Lambda terms practice — Solutions

In reductions below, we underline the *redex* — *red*ucible *ex*pression — which is being contracted in each term.

$$
\begin{aligned}
\mathtt{SKK} &= \underline{(\lambda xyz.xz(yz))\mathtt{K}}\mathtt{K} \\
&= \underline{(\lambda yz.\mathtt{K}z(yz))\mathtt{K}} \\
&= (\lambda z.\underline{\mathtt{K}z(\mathtt{K}z)}) \\
&= \lambda z.z \\
&= \mathtt{I} \\
\mathtt{BI} &= \underline{(\lambda xyz.x(yz))\mathtt{I}} \\
&= (\lambda yz.\underline{\mathtt{I}(yz)}) \\
&= \lambda yz.(yz) \\
&= \lambda yz.yz \\
&= \mathtt{1} \\
\mathtt{SII} &= \underline{(\lambda xyz.xz(yz))\mathtt{I}}\mathtt{I} \\
&= \underline{(\lambda yz.\mathtt{I}z(yz))\mathtt{I}} \\
&= (\lambda z.\underline{\mathtt{I}z}(\mathtt{I}z)) \\
&= (\lambda z.z(\underline{\mathtt{I}z})) \\
&= \lambda z.zz \\
&= \omega \\
\mathtt{BI} &= \underline{(\lambda xyz.x(yz))\mathtt{C}}\mathtt{S} \\
&= \underline{(\lambda yz.\mathtt{C}(yz))\mathtt{S}} \\
&= \lambda z.\mathtt{C}(\mathtt{S}z) \\
&= \lambda z.(\lambda xyz.xzy)(\mathtt{S}z) \\
&=_\alpha \lambda v.\underline{(\lambda xyz.xzy)(\mathtt{S}v)} \\
&= \lambda v.(\lambda yz.\mathtt{S}vzy) \\
&= \lambda vyz.\underline{\mathtt{S}vzy} \\
&= \lambda vyz.vy(zy) \\
&=_\alpha \lambda xyz.xy(zy)
\end{aligned}
$$

$$
\begin{aligned}
\mathtt{WWI} &= \underline{(\lambda xy.xyy)\mathtt{W}}\mathtt{I} \\
&= \underline{(\lambda y.\mathtt{W}yy)\mathtt{I}} \\
&= \mathtt{WII} \\
&= \underline{(\lambda xy.xyy)\mathtt{I}}\mathtt{I} \\
&= \underline{(\lambda y.\mathtt{I}yy)\mathtt{I}} \\
&= \underline{\mathtt{II}}\mathtt{I} \\
&= \underline{\mathtt{II}} \\
&= \mathtt{I} \\
\mathtt{WIW} &= \underline{(\lambda xy.xyy)\mathtt{I}}\mathtt{W} \\
&= \underline{(\lambda y.\mathtt{I}yy)\mathtt{W}} \\
&= \underline{\mathtt{IW}}\mathtt{W} \\
&= \mathtt{WW} \\
&= (\lambda xy.xyy)\mathtt{W} \\
&= \lambda y.\mathtt{W}yy \\
&= \lambda y.(\lambda x'y'.x'y'y')yy \\
&= \lambda y.yyy \\
&=_\alpha \lambda x.xxx \\
\mathtt{WWW} &= \underline{(\lambda xy.xyy)\mathtt{W}}\mathtt{W} \\
&= \underline{(\lambda y.\mathtt{W}yy)\mathtt{W}} \\
&= \mathtt{WWW} \\
(\lambda xy.y(xy))\omega(\lambda z.z\mathtt{I}) &= \underline{(\lambda xy.y(xy))\omega}(\lambda z.z\mathtt{I}) \\
&= \underline{(\lambda y.y(\omega y))(\lambda z.z\mathtt{I})} \\
&= \underline{(\lambda z.z\mathtt{I})(\omega(\lambda z.z\mathtt{I}))} \\
&= (\omega(\lambda z.z\mathtt{I}))\mathtt{I} \\
&= \omega(\lambda z.z\mathtt{I})\mathtt{I} \\
&= \underline{(\lambda x.xx)(\lambda z.z\mathtt{I})}\mathtt{I} \\
&= \underline{(\lambda z.z\mathtt{I})(\lambda z.z\mathtt{I})}\mathtt{I} \\
&= (\underline{(\lambda z.z\mathtt{I})\mathtt{I}})\mathtt{I} \\
&= (\underline{\mathtt{II}})\mathtt{I} \\
&= \mathtt{II} \\
&= \mathtt{I}
\end{aligned}
$$

$$
\begin{aligned}
(\lambda v.v\mathtt{K})[M,N] &= \underline{(\lambda v.v\mathtt{K})[M,N]} \\
&= [M,N]\mathtt{K} \\
&= (\lambda z.zMN)\mathtt{K} \\
&= \mathtt{K}MN && \text{(since } z \notin M,N) \\
&= M \\
(\lambda v.v\mathtt{c}_0)[M,N] &= \underline{(\lambda v.v\mathtt{c}_0)[M,N]} \\
&= [M,N]\mathtt{c}_0 \\
&= (\lambda z.zMN)\mathtt{c}_0 \\
&= \mathtt{c}_0MN && \text{(since } z \notin M,N) \\
&= \underline{(\lambda fz.z)M}N \\
&= \underline{(\lambda z.z)N} \\
&= N \\
\mathtt{c}_2\mathtt{K}xyz &= \underline{(\lambda fz.f(fz))\mathtt{K}}xyz \\
&= \underline{(\lambda z.\mathtt{K}(\mathtt{K}z))x}yz \\
&= \underline{\mathtt{K}(\mathtt{K}x)y}z \\
&= \underline{(\mathtt{K}x)z} \\
&= x \\
S^+\mathtt{c}_2 &= \underline{(\lambda nfz.f(nfz))\mathtt{c}_2} \\
&= \lambda fz.f(\mathtt{c}_2fz) \\
&= \lambda fz.f(\underline{(\lambda f'z'.f'(f'z'))f}z) \\
&= \lambda fz.f(\underline{(\lambda z'.f(fz'))z}) \\
&= \lambda fz.f(f(fz)) \\
&= \mathtt{c}_3
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{c}_{+}\mathsf{c}_2\mathsf{c}_3 &= \underline{(\lambda mnfz.mf(nfz))\mathsf{c}_2}\mathsf{c}_3 \\
&= \underline{(\lambda nfz.\mathsf{c}_2 f(nfz))\mathsf{c}_3} \\
&= (\lambda fz.\mathsf{c}_2 f(\mathsf{c}_3 fz)) \\
&= \lambda fz.\mathsf{c}_2 f(\underline{(\lambda f'z'.f'(f'(f'z')))f}z) \\
&= \lambda fz.\mathsf{c}_2 f(\underline{(\lambda z'.f(f(fz')))z}) \\
&= \lambda fz.\mathsf{c}_2 f(f(f(fz))) \\
&= \lambda fz.\underline{(\lambda f'z'.f'(f'z'))f}(f(f(fz))) \\
&= \lambda fz.\underline{(\lambda z'.f(fz'))(f(f(fz)))} \\
&= \lambda fz.f(f(f(f(fz)))) \\
&= \mathsf{c}_5 \\
\mathsf{c}_{\times}\mathsf{c}_2\mathsf{c}_3 &= \underline{(\lambda mnf.m(nf))\mathsf{c}_2}\mathsf{c}_3 \\
&= \underline{(\lambda nf.\mathsf{c}_2(nf))\mathsf{c}_3} \\
&= (\lambda f.\mathsf{c}_2(\mathsf{c}_3 f)) \\
&= \lambda f.(\lambda fz.f(fz))(\mathsf{c}_3 f) \\
&=_{\alpha} \lambda f.\underline{(\lambda gz.g(gz))(\mathsf{c}_3 f)} \\
&= \lambda f.(\lambda z.(\mathsf{c}_3 f)((\mathsf{c}_3 f)z)) \\
&= \lambda f.\lambda z.\mathsf{c}_3 f(\mathsf{c}_3 fz) \\
&= \lambda fz.\mathsf{c}_3 f(\underline{(\lambda f'z'.f'(f'(f'z')))f}z) \\
&= \lambda fz.\mathsf{c}_3 f(\underline{(\lambda z'.f(f(fz')))z}) \\
&= \lambda fz.\mathsf{c}_3 f(f(f(fz))) \\
&= \lambda fz.\underline{(\lambda f'z'.f'(f'(f'z')))f}(f(f(fz))) \\
&= \lambda fz.\underline{(\lambda z'.f(f(fz')))(f(f(fz)))} \\
&= \lambda fz.f(f(f(f(f(fz))))) \\
&= \mathsf{c}_6
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{c}_\wedge \mathsf{c}_2 \mathsf{c}_3 &= \underline{(\lambda mn.nm)\mathsf{c}_2}\mathsf{c}_3 \\
&= \underline{(\lambda n.n\mathsf{c}_2)\mathsf{c}_3} \\
&= \mathsf{c}_3\mathsf{c}_2 \\
&= (\lambda fz.f(f(fz)))\mathsf{c}_2 \\
&= \lambda z.\mathsf{c}_2\big(\mathsf{c}_2(\mathsf{c}_2 z)\big) \\
&=_\alpha \lambda x.\mathsf{c}_2\big(\mathsf{c}_2(\mathsf{c}_2 x)\big) \\
&= \lambda x.\underline{\Big(\lambda fz.f\Big(fz\Big)\Big)\big(\mathsf{c}_2(\mathsf{c}_2 x)\big)} \\
&= \lambda x.\Big(\lambda z.\big(\mathsf{c}_2(\mathsf{c}_2 x)\big)\Big(\big(\mathsf{c}_2(\mathsf{c}_2 x)\big)z\Big)\Big) \\
&= \lambda x.\lambda z.\big(\mathsf{c}_2(\mathsf{c}_2 x)\big)\Big(\mathsf{c}_2(\mathsf{c}_2 x)z\Big) \\
&= \lambda xz.\mathsf{c}_2(\mathsf{c}_2 x)\Big(\mathsf{c}_2(\mathsf{c}_2 x)z\Big) \\
&=_\alpha \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)\Big(\mathsf{c}_2(\mathsf{c}_2 x)y\Big) \\
&= \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)\Big(\underline{(\lambda fz.f(fz))(\mathsf{c}_2 x)}y\Big) \\
&= \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)\Big(\big(\mathsf{c}_2 x\big)\big((\mathsf{c}_2 x)y\big)\Big) \\
&= \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)\Big(\mathsf{c}_2 x\big(\mathsf{c}_2 xy\big)\Big) \\
&= \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)\Big(\mathsf{c}_2 x\big(\underline{(\lambda fz.f(fz))xy}\big)\Big) \\
&= \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)\Big(\mathsf{c}_2 x\big(x(xy)\big)\Big) \\
&= \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)\Big(\underline{\big(\lambda fz.f(fz)\big)x\big(x(xy)\big)}\Big) \\
&= \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)\Big(\underline{\big(\lambda z.x(xz)\big)\big(x(xy)\big)}\Big) \\
&= \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)\Big(x\big(x\big(x(xy)\big)\big)\Big) \\
&= \lambda xy.\mathsf{c}_2(\mathsf{c}_2 x)(x^4 y) \\
&= \lambda xy.\underline{\Big(\lambda fz.f(fz)\Big)(\mathsf{c}_2 x)(x^4 y)} \\
&= \lambda xy.\Big((\mathsf{c}_2 x)\big((\mathsf{c}_2 x)(x^4 y)\big)\Big) \\
&= \lambda xy.\mathsf{c}_2 x\big(\mathsf{c}_2 x(x^4 y)\big) \\
&= \lambda xy.\mathsf{c}_2 x\big(\underline{\big(\lambda fz.f(fz)\big)x(x^4 y)}\big) \\
&= \lambda xy.\mathsf{c}_2 x\big(x(x(x^4 y))\big) \\
&= \lambda xy.\underline{\big(\lambda fz.f(fz)\big)x(x^6 y)} \\
&= \lambda xy.\big(x(x(x^6 y))\big) \\
&= \lambda xy.x^8 y \\
&= \mathsf{c}_8
\end{aligned}
$$

$$
\begin{aligned}
\mathtt{c}_4\mathtt{CK} &= (\lambda f z. f(f(f(fz))))\mathtt{CK} \\
&= \mathtt{C}(\mathtt{C}(\mathtt{C}(\mathtt{CK}))) \\
&= \underline{(\lambda xyz.xzy)(\mathtt{C}(\mathtt{C}(\mathtt{CK})))} \\
&= \lambda yz.(\mathtt{C}(\mathtt{C}(\mathtt{CK})))zy \\
&= \lambda yz.\underline{\big(\lambda x'y'z'.x'z'y'\big)(\mathtt{C}(\mathtt{CK}))zy} \\
&= \lambda yz.(\mathtt{C}(\mathtt{CK}))yz \\
&= \lambda yz.\underline{\big(\lambda x'y'z'.x'z'y'\big)(\mathtt{CK})yz} \\
&= \lambda yz.(\mathtt{CK})zy \\
&= \lambda yz.\underline{\big(\lambda x'y'z'.x'z'y'\big)\mathtt{K}zy} \\
&= \lambda yz.\underline{\mathtt{K}yz} \\
&= \lambda yz.y \\
&= \mathtt{K}
\end{aligned}
$$