

# Exercise 1

$$\begin{aligned} 1. \mathbb{E} \left[ \sup_{\theta \in \Theta} \{ \psi(\theta) + \varepsilon_i \varphi(\theta) \} \right] &= \frac{1}{2} \sup_{\theta \in \Theta} \{ \psi(\theta) + \varphi(\theta) \} + \frac{1}{2} \sup_{\theta' \in \Theta} \{ \psi(\theta') - \varphi(\theta') \} \\ &= \frac{1}{2} \sup_{\theta, \theta' \in \Theta} \{ \psi(\theta) + \psi(\theta') + \varphi(\theta) - \varphi(\theta') \} , \text{ or } \sup_{\theta, \theta' \in \Theta} \varphi(\theta) - \varphi(\theta') = \sup_{\theta, \theta' \in \Theta} \varphi(\theta') - \varphi(\theta) \\ \text{donc} \quad &= \frac{1}{2} \left[ \sup_{\theta, \theta' \in \Theta} \{ \psi(\theta) + \psi(\theta') + |\varphi(\theta) - \varphi(\theta')| \} \right] \end{aligned}$$

$$\begin{aligned} 2. \mathbb{E} \left[ \sup_{\theta \in \Theta} \left\{ b(\theta) + \sum_{i=1}^{k+1} \varepsilon_i \varphi_i(a_i(\theta)) \right\} \right] &= \mathbb{E} \left[ \sup_{\theta \in \Theta} b(\theta) + \sum_{i=1}^k \varepsilon_i \varphi_i(a_i(\theta)) + \varepsilon_{k+1} \varphi_{k+1}(a_{k+1}(\theta)) \right] \\ &= \underbrace{\mathbb{E} \left[ \sup_{\theta \in \Theta} \{ b(\theta) + \sum_{i=1}^k \varepsilon_i \varphi_i(a_i(\theta)) \} \right]}_{\leq \mathbb{E} \left[ \sup_{\theta \in \Theta} b(\theta) + \sum_{i=1}^k \varepsilon_i a_i(\theta) \right]} + \underbrace{\mathbb{E} \left[ \sup_{\theta \in \Theta} \{ \varepsilon_{k+1} \varphi_{k+1}(a_{k+1}(\theta)) \} \right]}_{= \frac{1}{2} \left[ \sup_{\theta, \theta'} |\varphi_{k+1}(a_{k+1}(\theta)) - \varphi_{k+1}(a_{k+1}(\theta'))| \right]} \\ &\leq \frac{1}{2} \sup_{\theta, \theta'} |a_{k+1}(\theta) - a_{k+1}(\theta')| \\ &= \mathbb{E} \left[ \sup_{\theta \in \Theta} \varepsilon_{k+1} a_{k+1}(\theta) \right] \text{ par (2)} \end{aligned}$$

$$\leq \mathbb{E} \left[ \sup_{\theta \in \Theta} b(\theta) + \sum_{i=1}^{k+1} \varepsilon_i a_i(\theta) \right]$$

$$\begin{aligned} 3. \quad i=1: \mathbb{E} \left[ \sup_{\theta \in \Theta} b(\theta) + \varepsilon_1 \varphi_1(a_1(\theta)) \right] &= \frac{1}{2} \left[ \sup_{\theta, \theta' \in \Theta} b(\theta) + b(\theta') + \underbrace{|\varphi_1(a_1(\theta)) - \varphi_1(a_1(\theta'))|}_{\leq |a_1(\theta) - a_1(\theta')|} \right] \\ &\leq \mathbb{E} \left[ \sup_{\theta \in \Theta} b(\theta) + \varepsilon_1 a_1(\theta) \right] \end{aligned}$$

donc (1) est vrai par induction.