Chan and Khandani:
Combinatorial Analysis

An Introduction

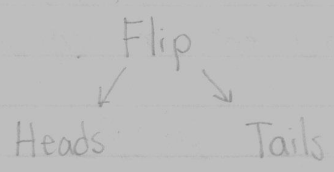
Probability, as a concept, is a very simple thing - what are the chances a certain event might occur? By simply counting the number of possibilities, we can find solutions to many probabalistic problems - this is called combinatorial analysis

ECE 103: A Review

There are a few concepts we learned approximately 15 years ago in 103 that rear their ugly heads again this term. As the foundation of knowledge for any course is important, we'll be going over them again.

The Basics of Counting

In the simplest case, we have a coin flip. Assuming the coin is fair, we have two possibilities:



The total number of possibilities is 2. Since we can assume there is an equal chance of each result occuring, we can say the ratio of getting a particular side is $1/2$.

But why is it that we can say that? It's definitely possible to get 100 heads in a row, albeit rare the more exact way to describe this is:

$$\text{Ratio} = \frac{\#\text{ Heads}}{\text{Tries}} \quad \text{as Tries} \to \infty$$

So the ratio of heads to tails will tend towards 1/2 as the number of attempts approaches infinity.

Ways to Perform Procedures in Succession

Let's face it, maybe flipping one coin is kind of boring. But what about TWO coins?

Coin 1 ↙ Heads ↘ Tails

Coin 2 ↙ Heads ↘ Tails

Each coin's result has absolutely zero effect on the second coin's result, and as such, we can denote the two events as independent.

Coin 1	Coin 2
Heads	Heads
Heads	Tails
Tails	Heads
Tails	Tails

We're left with 4 distinct results.

Each coin, originally, only had two possibilities, but performing each flip in succession has changed the number of possible results to the PRODUCT of the number of possibilities of each independent event.

As such, this can be generalized to:

$$\text{Possibilities} = (n_1)(n_2)(n_3)\dots(n_r)$$

Where n_i are the number of possibilities for each of r independent events. This, in addition to permutations, will be important building blocks of other counting methods.

Permutations

Permutations describe the number of ways you can order a set of elements.

The simplest case is how many ways we can order a set of distinct elements. Here we have a square, a circle, and a triangle.

Set: □ ○ △

Orderings:
□ ○ △
□ △ ○
○ □ △
○ △ □
△ □ ○
△ ○ □

6 ways

This can be denoted as:

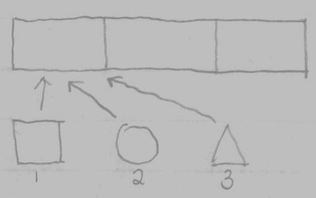
$$P^3_3 = (3)(2)(1)$$

or generally,

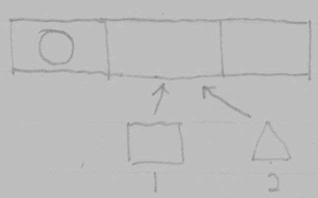
$$P^n_n = (n)(n-1)(n-2)\dots(2)(1)$$
$$= n!$$

The P^3_3 simply says our set contains 3 (top) objects, and we're trying to see how we can order 3 (bottom) of those 3 objects.

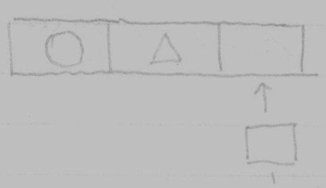
But what's the logic behind $n!$? We have 3 spaces to place our blocks.



We can arbitrarily place any shape into the first box. As such, there's 3 possibilities for the first box. Let's put the circle in.

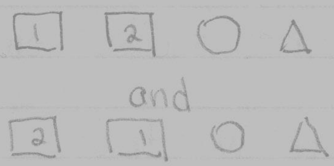


Now, since we've used up a shape, we only have 2 left to place in the second box; 2 possibilities. Let's put the triangle in.

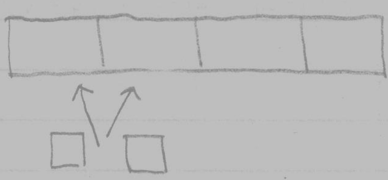


Now, only the square is left; 1 possibility for the last box. Remember what independent meant - events that don't change the number of possibilities for other events. As such, these are independent events, meaning again, the total possibilities are just the products of each.

So what happens if we have multiple indistinct elements, like two squares? Obviously we can't simply use $n!$ because permutations like



aren't distinct orderings, so we can't count that as 2 ways. So let's say we have r objects, n_1 of type 1, n_2 of type 2 ... n_r of type r. We still have n spots we need to fill.



□ = type 1, $n_1 = 2$
○ = type 2, $n_2 = 1$
△ = type 3, $n_3 = 1$

But now, we can put two squares in any two of 4 boxes. How many ways is that?

indistinct objects

$$6 = \frac{4!}{2!\,(4-2)!} = \binom{4}{2}$$

boxes

This notation is called "n choose k" or "4 choose 2", and the proof is somewhat arduous so we'll cover it in combinations instead

Following the logic before, we then have 3 separate events:

a) squares: $\binom{4}{2}$

b) circles: $\binom{2}{1}$
Since squares have chosen their spaces, there's only 2 left

c) triangles: $\binom{1}{1}$
Triangle takes the last empty space.

So the possibilities look like this:

$$P = \binom{4}{2}\binom{2}{1}\binom{1}{1} \quad \text{because of independence}$$

$$= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}$$

$$= \left[\frac{n!}{n_1!\,\cancel{(n-n_1)!}}\right]\left[\frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}}\right]\left[\frac{\cancel{(n-n_1-n_2)!}}{n_3!\,(n-n_1-n_2-n_3)!}\right]$$

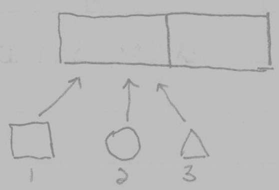
and remember $n-n_1-n_2-n_3=0$

$$= \frac{n!}{n_1!\,n_2!\,n_3!}$$

with the generalized form becoming

$$\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_r!}$$

In the final case, we have less boxes than distinct objects. For example, how many orderings are there of 2 objects out of a set of 3?



Same thing applies here: 3 possibilities for the first box, and 2 for the second.

$$P^3_2 = (3)(2) = 6$$

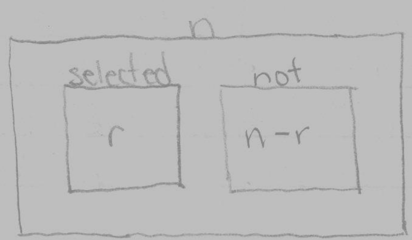
or generally,

$$P^n_r = (n)(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Another way to look at this is as such:

We've got n elements in our set. There are n! permutations if we have n boxes.

But we don't! We've only got r boxes. So we select r of them. The remaining (n-r) elements we don't care about. There are (n-r)! ways to permute the unselected ones.



So we take our total possibilities, then just chop off the (n-r)! possibilities we can never achieve.

Combinations

Where permutations were concerned about ordering, combinations are concerned with the ways to select a subset of r items out of a larger set of n items.

This comes back to our:

$$C_r^n = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

where in almost all cases, $r < n$. $\binom{n}{r}$ where $r > n$ simply gives 0 : you can't choose more elements out of a set than the set itself contains.

It's like asking a friend to pick up an extra copy of the assignment for you when there aren't any left. All you'll be left with is no assignment and a funny look from your buddy.

We can actually extrapolate the combination equation from permutations of P_r^n.

$$P_r^n = \frac{n!}{(n-r)!} \quad \text{and} \quad C_r^n = \frac{n!}{r!\,(n-r!)}.$$

So why do we have the extra $r!$? It's fairly simple - now, instead of just not giving a shit about the orderings of elements we DIDN'T choose, we also don't care about the order of the things we DID choose.

It's fairly intuitive that permutations of r elements from a set of n, ignoring order, is just combinations

There are a few other notable properties of combinations, the first of which is:

$$\binom{n}{0} = \binom{n}{n} = 1$$

If you don't want to choose anything, there's only one way to do this - leaving all the elements in the set. Note that this is the same if you want everything. Choosing elements and leaving them behind are really the same type of thing, which leads us to the next property:

$$\binom{n}{r} = \binom{n}{n-r}$$

You can look at this property as the difference between choosing r elements or leaving behind n-r. At the end of the day, if you've got a basket of 5 fruits, the number of ways to grab 2 of them is the same as the number of ways to leave behind 3.

Taking 2 (Yours)	Leaving 3 (Basket)
A, B	A, B, C
A, C	A, B, D
A, D	A, B, E
A, E	A, C, D
B, C	A, C, E
B, D	A, D, E
B, E	B, C, D
C, D	B, C, E
C, E	B, D, E
D, E	C, D, E

Finally, there's one last property that's not quite as intuitive

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Let's use a concrete example to help this along.

ex. Again, say we have 5 fruits, and we want to select 2.

Basket

$5 \left\{ \begin{array}{l} \text{Apple} \\ \text{Banana} \\ \text{Strawberry} \\ \text{Kiwi} \\ \text{Orange} \end{array} \right.$

Let's say we've got a raging hard-on for apples. Nothing quite tickles our fancy like apples do.

So we put the apple aside, shrinking the size of the basket to 4, knowing for sure we want the apple.

Basket

Banana
Strawberry
Kiwi
Orange

You

Apple
?

Now, we've reduced the problem to selecting 1 fruit from 4.

In other words, this is our $\binom{n-1}{r-1}$ different ways of selecting 2 fruits, where one is FOR SURE an apple.

But where do the remaining $\binom{n-1}{r}$ combinations come from?

Our possibilities look like this currently:

You		You		You		You
Banana	or	Strawberry	or	Kiwi	or	Orange
Apple		Apple		Apple		Apple

But here's where we've gone wrong - blinded by our greed, we've forgotten about combinations NOT containing apples!

So how many combinations of two fruits can we make WITHOUT apples?

basket w/o apple

$$\binom{4}{2}, \text{ or simply, } \binom{n-1}{r} \text{ ways.}$$

$$\therefore \quad \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

combinations that definitely contain a certain element

combinations definitely w/o that element

Nice! Time to move on.

Binomial Theorem

It's very tempting to say that the binomial theorem looks like a complex way to calculate the results of binomial expansion, but that's not necessarily correct.

It's more correct to say the binomial theorem IS the method of binomial expansion.

So let's do some grade school review.

$$(x_1 + y_1)^2 = (x_1 + y_1)(x_1 + y_1)$$

If you guys rememember FOIL (first outer inner last), let's break down that process.

$$\text{First: } (x_1 + y_1)(x_1 + y_1)$$
$$= x_1 x_1$$

$$\text{Outer: } (x_1 + y_1)(x_1 + y_1)$$
$$= x_1 y_1$$

$$\text{Inner: } (x_1 + y_1)(x_1 + y_1)$$
$$= y_1 x_1$$

$$\text{Last: } (x_1 + y_1)(x_1 + y_1)$$
$$= y_1 y_1$$

What are we actually doing here? In each step, we have a box that can only contain 2 elements. Note that this is exactly equal to how many terms there are in the binomial.

In each case, we can select EITHER x_1 or y_1. One of each is valid, and selecting x_1 or y_1 twice is valid, too.

Let's introduce part of the binomial theorem now.

$$x_1^{\,r}\, y_1^{\,n-r}$$

In this case, $n=2$.

So if we simply plug in some numbers,

*$r = 0$: selecting $\left.\begin{matrix} 0 \; x_1\text{s} \\ 2 \; y_1 \end{matrix}\right\}$ 1 way to do this : $y_1 y_1$

$r = 1$: selecting $\left.\begin{matrix} 1 \; x_1 \\ 1 \; y_1 \end{matrix}\right\}$ 2 ways: $x_1 y_1 \quad y_1 x_1$

$r = 2$: selecting $\left.\begin{matrix} 2 \; x_1 \\ 0 \; y_1 \end{matrix}\right\}$ 1 way : $x_1 x_1$

What we're doing is finding permutations of x_1 and y_1! So that means we need a choose:

$$\binom{n}{r} x_1^{\,r} y_1^{\,n-r} \quad **$$

But of course, with multiplication, we need EVERY valid r, which is 0 up to n.

Now, for the final part of FOIL, you need to add the individual terms together - a summation is perfect here.

$$\sum_{r=0}^{n} \binom{n}{r} x_1^{\,r} y_1^{\,n-r} = (x_1 + y_1)^n$$

And there we have it! We've derived the entire binomial theorem, simply by counting.

*technically we're just picking a number of x_1s, where the x_1 in each of $(x_1 + y_1)$ is considered a distinct entity, which is why $x_1 y_1$ and $y_1 x_1$ are considered different.

** $\binom{n}{r} = \binom{n}{r,\, n-r}$ in this case

Multinomial Theorem

So how do we expand this to accommodate multinomials? First, we're going to change all variables to x_i, and assume that when the subscripts aren't matching, they're totally different types of variables

$$(x_1 + x_2 + \ldots + x_r)^n = \sum \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

where there are n_1 type x_1s, n_2 type x_2s, and so on, such that $n_1 + n_2 + \ldots + n_r = n$.

Each x_1 is indistinguishable from each other x_1 - there are multiple indistinct instances of multiple distinct types. Does this sound like something we've done before? It should - permuting multiple indistinct objects

This is where the

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{(n_1!)(n_2!)\ldots(n_r!)} \quad \text{comes in.}$$

This isn't actually much different from the time before, we just can't reduce it to $\binom{n}{r}$. If you take the $r=2$ case, we simply revert back to the binomial theorem.

Number of Positive Integer Solutions Satisfying an Equation

Jesus, wordy section title alert.

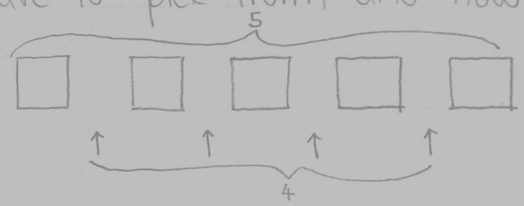
Honestly, this is more of an example than anything, so let's label it as such.

ex. Given an integer-valued vector $[x_1, \ldots, x_r]$ how many solutions exist that satisfy

$$x_1 + x_2 + \ldots + x_r = n \qquad (r \leq n)$$
$$x_i > 0 \,?$$

So in other words: how many WAYS can we assign positive values to each x_i such that their sum is n?

Since sums don't care about order, it must be a combination of some sort. But how many things do we have to pick from, and how many do we pick?



Let's say $n = 5$, and $r = 3$.

The boxes denote the total value we must achieve. Since we have 3 variables, we can divide these boxes* using 2 cuts - valid at any of the arrows (none on the sides as those wouldn't divide our set). These valid cutting spots must also be distinct as we can't cut in the same place.

$\therefore$ we are choosing $\underline{2}$ spots to cut from $\underline{4}$ valid spaces, meaning we can do this

$$\binom{n-1}{r-1} \text{ ways.}$$

*into 3 groups, each being the value of a variable

Number of Non-Negative Solutions Satisfying an Equation

So what if we took the previous question and wanted non-negative solutions, too?

Say we found every single combination in the strictly positive case.

One solution: $\left.\begin{matrix} x_1 = 1 \\ x_2 = 2 \\ x_3 = 2 \end{matrix}\right\} \Sigma = 5$

If we decrease every element by 1, we have now included 0 as a possibility.

New modified solution: $\left.\begin{matrix} x_1 = 0 \\ x_2 = 1 \\ x_3 = 1 \end{matrix}\right\} \Sigma = 2$

But our overall sum has decreased by r. Rectifying this is simple: we just need to change the sum we want to add up to to have an offset of +r.

$\therefore$ There are $\binom{n-1+r}{r-1}$ non-negative solutions to the same problem.