Chan and Mitran
Narrowband / Wideband FM

Narrowband FM

$$\phi_{NBFM}(t) = A\cos\left[\omega_c t + k_f \int_0^t f(\tau)\,d\tau\right]$$

where $\beta << 1$.

We can actually use this odd constraint to express this differently. Right now, it's extremely difficult to determine how to construct a signal like this, so we'll make some approximations in hopes of making our lives easier.

First, let's expand, by using $\cos(a+b)$.

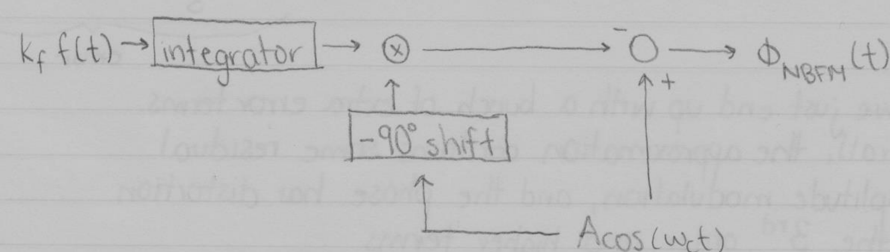
$$\phi_{NBFM}(t) = A\cos(\omega_c t)\cos\left[k_f \int_0^t f(\tau)\,d\tau\right] - A\sin(\omega_c t)\sin\left[k_f \int_0^t f(\tau)\,d\tau\right]$$

Here, we begin our approximations.

$$\sin(x) \approx x \text{ for small } x$$
$$\cos(x) \approx 1 \text{ for small } x$$

$$\approx A\cos(\omega_c t) - A\sin(\omega_c t)\, k_f \int_0^t f(\tau)\,d\tau$$

Now that the integral has been extracted from within the angle, it becomes much easier to build this signal



But what errors do we get by using the approximation instead of the actual thing?

We know that

$$A\cos(w_c t) - B\sin(w_c t) = \sqrt{A^2+B^2}\cos\left(w_c t + \tan^{-1}\left(\tfrac{B}{A}\right)\right)$$

So we can apply this to our NBFM approximation

$$\phi_{NBFM}(t) = \underbrace{A\left[1+\left(k_f\int_0^t f(\tau)d\tau\right)^2\right]^{1/2}}_{\text{ideally, this is equal to 1}} \cos\left(w_c t + \tan^{-1}\underbrace{\left[\frac{A\,k_f\int_0^t f(\tau)d\tau}{A}\right]}_{\text{ideally, this is equal to } k_f\int_0^t f(\tau)d\tau.}\right)$$

So the envelope is $A\left[1+\left(k_f\int_0^t f(\tau)d\tau\right)^2\right]^{1/2}$ instead of A. The integral itself ranges from $[0, \beta]$ so squaring changes the range to $[0, \beta^2]$.

As such, the envelope of the approximation can be $[A, A+\beta]$.

The phase $\theta_i(t) = w_c t + \tan^{-1}\left(k_f\int_0^t f(\tau)d\tau\right)$ instead of just $w_c t$. Since the expansion of $\tan^{-1}(x)$ is

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots$$

$$\theta_i(t) = w_c t + k_f\int_0^t f(\tau)d\tau \underbrace{- \frac{\left(k_f\int_0^t f(\tau)d\tau\right)^3}{3} \ldots}_{\text{error}}$$

So we just end up with a bunch of extra error terms. Overall, the approximation contains some residual amplitude modulation, and the phase has distortion in the 3rd order and higher terms.

The basic rule of thumb is if $\beta \leq 0.3$, we can ignore the errors.

So what is the bandwidth of NBFM?

$$\phi_{NBFM}(t) = A\cos(\omega_c t) + A k_f \int_0^t f(\tau)d\tau \sin(\omega_c t)$$

Compare this to DSB-LC.

$$\phi_{DSB-LC}(t) = A_c + A_c f(t)\cos(\omega_c t)$$

Looks kind of similar to it, does it not? There's some height increase followed by A × (integral instead of just f(t) × sin(wct).

Swapping out cos(wct) for sin(wct) in the second term doesn't actually change much as far as spectrum goes. Remember that DSB-LC's bandwidth was

$$2 \times \text{BW of } f(t)$$

Now, it's

$$2 \times \text{BW of } \int f(\tau)d\tau$$

What does the integral actually do

$$\int \cos(\omega_m t) = \omega_m \sin(\omega_m t)$$
$$= \omega_m \cos(\omega_m t + \pi/2)$$

It modifies the amplitude and shifts the phase. It doesn't touch the frequency whatsoever. Which means integration does NOT add or remove frequencies → meaning the spectrum is EXACTLY the same as before, given that f(t) is a simple sinusoid.

The conclusion we can draw here is that if all we care about is spectral efficiency, there's no reason to use NBFM over DSB-LC, as the latter is easier to demodulate.

We'll get into demodulation soon, but first, wideband.

Wideband FM

$$\phi_{WBFM}(t) = A\cos[\omega_c t + \beta \sin(\omega_m t)]$$

Wideband FM is a trickier problem, so we'll only look at a few simple cases. We can assume:

$$f(t) = a\cos(\omega_m t)$$
$$\Delta\omega = a k_f$$
$$\beta = \frac{\Delta\omega}{\omega_m} \quad \text{(not necessarily } > 1)$$

↑ which is really the general case

We need to know what $\Phi_{WBFM}(\omega)$ is, but taking the Fourier transform of $\phi_{WBFM}(t)$ is totally and completely fucked. Like in every scenario that we encounter weird shit in, we'll manipulate it

Using $\cos(\theta) = \mathrm{Re}\{e^{j\theta}\}$:

$$\phi_{WBFM}(t) = \mathrm{Re}\{ A e^{j\omega_c t} e^{j\beta \sin \omega_m t} \}$$
$$= \mathrm{Re}\{ \tilde{\Phi}(t) e^{j\omega_c t} \}$$

$\tilde{\Phi}(t)$ is the complex baseband equivalent of $\phi(t)$, but that's not actually important here, it's just kind of a coincidence.

Since $\tilde{\Phi}(t) = Ae^{j\beta \sin \omega_m t}$, it is a function of a periodic function. Therefore, $\tilde{\Phi}(t)$ itself is periodic. Since $\tilde{\Phi}(t)$ is periodic, it has a Fourier series equivalent given by

$$\tilde{\Phi}(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega_m t}$$

where $F_n = \frac{1}{T} \int_{-T/2}^{T/2} Ae^{j\beta \sin \omega_m t} e^{-jn\omega_m t} dt$

$\tilde{\Phi}(t)$'s period T is $2\pi/\omega_m$:

$$= \frac{\omega_m}{2\pi} \int_{-\frac{\pi}{\omega_m}}^{\frac{\pi}{\omega_m}} Ae^{j(\beta \sin \omega_m t - n\omega_m t)} dt$$

If we let $u = \omega_m t \rightarrow du = \omega_m dt$

$$= A \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(u) - n(u))} du \right]$$

new bounds = old bounds $\times \omega_m$

$$F_n = A J_n(\beta)$$

$J_n(\beta)$ is a new kind of function called an n^{th} order Bessel function of the 1^{st} kind. We'll talk more about the Bessel function later - for now, it's enough to know that you can read the result off a table given some n and β.

$$\therefore \tilde{\Phi}(t) = \sum_{n=-\infty}^{\infty} A J_n(\beta) e^{jn\omega_m t}$$

and

$$\Phi_{WBFM}(t) = Re\left\{ \left(\sum_{n=-\infty}^{\infty} A J_n(\beta) e^{jn\omega_m t} \right) e^{j\omega_c t} \right\}$$

The Bessel function is always real-valued so we can pull it and A outside of the $Re\{\}$.

$$= A \sum_{n=-\infty}^{\infty} J_n(\beta)\, Re\{ e^{j(n\omega_m t + \omega_c t)} \}$$

Since $Re\{e^{jt}\} = \cos(t)$,

$$= A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[(\omega_c + n\omega_m)t]$$

Now, we have some constants multiplied by a cosine.

$$\mathcal{F}\{\cos(\omega_c t)\} = \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$$

As such, we can extrapolate that

$$\mathcal{F}\{\cos[(\omega_c + n\omega_m)t]\} = \pi[\delta(\omega - [\omega_c + n\omega_m]) + \delta(\omega + [\omega_c + n\omega_m])]$$

Finally,

$$\mathcal{F}\{A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[(\omega_c + n\omega_m)t]\} = \pi A \sum_{n=-\infty}^{\infty} J_n(\beta) \times [\delta(\omega - [\omega_c + n\omega_m]) + \delta(\omega + [\omega_c + n\omega_m])]$$
$$= \Phi_{WBFM}(\omega)$$

So we've managed to calculate the Fourier transform of $\phi(t)$. What does this actually look like? Let's look at the $n=0$ case.

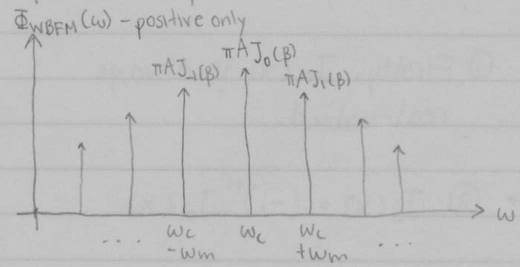
$$\Phi_{WBFM}(\omega)\Big|_{n=0} = \pi A J_0(\beta)[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$$

So we've got two delta functions at $\pm\omega_c$ of height $\pi A J_0(\beta)$. How about $n=1$?

$$\Phi_{WBFM}(\omega)\Big|_{n=1} = \pi A J_1(\beta)[\delta(\omega - [\omega_c + n\omega_m]) + \delta(\omega + [\omega_c + n\omega_m])]$$

Again, two delta functions. This time, their amplitudes are $\pi A J_1(\beta)$ and they're located at $\omega_c \pm \omega_m$.

If we continue calculation, we find the overall spectrum consists of a train of delta functions of height $A\pi J_n(\beta)$, and spread away from ω_c at intervals of ω_m.



This is our overall spectrum. The further we get, the smaller $J_n(\beta)$ becomes.

So the bandwidth of this is ∞ as $J_n(\beta)$ approaches 0. But obviously this can't be - we can't feasibly use the entire frequency spectrum to communicate a message.

Which means we have to say at some point the impulse is too small and no longer is negligible, allowing us to constrain ourselves to a finite bandwidth.

So when is this? There are two methods of approximating the bandwidth:

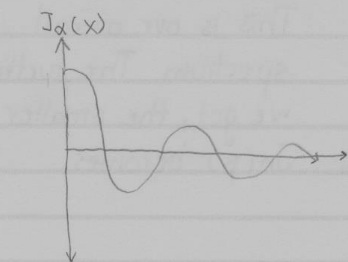
1. The 1% Rule
2. Carson's Rule

Let's explore them in order, but first, let's take a look at the Bessel function

The Bessel Function and its Properties

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

This equation for Bessel functions of the first kind is included purely for completion's sake. You may as well not even read it.



① Firstly, $J_\alpha(x)$ is always real-valued.

② $J_\alpha(x) = (-1)^n J_\alpha(x)$

It's even.

③ For $\beta << 1$ (NBFM): $J_0(\beta) \approx 1$

$J_1(\beta) \approx \beta/2$

$J_n(\beta) \approx 0$ for $|n| \geq 2$

The Bessel function shrinks in height as n increases, so it becomes essentially negligible for NBFM @ $n = \pm 2$. No such claim can be made for WBFM.

④ $\sum_{n=-\infty}^{\infty} (J_n(\beta))^2 = 1$

Alright, that's it for the Bessel function, back to our regularly scheduled programming.

Approximating FM's Bandwidth

① The 1% Rule

No, this is not a socioeconomic commentary. Basically, given β, find

$$|J_n(\beta)| \leq 0.01$$

and then take the n right before it hits or goes lower than 0.01; we'll call this n_0.

Now, since the frequency is from $\omega_c - n\omega_m$ to $\omega_c + n\omega_m$, the overall bandwidth is

$$BW = 2n_0\omega_m$$

② Carson's Rule

$$BW = 2\omega_m(1+\beta)$$

That's literally it. This is less exact than the 1% rule but obviously is simpler to calculate. Let's do some quick examples.

ex. Carrier $c(t) = 100\cos(100\times10^3 t)$. Message $f(t) = 100\cos(10^3 t)$. $k_f = 16$. Use both the 1% rule and Carson's rule to determine the bandwidth of the modulated signal.

a)

n	0	1	2	3	4	5
$J_n(\beta)$	0.455	0.570	0.257	0.073	0.015	0.002

where $\beta = \frac{\Delta\omega}{\omega_m} = \frac{a\,k_f}{\omega_m} = 1.6$

Since $0.002 \leq 0.01$, we choose $n_0 = 4$.

$$\therefore BW = 2(4)\,\omega_m = 8\,\omega_m$$

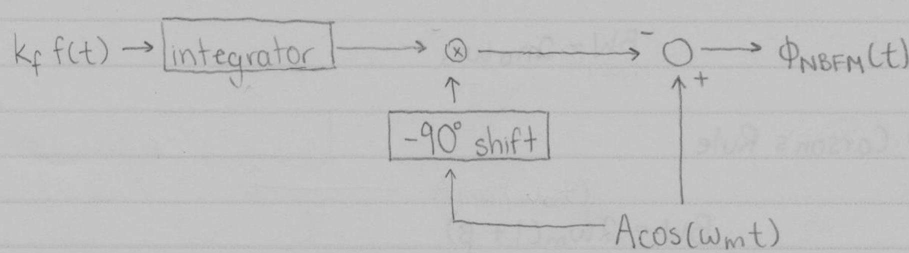
b) We've found β as 1.6.

$$\therefore BW = 2\omega_m(1+\beta)$$
$$= 5.2\,\omega_m$$

Now, what about generation and demodulation of FM signals?

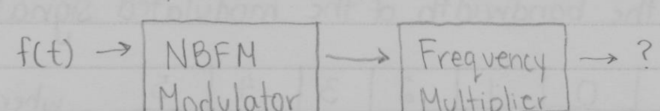
Generating FM Signals

We already have a system for generating approximations of NBFM signals. Here it is again:



$$\phi_{NBFM}(t) = A\cos(\omega_m t) - k_f \int_0^t f(\tau)\,d\tau\,(\sin(\omega_m t)A)$$

NBFM is defined by $\beta \ll 1$, where $\beta = \frac{A k_f}{\omega_m}$. So what if we wanted to make WBFM? We'd need to increase β somehow. Let's propose this:



This particular "indirect" FM transmission block sequence is known as an Armstrong phase modulator.

What happens when we send $\Phi_{NBFM}(t)$ into the frequency multiplier?

We know that for $f(t) = a\cos(\omega_m t)$,

$$\Phi_{FM}(t) = A\cos\left(\omega_c t + \frac{k_f a}{\omega_m}\sin(\omega_m t)\right)$$

Multiplying the frequency by some n means the entire inside of the cos is multiplied by n.

$$= A\cos\left(n\left[\omega_c t + \frac{k_f a}{\omega_m}\sin(\omega_m t)\right]\right)$$

$$= A\cos\left(n\omega_c t + \frac{nk_f a}{\omega_m}\sin(\omega_m t)\right)$$

β is originally given as $\frac{k_f a}{\omega_m}$, this. But now that it has become $\frac{nk_f a}{\omega_m}$,

$$\beta' = \frac{nk_f a}{\omega_m}$$

So as long as $n > 1$, we can feasibly create WBFM from NBFM indirectly this way. Now the question becomes: how do we implement a frequency multiplier?

The answer to this (and many other questions in this course) is to use trig identities and filters.

The Frequency Multiplier

$$\phi_1(t) \rightarrow \boxed{\text{Nonlinear Memoryless Device}} \xrightarrow{\phi_1(t)'} \boxed{\text{BPF @ } n\omega_c} \rightarrow \phi_2(t)$$

Let's see how this works. A non-linear memoryless system makes

$$\phi_1(t)' = b_0 + b_1\,\phi_1(t) + b_2[\phi_1(t)]^2 + \ldots + b_m[\phi_1(t)]^m$$

I don't exactly remember why this is the case. But let's look at each individual term if $\phi_1(t)$ was a simple sinusoid.

$$\begin{aligned}
\phi_1(t) &= \cos(x) \\
\phi_1(t)^2 &= \cos^2(x) \\
&= 1/2 + 1/2\cos(2x) \\
\phi_1(t)^3 &= \cos^3(x) \\
&= \cos^2(x)\cos(x) \\
&= [1/2 + 1/2\cos(2x)]\cos(x) \\
&= 1/2\cos(x) + 1/2\cos(2x)\cos(x) \\
&= 1/2\cos(x) + 1/4\cos(x) + 1/4\cos(3x) \\
&= 3/4\cos(x) + 1/4\cos(3x) \\
\phi_1(t)^4 &= \cos^2(x)\cos^2(x) \\
&= [1/2 + 1/2\cos(2x)][1/2 + 1/2\cos(2x)] \\
&= 1/4 + 1/4\cos(2x) + 1/4\cos(2x) + 1/4\cos^2(2x) \\
&= 1/4 + 1/2\cos(2x) + 1/4[1/2 + 1/2\cos(4x)] \\
&= 1/4 + 1/8 + 1/2\cos(2x) + 1/8\cos(4x) \\
&= 3/8 + 1/2\cos(2x) + 1/8\cos(4x)
\end{aligned}$$

etc, etc.

So conceivably, we can express $\phi_1(t)'$ as a sum of sinusoids at different frequencies, by cutting out the frequencies we don't want with the BPF.

$$\cos^2(x) = \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)$$
$$\downarrow \text{BPF @ 2}$$
$$= \tfrac{1}{2}\cos(2x)$$
$$\cos^3(x) = \tfrac{3}{4}\cos(x) + \tfrac{1}{4}\cos(3x)$$
$$\downarrow \text{BPF @ 3}$$
$$= \tfrac{1}{4}\cos(3x)$$
$$\cos^4(x) = \tfrac{3}{8} + \tfrac{1}{2}\cos(2x) + \tfrac{1}{8}\cos(4x)$$
$$\downarrow \text{BPF @ 4}$$
$$= \tfrac{1}{8}\cos(4x)$$

Since $\phi_1(t) = A\cos(x)$ where $x = \omega_c t + k_f \int_0^t f(\tau)\,d\tau$,

$$\phi_1(t)^2 \text{ BPF'd @ } 2\omega_c = \tfrac{A}{2}\cos\left(2\omega_c t + 2k_f \int_0^t f(\tau)\,d\tau\right)$$

$$\phi_1(t)^3 \text{ BPF'd @ } 3\omega_c = \tfrac{A}{4}\cos\left(3\omega_c t + 3k_f \int_0^t f(\tau)\,d\tau\right)$$

and so on and so forth. So we can say

$$\phi_1(t)' = \sum_{n=0}^{\infty} d_n \cos\left(n\omega_c t + nk_f \int_0^t f(\tau)\,d\tau\right)$$

and bandpass-filtering $\phi_1(t)'$ at a particular n gives us

$$\phi_2(t) = d_n \cos\left(n\omega_c t + nk_f \int_0^t f(\tau)\,d\tau\right)$$

So if we wanted to increase $\phi_1(t)$'s frequency by 3 times, we'd pass it through the non-linear memoryless system, then set up a BPF centered at $3\omega_c$, yielding

$$\tfrac{A}{4}\cos\left(3\omega_c t + 3k_f \int_0^t f(\tau)\,d\tau\right)$$

Let's do an example or two to illustrate.

ex. $f(t)$ w/ $f_m = 100\text{ Hz}$ → $\boxed{\text{NBFM w/ } \beta = 0.2}$ → $\boxed{\text{Freq Mult w/ } n_1}$ → $g(t)$

↑

$c(t)$ w/ $f_c = 0.1\text{ MHz}$

We want $g(t)$ to have frequency deviation $\Delta f = 75\text{ kHz}$
frequency $f_c = 100\text{ MHz}$.
What would n_1 need to be to yield this result?

The output of the NBFM is

$$\phi_{NBFM}(t) = A\cos(0.1\text{ MHz}\,t + 0.2\sin(100\text{Hz}\,t))$$
$$\Rightarrow f_{NBFM} = 0.1\text{ MHz}$$
$$\beta_{NBFM} \triangleq \frac{\Delta f}{f_m} \Rightarrow \Delta f = (0.2)(100\text{ Hz}) = 20\text{ Hz}$$

$\beta \to$ Freq Mult w/ $n \to n\beta$ $\therefore$ $\Delta f \to$ Freq Mult w/ $n \to n\Delta f$
as f_m, which is contained in the message, does not (and cannot, as you'd fuck up the message itself) change.

$$\therefore \Delta f_{final} = n_1 \Delta f$$
$$75\text{ kHz} = n_1\, 20\text{ Hz}$$
$$n_1 = 3750$$

f_c does become $n f_c$, though, meaning

$$f_{c,final} = (3750)(0.1\text{ MHz}) = 375\text{ MHz}$$

So getting both Δf and f_c to 75kHz and 100MHz is impossible in this specific configuration. How do we get around this?

We simply add more blocks.

$$f(t) \rightarrow \boxed{\text{NBFM w/ } \beta = 0.2} \rightarrow \boxed{\text{Fmult w/ } n_1} \rightarrow \boxed{\text{Mixer}} \rightarrow \boxed{\text{Fmult w/ } n_2} \rightarrow g(t)$$

with $c(t)$ feeding the NBFM block and $c_2(t) = \cos(\omega_{c2} t)$ feeding the Mixer.

Now, in this case, we add a mixer (a multiplier) that mixes another carrier signal $c_2(t)$ into the modulated signal.

$g(t)$'s Δf is now:

$$\Delta f = n_2 n_1 \Delta f_{original}$$
$$\Rightarrow n_2 n_1 = 3750$$

The output of the mixer becomes

$$= A\cos(n_1 \omega_c t + n_1 \beta \sin(\omega_m t)) \cos(\omega_{c2} t)$$
$$\downarrow \text{trig identity}$$
$$= \frac{A}{2}\left[\cos(n_1 \omega_c t + \omega_{c2} t + n_1 \beta \sin(\omega_m t)) + \cos(n_1 \omega_c t - \omega_{c2} t + n_1 \beta \sin(\omega_m t))\right]$$

So it shifts the frequency of the signal $\pm \omega_{c2}$. We'll say $\omega_{c2} = 9.5$ MHz. We want the final f_c to be 100 MHz, and the final Δf to be 75 kHz.

① $n_1 n_2 = 3750$

② $n_2 |n_1 \omega_c - \omega_{c2}| = 100$ MHz
$n_2 |n_1 (100\text{kHz}) - 9.5\text{MHz}| = 100$ MHz

③ $n_2 (n_1 \omega_c + \omega_{c2}) = 100$ MHz
$n_2 (n_1 (100\text{ kHz}) + 9.5\text{MHz}) = 100$ MHz

We now have a system of three equations with two unknowns. But do we really need ③?

③ $n_1 n_2 (100 kHz) + n_2 (9.5 MHz) = 100 MHz$

Since $n_1 n_2 = 3750$,

$$375 MHz + n_2(9.5 MHz) = 100 MHz$$

This simply isn't possible. There's no positive value of n_2 that makes this statement true. The frequency of

$$\cos(n_1 \omega_c t + \omega_{c2} t + n_1 k_f \sin(\omega_m t))$$

will be way higher than what we want, so we'll just imagine it'll get cut out by the BPF inside the first frequency multiplier. That leaves us with:

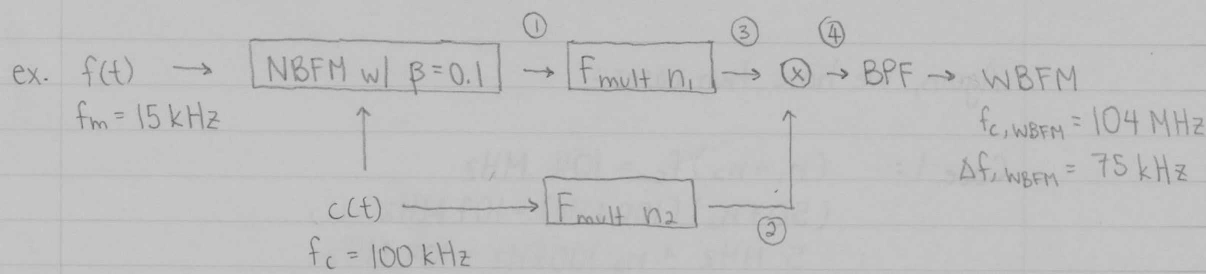
① $n_1 n_2 = 3750$
② $n_2 | n_1 (100 kHz) - 9.5 MHz | = 100 MHz$

$$\Rightarrow |n_2 n_1 (100 kHz) - n_2 9.5 MHz| = 100 MHz$$
$$|375 MHz - n_2 9.5 MHz| = 100 MHz$$
$$\therefore \begin{matrix} n_2 = \\ n_1 = \end{matrix} \begin{bmatrix} 28.947 \\ 129.545 \end{bmatrix} \begin{matrix} \text{or} \\ \text{or} \end{matrix} \begin{bmatrix} 50 \\ 75 \end{bmatrix}$$

So in order to get $g(t)$'s Δf to be 75 kHz and its f_c to be 100 MHz, $n_2 = 28.947$ and $n_1 = 129.545$ or $n_2 = 50$ and $n_1 = 75$.

We'll do one more, more difficult example before moving to demodulation.

ex.



a) What is n_1 and n_2, and the center frequency of the BPF to produce a WBFM signal with $\Delta f = 75\text{ kHz}$
$f_c = 104\text{ MHz}$?

① Produces $\phi_{NBFM}(t) = A\cos(100\text{kHz}\,t + 0.1\sin(15\text{ kHz}\,t))$

③ produces $A\cos(n_1 100\text{kHz}\,t + n_1 0.1\sin(15\text{ kHz}\,t))$

Again, $\beta' = n_1\beta = n_1 \dfrac{\Delta f_{original}}{\Delta f_m} \Rightarrow \Delta f_{original} = (0.1)(15\text{ kHz})$
$= 1.5\text{ kHz}$
$\Rightarrow \beta' = n_1 1.5\text{ kHz}$

② produces $\cos(n_2 100\text{ kHz}\ t)$.

Multiplying ② with ③ changes f_c, but not Δf. The BPF also does not change Δf. That means at step ③, we must already have our final Δf value:

$$\Delta f_{final} = n_1 \Delta f$$
$$75\text{ kHz} = n_1\ 1.5\text{kHz}$$
$$n_1 = 50$$

Now, let's figure out our f_c.

At ④, we have $\frac{A}{2}\left[\cos([n_1+n_2]f_c t + n_1\beta\sin(f_m t)) + \cos([n_1-n_2]f_c t + n_1\beta\sin(f_m t))\right]$

using our handy dandy trig identities

Again, we have two cases:

Case 1: $(n_1 + n_2) f_c = 104 \text{ MHz}$

$$(50 + n_2)(100 \text{ kHz}) = 104 \text{ MHz}$$
$$5 \text{ MHz} + n_2 \, 100 \text{kHz} = 104 \text{ MHz}$$
$$n_2 = 990$$

Case 2: $|(n_1 - n_2)| f_c = 104 \text{ MHz}$

$$|(50 - n_2)|(100k) = 104 \text{ MHz}$$
$$|(50 - n_2)| = 1040$$
$$n_2 = -990 \text{ or } 1090$$

Obviously $n_2 = -990$ won't do as n_2 must be positive. But what about the other two? Remember $\phi_{FM}(t)$ is DEFINED as

$$\phi_{FM}(t) = A\cos(w_c t + \beta \sin(w_m t))$$

In the second case, plugging $(n_1 - n_2)$ back in, we get

$$\cos(-1040 f_c t + 50 \beta \sin(f_m t))$$

Since we can't create negative frequencies, we can use cosine's even property - $\cos(x) = \cos(-x)$ - to rewrite this:

$$\cos(1040 f_c t - 50 \beta \sin(f_m t))$$

This is no longer a valid FM signal as the frequency deviation is negative instead of positive. Because of that, we have to veto $n_2 = 1090$.

As such, our only option becomes $n_2 = 990$.

b) If the carrier frequency is ± 2 kHz of its intended value, find the maximum allowable error of the 100 kHz oscillator.

We know from part a) that

$$(n_1 + n_2) f_c = f_{c,WBFM}$$
$$(50 + 990)(100K \pm \Delta) = 104 \text{ MHz} \pm 2 \text{ kHz}$$
$$104 \text{ MHz} \pm 1040\Delta = 104 \text{ MHz} \pm 2\text{kHz}$$
$$\pm 1040\Delta = \pm 2\text{kHz}$$
$$\Delta = \pm 1.92 \text{ Hz}$$

So in order for $f_{c,WBFM}$ to be within 104 MHz $\pm$ 2kHz, the oscillator's maximum error must be ± 1.92 Hz.

Demodulation of FM Signals

There are two methods of demodulation: indirect and direct.

Indirect uses the standard phase-locked loop to exactly match the modulated wave used. This is similar to the demodulation strategies used before.

The direct demodulation is new for us.

$$\begin{array}{c}\Phi(t) \text{ with} \\ \omega_i(t) = \omega_c + k_f f(t)\end{array} \longrightarrow \boxed{\begin{array}{c}\text{Frequency} \\ \text{Discriminator}\end{array}} \longrightarrow \begin{array}{l}\text{output proportional} \\ \text{to } \omega_i(t)\end{array}$$

Alright, so what exactly is happening here?

A frequency discriminator is an idealized block (like how a perfect low-pass filter is an ideal LPF) that converts (changes in frequency) → (changes in amplitude).

It essentially takes the derivative of $\phi(t)$:

$$\frac{d}{dt}\,\phi(t) = \frac{d}{dt}\,A\cos\left[\omega_c t + k_f\int_0^t f(\tau)\,d\tau\right]$$

$$= \left[\omega_c + k_f f(t)\right] A\left[-\sin\left(\omega_c t + k_f\int_0^t f(\tau)\,d\tau\right)\right]$$

$$= \underbrace{-A\left[\omega_c + k_f f(t)\right]}_{\text{amplitude change proportional to frequency } \omega_i(t)} \sin\left(\omega_c t + k_f\int_0^t f(\tau)\,d\tau\right)$$

Now, if $|k_f f(t)| < \omega_c$, we can now use envelope detection to get just

$$A\left[\omega_c + k_f f(t)\right]$$

Which is simply $f(t)$ scaled by Ak_f and DC shifted by $A\omega_c$. So to finally retrieve $f(t)$ back, we'd shift by $-A\omega_c$ and scale by $\frac{1}{Ak_f}$.

It's been a long chapter, we really hung in there. Next up, sampling.