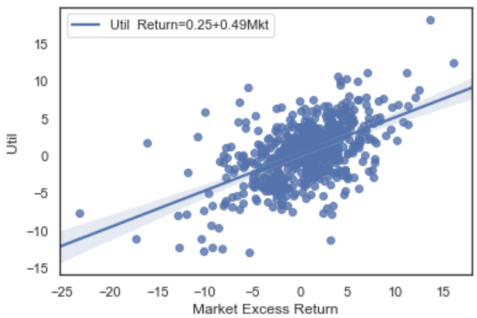
(d)



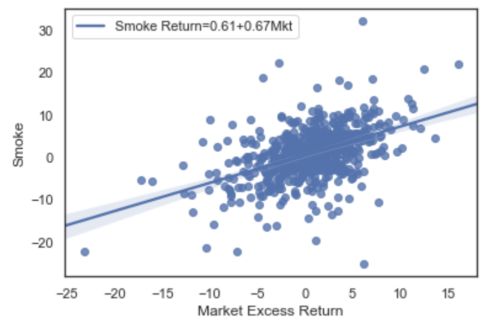
(e)

Figure 1: Industries with Top 5 Market Betas

The industries with bottom 5 betas were "Utilities", "Tobacco Products", "Food Products", "Beer & Liquor" and "Communication".



(a)



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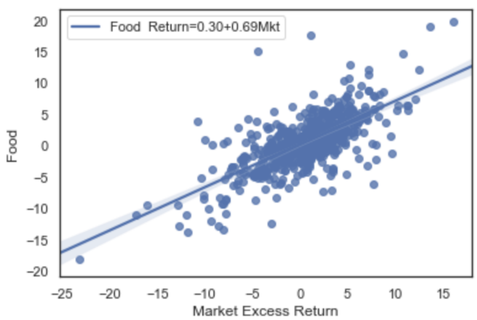
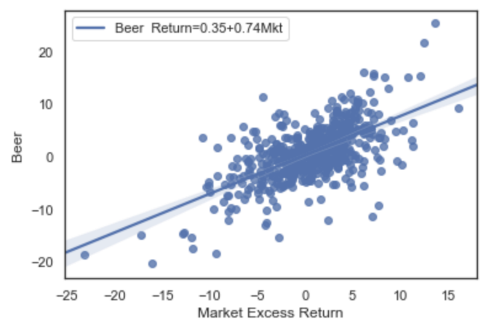
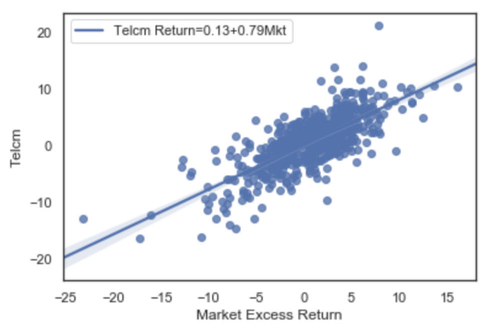


(c)



(d)

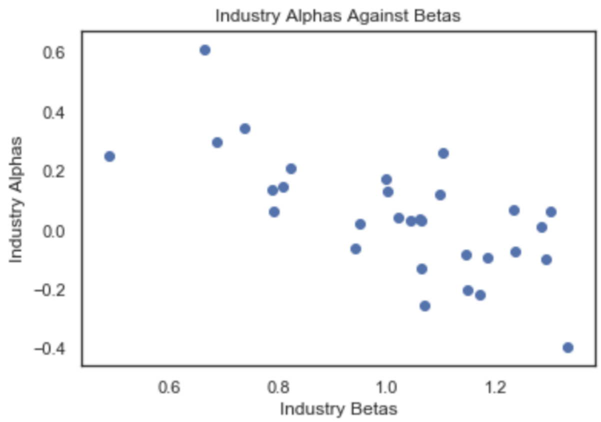


(e)

Figure 2: Industries with Bottom 5 Market Betas

We observe that, the top 5 betas are around 1.3 and the bottom 5 betas are around 0.6. Overall, the regression lines seem to fit better the top 5 industries compared to the bottom ones.

The graph below plots the alphas of industries against their respective betas.



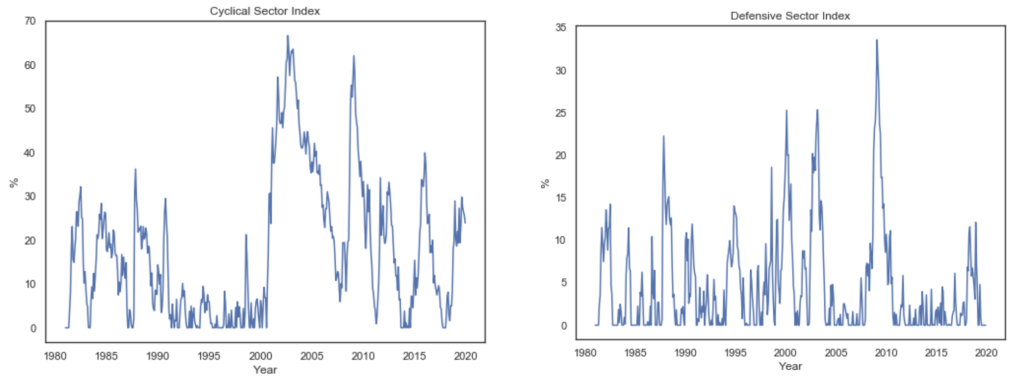
As can be seen from the graph above, the estimated industry alphas and betas seem to have a negative relationship, which is surprising, since we would expect riskier stocks with high betas to to also have high alphas to compensate investors for taking on the risk.

c) To construct the Cyclical and Defensive sector indexes, we regress each industries' monthly excess returns on the market excess return for 10 years period, obtain the betas of regressions, and assign industries with top 5 and bottom 5 betas to Cyclical and Defensive sectors respectively. We equally weight those industries in the sector index(assigning 20% weight to each) for the next month. We perform these regressions on a rolling basis using 10 years data to obtain weights for 1 month. After obtaining the monthly weights of industries, we calculate the returns for Cyclical and Defensive sector indexes for each month starting 01/1981 until 12/2019.

	Average Excess Returns	Alpha (%)	Beta	Sharpe Ratio	IR
Cyclical Portfolio	6.119	-0.333	1.29	0.286	-0.174
Defensive Portfolio	9.883	0.426	0.608	0.77	0.19

The average excess returns, Sharpe ratios and information ratios reported above are on annualized basis. The table shows that compared to the Defensive portfolio, Cyclical portfolio has on average lower excess returns, lower alpha, higher beta, hence lower Sharpe ratio and information ratio. These results are also surprising, since stocks with lower beta have both higher excess returns, alpha and Sharpe ratio, meaning they are low in volatility and risk and high in returns.

To understand even further, we plot the graphs for drawdown of both Cyclical and Defensive portfolios.



(a) (b)

Figure 3: Drawdowns of Cyclical and Defensive Portfolios

The drawdown of Cyclical portfolio is higher throughout the sample period compared with the Defensive one, which is expected since they are high beta stocks. Maximum drawdown of Cyclical portfolio for the period 01/1981 until 12/2019 was 66.69% and for Defensive portfolio it was 33.57%, almost twice as low.

Industries which were included in Cyclical portfolio throughout the entire period were: "Recreation", "Apparel, Textiles", "Construction and Construction Materials", "Steel Works Etc", "Fabricated Products and Machinery", "Electrical Equipment", "Automobiles and Trucks", "Aircraft ships and railroad equipment", "Precious Metals, Non-Metallic, and Industrial Metal Mining", "Coal", "Personal and Business Services", "Business Equipment", "Wholesale", "Retail", "Restaurants, Hotels, Motels", "Banking, Insurance, Real Estate, Trading" and "Other". These are industries with volatile returns as expected.

Industries which were included in Defensive portfolio for the whole period were: "Food Products", "Beer & Liquor", "Tobacco Products", "Consumer Goods", "Textiles", "Automobiles and Trucks", "Precious Metals, Non-Metallic, and Industrial Metal Mining", "Coal", "Petroleum and Natural Gas", "Utilities" and "Communication". Here, we have industries that are usually less volatile through time, since they either mostly include necessities.

Industries which were included both in Cyclical and Defensive portfolios at any time throughout the sample were: "Textiles", "Automobiles and Trucks", "Precious Metals, Non-Metallic, and Industrial Metal Mining" and "Coal". Given that the time period is 40 years, it makes sense that these industries produced volatile returns in some periods compared to others, due to shifts in demand and production.

More specifically, Textiles were included in Cyclical portfolio for 10 years, around 2010 to 2019 compared to almost 1 year included in Defensive(2000-2001). Automobiles and trucks were less risky in the beginning of the period from 1980 to 2000 and were included in Cyclical portfolio from the last 9 years. Mines and Coal were included in the Cyclical and Defensive portfolios around same time periods in our sample.

Overall, out of 30 industries 18 were used in the Cyclical sector index, 12 for Defensive sector index and 4 were used in both.

d) We assume that Cyclical and Defensive portfolio returns follow an AR(1) process

$$r_{t+1}^e = \alpha_0 + \alpha_1 r_t^e + \epsilon_{t+1} \tag{10}$$

where $\epsilon_{t+1} \sim N(0, \sigma_e^2)$

From the previous problem set, we obtained that $\mu_{r,t} = \alpha_0 + \alpha_1 r_t^e$, which are our one-month ahead returns predicted from the model (10). The variance of portfolio returns are $\hat{\sigma}_{r,t}^2 = \sigma_e^2$.

AR(1) regression results for the Cyclical and Defensive sector indexes are reported below:

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    ret   R-squared:                       0.006
Model:                            OLS   Adj. R-squared:                  0.003
Method:                 Least Squares   F-statistic:                     2.341
Date:                Tue, 03 Mar 2020   Prob (F-statistic):              0.127
Time:                        00:47:05   Log-Likelihood:                -1285.5
No. Observations:                 395   AIC:                             2575.
Df Residuals:                     393   BIC:                             2583.
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.5641      0.318      1.777      0.076      -0.060       1.188
lag_ret        0.0769      0.050      1.530      0.127      -0.022       0.176
==============================================================================
Omnibus:                       40.479   Durbin-Watson:                   1.988
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               95.476
Skew:                          -0.524   Prob(JB):                     1.85e-21
Kurtosis:                       5.168   Cond. No.                         6.34
==============================================================================
```

For Cyclical sector index, $\alpha_0 = 0.56, \alpha_1 = 0.08$. $(\sigma_e^c)^2 = 0.39\%$

OLS Regression Results

Dep. Variable:	ret	R-squared:	0.000
Model:	OLS	Adj. R-squared:	-0.002
Method:	Least Squares	F-statistic:	0.1083
Date:	Tue, 03 Mar 2020	Prob (F-statistic):	0.742
Time:	00:23:29	Log-Likelihood:	-1089.9
No. Observations:	395	AIC:	2184.
Df Residuals:	393	BIC:	2192.
Df Model:	1		
Covariance Type:	nonrobust		

	coef	std err	t	P>|t|	[0.025	0.975]
const	0.8213	0.197	4.164	0.000	0.434	1.209
lag_ret	0.0166	0.050	0.329	0.742	-0.083	0.116

Omnibus:	23.563	Durbin-Watson:	2.001
Prob(Omnibus):	0.000	Jarque-Bera (JB):	62.150
Skew:	-0.214	Prob(JB):	3.19e-14
Kurtosis:	4.896	Cond. No.	4.01

For Defensive sector index, $\alpha_0 = 0.82, \alpha_1 = 0.02$. $(\sigma_e^d)^2 = 0.15\%$ and $Cov(\epsilon^c, \epsilon^d) = 0.14\%$

In the same way since at time t, the returns are already realized(treated as constants), we obtain the covariance between two portfolios(Cyclical and Defensive) as

$$Cov(\alpha_o^c + \alpha_1^c r_t^c + \epsilon^c, \alpha_o^d + \alpha_1^d r_t^d + \epsilon^d) = Cov(\epsilon^c, \epsilon^d) \tag{11}$$

where 'c' stands for the Cyclical sector index and 'd' for Defensive.

Our portfolio which consists of the Cyclical and Defensive sector indexes has the return equal

$$R_{t+1}^p = wR_{t+1}^c + (1-w)R_{t+1}^d \tag{12}$$

where w is the weight on the Cyclical sector index.

Hence, we need to maximize

$$\begin{aligned} E_t(R_{t+1}^p) - \frac{\alpha}{2}Var_t(R_{t+1}^p) &= w\mu^c + (1-w)\mu^d - \frac{\alpha}{2}w^2(\sigma_e^c)^2 \\ &- \frac{\alpha}{2}(1-w)^2(\sigma_e^d)^2 - \alpha w Cov(\epsilon^c, \epsilon^d) + \alpha w^2 Cov(\epsilon^c, \epsilon^d) \end{aligned}$$

From here we obtain the F.O.C.

$$\begin{aligned} &\mu^c - \mu^d - \alpha w(\sigma_e^c)^2 + \alpha(1-w)(\sigma_e^d)^2 \\ &- \alpha Cov(\epsilon^c, \epsilon^d) + 2\alpha w Cov(\epsilon^c, \epsilon^d) = 0 \end{aligned}$$

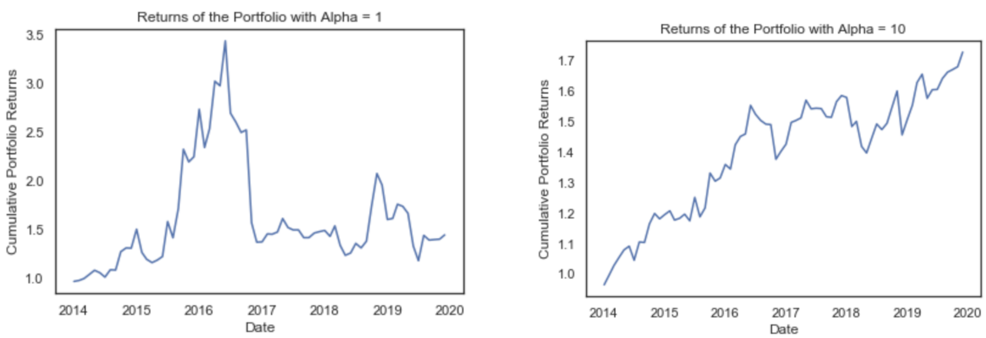
This gives us the optimal weight for the Cyclical sector index w as:

$$w = \frac{\alpha(Cov(\epsilon^c, \epsilon^d) - (\sigma_e^d)^2) + \mu^d - \mu^c}{\alpha(2Cov(\epsilon^c, \epsilon^d) - (\sigma_e^d)^2) - (\sigma_e^c)^2)} \tag{13}$$

and for the Defensive index the weight is:

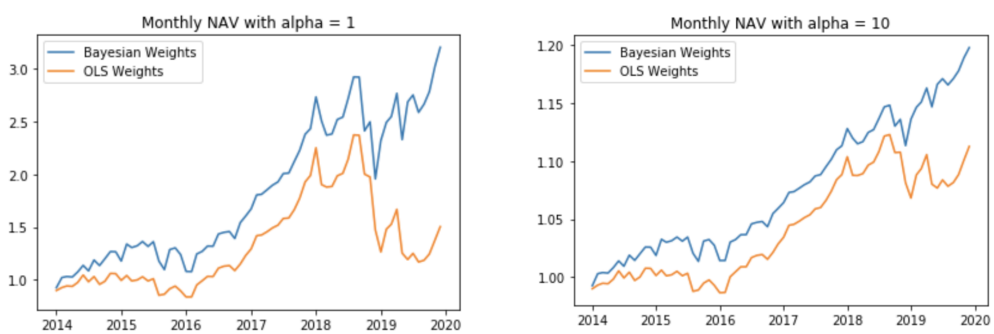
$$1 - w = \frac{\alpha Cov(\epsilon^c, \epsilon^d) - \alpha(\sigma_e^c)^2 - \mu^d + \mu^c}{\alpha(2Cov(\epsilon^c, \epsilon^d) - (\sigma_e^d)^2) - (\sigma_e^c)^2)} \tag{14}$$

Using the weight calculated above, we obtain the following portfolio returns for the period 01/2014 until 12/2019:



(a) (b)

Figure 4: Portfolio Cumulative Returns



(a) (b)

Figure 5: Previous Portfolio Cumulative Returns

As can be seen from figures (4) and (5), when risk aversion parameter is 1, the new portfolio produces highest returns of 350% near the period of 2016 to 2017, whereas the old portfolio peaks at 240% around 2018-2019. We also observe that, the new portfolio is more volatile during the whole period and produces cumulative returns around 150% at the end of the period similar to the one produces by the old portfolio. The higher volatility comes from the fact, that in this case both the indexes that we invest in are risky, whereas in the case of the old portfolio one of the assets is the risk-free rate.

In contrast, when risk aversion parameter is higher, the new portfolio is steadily increasing throughout the whole period, whereas the old portfolio starts increasing after

2016. This is due to the fact that with high risk aversion, we would invest in Defensive sector index more than the Cyclical index. In case of the old portfolio, we invest more in the risk-free and hence the overall level of cumulative returns is lower and produces 110% returns at the end of the period compared with 170% given by the new portfolio.