Exercice 1

1. $\mathbb{E}\left[\sup_{\theta\in\Theta}\{\psi(\theta)+\varepsilon_i\varphi(\theta)\}\right] = \frac{1}{2}\sup_{\theta\in\Theta}\{\psi(\theta)+\varphi(\theta)\} + \frac{1}{2}\sup_{\theta'\in\Theta}\{\psi(\theta')-\varphi(\theta')\}$

$= \frac{1}{2}\sup_{\theta,\theta'\in\Theta}\{\psi(\theta)+\psi(\theta')+\varphi(\theta)-\varphi(\theta')\}$, or $\sup_{\theta,\theta'\in\Theta}\varphi(\theta)-\varphi(\theta') = \sup_{\theta,\theta'\in\Theta}\varphi(\theta')-\varphi(\theta)$

donc

$= \frac{1}{2}\left[\sup_{\theta,\theta'\in\Theta}\{\psi(\theta)+\psi(\theta')+|\varphi(\theta)-\varphi(\theta')|\}\right]$

2. $\mathbb{E}\left[\sup_{\theta\in\Theta}\{b(\theta)+\sum_{i=1}^{k+1}\varepsilon_i\varphi_i(a_i(\theta))\right] = \mathbb{E}\left[\sup_{\theta\in\Theta} b(\theta)+\sum_{i=1}^{k}\varepsilon_i\varphi_i(a_i(\theta))+\varepsilon_{k+1}\varphi_{k+1}(a_{k+1}(\theta))\}\right]$

$$= \underbrace{\mathbb{E}\left[\sup_{\theta\in\Theta}\{b(\theta)+\sum_{i=1}^{k}\varepsilon_i\varphi_i(a_i(\theta))\}\right]}_{\leq \mathbb{E}\left[\sup_{\theta\in\Theta} b(\theta)+\sum_{i=1}^{k}\varepsilon_i a_i(\theta)\right.} + \underbrace{\mathbb{E}\left[\sup_{\theta\in\Theta}\{\varepsilon_{k+1}\varphi_{k+1}(a_{k+1}(\theta))\}\right]}_{=\frac{1}{2}\left[\sup_{\theta,\theta'}|\varphi_{k+1}(a_{k+1}(\theta))-\varphi_{k+1}(a_{k+1}(\theta'))|\right]}$$

$\leq \frac{1}{2}\sup_{\theta,\theta'}|a_{k+1}(\theta)-a_{k+1}(\theta')|$

$= \mathbb{E}\left[\sup_{\theta\in\Theta}\varepsilon_{k+1}a_{k+1}(\theta)\right]$ par (2)

$\leq \mathbb{E}\left[\sup_{\theta\in\Theta} b(\theta)+\sum_{i=1}^{k+1}\varepsilon_i a_i(\theta)\right]$

3. $i=1$: $\mathbb{E}\left[\sup_{\theta\in\Theta} b(\theta)+\varepsilon_1\varphi_1(a_1(\theta))\right] = \frac{1}{2}\left[\sup_{\theta,\theta'\in\Theta} b(\theta)+b(\theta')+\underbrace{|\varphi(a_1(\theta))-\varphi(a_1(\theta'))|}_{\leq |a_1(\theta)-a_1(\theta')|}\right]$

$\leq \mathbb{E}\left[\sup_{\theta\in\Theta} b(\theta)+\varepsilon_1 a_1(\theta)\right]$

donc (1) est vrai par induction.