Methods for Data Driven Modelling: Tutorial 4 S.C., J.C. R.M., Eric Chassinde-Mottin

Monte-Carlo Sampling for Bayesian inference, application to Gravitational Waves

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Gravitational waves are traveling perturbations in the shape of space-time, generated by cataclysmic events in distant galaxies. The detection of gravitational waves has been a long-held goal of the physics community, and represents a further confirmation of Einstein's Theory of General Relativity. On September 14, 2015, the two detectors Laser Interferometer Gravitational-wave Observatory (LIGO) observed a gravitational wave signal generated by the merger of two black holes (BH). The LIGO detectors had been recently upgraded with new instrumentation and operating as Advanced LIGO, able to measure changes in the local shape of space-time with a precision better than one-thousandth the diameter of a proton. This discovery was awarded by a nobel prize in 2017.

The L-shaped LIGO instruments consist of two perpendicular arms, each two of 4 kilometers long. A passing gravitational wave will alternately stretch one arm and squeeze the other, and then vice-versa; this generates an interference pattern at the readout port of the interferometer, which is measured by photodetectors (Fig.1). Nice divulgation tutorials can be viewed in https://www.gw-openscience.org/path/.

Since this first detection, different detectors operating in the word, including Ligo and Virgo, and have detected an increasing number of events, see [4].

The goal of this tutorial is to use Monte-Carlo Sampling and Bayes Inference to measure the parameters of the binary black holes (BBH) collision event detected by the gravitational wave GW150914, focusing on the inference of the two BBH masses and their lumonisity distance. Eric Chassinde-Mottin (laboratoire AstroParticule et Cosmologie (APC) Paris) has kindly provided us the access to the data and to the signal pre-traitement.

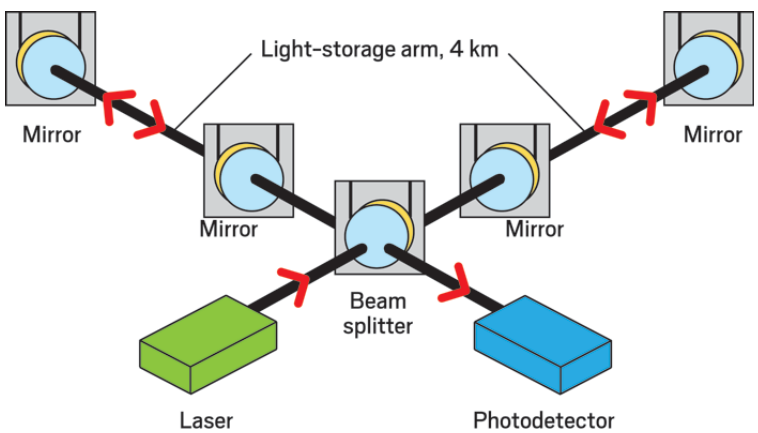

Figure 1: Sketch and description of the experiment from [5]. As laser light bounces back and forth along arms between its mirrors, LIGO senses gravitational waves as minuscule fluctuations in the lengths of the arms.

1 Questions

This tutorial is divided in two parts, the first on the analysis of artificial data and the second on the analysis of GW150914.

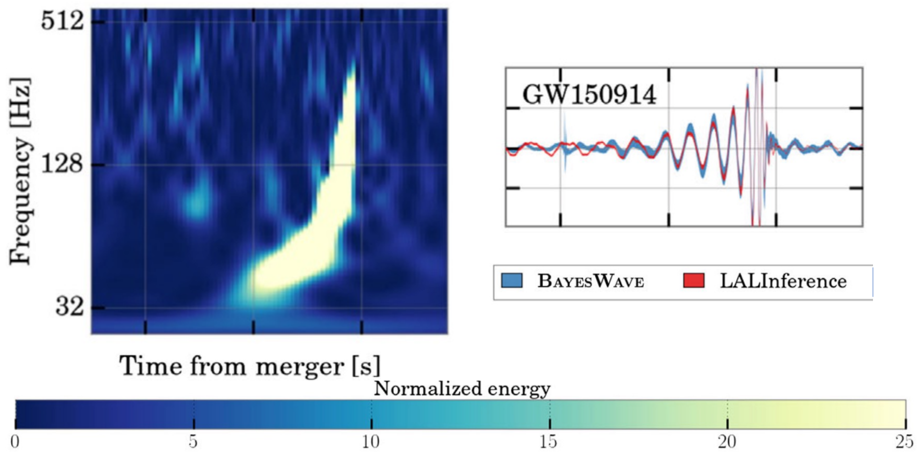

Figure 2: Detected signal for GW150914: The first panel shows a normalized time- frequency power map of the GW strain. The remaining pair of panels shows time-domain reconstructions of the whitened signal, in units of the standard deviation of the noise. The upper panels show the 90% credible intervals from the posterior probability density functions of the waveform time series, inferred using CBC waveform templates from Bayesian inference (LALINFERENCE) with the PhenomP model (red band) and by the BAYESWAVE wavelet model (blue band) [4].

First Part: Artificial Data

We start by using artificial data, given in the starting python notebook, to build and test the inference procedure. We have sample the data from a simple linear model, where the model is a linear function of a variable x_n in $[0, 1]$, depending on two parameters (denoted by $\boldsymbol{\theta}$): a slope a_1 and an intercept a_2.

$$f(x_n, \theta) = a_1^*\, x_n + a_2^* \tag{1}$$

The parameters $\boldsymbol{\theta}^* = (a_1*, a_2*)$ have been fixed to $a_1^* = 30$, $a_2^* = 800$. The sampled signal is the sum of the deterministic term given by the model and a Gaussian noise term (ϵ_n) with standard deviation which largely depend on the frequency bin n, σ_n. This Gaussian noise and its large dependence on the frequency bin is a key component of the detection of the gravitational waves, very sensible to noises at specific frequencies see Fig.3.

$$d_n = f(x_n, \theta) + \epsilon_n \tag{2}$$

To reproduce a large variability in the noise variances depending in the frequency bin we have extracted the bin (n) dependent standard deviations σ_n from a log-normal distribution : $\sigma_n = e^{\alpha}$ with α uniformly distributed in the interval $[-2, 6]$. In the experiment $\sigma_n$is a very well characterized function see Fig.3. The signal d_n given in the starting notebook as your initial data is sampled for x_n at interval $dx = 5\,10^{-3}$ on ,n=1,..M bins with $M = 200$, and for a given random realisation of the noise is . The aim of the first part of the tutorial will be to infer the parameters a_1 and a_2 from the measures of d_n knowing the standard deviations σ_n, by maximizing the Gaussian distribution for the data $\mathbf{d}$ given the linear model of Eq. 1 and the gaussian noise ϵ_n.

1. The Log-Likelihood distribution of the signal $\mathbf{d}$ given the parameters $\boldsymbol{\theta}$ of the model and the gaussian

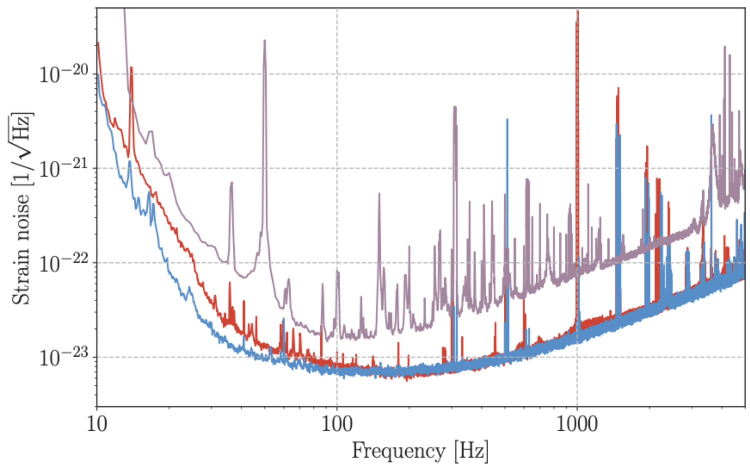

Figure 3: Amplitude spectral density of the total strain noise of the Virgo (violet), Ligo-Hanford LHO (red), and Ligo-Livingston LLO (blue) detectors. The curves are representative of the best performance of each detector during O2.[4]. Peaks correspond to known noises sources such as seismic (from ground vibrations), thermal noise (from the microscopic fluctuations of the individual atoms in the mirrors), quantum noise (from photon counting in the photodetector),laser noises (small variations in the laser intensity and frequency), beam jitter (misalign the laser beam with respect to the optical cavities),Scattered light, generated by tiny imperfections in the mirrors of the interferometers, etc.. [4]

noise is given by the gaussian distribution

$$\log \mathcal{L}(\{d_n|\boldsymbol{\theta}\}) = \sum_{n=0}^{M} \log \mathcal{L}(d_n|\boldsymbol{\theta}) = -\frac{1}{2}\sum_n log(2\pi\sigma_n^2) - \sum_n [d_n - f(\boldsymbol{\theta})]^2/(2\sigma_n^2) \tag{3}$$

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2. Write the probability for the parameters $P_{post}\,(\boldsymbol{\theta}|\mathbf{d},\boldsymbol{\sigma})$ from the data $\mathbf{d}$. Use two type of priors: Non-negative priors and more stringent uniform priors in a given interval. Consider for example that a_1 must be comprise in the interval between 20 and 100 and a_2 must be comprise in the interval between 300 and 3000.

3. Write a Monte-Carlo algorithm to sample the probability distribution for the parameters. The log-probability, given by the sum of the log likelihood and the log prior, is minus the energy in a Monte-Carlo sampling at inverse temperature $\beta = 1$. Start for example from the minimal possible value of the parameters and use Monte-Carlo steps in parameters proposing moves in parameters with amplitudes chosen from a Gaussian distribution with standard deviation $\sigma_{a_1} = 1$ for the mass and $\sigma_{a_2} = 10$ for distance (such choice depends only on the order of magnitude of such parametrs). Plot the log-probability as a function of Monte-Carlo Step.

4. Once reached the stationary distribution for the log-probability, estimate with the Monte-Carlo algorithm the values of the parameters; plot their distribution and compare their average value and standard variation with the true values. Use a corner plot to plot the distribution of on parameter as a function of the other, during the optimization.

5. Compare the results for the non-negative prior and the uniform prior in the given interval

Second Part: Analysis of the GW150914

The second part of the tutorial consists in analysing the first Binary Black hole detection, GW150914 with the Template IMRPhenomPv2 and SEOBNRv4 and to find the masses for the two black holes and their luminosity distance from earth.

Following [1] (see Appendix B of the paper) we introduce here the Gaussian noise likelihood used in gravitational-wave astronomy. $\mathbf{d}$ represent the data, more precisely the Fourier transform of the strain time series d(t) measured by a gravitational-wave detector (see Fig. 2): $\mathbf{d} = fft(d(t))/f_s$, where f_s is the sampling frequency and fft is a Fast Fourier transform . The fft signal can be decomposed by a sum of signals d_n in each frequency bin n. The signal is theoretically described by a template $\mu_j(\boldsymbol{\theta})$ based on Einstein's general relativity theory and related to the metric perturbation due to a collision event [1]. Parameters $\boldsymbol{\theta}$ are both extrinsic depending on the detector, *eg* its positions, and intrinsic depending on the collision event itself *eg.* the masses and distance to the earth of the colliding objects. The parameters for a BBH are described below in more detail. The detection is affected by noise depending on the frequency as illustrated in Fig.3, which can be described by a gaussian noise, with standard deviation σ_n depending on the frequency bin. Given the model, described by the template $\mu(\boldsymbol{\theta})$, the likelihood for the data d_n in a single frequency bin n is:

$$\mathcal{L}(d_n|\boldsymbol{\theta}) = \frac{1}{2\pi\sigma_n} e^{-2\Delta f (d_n - \mu_n(\boldsymbol{\theta}))^2/\sigma_n^2} \tag{4}$$

where Δf is the frequency resolution. Note that the factor $2\Delta f$ comes from a factor of 1/2 in the normal distribution and a factor of $4\Delta f$ needed to convert the square of the Fourier transforms into units of one-sided (having only positive frequencies) power spectral density. Moreover the normalisation factor does not contain a square root because the data are complex, and so the Gaussian is a two-dimensional likelihood. Gravitational-wave signals are typically spread over many (M) frequency bins. Assuming the noise in each bin is independent, the combined likelihood is a product of the likelihoods for each bin which is written after taking its logarithm, to avoid to multiply small numbers, as the following log-likelihood

$$\log \mathcal{L}(\{d_n|\boldsymbol{\theta}\}) = \sum_{n=0}^{M} \log \mathcal{L}(d_n|\boldsymbol{\theta})\} = -\sum_n log(2\pi\sigma_n) - 2\Delta f \sum_n (d_n - \mu_n(\boldsymbol{\theta}))^2/\sigma_n \tag{5}$$

Where the first term is a constant term.

Parameters $\boldsymbol{\theta}$ for the BBH coalescence: The GW signal emitted from a BBH coalescence depends on several intrinsic parameters that directly characterize the binary's dynamics and emitted waveform, and extrinsic parameters that encode the relation of the source to the detector network.

In general relativity, an isolated BH is uniquely described by its mass, spin, and electric charge. For astrophysical BHs, we assume the electric charge to be negligible. A BBH undergoing quasicircular inspiral can be described by eight intrinsic parameters, the two masses m_1, m_2 and the two three-dimensional spin vectors $\mathbf{s}_i$ of its component BHs defined at a reference frequency. Seven additional extrinsic parameters are needed to describe a BH binary: the sky location (right ascension and declination), the luminosity distance, the orbital inclination and polarization angle, the time, and phase at coalescence.

We will focus here on the inference of the detector-frame component masses $m_1(M_{sun})$ and $m_2(M_{sun})$, in units of solar mass (M_{sun}) and the the luminosity distance *distance* in MegaParsecs (Mpc) (where $1Mpc \sim 3\,10^{22}$ meters). As it is explained in detail in [1] the posterior for such parameters can be obtained by marginalising over all the other parameters (eg. see Appendix C.6 for the reconstruction for the luminosity distance parameter D_L). For sake of simplicity here we will fix all parameters to the optimal ones except the 3 parameters we are interested on and to simplify we will assume that the two masses are equal $m_1 = m_2 = m$, we will only infer therefore 2 parameters the mass m and the distance *distance*

Question 6. Look at the notebook Tutorial 4 The starting code for signal pre-traitement is taken form gw-odw.

Follow previous steps implemented for artificial data on the real signal, in particular use your own Monte-Carlo to sample the posterior probability, and find distribution of the parameters $\theta = \{m, distance\}$

at equilibrium and their average values and variance. Plot the corner plot of distribution of parameters during the MC sampling.

Include both non-negative priors and priors given by variability intervals form BBH detected in the first and second run of the gravital waves detections [4]: BBH masses between 20 and 100 M_{sun} and Distances between 300 and 3000 M_{pc}.

References

[1] Eric Thrane and Colm Talbot An introduction to Bayesian inference in gravitational-wave astronomy: parameter estimation, model selection, and hierarchical models Astronomical Society of Australia (PASA) doi: 10.1017/pas.2020.xxx. (2020)

[2] Anderson W. G., Brady P. R., Creighton J. D. E., Flana- gan E. E., 2001, Phys. Rev. D, 63, 042003

[3] Codes available at bilby example and gw-odw.

[4] B. P. Abbott et al. GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs. arXiv. This should lead to discussion and interpretation. The data used in these tutorials will be downloaded from the public DCC page LIGO-P1800370. Phys.Review X 9, 031040 (2019)

[5] see ligo.org and Nobek prize description for a brief description of the experiment.