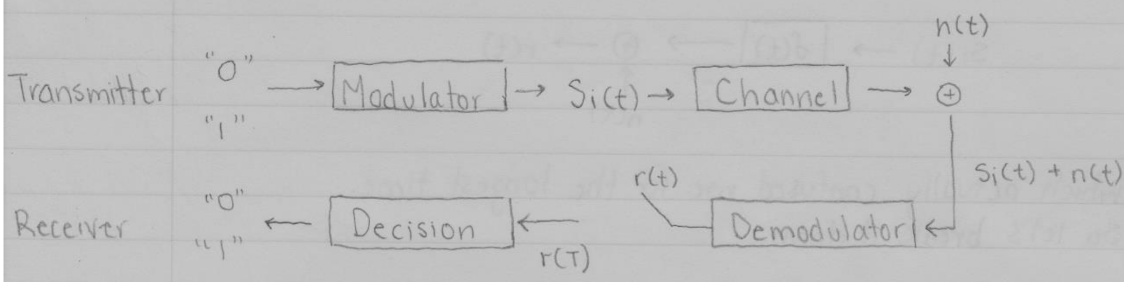
ECE 358 Examples / Derivations
Physical Layer
(Pre-Midterm)

<u>A Model for Physical Layer Communications</u>



What the fuck is this?

This is a model for an entire physical transmitter/receiver system. We're going to learn how to analyze it.

We want to look at:

- the probability of transmission error
- the bandwidth required
- the signal power/energy

This model differs from the pre-midterm model in 318 in two ways. Firstly, we have some noise signal $n(t)$. Secondly, after the demodulator, we have a decision machine that samples $r(t)$ at some time T, so it can then reconstruct the binary signal we originally sent.

Firstly, let's take a look at the noise signal.

Additive White Gaussian Noise

The slides have this diagram:

$$
\begin{tikzcd}
s_i(t) \arrow[r] & \boxed{\delta(t)} \arrow[r] & \oplus \arrow[r] & r(t) \\
 & & n(t) \arrow[u] &
\end{tikzcd}
$$

which actually confused me for the longest time. So let's break it down.

$s_i(t)$ is our information signal. It can be 1, 0, or something more complex.

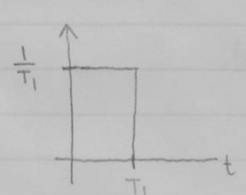
$s_i(t)$ is then sent into some filtering block with transfer function $h(t) = \delta(t)$, the impulse function. Remember that:

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases}$$

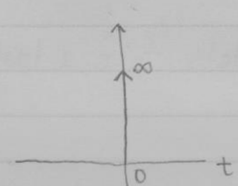
and its area,

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

We can create the delta function from the unit pulse with area 1:



and reducing its width, or pulse length to 0



So, given than $s_i(t) = 1$ from $[0, T]$, what happens if we let $h(t)$ be the unit pulse of length $\frac{3T}{2}$?

Remember that sending something into a filtering block means that the result is:

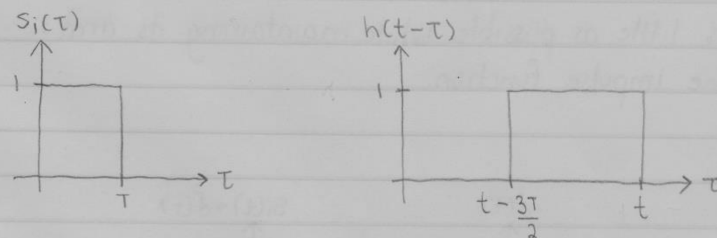
$$\mathcal{F}^{-1}\{\mathcal{F}\{s_i(t)\}\,\mathcal{F}\{h(t)\}\}$$
$$=$$
$$s_i(t) * h(t)$$

the inverse Fourier of the product of the Fourier-transformed inputs, which is the convolution of the two inputs in the time domain.

$$s_i(t) * h(t) = \int_{-\infty}^{\infty} s_i(\tau)\, h(t-\tau)\, d\tau$$

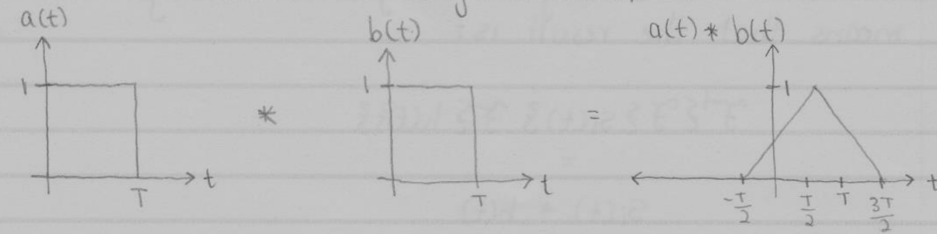
$$\text{where } s_i(\tau) = \begin{cases} 1 & \text{for } \tau \in [0, T] \\ 0 & \text{otherwise} \end{cases}$$

$$h(t-\tau) = \begin{cases} 1 & \text{for } (t-\tau) \in [0, 3T/2] \\ 0 & \text{otherwise} \end{cases}$$

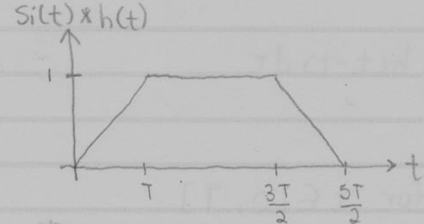


In convolution, we start $h(t-\tau)$ at $\tau = -\infty$, and we increase τ. Whenever $h(t-\tau)$ intersects with $s_i(\tau)$ we create a new graph that shows <u>how much intersected area there is as we vary τ</u>. Wikipedia's convolution page has a FANTASTIC animation of the exact same case we're trying to do here except where both rectangles are the same width.

Their case constructs a triangle of width 2T.



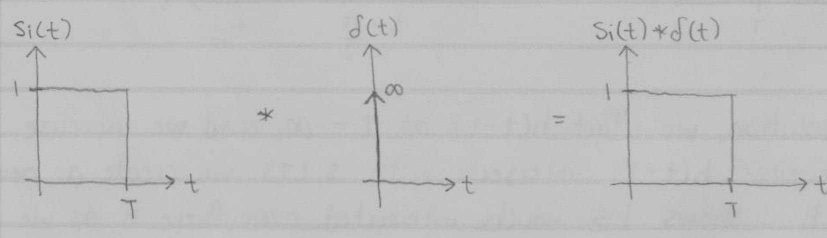
In our case, it makes a trapezoid with a constant top width of $3T/2 - T = T/2$, and overall width of $3T/2 + T = 5T/2$.



This "stretching" of our original signal is called inter-symbol interference.

The stretch occurs because $h(t)$ is wider than $s_i(t)$. So how do we lessen the effects of $h(t)$? We shrink its width to as little as possible while maintaining its area: creating the impulse function.

So:



So we've just gotten back exactly what we've started with. So why is this useful?

Imagine $s_i(t)$ isn't a constant but rather a continuous time signal. Sending it into a block with transfer function with width T is very similar (if not exactly the same) as sampling $s_i(t)$ for a duration T. It constructs an approximation of $s_i(t)$. So by letting $T \to 0$, we build a better and better approximation.

$$s_i(t) \longrightarrow \boxed{\delta(t)} \longrightarrow x(t) \qquad \begin{aligned} x(t) &= s_i(t) * \delta(t) \\ &= \mathcal{F}^{-1}[\,S_i(\omega)\,(1)\,] \\ &= s_i(t) \end{aligned}$$

This is simply an idealized sampling of our signal $s_i(t)$. As far as the overall picture goes, this mechanism would appear inside the Channel block itself.

Alright, enough deviation, as past this point, knowing this isn't actually terribly important. Back to AWGN. It is named this way purposefully.

Additive, like the $\overset{n(t)}{\downarrow}\!\!\oplus$ block, simply means it distorts $s_i(t)$ by adding or subtracting some amount from the amplitude of $s_i(t)$.

Gaussian, because the amplitude of the noise* follows a Gaussian distribution:

$$n(t_0) = N(t_0 \mid \underset{\text{mean}}{\mu}, \underset{\text{variance}}{\sigma^2}) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

White, because its power over the entirety frequency spectrum is constant. That is, its power spectral density, $S_n(f)$, is equal to $\frac{N_0}{2}$,

$$S_n(f) = \frac{N_0}{2}$$

for all frequencies f.

*at some t_0

How we arrive at this requires some review of some probability concepts.

Random Processes

A random process $X(t,S)$ yields a variety of results for some value of t, and S.

So for example,

$$X(t_0, S_3) = X_3(t_0)$$

some time t_0 — some state S_3 — the resulting time-dependent variable using $t = t_0$

There are a few cases:

t	S	$X(t,S)$
variable	variable	family of deterministic functions
fixed	variable	a random variable
variable	fixed	a time-dependent function
fixed	fixed	a real number

We can then define covariance, as

$$\mathrm{Cov}(X,Y) = E[(X-E[X])(Y-E[Y])]$$

which gives some idea about how two random variables change together. (X and Y can only be random variables).

We can also define the correlation as

$$\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

which is a direct measure of the linear relationship between two random variables, on [-1, 1].

What is perhaps more important is autocovariance and autocorrelation - the relationship between two specific instances of a random process.

Autocovariance:

$$\mathrm{Cov}[X(t_1,S), X(t_2,S)] = E[(X_1 - E[X_1])(X_2 - E[X_2])]$$
$$= E[X_1X_2 - X_2E[X_1] - X_1E[X_2] + E[X_1]E[X_2])$$

Since the average of (X) + something is the same as average of (X + something),

$$= E(X_1X_2 - X_2E[X_1] - X_1E[X_2]) + \mu_{X_1}\mu_{X_2}$$

The expected value $E(aX) = aE(X)$, and $E(a+b) = E(a) + E(b)$. So we can extract the constants E(single thing) and break it up

$$= E(X_1X_2) - \mu_{X_1}\cancel{E(X_2)} - \mu_{X_2}\cancel{E(X_1)} + \mu_{X_1}\mu_{X_2}$$
$$= \boxed{E(X_1X_2) + \mu_{X_1}\mu_{X_2}}$$

Now, here's something kind of fucked: we're only particularly interested in autocorrelation (which for some reason is used interchangeably with regular correlation). We also have a definition for correlation I have NEVER seen, nor am I able to derive it.

Sorry guys, I just don't know where it comes from or how to get to it, but here it is:

$$R_X(t_1, t_2) = E[X_1 X_2]^{*}$$ (on next page)

Then, we have wide-sense stationary, which is a name that describes anything with:

$$R_X(\tau) = E[X(t,S)\,X(t-\tau,S)]$$
and
$$E[X(t,S)] = \text{constant}$$

Wide-sense stationary is used to describe random processes where the mean (and variance, though I guess we don't care about it?) is independent of time.

The only thing autocorrelation depends on is the difference between t_1 and $t_2 = t_1 - \tau$, which is why we can write $R_X(\tau)$ instead of $R_X(t, t-\tau)$.

It also helps that autocorrelation is an even function (we won't do the proof here but it's easy as far as proofs go), so

$$R_X(\tau) = R_X(-\tau)$$

and so it doesn't matter if it's $t-\tau$ or $t+\tau$. One thing that is interesting to see is that the maximum autocorrelation occurs at $\tau = 0$. Intuitively, this makes sense because $X(t,S)$ and $X(t+0,S)$ is the exact same thing so it makes sense that they're perfectly correlated. That is,

$$|R_X(\tau)| \leq R_X(0)$$

* I figured it out. This is the definition for autocorrelation which <u>IS NOT</u> analagous to the correlation coefficient.

$$R_X(t_1, t_2) = E[X(t_1)\, X(t_2)]$$
$$= \operatorname{cov}(X(t_1), X(t_2)) + \mu_{X(t_1)}\mu_{X(t_2)}$$

This is a completely separate thing, which is more analagous to covariance than it is correlation. Alright, back to our regular scheduled programming.

How we prove $|R_X(\tau)| \le R_X(0)$ is a little odd. Let's start with this:

$$E[(X(t) \pm X(t-\tau))^2] \ge 0$$

While I can't imagine exactly why you'd ever want to calculate something like this, let's expand the inside.

$$E[X^2(t) \pm 2X(t)X(t-\tau) + X^2(t-\tau)] \ge 0$$
$$E[X^2(t)] \pm 2E[X(t)X(t-\tau)] + E[X^2(t-\tau)] \ge 0$$

The middle term, given that X is a WSS process, is exactly the definition of $R_X(\tau)$.

$$E[X^2(t)] \pm 2R_X(\tau) + E[X^2(t-\tau)] \ge 0$$

Remember that τ is the difference of t. In $E[X^2(t)]$ and $E[X^2(t-\tau)]$, the difference is 0. So both create $R_X(0)$.

$$2R_X(0) \pm 2R_X(\tau) \ge 0$$

From here, it's pretty simple:

$$R_X(0) \ge \mp R_X(\tau)$$

Now, for something more interesting: $R_X(0)$ is the average power of the random process X. Surprisingly, this one is actually quite easy to prove. We know:

$$P_{avg} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 \, dt$$

Though it doesn't make much sense to find $E[P_{avg}]$, the average of average power, we're going to do it anyway.

$$E[P_{avg}] = E\left[\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 \, dt \right]$$

$E[P_{avg}]$ obviously is still P_{avg}. What parts does the $E[\,]$ apply to on the right side? Only $f(t)$, right?

$$P_{avg} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} E[f(t)^2] \, dt$$

And we just saw $R_X(0)$ is $E[X^2(t)]$, so:

$$P_{avg} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_X(0) \, dt$$

Now, $R_X(0)$ is NOT time-dependent, so the limit and integral just resolves $R_X(0)$ to itself.

$$\therefore P_{avg} = R_X(0)$$

Finally, as if things didn't get weird enough, the Wiener-Khinchin theorem says:

$$S_X(f) = \mathcal{F}\{R_X(\tau)\}$$
$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\}$$

Autocorrelation and power spectral density are Fourier transform pairs.

I'm going to prove this, but not rigorously (that's too much work, you bastards).

$$\mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} \, d\tau$$

This is by definition. However, iterating τ from $-\infty$ to ∞ should produce an infinite number of random processes $x(t)$ since

$$R_X(\tau) = E[\, x(t)\, x(t-\tau)\,]$$

For each $x(t)$, we can say its Fourier transform $X_T(f)$ is:

$$X_T(f) \triangleq \int_{-T/2}^{T/2} x(t) e^{-j2\pi f t} \, dt$$

And for each $X_T(f)$, we can define its power spectral density

$$S_T(f) \triangleq \frac{1}{T} | X_T(f) |^2$$

But since $x(t)$ is a random process, <u>for EVERY f</u>, $X_T(f)$ becomes a random variable and $\frac{1}{T}|X_T(f)|^2$ is just some transformation of a random variable.

So we have to amend our definition of $S_T(f)$ because $S_T(f)$ can't be a bunch of random variables - it should be a number for some f instead. So let's re-define it:

$$S_T(f) \triangleq E[\, \tfrac{1}{T} | X_T(f) |^2 \,]$$

where we just take the average. Why is this okay? Because we're working with white Gaussian noise, which by definition, has a constant power spectral density $\forall f$.

So for one out of infinite $x(t)$, $x(t)$'s power spectral density

$$S_X(f) = \lim_{T\to\infty} S_T(f)$$

$$= \lim_{T\to\infty} E\left[\frac{1}{T}|X_T(f)|^2\right]$$

as we need to account for all of the processes' random variables. How do we calculate this? Let's look at $E[|X_T(f)|^2]$ first.

$$E[|X_T(f)|^2] = E\left[\left|\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt\right|^2\right]$$

Here we've just expanded for the definition of $\mathcal{F}$.

$$= E\left[\left(\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt\right)\left(\int_{-T/2}^{T/2} x(\tau)e^{-j2\pi f\tau}d\tau\right)\right]$$

Expanded the square. We could use τ or α or whatever, it's just a dummy variable of integration

$$= E\left[\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} x(t)x(\tau)e^{-j2\pi f(t-\tau)}dt\,d\tau\right]$$

Multiplied the two integrals

$$= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} E[x(t)x(\tau)]e^{-j2\pi f(t-\tau)}dt\,d\tau$$

As we've done in the example before, we move the average to only affect the things it actually can. By definition, this is $R_X(t-\tau) = R_X(\tau-t)$, whichever we want.

$$= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} R_X(t-\tau)e^{-j2\pi f(t-\tau)}dt\,d\tau$$

Since

$$\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} f(t-\tau)\,dt\,d\tau = \int_{-T}^{T} (T-|\tau|)\,f(\tau)\,d\tau^{*},$$

we can apply this to our own scenario, yielding

$$E[|X_T(f)|^2] = \int_{-T}^{T} (T-|\tau|)\,R_x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

and as such

$$E[\tfrac{1}{T}|X_T(f)|^2] = \int_{-T}^{T} \frac{T-|\tau|}{T} R_x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

$$= \int_{-T}^{T} (1-\tfrac{|\tau|}{T})\,R_x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

But the $(1-\frac{|\tau|}{T})$ is ugly, so:

$$= \int_{-\infty}^{\infty} R_{x,T}(\tau)\,e^{-j2\pi f\tau}\,d\tau \quad \text{where } R_{x,T}(\tau) = \begin{cases} (1-\frac{|\tau|}{T})\,R_x(\tau), & |\tau|<T \\ 0 & |\tau|\geq T \end{cases}$$

So finally,

$$S_x(f) = \lim_{T\to\infty} E[\tfrac{1}{T}|X_T(f)|^2]$$

$$= \lim_{T\to\infty} \int_{-\infty}^{\infty} R_{x,T}(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

where $T\to\infty$ meaning $R_{x,T}(\tau) = (1-\frac{|\tau|}{\infty})\,R_x(\tau) = R_x(\tau)$

$$= \int_{-\infty}^{\infty} R_x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

$$= \mathcal{F}\{R_x(\tau)\}$$

Phew. Rough shit, boys. Let's summarize these properties.

* don't ask. I don't know either. I'm bad at math.

Summary of Autocorrelation Properties

① $R_X(\tau) = E[X(t)X(t-\tau)]$ for WSS process X

② $R_X(\tau) = R_X(-\tau)$

③ $|R_X(\tau)| \leq R_X(0)$

④ $P_{avg}(X) = R_X(0)$

⑤ $S_X(f) = \mathcal{F}\{R_X(\tau)\}$ and $R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\}$

Summary of White Gaussian Noise Properties

① $n(t)$ is a random process that mimics thermal noise, where $n(t_0)$ is a random Gaussian variable

② $E[n(t)] = 0$

③ $S_n(f) = N_0/2$, where N_0 is the noise power per unit frequency, and $\frac{1}{2}$ because we're considering all $f \in (-\infty, \infty)$ instead of $[0, \infty)$.

④ $R_N(\tau) = \frac{N_0}{2}\delta(\tau)$, since $R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\}$.

⑤ $P_{N_{avg}} = R_N(0) = \infty$ as an impulse is infinite height.

⑥ $S_{n_o}(f) = |H(f)|^2 S_n(f)$ if n_o is produced by filtering $n(t)$ through a block with transfer function $h(t)$

The last one isn't WGN-specific, but is useful. Not going to prove it as it's a concept from our math courses.

The Decision Device

Okay, so approximately 100 pages later, we know that

$$s_i(t) + n(t)$$

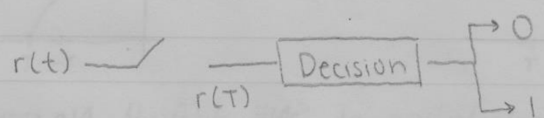
goes into the demodulator, and that $n(t)$ is AWGN with power spectral density $S_n(f)$. The demodulator

$$s_i(t) + n(t) \rightarrow \boxed{\begin{array}{c}\text{Demodulator}\\ h(t)\end{array}} \rightarrow r(t)$$

is just another block with transfer function $h(t)$, so as per property ⑥ :

$$r(t) = |H(f)|^2 s_i(t) + |H(f)|^2 n(t)$$

What happens from here?



We sample $r(t)$ at $t = T$:

$$r(T) = |H(f)|^2 s_i(T) + |H(f)|^2 n(T)$$

In order to optimize this detector, we want to maximize the valve of $|H(f)|^2 s_i(T)$ and minimize $|H(f)|^2 n(T)$, increasing our signal-to-noise ratio, by choosing the correct $H(f)$, and as such, the correct $h(t)$.

The signal-to-noise ratio is given as

$$(SNR)_o = \frac{[S_o(T)]^2}{E[n_o^2(T)]}$$

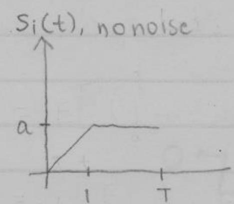
where $S_o(T) = h(T) * s_i(T) = |H(f)|^2 s_i(T)$ and
$n_o(T) = h(T) * n(T) = |H(f)|^2 n(T)$

So how do we maximize this ratio? We let

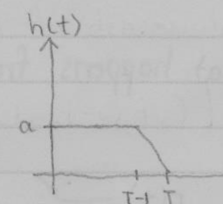
$$h(t) = A \times s_i(T-t) \ , \ A = \left(\frac{2}{N_0}\right)$$

which is called a "matched filter". Why that? Great news, we don't need to know why. All we need to know is that making the response $h(t)$ a flipped version of $s_i(t)$ maximizes SNR.

So if:



then $h(t)$ should be



But that still makes calculation of SNR difficult. No one wants to do convolution integrals (least of all me).

So is there another way we can express it? Something a bit simpler?

There is: we can express it using signal energy. Let's take a look.

Remember

$$E_s = \int_0^T |s(t)|^2 \, dt$$

Let's make an assumption. Let's say

$$S(t) = \begin{cases} a & \text{when } t \in [0,T] \\ 0 & \text{otherwise} \end{cases}$$

We can then define $S_1(t)$ and $S_2(t)$ as

$$S_1(t) = S(t) = \begin{cases} a & t \in [0,T] \\ 0 & t \notin [0,T] \end{cases}$$

$$S_2(t) = -S(t) = \begin{cases} -a & t \in [0,T] \\ 0 & t \notin [0,T] \end{cases}$$

Using what we've found from before, letting $h(t) = s_i(T-t)$,

$$h_1(t) = S_1(T-t) = \begin{cases} a & t \in [0,T]^* \\ 0 & \text{otherwise} \end{cases}$$

$$h_2(t) = S_2(T-t) = \begin{cases} -a & t \in [0,T] \\ 0 & \text{otherwise} \end{cases}$$

$* \; 0 \le T-t \le T$

$-T \le -t \le 0$

$T \ge t \ge 0$

Now, what is $S_0(T) = S_i(t) * h(t)$? We'll do type 1 first.

$$S_0(T)\Big|_{\text{type 1}} = S_1(t) * h_1(t)$$

$$= \int_{-\infty}^{\infty} h_1(t)\, S_1(T-t)\, dt$$

or

$$= \int_{-\infty}^{\infty} S_1(t)\, h_1(T-t)\, dt$$

Since these only exist on $[0, T]$

$$= \int_0^T S_1^2(t)\, dt$$

or

$$= \int_0^T S_1^2(T-t)\, dt$$

which $\underline{\text{IS}}$ the signal energy of S_1. So by selecting our h(t) as a matched filter of $s_1(t)$, the created output of the demodulator (excluding the noise) is exactly Es.

But obviously the constant $A = 2/N_0$ changes it. As such, $S_0(t)$ isn't exactly Es, but rather:

$$S_0(t)\big|_{type\,1} = \frac{2E_s}{N_0}$$

So as such, the average of $s_1(t)$ and $s_2(t)$ is:

$$S_0(t) = \frac{S_0(t)|_1 - S_0(t)|_2}{2}$$

$$= \frac{\frac{2E_s}{N_0} - \left[-\frac{2E_s}{N_0}\right]}{2}$$ ← as type 2 is negative

$$= \frac{2E_s}{N_0}$$

Now we have an idea of our numerator. What about the denominator?

$$E[n_0^2(t)] = R_{n_0}(0)$$

by definition. We can expand $R_{n_0}(0)$ to be

$$= \int_{-\infty}^{\infty} S_{n_0}(f)\,df$$

Note we CANNOT say that $R_{n_0}(0) = N_0/2$ because we don't know what happened to the noise after it went through the filter

Remember that $S_{no}(f) = |H(f)|^2 S_n(f)$

$$\therefore \quad = \int_{-\infty}^{\infty} |H(f)|^2 S_n(f)\, df$$

where now, we can say $S_n(f) = N_0/2$.

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df$$

Parseval's theorem states that $\int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$
so we can apply that:

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(t)\, dt$$

We defined $h(t)$ to only exist on $[0,T]$ so

$$= \frac{N_0}{2} \int_{0}^{T} h^2(t)\, dt$$

$$= \frac{N_0}{2} \int_{0}^{T} \left(\frac{2}{N_0} s(T-t) \right)^2 dt$$

$$= \frac{N_0}{2} \left[\frac{2}{N_0} \right]^2 \underbrace{\int_{0}^{T} s^2(T-t)\, dt}_{E_s}$$

$$E[n_o^2(t)] = \frac{2E_s}{N_0} \quad \text{again!}$$

So, let's put all of our puzzle pieces together.

$$(SNR)_o = \frac{\left[\frac{2E_s}{N_0} \right]^2}{\frac{2E_s}{N_0}}$$

$$= \frac{2E_s}{N_0}$$

So if we know the power spectral density of $n(t)$ and the energy of $s(t)$, finding $(SNR)_o$ is trivial.

If we divide both the numerator and denominator by T, we get another variation:

$$(SNR)_0 = \frac{E_s/T}{\frac{N_0}{2}/T} = \frac{\text{signal power}}{\text{noise power}}$$

So up to this point, we've learned how to minimize the effects of noise on our signal. So how do we actually determine the bit sent?

$$r(T)|_1 \sim N\left(\frac{2E_s}{N_0}, \frac{2E_s}{N_0}\right) \qquad r(T)|_2 \sim N\left(-\frac{2E_s}{N_0}, \frac{2E_s}{N_0}\right)$$

The samples at time T give us normal distributions with mean $\pm 2E_s/N_0$ and variance $2E_s/N_0$ depending on which type we managed to pick up at the time of sampling.

First, we average the means:

$$\mu = \frac{\frac{2E_s}{N_0} + \left(-\frac{2E_s}{N_0}\right)}{2}$$
$$= 0$$

Then, our decision is decided as such:

$$\begin{cases} 1 & \text{if } r(T) \geq \mu \\ 0 & \text{if } r(T) < \mu \end{cases}$$

This is called the decision threshold. But of course, the noise must cause errors. When do these happen? What are the chances of it happening?

Obviously, an error is when we decide that we see a 1 but it was actually a 0 and vice versa.

$$P\{\text{error}\} = P\{r(T) < \mu \mid 1\}P\{1\} + P\{r(T) \geq \mu \mid 0\}P\{0\}$$

Generally we assume the probability of sending a 1 or 0 is 1/2 for each.

$$= \frac{1}{2}\left[P\{r(T) < \mu \mid 1\} + P\{r(T) \geq \mu \mid 0\}\right]$$

μ for us is 0, and will generally be 0 as we'll usually define a "1" to be some $s(t)$ and "0" to be $-s(t)$.

$$= \frac{1}{2}\left[P\{r(T) < 0 \mid 1\} + P\{r(T) \geq 0 \mid 0\}\right]$$

where by some rather convoluted proof,

$$P\{r(T) < 0 \mid 1\} = \Phi\left(\frac{0 - \frac{2E_s}{N_0}}{\sqrt{\frac{2E_s}{N_0}}}\right) \leftarrow \text{type 1}$$

where Φ is the cumulative distribution function of the standard normal distribution, $N(0,1)$. This applies to the other case as well:

$$P\{r(T) \geq 0 \mid 1\} = 1 - \Phi\left(\frac{0 - \left(-\frac{2E_s}{N_0}\right)}{\sqrt{\frac{2E_s}{N_0}}}\right) \leftarrow \text{type 2}$$

$$\therefore \quad P\{\text{error}\} = \frac{1}{2}\left[\Phi\left(\frac{-\frac{2E_s}{N_0}}{\sqrt{\frac{2E_s}{N_0}}}\right) + 1 - \Phi\left(\frac{\frac{2E_s}{N_0}}{\sqrt{\frac{2E_s}{N_0}}}\right)\right]$$

$$= \frac{1}{2}\left[\Phi\left(-\frac{2E_s}{N_0}\right) + 1 - \Phi\left(\frac{2E_s}{N_0}\right)\right]$$

Again, by some magic this is

$$= 1 - \Phi\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

which is by definition

$$= Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$ (there's a table with what Q gives for some input)

Where $2E_s/N_0$ is the SNR, so

$$= Q(\sqrt{SNR})$$

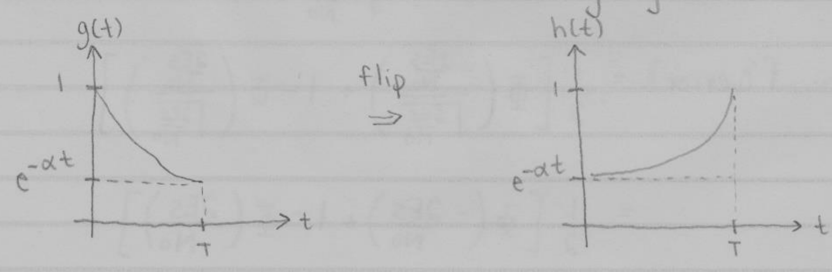
To finish this off, let's walk through the first question in Assignment 2.

Question 1

$g(t) = e^{-\alpha t}$ on $(0, T)$ is corrupted by white Gaussian noise with mean 0 and $S_n(f) = N_0/2$. The optimum detector is a matched filter.

a) Find and sketch the impulse response for the matched filter.

In order to do that, we first have to get $g(t)$.



We just flip it on the range that $g(t)$ exists!

b) Derive an expression for $(SNR)_{0_{t_{max}}}$ using α, T, and N_0.

$$(SNR_0) = \frac{2E_s}{N_0}$$

So we need to find what E_s is.

$$E_s = \int_{-\infty}^{\infty} |g(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} (e^{-\alpha t})^2 \, dt$$

$$= \int_{-\infty}^{\infty} e^{-2\alpha t} \, dt$$

Since this only exists on $(0, T)$,

$$= \int_{0}^{T} e^{-2\alpha t} \, dt$$

$$= \frac{1-e^{-2\alpha t}}{2\alpha} \Big|_0^T$$

So

$$(SNR)_0 = \frac{\frac{(1-e^{-2\alpha t})}{2\alpha} 2}{N_0}$$

$$= \frac{1-e^{-2\alpha t}}{N_0 \alpha}$$

So when is the energy greatest? Energy is an integral, so the more area we accumulate the better. As such, its maximum occurs at $t = T$

$$\therefore (SNR)_{0,max} = \frac{1-e^{-2\alpha T}}{N_0 \alpha}$$

c) $s_0(t) = g(t)$ or $s_1(t) = 0$ is transmitted every T seconds, with equal probability, over a WGN channel with mean 0 and PSD $N_0/2$. Sketch the decision variable corresponding to $s_0(t)$ and $s_1(t)$ and determine the probability of error.

At the decision device,

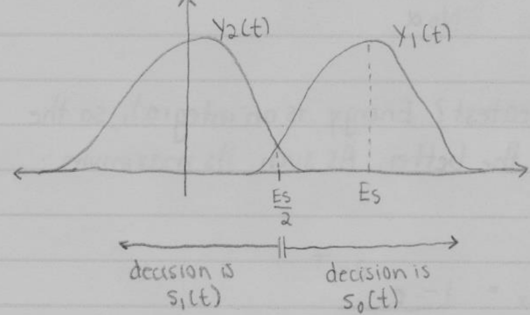
$$r(T) = s_i(T) + n_0(T)$$

$n_0(T)$ is a Gaussian variable, so so is $r(T)$. $n_0(T)$ has mean 0. $s_i(T)$, while not being a random variable, is equivalent to its energy E_s, if $s_0(t)$ was sent, and 0 if $s_1(t)$ was sent.

The variance of $n_0(T)$ is $\frac{N_0}{2} E_s$. E_s itself is a constant, so regardless of what was sent, the variance of $r(T)$ is $\frac{N_0}{2} E_s$.

$$\Rightarrow r(T) \sim N\left(E_s, \frac{N_0 E_s}{2}\right) \quad \text{if } s_0(t) = y_1(t)$$
$$\sim N\left(0, \frac{N_0 E_s}{2}\right) \quad \text{if } s_1(t) = y_2(t)$$

The decision threshold occurs at the average of the means: in this case it is $E_s/2$.



Unfortunately since this case isn't the same as the one we did before, we can't just say $P\{error\} = Q(\sqrt{SNR})$ and call it a day, we have to re-derive it again.

$$P\{error\} = P\{r(T) < \tfrac{E_s}{2} \mid s_0(t)\}\, P\{s_0(t)\} + P\{r(T) \geq \tfrac{E_s}{2} \mid s_1(t)\}\, P\{s_1(t)\}$$
$$= \tfrac{1}{2}\left[P\{r(T) < E_s/2 \mid s_0(t)\} + P\{r(T) \geq E_s/2 \mid s_1(t)\}\right]$$

Up until now I was hoping we didn't need to know the definition of the Gaussian, but we do. It's this:

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\therefore P\{error\} = \frac{1}{2}\left[\int_{-\infty}^{\frac{E_s}{2}} \frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0E_s}{2}}}\, e^{-\frac{(r-E_s)^2}{E_sN_0}}\, dr + \int_{\frac{E_s}{2}}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0E_s}{2}}}\, e^{-\frac{(r-0)^2}{E_sN_0}}\, dr\right]$$

Great. Exactly what I wanted to see. Let's clean up a bit.

$$= \frac{1}{2}\left[\int_{-\infty}^{E_s/2} \frac{e^{-\frac{(r-E_s)^2}{E_sN_0}}}{\sqrt{\pi N_0 E_s}}\, dr + \int_{E_s/2}^{\infty} \frac{e^{-\frac{r^2}{E_sN_0}}}{\sqrt{\pi N_0 E_s}}\, dr\right]$$

The Q function itself is $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{\left(-\frac{u^2}{2}\right)}\, du$

so let's try our best to manipulate into this form. We'll just do the first integral for simplicity's sake.

$$\int_{-\infty}^{E_s/2} \frac{e^{-\frac{(r-E_s)^2}{E_sN_0}}}{\sqrt{\pi N_0 E_s}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{E_s/2} \frac{e^{-\frac{[r^2-2rE_s+E_s^2]}{E_sN_0}}}{\sqrt{\frac{E_sN_0}{2}}}$$

Okay, I'm stuck. This is the limit of my terrible mathematical abilities. I'll write out the rest for the sake of completion but know that I don't know what they're doing.

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{\frac{Es}{2No}}} e^{\frac{-v^2}{2}} dv + \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{Es}{2No}}}^{\infty} e^{-\frac{u^2}{2}} du \right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{Es}{2No}}}^{\infty} e^{\frac{-v^2}{2}} dv + \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{Es}{2No}}}^{\infty} e^{-\frac{u^2}{2}} du \right]$$

$$= \frac{1}{2}\left[Q\left(\sqrt{\frac{Es}{2No}}\right) + Q\left(\sqrt{\frac{Es}{2No}}\right) \right]$$

$$= Q\left(\sqrt{\frac{Es}{2No}}\right)$$

Well, that's it. What a fucking ride, am I right? Jesus

HOLY SHIT, two days later, I finally figured it out. What we have is a NON-standard normal distribution and we need to express this in terms of the standard normal.

Thank god I remembered this property from 316 101. It says

$$F_X(a) = CDF_X(a) = \Phi\left(\frac{a-\mu}{\sigma}\right)$$

With this, we can express ANY normal distribution's CDF in terms of the standard normal. We want a to be $Es/2$ in type 1, so:

$$CDF_1(Es/2) = \Phi\left[\frac{Es/2 - Es}{\sqrt{\frac{NoEs}{2}}}\right] = \Phi\left[\left(-\frac{1}{2}Es\right)\left(\sqrt{\frac{2}{NoEs}}\right)\right]$$

should be Φ here but I forgot

$$CDF_1(a) = \left[-\frac{\frac{\sqrt{2}}{2}E_s}{\sqrt{N_oE_s}}\right]$$

$$= \left[-\frac{\sqrt{2}\,E_s}{\sqrt{4}\sqrt{N_oE_s}}\right]$$

$$= \left[-\frac{E_s}{\sqrt{2N_oE_s}}\right]$$

$$= \left[-\sqrt{\frac{E_s}{2N_o}}\right]$$

So as such, we can now say:

$$CDF_1(E_s/2) = \int_{-\infty}^{-\sqrt{\frac{E_s}{2N_o}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 1 - \int_{-\sqrt{\frac{E_s}{2N_o}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

by the virtue of this being a probability so it needs to add up to 1.

$$= 1 - Q\left(-\sqrt{\tfrac{E_s}{2N_o}}\right)$$

and by definition of how the Q function works,

$$Q(x) = 1 - Q(-x)$$
$$= 1 - \Phi(x)$$

$$\therefore\ CDF_1(E_s/2) = Q\left(\sqrt{\tfrac{E_s}{2N_o}}\right)$$

and we can easily derive the rest of the solution from this.