

Find a type $A \in \mathbb{T}$ for each of the following terms $M \in \Lambda$ and derive the typing judgment $\vdash M : A$ using the inference rules given above.

Solutions

1. $\lambda xy.xyy$

$$\begin{array}{c}
\frac{\frac{A_1 = C \rightarrow B \rightarrow A_4}{(x : C \rightarrow B \rightarrow A_4) \in \{x : A_1, y : A_3\}}}{x : A_1, y : A_3 \vdash x : C \rightarrow B \rightarrow A_4} \quad \frac{\frac{C = A_3}{(y : C) \in \{x : A_1, y : A_3\}}}{x : A_1, y : A_3 \vdash y : C} \quad \frac{B = A_3}{(y : B) \in \{x : A_1, y : A_3\}} \\
\hline
x : A_1, y : A_3 \vdash xy : B \rightarrow A_4 \quad x : A_1, y : A_3 \vdash y : B \\
\hline
x : A_1, y : A_3 \vdash xyy : A_4 \\
\hline
x : A_1 \vdash \lambda y.xyy : A_2 \quad A_2 = A_3 \rightarrow A_4 \\
\hline
\vdash \lambda x.\lambda y.xyy : A \quad A = A_1 \rightarrow A_2
\end{array}$$

$$\begin{aligned}
A &= A_1 \rightarrow A_2 \\
&= (C \rightarrow B \rightarrow A_4) \rightarrow (A_3 \rightarrow A_4) \\
&= (A_3 \rightarrow A_3 \rightarrow A_4) \rightarrow (A_3 \rightarrow A_4) \\
&\simeq (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta
\end{aligned}$$

$$\boxed{\lambda x \lambda y. xyy : (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta}$$

2. $\lambda xy.yx$

$$\begin{array}{c}
\frac{\frac{B_1 = C \rightarrow B_2}{(y : C \rightarrow B_2) \vdash \{x : A_1, y : B_1\}}}{x : A_1, y : B_1 \vdash y : C \rightarrow B_2} \quad \frac{\frac{C = A_1}{(x : C) \in \{x : A_1, y : B_1\}}}{x : A_1, y : B_1 \vdash x : C} \\
\hline
x : A_1, y : B_1 \vdash yx : B_2 \\
\hline
x : A_1 \vdash \lambda y.yx : A_2 \quad A_2 = B_1 \rightarrow B_2 \\
\hline
\vdash \lambda x.\lambda y.yx : A \quad A = A_1 \rightarrow A_2
\end{array}$$

$$\begin{aligned}
A &= A_1 \rightarrow A_2 \\
&= C \rightarrow (B_1 \rightarrow B_2) \\
&= C \rightarrow ((C \rightarrow B_2) \rightarrow B_2) \\
&\simeq \alpha \rightarrow (\alpha \rightarrow \beta) \rightarrow \beta
\end{aligned}$$

$$\boxed{\lambda xy.yx : \alpha \rightarrow (\alpha \rightarrow \beta) \rightarrow \beta}$$

3. $\lambda xyz.zyx$

$$\begin{array}{c}
\frac{\frac{E \rightarrow D \rightarrow C_2 = C_1}{\Gamma \vdash z : E \rightarrow D \rightarrow C_2} \quad \frac{\frac{E = B_1}{(y : E) \in \Gamma}}{\Gamma \vdash y : E} \quad \frac{D = A_1}{x : D \in \Gamma}}{\Gamma \vdash zy : D \rightarrow C_2} \quad \frac{\Gamma \vdash x : D}{\Gamma \vdash x : D} \\
\frac{\Gamma := \{x : A_1, y : B_1, z : C_1\} \vdash zyx : C_2}{x : A_1, y : B_1 \vdash \lambda z.zyx : B_2} \quad \frac{B_2 = C_1 \rightarrow C_2}{x : A_1 \vdash \lambda y.\lambda z.zyx : A_2} \quad \frac{A_2 = B_1 \rightarrow B_2}{\lambda x.\lambda y.\lambda z.zyx : A} \quad \frac{A = A_1 \rightarrow A_2}{A = A_1 \rightarrow A_2}
\end{array}$$

$$\begin{aligned}
A &= A_1 \rightarrow A_2 \\
&= D \rightarrow (B_1 \rightarrow B_2) \\
&= D \rightarrow (E \rightarrow (C_1 \rightarrow C_2)) \\
&= D \rightarrow (E \rightarrow ((E \rightarrow D \rightarrow C_2) \rightarrow C_2)) \\
&\simeq \alpha \rightarrow \beta \rightarrow (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow \gamma
\end{aligned}$$

$$\boxed{\lambda xyz.zyx : \alpha \rightarrow \beta \rightarrow (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow \gamma}$$

4. $\lambda xy.x(xy)$

$$\begin{array}{c}
\frac{\frac{A_1 = C \rightarrow B_2}{(x : C \rightarrow B_2) \in \{x : A_1, y : B_1\}}}{x : A_1, y : B_1 \vdash x : C \rightarrow B_2} \quad \frac{\frac{A_1 = D \rightarrow C}{(x : D \rightarrow C) \in \{x : A_1, y : B_1\}}}{x : A_1, y : B_1 \vdash x : D \rightarrow C} \quad \frac{\frac{D = B_1}{(y : D) \in \{x : A_1, y : B_1\}}}{x : A_1, y : B_1 \vdash y : D} \\
\frac{x : A_1, y : B_1 \vdash x : C \rightarrow B_2 \quad x : A_1, y : B_1 \vdash x : D \rightarrow C \quad x : A_1, y : B_1 \vdash y : D}{x : A_1, y : B_1 \vdash x(xy) : B_2} \\
\frac{x : A_1 \vdash \lambda y.x(xy) : A_2 \quad A_2 = B_1 \rightarrow B_2}{\vdash \lambda x.\lambda y.x(xy) : A} \quad A = A_1 \rightarrow A_2
\end{array}$$

$$\begin{aligned}
C \rightarrow B_2 &= A_1 = D \rightarrow C \\
C \rightarrow B_2 &= D \rightarrow C \\
C = D \quad \& \quad B_2 = C \\
B_1 &= D = C \\
B_1 &= C
\end{aligned}$$

$$\begin{aligned}
A &= A_1 \rightarrow A_2 \\
&= (C \rightarrow B_2) \rightarrow (B_1 \rightarrow B_2) \\
&= (C \rightarrow C) \rightarrow (C \rightarrow C) \\
&\simeq (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha
\end{aligned}$$

$$\boxed{\lambda xy.x(xy) : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha}$$

5. $\lambda x.x(\lambda y.y)$

$$\frac{\frac{\frac{A_1 = B \rightarrow A_2}{(x : B \rightarrow A_2) \in \{x : A_1\}}}{x : A_1 \vdash x : B \rightarrow A_2} \quad \frac{\frac{\frac{B_2 = B_1}{(y : B_2) \in \{x : A_1, y : B_1\}}}{x : A_1, y : B_1 \vdash y : B_2}}{x : A_1 \vdash \lambda y.y : B \quad B = B_1 \rightarrow B_2}}{x : A_1 \vdash x(\lambda y.y) : A_2} \quad \frac{}{\vdash \lambda x.x(\lambda y.y) : A \quad A = A_1 \rightarrow A_2}$$

$$\begin{aligned} A &= A_1 \rightarrow A_2 \\ &= (B \rightarrow A_2) \rightarrow A_2 \\ &= ((B_1 \rightarrow B_2) \rightarrow A_2) \rightarrow A_2 \\ &= ((B_1 \rightarrow B_1) \rightarrow A_2) \rightarrow A_2 \\ &\simeq ((\beta \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha \end{aligned}$$

$$\boxed{\lambda x.x(\lambda y.y) : ((\beta \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha}$$

6. $\lambda xyz.xz(yz)$

$$\frac{\frac{\frac{A_1 = E \rightarrow D \rightarrow C_2}{(x : E \rightarrow D \rightarrow C_2) \in \Gamma}}{\Gamma \vdash x : E \rightarrow D \rightarrow C_2} \quad \frac{\frac{E = C_1}{(z : E) \in \Gamma}}{\Gamma \vdash z : E} \quad \frac{\frac{B_1 = F \rightarrow D}{(y : F \rightarrow D) \in \Gamma}}{\Gamma \vdash y : F \rightarrow D} \quad \frac{\frac{F = C_1}{(z : F) \in \Gamma}}{\Gamma \vdash z : F}}{\Gamma \vdash xz : D \rightarrow C_2 \quad \Gamma \vdash yz : D} \quad \frac{\Gamma := \{x : A_1, y : B_1, z : C_1\} \vdash xz(yz) : C_2}{x : A_1, y : B_1 \vdash \lambda z.xz(yz) : B_2} \quad \frac{B_2 = C_1 \rightarrow C_2}{x : A_1 \vdash \lambda y.\lambda z.xz(yz) : A_2} \quad \frac{A_2 = B_1 \rightarrow B_2}{\vdash \lambda x.\lambda y.\lambda z.xz(yz) : A \quad A = A_1 \rightarrow A_2}$$

$$\begin{aligned} A &= A_1 \rightarrow A_2 \\ &= (E \rightarrow D \rightarrow C_2) \rightarrow (B_1 \rightarrow B_2) \\ &= (E \rightarrow D \rightarrow C_2) \rightarrow (F \rightarrow D) \rightarrow (C_1 \rightarrow C_2) \\ &= (E \rightarrow D \rightarrow C_2) \rightarrow (C_1 \rightarrow D) \rightarrow (C_1 \rightarrow C_2) \\ &= (E \rightarrow D \rightarrow C_2) \rightarrow (E \rightarrow D) \rightarrow (E \rightarrow C_2) \\ &\simeq (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma \end{aligned}$$

$$\boxed{\lambda xyz.xz(yz) : (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma}$$

7. $\lambda xy.y(xy)$

$$\begin{array}{c}
\frac{B_1 = C \rightarrow B_2}{(y : C \rightarrow B_2) \in \{x : A_1, y : B_1\}} \quad \frac{A_1 = D \rightarrow C}{(x : D \rightarrow C) \in \{x : A_1, y : B_1\}} \quad \frac{D = B_1}{(y : D) \in \{x : A_1, y : B_1\}} \\
\frac{x : A_1, y : B_1 \vdash y : C \rightarrow B_2}{x : A_1, y : B_1 \vdash y : C \rightarrow B_2} \quad \frac{x : A_1, y : B_1 \vdash x : D \rightarrow C}{x : A_1, y : B_1 \vdash x : D \rightarrow C} \quad \frac{x : A_1, y : B_1 \vdash y : D}{x : A_1, y : B_1 \vdash y : D} \\
\hline
x : A_1, y : B_1 \vdash y(xy) : B_2 \\
\hline
x : A_1 \vdash \lambda y.y(xy) : A_2 \quad A_2 = B_1 \rightarrow B_2 \\
\hline
\vdash \lambda x.\lambda y.y(xy) : A \quad A = A_1 \rightarrow A_2
\end{array}$$

$$\begin{aligned}
A &= A_1 \rightarrow A_2 \\
&= (D \rightarrow C) \rightarrow (B_1 \rightarrow B_2) \\
&= (B_1 \rightarrow C) \rightarrow (B_1 \rightarrow B_2) \\
&= ((C \rightarrow B_2) \rightarrow C) \rightarrow (C \rightarrow B_2) \rightarrow B_2 \\
&\simeq ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \beta
\end{aligned}$$

$$\boxed{\lambda xy.y(xy) : ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \beta}$$

8. $\lambda xyz.xy(yz)$

$$\begin{array}{c}
\frac{A_1 = E \rightarrow D \rightarrow C_2}{(x : E \rightarrow D \rightarrow C_2) \in \Gamma} \quad \frac{E = B_1}{(y : E) \in \Gamma} \quad \frac{B_1 = F \rightarrow D}{(y : F \rightarrow D) \in \Gamma} \quad \frac{F = C_1}{(z : F) \in \Gamma} \\
\frac{\Gamma \vdash x : E \rightarrow D \rightarrow C_2}{\Gamma \vdash x : E \rightarrow D \rightarrow C_2} \quad \frac{\Gamma \vdash y : E}{\Gamma \vdash y : E} \quad \frac{\Gamma \vdash y : F \rightarrow D}{\Gamma \vdash y : F \rightarrow D} \quad \frac{\Gamma \vdash z : F}{\Gamma \vdash z : F} \\
\hline
\Gamma \vdash xy : D \rightarrow C_2 \quad \Gamma \vdash yz : D \\
\hline
\Gamma := \{x : A_1, y : B_1, z : C_1\} \vdash xy(yz) : C_2 \\
\hline
x : A_1, y : B_1 \vdash \lambda z.xy(yz) : B_2 \quad B_2 = C_1 \rightarrow C_2 \\
\hline
x : A_1 \vdash \lambda y.\lambda z.xy(yz) : A_2 \quad A_2 = B_1 \rightarrow B_2 \\
\hline
\vdash \lambda x.\lambda y.\lambda z.xy(yz) : A \quad A = A_1 \rightarrow A_2
\end{array}$$

$$\begin{aligned}
A &= A_1 \rightarrow A_2 \\
&= (E \rightarrow D \rightarrow C_2) \rightarrow (B_1 \rightarrow B_2) \\
&= (E \rightarrow D \rightarrow C_2) \rightarrow (B_1 \rightarrow (C_1 \rightarrow C_2)) \\
&= (B_1 \rightarrow D \rightarrow C_2) \rightarrow (F \rightarrow D) \rightarrow F \rightarrow C_2 \\
&= ((F \rightarrow D) \rightarrow D \rightarrow C_2) \rightarrow (F \rightarrow D) \rightarrow F \rightarrow C_2 \\
&\simeq ((\alpha \rightarrow \beta) \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma
\end{aligned}$$

$$\boxed{\lambda xyz.xy(yz) : ((\alpha \rightarrow \beta) \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma}$$

9. $\lambda xy.x(\lambda z.yz)$

$$\begin{array}{c}
\frac{A_1 = C \rightarrow B_2}{(x : C \rightarrow B_2) \in \{x : A_1, y : B_1\}} \quad \frac{\frac{B_1 = D \rightarrow C_2}{(y : D \rightarrow C_2) \in \Gamma} \quad \frac{C_1 = D}{(z : D) \in \Gamma}}{\Gamma \vdash y : D \rightarrow C_2 \quad \Gamma \vdash z : D} \\
\frac{x : A_1, y : B_1 \vdash x : C \rightarrow B_2}{x : A_1, y : B_1 \vdash \lambda z.yz : C} \quad \frac{\Gamma := x : A_1, y : B_1, z : C_1 \vdash yz : C_2}{C = C_1 \rightarrow C_2} \\
\frac{x : A_1, y : B_1 \vdash x(\lambda z.yz) : B_2}{x : A_1 \vdash \lambda y.x(\lambda z.yz) : A_2} \quad \frac{A_2 = B_1 \rightarrow B_2}{\vdash \lambda x.\lambda y.x(\lambda z.yz) : A} \quad A = A_1 \rightarrow A_2
\end{array}$$

$$\begin{aligned}
A &= A_1 \rightarrow A_2 \\
&= (C \rightarrow B_2) \rightarrow (B_1 \rightarrow B_2) \\
&= ((C_1 \rightarrow C_2) \rightarrow B_2) \rightarrow ((D \rightarrow C_2) \rightarrow B_2) \\
&= ((D \rightarrow C_2) \rightarrow B_2) \rightarrow (D \rightarrow C_2) \rightarrow B_2 \\
&\simeq ((\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \gamma
\end{aligned}$$

$$\boxed{\lambda xy.x(\lambda z.yz) : ((\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \gamma}$$

10. $\lambda xy.x(\lambda z.z(yx))$

$$\begin{array}{c}
\frac{A_1 = C \rightarrow B_2}{(x : C \rightarrow B_2) \in \{x : A_1, y : B_1\}} \quad \frac{\frac{C_1 = D \rightarrow C_2}{(z : D \rightarrow C_2) \in \Gamma} \quad \frac{B_1 = E \rightarrow D}{(y : E \rightarrow D) \in \Gamma} \quad \frac{E = A_1}{(x : E) \in \Gamma}}{\Gamma \vdash z : D \rightarrow C_2 \quad \Gamma \vdash yx : D} \\
\frac{x : A_1, y : B_1 \vdash x : C \rightarrow B_2}{x : A_1, y : B_1 \vdash \lambda z.z(yx) : C} \quad \frac{\Gamma := x : A_1, y : B_1, z : C_1 \vdash z(yx) : C_2}{C = C_1 \rightarrow C_2} \\
\frac{x : A_1, y : B_1 \vdash x(\lambda z.z(yx)) : B_2}{x : A_1 \vdash \lambda y.x(\lambda z.z(yx)) : A_2} \quad \frac{A_2 = B_1 \rightarrow B_2}{\vdash \lambda x.\lambda y.x(\lambda z.z(yx)) : A} \quad A = A_1 \rightarrow A_2
\end{array}$$

$$\begin{aligned}
E &= A_1 = C \rightarrow B_2 = (C_1 \rightarrow C_2) \rightarrow B_2 = ((D \rightarrow C_2) \rightarrow C_2) \rightarrow B_2 \\
B_1 &= E \rightarrow D = (((D \rightarrow C_2) \rightarrow C_2) \rightarrow B_2) \rightarrow D \\
A &= A_1 \rightarrow A_2 \\
&= E \rightarrow (B_1 \rightarrow B_2) \\
&= (((D \rightarrow C_2) \rightarrow C_2) \rightarrow B_2) \rightarrow (((((D \rightarrow C_2) \rightarrow C_2) \rightarrow B_2) \rightarrow D) \rightarrow B_2) \\
&\simeq (((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \gamma) \rightarrow (((((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \gamma) \rightarrow \alpha) \rightarrow \gamma)
\end{aligned}$$

$$\boxed{\lambda xy.x(\lambda z.z(yx)) : (((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \gamma) \rightarrow (((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma}$$