ECE 358 Example / Derivations
Chapter 2 Walkthrough

ex. Every packet has n bits. There is a probability p_B that a bit gets corrupted. <u>What is the probability a packet has exactly one corrupted bit?</u>

We're looking for one "success" with probability p_B in n trials. This indicative of the binomial distribution.

$$P\{X=1\} = \binom{\overset{\text{trials}}{n}}{\underset{\text{successes}}{1}} \underbrace{p_B^{\,1}}_{1 \text{ success}} \underbrace{(1-p_B)^{n-1}}_{n-1 \text{ failures}}$$

$$= \frac{n!}{1!\,(n-1)!}\, p_B (1-p_B)^{n-1}$$

$$= \frac{n!}{(n-1)!}\, p_B (1-p_B)^{n-1}$$

<u>What is the probability no packets are corrupted?</u> This case is where X, the number of corrupt packets, is zero.

$$P\{X=0\} = \binom{n}{0} p_B^{\,0} (1-p_B)^n$$

$$= \frac{\cancel{n!}}{\underbrace{0!}_{1}\,\cancel{n!}} (1)(1-p_B)^n$$

$$= (1-p_B)^n$$

<u>What is the probability the packet has at least one corrupted bit?</u> We can do this two ways: use the cumulative distribution function, or simply find the opposite of no errors occurring.

We already know the second, so we'll use that instead.

$$P\{X > 0\} = 1 - P\{X = 0\}$$
$$= 1 - (1-p_B)^n$$

ex. A wireless network protocol uses a stop-and-wait transmission policy. Each packet has probability p_E of being corrupted or lost. What is the probability that the protocol will need 3 transmissions to send a packet successfully?

We're looking for 2 failures followed by 1 success at the end. This is indicative of the geometric distribution

$$P\{X=3\} = \underbrace{p_E^{\,2}}_{\text{2 failures}} \underbrace{(1-p_E)}_{\text{1 success}}$$

What is the average number of transmissions needed per packet? If we already knew the expression for $E(X)$ for geometric distributions we could invoke it here, but we don't. So we'll have to solve for it ourselves.

$$E(X) = \sum_{i=1}^{\infty} i \, P\{X = i\}$$
$$= \sum_{i=1}^{\infty} i \, p_E^{\,i-1} (1-p_E)$$

$1-p_E$ has no dependence on i so we'll pull it out.

$$= (1-p_E) \sum_{i=1}^{\infty} i \, p_E^{\,i-1}$$

Remember the rules for infinite sums:

$$\sum_{i=1}^{\infty} i \, x^{i-n} = \frac{x}{(1-x)^2 x^n} \quad \text{for } |x| < 1$$

Since x in our case is < 1 (it's a probability):

$$\therefore\ (1-p_E)\sum_{i=1}^{\infty} i\, p_E^{\,i-1} = (1-p_E)\left[\frac{\cancel{p_E}}{(1-p_E)^2\,\cancel{p_E^{\,1}}}\right]$$

$$= \cancel{(1-p_E)}\left[\frac{1}{(1-p_E)^{\cancel{2}}}\right]$$

$$= \frac{1}{1-p_E}$$

ex. Analysis of an M/M/1 queue with arrival rate λ and departure rate μ.

<u>Show state transitions from S_n in some time Δt.</u>

We have three possible transitions that stem from the following cases:

1) We finish servicing a packet and it departs. Departure rate is μ so the probability of this occurring is $\mu\Delta t$.
2) A new packet arrives. Arrival rate is λ, so the probability of an arrival is $\lambda\Delta t$.
3) Packets do not finish servicing and no new packets arrive. The probability is no arrivals $\cup$ no departures, which is just the two multiplied as they are independent: $(1-\lambda\Delta t)(1-\mu\Delta t)$

$$\begin{tikzcd} S_{n-1} & S_n \arrow[l, "\mu\Delta t"'] \arrow[r, "\lambda\Delta t"] \arrow[loop below, "(1-\lambda\Delta t)(1-\mu\Delta t)"] & S_{n+1} \end{tikzcd}$$

where $(1-\lambda\Delta t)(1-\mu\Delta t) = 1-\mu\Delta t-\lambda\Delta t+\lambda\mu\Delta t^2$
$\approx 1-\mu\Delta t-\lambda\Delta t$ if $\Delta t =$ small

So what is the probability $P_n(t+\Delta t)$ that the queue has n packets in it at time $t+\Delta t$?

We can enter S_n by:

- new arrival when in S_{n-1}, probability $\lambda \Delta t$
- departure when in S_{n+1}, probability $\mu \Delta t$
- neither when in S_n, probability $(1-\lambda\Delta t)(1-\mu\Delta t)$

$$\begin{aligned} \therefore \quad P_n(t+\Delta t) &= P_{n-1}(t)\lambda\Delta t \\ &\quad + P_{n+1}(t)\mu\Delta t \\ &\quad + P_n(t)(1-\lambda\Delta t)(1-\mu\Delta t) \\ &\approx P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t \\ &\quad + P_n(t)[1-\lambda\Delta t-\mu\Delta t] \qquad (1) \end{aligned}$$

← all unions of independent events

How about $P_0(t+\Delta t)$?

This case only has two options

- departure when in S_1, probability $\mu\Delta t$
- no arrival when in S_0, probability $(1-\lambda\Delta t)$

$$\begin{aligned} \therefore \quad P_0(t+\Delta t) &= P_1(t)\mu\Delta t + P_0(t)(1-\lambda\Delta t) \\ &= P_1(t)\mu\Delta t + P_0(t) - P_0(t)\lambda\Delta t \qquad (2) \end{aligned}$$

Gather the Δt terms. What happens when $\Delta t \to 0$?

$$\begin{aligned} (1) \quad P_n(t+\Delta t) &= P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)\mu\Delta t + P_n(t) \\ &\quad - P_n(t)\lambda\Delta t - P_n(t)\mu\Delta t \end{aligned}$$

$$\begin{aligned} P_n(t+\Delta t) - P_n(t) &= \Delta t\,[P_{n-1}(t)\lambda + P_{n+1}(t)\mu - P_n(t)\lambda \\ &\quad - P_n(t)\mu] \end{aligned}$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = (-\lambda-\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

$P_n(t+\Delta t)$ approaches $P_n(t)$ so the left side becomes zero.

② $P_0(t+\Delta t) = P_1(t)\mu\Delta t + P_0(t) - P_0(t)\lambda\Delta t$

$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t)$$

Same thing here, left side also becomes zero.

<u>Use the two equivalent expressions to derive expressions for P_1 up to P_n.</u>

① $$0 = (-\lambda-\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

We want to isolate P_n and related terms.

$$(\lambda+\mu)P_n(t) = \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

However, we still have the functions of t, whereas we just want some concrete value. How do we get rid of it? Well, let's consider something else.

Let's say we have a coin flip, but the probability of each side is dependent on time. So at $t = t_0$, perhaps $P\{\text{heads}\}$ is 30% and $P\{\text{tails}\} = 70\%$. But if we wanted to find the overall probability of the flips, what would we do? Do the experiment infinite times: let $t \to \infty$. In that case $P\{\text{heads}\}$ and $P\{\text{tails}\}$ should approach a true value.

By the same analogy, letting $t \to \infty$ for $P_n(t)$ should reduce it to simply P_n, the final overall chance.

$$\Rightarrow \quad (\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$$

This is the same for any n:

② $$\lambda P_0 = \mu P_1$$

We can then begin to express overall probabilities:

$$② \quad P_1 = \lambda/\mu \, P_0$$

Subbing $n=1$ in the P_n equation:

$$(\lambda+\mu)P_1 = \lambda P_0 + \mu P_2$$
$$\lambda P_1 + \mu P_1 = \lambda P_0 + \mu P_2$$

where we can express $P_0 = \mu/\lambda \, P_1$

$$\lambda P_1 + \mu P_1 = \lambda\left(\frac{\mu}{\lambda} P_1\right) + \mu P_2$$

$$\lambda P_1 + \mu P_1 = \mu P_1 + \mu P_2$$
$$\lambda P_1 = \mu P_2$$
$$P_2 = \frac{\lambda}{\mu} P_1$$

We'll continue this with $n=2$:

$$(\lambda+\mu)P_2 = \lambda P_1 + \mu P_3$$
$$\lambda P_2 + \mu P_2 = \lambda\left(\frac{\mu}{\lambda} P_2\right) + \mu P_3$$
$$\lambda P_2 + \mu P_2 = \mu P_2 + \mu P_3$$
$$\lambda P_2 = \mu P_3$$
$$P_3 = \frac{\lambda}{\mu} P_2$$
$$= \frac{\lambda^2}{\mu^2} P_1$$
$$= \frac{\lambda^2}{\mu^2} P_0$$

As such, we can say that $P_n = \frac{\lambda}{\mu} P_{n-1}$

$$= \rho^n P_0$$

Find an expression for the M/M/1 queue's P_0 that is not reliant on other P_n.

We do know that the sum of all P_n is 1, as it's a probability.

$$\sum_{i=0}^{\infty} P_i = 1$$

$$= P_0 + P_1 + P_2 + \ldots + P_{n-1} + P_n$$

However, each $P_{n \neq 0}$ can be expressed as P_0 times some power of (λ/μ).

$$= P_0 \left(1 + \lambda/\mu + \lambda^2/\mu^2 + \ldots + \lambda^n/\mu^n\right)$$

An infinite series $1 + x + x^2 + \ldots + x^n$ converges on $\frac{1}{1-x}$

$$\therefore \underset{\text{total probability}}{1} = P_0 \left(\frac{1}{1-\rho}\right)$$

$$\Rightarrow P_0 = \frac{1}{\left(\frac{1}{1-\rho}\right)}$$

$$= 1 - \rho$$

What is the average number of packets in the queue?

$$E(N) = \sum_{i=0}^{\infty} i P_i$$

$$= \sum_{i=0}^{\infty} i (\rho^i) P_0$$

$$= \sum_{i=0}^{\infty} i (\rho^i)(1-\rho)$$

$$= (1-\rho) \sum_{i=0}^{\infty} i (\rho^i)$$

$$= (1-\rho) \left[\frac{\rho}{(1-\rho)^2 \rho^0}\right]$$

$$= (1-\rho)\left[\frac{\rho}{(1-\rho)^2}\right]$$

$$E(N) = \frac{\rho}{1-\rho}$$

The average number of packets can be expressed directly as a function of ρ, queue utilization.

<u>What is the average number of packets in the queue (1 less than system)?</u>

$$E(N_q) = \sum_{i=1}^{\infty} (i-1) P_i$$

The summation starts from 1 because there must be a packet in the system for a new one to arrive and get put in the queue. As such we also use $i-1$ to denote that we're looking at (system - 1).

$$= \sum_{i=1}^{\infty} (i-1)\,\rho^i\,(1-\rho)$$

$$= (1-\rho) \sum_{i=1}^{\infty} \rho^i (i-1)$$

How do we determine the result of the summation?

$$\sum_{i=1}^{\infty} \rho^i (i-1) = \sum_{i=1}^{\infty} i\rho^i - \sum_{i=1}^{\infty} \rho^i$$

We know both of these sums.

$$= \frac{\rho}{(1-\rho)^2} - \left[-\frac{\rho}{\rho-1}\right]$$

$$= \frac{\rho}{(1-\rho)^2} + \frac{\rho}{\rho-1}$$

The denominators aren't the same so let's fix them up.

$$= \frac{\rho}{(1-\rho)^2} - \frac{\rho}{1-\rho}$$

$$= \frac{\rho}{(1-\rho)^2} - \frac{\rho(1-\rho)}{(1-\rho)^2}$$

$$= \frac{\rho - [\rho - \rho^2]}{(1-\rho)^2}$$

$$\sum_{i=1}^{\infty} \rho^i (i-1) = \frac{\rho^2}{(1-\rho)^2}$$

Finally we can plug this back into the expected value equation:

$$E(N_q) = (1-\rho) \sum_{i=1}^{\infty} \rho^i (i-1)$$

$$= (1-\rho)\left[\frac{\rho^2}{(1-\rho)^2}\right]$$

$$= \frac{\rho^2}{(1-\rho)}$$

What is the probability that the queue has more than N packets in it?

Here, we can add up the probabilities where $n > N$.

$$P\{n > N\} = \sum_{i=N+1}^{\infty} P_i$$

$$= \sum_{i=N+1}^{\infty} (1-\rho)\rho^i$$

$$= (1-\rho) \sum_{i=N+1}^{\infty} \rho^i$$

where $\sum_{i=m}^{\infty} x^i = \frac{-x^m}{x-1}$:

$$= (1-\rho)\left[\frac{-\rho^{N+1}}{\rho-1}\right]$$

$$= (\cancel{1-\rho})\left[\frac{\rho^{N+1}}{\cancel{1-\rho}}\right]$$

$$= \rho^{N+1}$$

ex. M/M/1 queue with $\lambda = 15$ per hour. What is the minimum μ required for

i) having the server idle at least 10% of the time

ii) expected queue length not exceeding 10

iii) probability at least 20 people in the queue is at most 50%

i) The server is idle when we're in state 0. The probability of this is P_0. P_0, as before, can be expressed as:

$$P_0 = 1 - \rho$$
$$= 1 - \lambda/\mu$$

We want $P_0 \geq 0.1$,

$$\therefore \quad 0.1 \leq 1 - \frac{15}{\mu}$$
$$-0.9\mu \leq -15$$
$$0.9\mu \geq 15$$
$$\mu \geq \frac{15}{0.9}$$
$$\geq 16.667$$

So μ must be ≥ 16.667 to achieve at least 10% idle time.

ii) We solved for $E(N_q)$ previously as:

$$E(N_q) = \frac{\rho^2}{1-\rho}$$

$$\Rightarrow \quad 10 \geq \frac{\rho^2}{1-\rho}$$

$$10(1-\rho) \geq \rho^2$$

$$10 - \frac{150}{\mu} \geq \frac{225}{\mu^2}$$

$$10\mu^2 - 150\mu - 225 \geq 0$$

Quadratic equation.

$$\mu = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-150) \pm \sqrt{(-150)^2 - 4(10)(-225)}}{2(10)}$$

$$= \frac{150 \pm 177.48}{20}$$

$$= 16.3741 \text{ and } -1.374$$

Departure rate can't be negative, so $\mu \geq 16.3741$ for $E(N_q)$ to be ≤ 10.

iii) In this case, we want $P(n > 20) \leq 0.5$.

$$P(n > 20) = \sum_{n=21}^{\infty} P_n$$

$$= \sum_{n=21}^{\infty} (1-\rho)(\rho^n)$$

$$= (1-\rho)\left[-\frac{\rho^{21}}{\rho - 1}\right]$$

$$\therefore \quad \rho^{21} \leq 0.5$$

$$\frac{15^{21}}{\mu^{21}} \le 0.5$$

$$\frac{15^{21}}{0.5} \le \mu^{21}$$

$$\mu^{21} \ge 9.976 \times 10^{24}$$

$$\mu \ge 15.503$$

To ensure that $P\{n > 20\}$ does not exceed 50%, μ must be ≥ 15.503

ex. Consider a system with arrival rate $k\lambda$ packets per second $(k > 1)$. Departure is $k\mu$ (service time is $(k\mu)^{-1}$. What is the average number of packets in the system?

We can use Little's Law for this:

$$L = \frac{\text{arrival}}{\text{departure} - \text{arrival}}$$

$$= \frac{k\lambda}{k\mu - k\lambda}$$

$$= \frac{\lambda}{\mu - \lambda}$$

What is the average delay per packet?

$$W = \frac{L}{\text{arrival}}$$

$$= \frac{L}{k\lambda}$$

$$= \left[\frac{\lambda}{\mu - \lambda}\right](k\lambda)^{-1}$$

$$= \frac{1}{k(\mu - \lambda)}$$

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$$E(N_q) = \frac{\rho^2}{1-\rho}$$

$$\Rightarrow \quad 10 \geq \frac{\rho^2}{1-\rho}$$

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iii) In this case, we want $P(n > 20) \leq 0.5$.

$$P(n > 20) = \sum_{n=21}^{\infty} P_n$$

$$= \sum_{n=21}^{\infty} (1-\rho)(\rho^n)$$

$$= (1-\rho)\left[-\frac{\rho^{21}}{\rho - 1}\right]$$

$$\therefore \quad \rho^{21} \leq 0.5$$

ex. Analysis of a M/M/1/N queve with arrival λ and departure μ.

What is the probability P_B that the queve is full and cannot accept another packet?

Unlike an infinite queve, our buffer size is N, so our summation for total probability changes. Now,

$$\sum_{i=0}^{N} P_i = 1$$

However, the relationship between P_0 and P_n doesn't change: it's still $\rho^n P_0$. The only thing that has changed is our maximum n, and as such: what P_0 is expressed by.

$$= P_0 \sum_{i=0}^{N} \rho^i$$

$$1 = P_0 \left[\frac{1-\rho^{N+1}}{1-\rho} \right]$$

$$\therefore \; P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

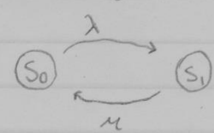
So now, we can express P_N, the probability the queve is blocked: aka P_B.

$$P_B = P_N = \left[\frac{1-\rho}{1-\rho^{N+1}} \right] \rho^N$$

ex. Derive P_0 up to P_n for a M/M/N queue with arrival rate λ and departure rate $\mu = \begin{cases} n\mu & \text{for } n \leq N \\ N\mu & \text{for } n > N \end{cases}$

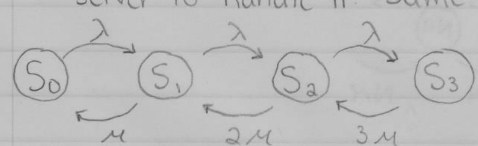
In the general sense, we're just increasing μ by a factor of the number of servers we're using concurrently. Obviously since we have a finite number, it caps out at N.

Let's try to figure out our state transitions. For P_0 and P_1, this is the same as always: one packet coming in, one server to serve it.



$\Rightarrow \quad \lambda P_0 = \mu P_1 \quad \therefore \quad P_1 = \frac{\lambda}{\mu} P_1$

How about if another packet arrives? We've got another server to handle it. Same with the next. Each time our departure rate increases.



Let's look at S_1:

$$\Rightarrow \quad \lambda P_0 + 2\mu P_2 = \lambda P_1 + \mu P_1$$

$$\cancel{\lambda}\left(\frac{\mu}{\cancel{\lambda}} P_1\right) + 2\mu P_2 = \lambda P_1 + \mu P_1$$

$$2\mu P_2 = \lambda P_1$$

$$P_2 = \frac{\lambda}{2\mu} P_1$$

It's a bit difficult to tell what the pattern is here so let's do another step, n=2.

$$\lambda P_2 + 2\mu P_2 = \lambda P_1 + 3\mu P_3$$

$$\lambda\left[\frac{\lambda}{2\mu} P_1\right] + \cancel{2\mu}\left[\frac{\lambda}{\cancel{2\mu}} P_1\right] = \lambda P_1 + 3\mu P_3$$

$$\frac{\lambda^2}{2\mu} P_1 + \cancel{\lambda P_1} = \cancel{\lambda P_1} + 3\mu P_3$$

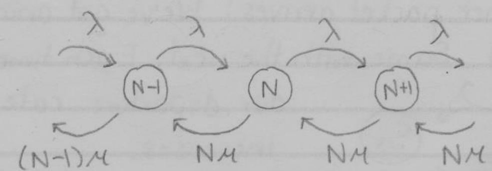
$$\frac{\lambda^2}{2\mu} P_1 = 3\mu P_3$$

$$P_3 = \frac{\lambda^2}{6\mu^2} P_1$$

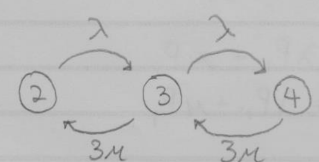
From here it's much clearer. For $n \leq N$,

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{P_0}{n!}$$

But what about $n > N$?



At N, the departure rate stalls. This is a bit too general, though. Let's say $N = 3$ and see if this makes it easier to solve.



So for $n = 3$:

$$\lambda P_3 + 3\mu P_3 = \lambda P_2 + 3\mu P_4$$

We know $P_3 = \frac{1}{6}\left(\frac{\lambda}{\mu}\right)^3 P_0$, and $P_2 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 P_0$.

$$\lambda\left(\frac{\lambda^3}{6\mu^3}\right)P_0 + 3\mu\left(\frac{\lambda^3}{6\mu^3}\right)P_0 = \lambda\left(\frac{\lambda^2}{2\mu^2}\right)P_0 + 3\mu P_4$$

$$\frac{\lambda^4}{\mu^3}P_0 + \frac{\lambda^3}{2\mu^2}P_0 = \frac{\lambda^3}{2\mu^2}P_0 + 3\mu P_4$$

$$P_0\left[\frac{\lambda^4}{\mu^3} + \frac{\lambda^3}{2\mu^2} - \frac{\lambda^3}{2\mu^2}\right] = 3\mu P_4$$

$$\frac{\lambda^4}{\mu^3}P_0 = 3\mu P_4$$

$$P_4 = \frac{\lambda^4}{3\mu^4}P_0$$

To avoid doing this again and again, I hope you can believe it when I say:

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{P_0}{N!} N^{N-n} \quad \text{for } n > N$$

Now, from these, how do we solve for an expression for P_0? We do know $\sum_0^\infty P_n = 1$. So we can just sum up our two cases.

$$\sum_{n=0}^{\infty} P_n = 1$$

$$1 = \underbrace{\sum_{n=0}^{N} (\lambda/\mu)^n \, P_0/n!}_{n \leq N} + \underbrace{\sum_{i=N}^{\infty} (\lambda/\mu)^i \, N^{N-i} \, P_0/N!}_{n > N}$$

Since P_0 isn't reliant on the iterator, we can pull it out.

$$1 = P_0\left[\sum_{n=0}^{\infty} (\lambda/\mu)^n \frac{1}{n!} + \frac{1}{N!}\sum_{i=N}^{\infty} (\lambda/\mu)^i N^{N-i}\right]$$

Now it's a rather simple matter to isolate for P_0.

$$1/P_0 = \sum_{n=0}^{N} (\lambda/\mu)^n \frac{1}{n!} + \frac{1}{N!}\sum_{i=N}^{\infty} (\lambda/\mu)^i N^{N-i}$$

$$\therefore \quad P_0 = \left(\sum_{n=0}^{N} (\lambda/\mu)^n \frac{1}{n!} + \frac{1}{N!}\sum_{i=N}^{\infty} (\lambda/\mu)^i N^{N-i} \right)^{-1}$$

<u>What happens as $N \to \infty$?</u>

In this case, we only have one version of P_n:

$$P_n = P_0 \, (\lambda/\mu)^n \frac{1}{n!}$$

as n will always be less than N. So what is P_0?

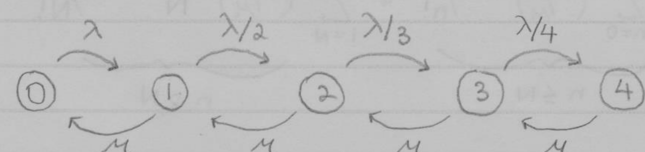
$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} P_0 \, (\lambda/\mu)^n \frac{1}{n!}$$

$$= P_0 \sum_{n=0}^{\infty} \frac{\rho^n}{n!}$$

$$1 = P_0 e^{\rho}$$

$$\therefore P_0 = e^{-\rho}$$

ex. Analyze (find P_0 up to P_n) for an M/M/1 queue where arrival is $\lambda_n = \frac{\lambda}{n+1}$ and $\mu_n = \mu$, so arrivals slow the more stuff in the queue there is.



The P_0 case is exactly the same as before.

$$\lambda P_0 = \mu P_1$$

Let's see what happens for $n = 1$.

$$\lambda/2\, P_1 + \mu P_1 = \lambda P_0 + \mu P_2$$
$$\frac{\lambda}{2}\left(\frac{\lambda}{\mu} P_0\right) + \mu\left(\frac{\lambda}{\mu}\right) P_0 = \lambda P_0 + \mu P_2$$
$$\frac{\lambda^2}{2\mu} P_0 + \cancel{\lambda P_0} = \cancel{\lambda P_0} + \mu P_2$$
$$\frac{P_0}{2}\left(\frac{\lambda^2}{\mu^2}\right) = P_2$$

Looks pretty similar to our $n \le N$ case in the question before. Let's do one more to confirm: $n=2$.

$$\lambda/3\, P_2 + \mu P_2 = \lambda/2\, P_1 + \mu P_3$$
$$\frac{\lambda}{3}\left(\frac{\lambda^2}{2\mu^2}\right) P_0 + \mu\left(\frac{\lambda^2}{2\mu^2}\right) P_0 = \left(\frac{\lambda}{2}\right)\left(\frac{\lambda}{\mu}\right) P_0 + \mu P_3$$
$$\frac{\lambda^3}{6\mu^2} P_0 + \cancel{\frac{\lambda^2}{2\mu} P_0} = \cancel{\frac{\lambda^2}{2\mu} P_0} + \mu P_3$$
$$P_3 = \frac{\lambda^3}{6\mu^3} P_0$$

Yep, it's exactly the same. $P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{P_0}{n!}$. What's P_0?

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} (\lambda/\mu)^n P_0/n!$$
$$1 = P_0 e^{\rho}$$
$$P_0 = e^{-\rho}$$

Coincidentally, it's exactly the $n \le N$ case where $N \to \infty$.

ex. Analyze an M/M/1/N queue with $\mu_n = \mu$ and $\lambda_n = \frac{\lambda}{n+1}$. Find the blocking probability.

There's no reason for our expression of P_n to change. Only P_0 may possibly change.

$$\sum_{n=0}^{N} P_n = 1$$
$$= \sum_{n=0}^{N} (\lambda/\mu)^n P_0/n!$$
$$1 = P_0 \sum_{n=0}^{N} (\lambda/\mu)^n \frac{1}{n!}$$

As $N \to \infty$, $\sum_{n=0}^{N}$ approximates e^{ρ} better and better. So we can say it $\approx e^{\rho}$, and as such:

$$P_0 = e^{-\rho}$$

The blocking probability occurs at $n=N$, $\therefore$

$$P_B = P_N = \left(\frac{\lambda}{\mu}\right)^N \frac{P_0}{N!}$$
$$= \left(\frac{\lambda}{\mu}\right)^N \frac{e^{-\rho}}{N!}$$

ex. Consider an M/M/m/m system. $\lambda_n = \lambda$, $\mu_n = n\mu$ for $n \leq m$. Analyze this system.

Obviously the $n > m$ case doesn't exist. We've solved this type before,

$$\lambda P_0 = \mu P_1$$
$$P_n = \frac{P_0}{n!}\left(\frac{\lambda}{\mu}\right)^n \quad \forall n \leq m$$

Determining P_0 using the summation:

$$1 = \sum_{n=0}^{m} \frac{P_0}{n!} (\lambda/\mu)^n \implies P_0 = \left[\sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right]^{-1}$$

And as such the blocking probability

$$P_B = P_m = \left(\frac{\lambda}{\mu}\right)^m \frac{1}{m!} \left[\sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right]^{-1}$$