Math 137 Lecture Notes.
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Spring 2023

1. Monday January 23

1.1. **Bookkeeping.** Lecture Times: MW 10:30-11:45. Location: SC 309a. Office Hours, CAs, etc. to be discussed next class.

1.2. **Historical Note.** Historically, algebraic geometry was the study of solutions to systems of polynomial equations. Let's consider the one-variable case: $f(x) = 0$. If $\deg(f) \in \{1, 2\}$, then we have formulas for the solutions. In particular, for degree 2, we have

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $\deg(f) = 3$, there also exists such a formula (Cardano's formula, 1545), and such a formula also exists for degree 4 polynomials. However, for $\deg(f) \geq 5$, it is not possible to write an explicit formula for a general polynomial (early 19th century: Abel-Ruffini Theorem; but can be proved using *Galois theory*, and in particular using the fact that A_n and S_n are not solvable for $n \geq 5$).

Now let's consider solutions of polynomials in two variables $f(x, y) = 0$, and, more generally, solutions to systems of polynomial equations $f_1(x, y) = \cdots = f_m(x, y) = 0$ with coefficients in $\mathbb{R}$. Goal: relate the "algebra" of f to the geometry of the zero locus of f. This is the start of algebraic geometry!

Two major observations due to Poncelet in 1812-1814 (in jail) :

(1) Work over $\mathbb{C}$ instead of $\mathbb{R}$. More generally, we'll want to work over algebraically closed fields.

(2) Instead of working in $\mathbb{C}^2$ (or $\mathbb{R}^2$), work in projective plane $\mathbb{P}^2$.

Reasons for (1):

- One variable: We may not have solutions over $\mathbb{R}$, e.g., $f(x) = x^2 + 1$. Moreover, the Fundamental Theorem of Algebra tells us that if $f \in \mathbb{C}[x]$ is of degree n, then f has exactly n solutions (counted with multiplicity). Note that $X^2 + Y^2$ and $X^2 - Y^2$ are very different polynomials over $\mathbb{R}$, but that over $\mathbb{C}$, we can get the second via the first by performing the change of variables $Y \mapsto iY$.
- Two variables: For example, take $f(x, y) = x^2 + y^2$. Over $\mathbb{R}$, the only solution is the point $(0, 0)$. But over $\mathbb{C}$, $f(x, y) = (x + iy)(x - iy)$, and the zero locus "looks like" two lines passing through the origin (in $\mathbb{C}^2$).

Notation 1.1. *Let $f \in k[X, Y]$, where $k = \mathbb{R}, \mathbb{C}, \ldots$. Let $C = Z(f) = \{(x, y) \mid f(x, y) = 0\} \subset k^2 = \mathbb{A}^2$ denote the* zero locus *of f. We use C because $Z(f)$ is a* plane algebraic curve. *In algebraic geometry, we often denote k^2 by $\mathbb{A}^2$, which is called the* affine plane.

Remark 1.2. Let $f(X, Y) = X^{2n} + Y^{2n} \in \mathbb{R}[X, Y]$, $n \geq 1$. Then $Z(f) = \{(0, 0)\}$. But we see that $X^6 + Y^6 = (X^2 + Y^2)(X^4 - X^2Y^2 + Y^4)$ is reducible.

Moreover, we also see that $f(X, Y) = X^2 + Y^2 + 1$ has no solutions over $\mathbb{R}$, but over $\mathbb{C}$ is a *conic*.

Therefore, we see that it is better to work with coefficients in an algebraically closed field $k = \overline{k}$.

Remark 1.3. Let $f \in k[X,Y]$, where $k = \overline{k}$. Then $Z(f)$ is always an infinite set.

Reasons for (2): "constancy of intersection numbers." Some motivation: suppose we want to study the solutions of $f(x) = 0$. To do so, we may consider the zero locus of $y - f(x) \in k[X,Y]$ and intersect with the line $y = 0$. Over $\mathbb{R}$, consider intersecting a conic with a line. We can have 0, 1, or 2 intersection points. If we consider two conics, we have even more variation: we can have 0, 1, 2, 4, etc. intersection points. But over $\mathbb{P}^2$, there are always four intersection points (counted with multiplicity). Consider a simpler example: two random lines in the plane almost always intersect in exactly one point, except if they are parallel. We fix this by "adding a point at infinity" that is the "intersection point" of two parallel lines. In $\mathbb{C}^2$, we can introduce a point at infinity for each direction through the origin. The resulting space is the projective plane $\mathbb{P}^2$, and this gives us the following theorem:

Theorem 1.4 (Bézout's Theorem)**.** *Suppose $C = Z(f)$, $D = Z(g)$, and let* $\deg(f) = d$ *and* $\deg(g) = e$. *Assume that $C \not\subset D$, and vice versa. Then the number of intersection points of C and D in $\mathbb{P}^2$ is always equal to de (counted with multiplicity).*

For example, two conics will always intersect in 4 points. Going forward, the following terminology will be useful: f where $\deg(f) = 1$ is called a line; f where $\deg(f) = 2$ is called a conic; f where $\deg(f) = 3$ is a cubic; ... quartic; ... quintic; etc.

It makes sense to study curves of non-algebraic fields (especially in number theory, arithmetic geometry, etc.). In number theory, often we'll want to take $k = \mathbb{Q}$, $f \in \mathbb{Q}[X,Y]$, and look for rational solutions to f. For example, Fermat's Last Theorem asks about solutions to $f(X,Y) = X^n + Y^n - 1$ in $\mathbb{Q}$. The proof uses elliptic curves, lots of algebraic geometry, etc. We can also set $k = \mathbb{F}_p$, $f(x,y) \in \mathbb{F}_p[X,Y], \mathbb{Z}[X,Y]$. The solutions of f are just the (X,Y) such that $f(X,Y) \equiv 0 \mod p$.

Then: in the early 20th century, Zariski, Weil, etc., made algebraic geometry made algebraic geometry "more algebraic." In other words, they made more connections to commutative algebra. After this, starting around 1950 (232ab): Grothendieck, Serre, Mumford, etc., introduced sheaves, schemes, cohomology, categorical language, stacks, etc. Why not go straight to scheme theory? Without the classical picture, you lose lots of great geometric intuition!

Some interesting open problems: the Hodge conjecture (Millennium Problem); classify all varieties (classification theory).

1.3. Plane Algebraic Curves.

Definition 1.5. Fix an algebraically closed field $k = \overline{k}$. The affine plane $\mathbb{A}^2$ (sometimes denoted $\mathbb{A}^2_k$ when the underlying field is not clear) is $\{(x,y) \mid x, y \in k\}$. For any nonconstant polynomial $f \in k[X,Y]$, $C = Z(f) \subset \mathbb{A}^2$ is the zero locus of f (defined earlier). The *degree* of C, denoted $\deg(C)$, is just the degree of the polynomial f.

Basic algebraic fact: The ring $k[X_1, \ldots, X_n]$ is a UFD. In other words, for any element f in this ring, we may write

$$f = \alpha f_1^{m_1} \cdots f_p^{m_p}$$

for irreducible polynomials f_i and $\alpha \in k$. The m_i's are called the multiplicities of the corresponding f_i's. For example, $f = X^2 + Y^2 = (X + iY)(X - iY)$. In general, we say that $C_i = Z(f_i)$; the C_i are called the *irreducible components* of C. An *irreducible curve* is the zero locus of an irreducible polynomial.

Some curves are easy to describe using parameterizations (differential geometry). For example, $f(x,y) = y - x^2$. Consider the map $k \to k^2$ given by $t \mapsto (t, t^2)$. This map will establish an isomorphism $\mathbb{A}^1 \to C = Z(f)$.

Definition 1.6. An affine plane curve C is *rational* if there exist rational functions in one variable $u(t), v(t)$ (recall rational functions in t are quotients of polynomials in t: $u(t) = p(t)/q(t)$ for $p, q \in k[t]$) such that $f(u(t), v(t)) \cong 0$ as a function of t.

Example 1.7. Here are some simple examples of rational curves:

(1) Every line is rational.

(2) Every conic is rational.

Fix a point $(x_0, y_0) \in C$ a conic. We'll parameterize the points on the conic using the slopes of lines through P. Let ℓ_t be the line intersecting C with slope t, $y - y_0 = t(x - x_0)$. Consider $C \cap \ell_t$. We know that $f(x,y) = 0$ if and only if $g(x) := f(x, t(x - x_0) + y_0) = 0$. This is a quadratic for almost all t: one root is x_0; other root is x_1. We have $x_1 + x_0 = -A(t)$, where $A(t)$ denotes the coefficient of x in monic expression of g. So, $x = -x_0 - A(t)$ and $y = y_0 + t(x - x_0)$ is our rational parameterization.

2. Wednesday January 25

2.1. **More Bookkeeping and Logistics.** CAs: Hahn Lheem and Eliot Hodges.
Class notes: posted on Canvas.
Homework: Posted each Wednesday; due the following Wednesday by the end of the day. Email Prof. Popa if there are extenuating circumstances preventing you from submitting on time, but, in general, late homework will not be accepted :(
Grading: 60% Homework, 40% Final. Lowest homework grade is dropped.
Collaboration: Use whatever sources you like, just make sure you learn the material and cite what you use and acknowledge who you worked with!
Office Hours:

2.2. **Plane Algebraic Curves.** Let $k = \overline{k}$ be an algebraically closed field. Let $f(x,y) \in k[x,y]$, and let $C = Z(f) \subset \mathbb{A}^2$. Recall that C is rational if and only if there exist rational functions $u(t), v(t)$ such that $f(u(t), v(t)) \cong 0$.

Example 2.1. Some examples of rational curves that we saw last time: lines and conics.

Exercise 2.2. Let $f, g \in k[x,y]$ so that f is irreducible and $f \nmid g$. Show that $Z(f)$ intersects $Z(g)$ in at most finitely many points.

Say that C is irreducible (i.e., f is an irreducible polynomial). Take polynomials $P(x,y), Q(x,y)$ such that $f \nmid Q$. Let $w(x,y) = \frac{P(x,y)}{Q(x,y)}$. This motivates the following definition:

Definition 2.3. The *field of rational functions* on C (or the *function field* of C) is taken to be

$$k(C) := \left\{ w = \frac{P}{Q} \mid f \nmid Q \right\} / \sim,$$

where $\sim$ is the equivalence relation given by $P_1/Q_1 \sim P_2/Q_2$ if and only if $f | P_1Q_2 - P_2Q_1$.

You should convince yourself that this indeed is a field, generated by two elements (the images of x and y). Note that x and y are no longer algebraically independent: they satisfy the algebraic relation $f(x, y) = 0$. It follows that $\operatorname{trdeg}_k(k(C)) = 1$.

Theorem 2.4 (Lüroth)**.** *Suppose $k \hookrightarrow L \hookrightarrow k(t)$ and that $k \neq L$. Then $L \cong k(t)$.*

Proposition 2.5. *C is rational if and only if $k(C) \cong k(t)$ (the field of rational functions in 1 variable).*

Proof. "$\Longleftarrow$": let $\phi : k(C) \to k(t)$ be a field isomorphism, and suppose that $x \mapsto u(t)$ and $y \mapsto v(t)$. By assumption, $f(x, y) = 0$ in $k(C)$. Applying ϕ to this expression, we get $f(u(t), v(t)) = 0$, and therefore C is rational. In other words, there exist such $u(t), v(t) \in k(t)$ such that $f(u(t), v(t)) = 0$.

"$\Longrightarrow$": Define $\phi : k(C) \to k(t)$ by

$$w(x, y) \mapsto w(u(t), v(t)) = \frac{P(u(t), v(t))}{Q(u(t), v(t))},$$

where u, v are the rational functions such that $f(u(t), v(t)) = 0$. We need to check a few things: (1) well-definededness; (2) that the denominator $Q(u(t), v(t)) \not\cong 0$. We see immediately that (2) holds by Exercise 2.2. To see (1), suppose P' and Q' are such that $w = P'/Q'$ (i.e., $f | P/Q - P'/Q'$). Say $w' = P'(u(t), v(t))/Q'(u(t), v(t))$. Since $f(u(t), v(t)) = 0$, it follows that $w(u(t), v(t)) = w'(u(t), v(t))$, and thus the map is well-defined.

So far, we have a field homomorphism $\phi : k(C) \to k(t)$, where ϕ takes $k \subset k(C)$ to $k \subset k(t)$. We'll apply Theorem 2.4 in the following way: take $L = k(C)$, and consider the following sequence of maps:

$$k \hookrightarrow L = k(C) \hookrightarrow k(X, Y).$$

Because the transcendence degree of L over k is 1, there must exist f such that $f(x, y) = 0$. Then, $L \cong k(C)$ and we are done.[1] $\square$

Exercise 2.6. Say C is a rational curve, parameterized by $u(t), v(t)$ (coming from $\phi : k(C) \to k(t)$). Then, except for a finite set of points on C, every $(x_0, y_0) \in C$ has a unique representative $(x_0, y_0) = (u(t_0), v(t_0))$.

2.3. **Local Study of Plane Curves.**

Definition 2.7. Let $f \in k[x, y]$ be nonconstant, and let $C = Z(f) \subset \mathbb{A}^2$. A point $p \in C$ is *singular* if

$$f(p) = \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0.$$

Otherwise, we say p is *nonsingular*, or *smooth*. When every point in C is nonsingular (resp., smooth), then we say C is nonsingular (resp., smooth).

Interpretation: without loss of generality, assume $p = (0, 0)$ by translation. Then

$$C \text{ is singular at } P \iff f(x, y) = ax^2 + bxy + cy^2 + dx^3 + ex^2y + \dots$$

[1] To be honest, I didn't fully understand this part of the proof in class. So my apologies if what I wrote down makes no sense. Here's another proof: By Lüroth's Theorem, it suffices to show that the map we constructed from $k(C) \to k(t)$ is injective. Recall that all nontrivial field maps are injective (fields have two ideals; the trivial ideal and the entire field, and the kernel of the map is an ideal). Thus, it follows that ϕ gives us the sequence of field maps $k \hookrightarrow L = k(C) \hookrightarrow k(t)$. The first map is not an isomorphism because $\operatorname{trdeg}_k L = 1$, and we may indeed apply Lüroth's theorem.

Example 2.8. Let's consider conics. C is singular if and only if $f = ax^2 + bxy + cy^2$. Note that f is a homogeneous polynomial, and therefore, f is a product of linear factors. We have two cases:

(1) $f = gh$ for $g \neq h$ (two lines; sorry I can't draw this. The point of intersection of the lines is called the node).

(2) $f = g^2$. This is called the *double line*. If you were to draw f, you would only draw one line, but you should think of f has a "double line."

Note that in both cases, the polynomial is reducible. It follows that an irreducible conic is nonsingular.

Example 2.9. Some cubics:

(1) $f = y^2 - x^3$. This has a singularity called a *cusp*. (Come back to later when we study tangents!)

(2) $f = y^2 - x^2 - x^3$ (or $y^2 = x^3 + x^2$). (This is really close to being an elliptic curve, but it's not because it's singular: $\frac{\partial f}{\partial y} = 2y = \frac{\partial f}{\partial x} = 3x^2 + 2x$.) This curve intersects itself! In a neighborhood of the singularity, this singularity "looks" exactly like the node from (1) in the previous example. But note that the curve is irreducible.

Definition 2.10. $P = (0, 0)$ is a *point of multiplicity* r on C if the lowest degree of a nontrivial term in f is r. I.e.,

$$f = f_r + f_{r+1} + \ldots + f_d,$$

where f_i is the homogeneous factor of degree i and $f_r \neq 0$. We'll denote this by $r = \mathrm{mult}_C(p)$.

Remark 2.11. (1) $p \in C$ is singular if and only if $\mathrm{mult}_C(p) \geq 2$;

(2) $\mathrm{mult}_C(p) \leq \deg(C)$ for all $p \in C$.

Proposition 2.12. *An irreducible plane curve has only finitely many singular points.*

Proof. Say that $C = Z(f)$. Consider $\frac{\partial f}{\partial x}$, and let $C_x = Z(\frac{\partial f}{\partial x})$. Note that if P is singular, then $P \in C \cap C_x$. We have a couple of cases: either $|C \cap C_x| < \infty$ and we are done, or $|C \cap C_x| = \infty$. Suppose $|C \cap C_x| = \infty$. By Exercise 2.2, it follows that $f | \frac{\partial f}{\partial x}$. But, we have that $\deg f > \deg \frac{\partial f}{\partial x}$, and it follows that $\frac{\partial f}{\partial x} = 0$. By symmetry, $\frac{\partial f}{\partial y} = 0$. Now we have to be careful:

(1) If $\mathrm{char}(k) = 0$, the above implies f is constant, which is a contradiction.

(2) If $\mathrm{char}(k) = p > 0$, then $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ imply $f(x, y) = g(x^p, y^p)$. Recall that, in characteristic p, $(a + b)^p = a^p + b^p$. It follows that $f(x, y) = g(x^p, y^p) = (h(x, y))^p$, and f must be reducible. This is a contradiction, since we assumed that f was irreducible.

In either case, we arrive at a contradiction. $\square$

2.4. **Rational Maps.** Let $C = Z(f) \subset \mathbb{A}^2$. Consider two rational functions $u, v \in k(C)$. Using these, we can define a map $C \to \mathbb{A}^2$ by sending $P \in C$ to $(u(P), v(P))$. This makes sense only where u and v are defined (usually not everywhere, since u and v are rational functions). We'll denote such a *rational map* using a dashed arrow: $C \dashrightarrow \mathbb{A}^2$.

Say we have a curve $B \subset \mathbb{A}^2$ given by $g(x, y) = 0$ and have $g(u(t), v(t)) \equiv 0$. Then in fact we'll get $C \dashrightarrow B \subset \mathbb{A}^2$. This is a *rational map* between the plane curves C and B.

For example, the parameterization of a line is a map $\mathbb{A}^1 \to L$ (this will actually be an isomorphism, and is defined everywhere). Parameterization of a conic: recall that to parameterize a conic

C, we fixed a point P on the conic and considered the lines through P and the x-axis. This gives us a rational map between $\mathbb{A}^1$ and C, we also get an inverse map from $C \to \mathbb{A}^1$. This is an example of a *birational map*.

Definition 2.13. A rational map $\varphi : C \dashrightarrow B$ is called *birational* if there exists a rational map $\psi : B \dashrightarrow C$ such that $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$ (wherever defined!). Then B and C are called birational.

Remark 2.14. C is rational if and only if C is birational to $\mathbb{A}^1$.

3. Monday January 30

Last time, we talked about smooth points on plane curves, which is a "local" property—i.e., we're asking about specific points on the curve. Here's another such local property:

3.1. **Intersection Multiplicities.** Let $C = (f = 0)$ and $D = (g = 0)$ be irreducible plane curves in $\mathbb{A}^2$. As always, assume that $C \not\subset D$

Definition 3.1. Suppose $p \in C$ is a smooth point. Then the *intersection multiplicity* of C and D at p, denoted $i_p(C, D)$, is defined as the multiplicity of p as a zero of $g|_C$.

It is clear what this means when $f(x, y) = y + h(x)$: replace y by $-h(x)$ so that $g(x, y)|_C = g(x, -h(x))$. But can we put f in this form in general? Because p is smooth, $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ is nonzero; we may then use the implicit function theorem to put f in this form. However, this is not exactly the "correct" algebraic definition; we'll see in section how to define intersection multiplicity algebraically. For now, we'll stick to intersections with lines, which can clearly be put into the desired form: $ax + by + x = 0$ can be rearranged to give y in terms of x.

Example 3.2. Let $C = (y = 0)$ and $D = (y - x^2 = 0)$. (We are intersecting the line and the parabola; something interesting is happening here because the line is tangent to the parabola at $(0, 0)$.) By definition: $i_{(0,0)}(C, D)$ is the multiplicity of 0 as a root of x^2; thus, $i_0(C, D) = 2$.

Definition 3.3. Suppose $p = (x_0, y_0) \in C = (f = 0)$ is a smooth point. Claim: There exists a unique line L through p such that $i_p(L, C) \geq 2$. This will be called the *tangent line* at p.

Proof of Claim. Let's parameterize our line L: $x = x_0 + \lambda t$ and $y = y_0 + \mu t$, where $t \in k$. Note that λ/μ completely determines our line (this gives its slope).

Write
$$f(x, y) = a(x - x_0) + b(y - y_0) + g(x, y),$$
where g corresponds to the terms of degree 2. Recall that because p is a smooth point, at least one of a and b is nonzero. Restricting f to the line L: $f|_L = (a\lambda + b\mu)t + t^2\varphi(t)$, where φ is some function of t. By definition, $i_p(L, C) \geq 2$ if and only if $a\lambda + b\mu = 0$. This implies that

$$L = a(x - x_0) + b(y - y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \tag{1}$$

which is the formula we know from calculus. □

Remark 3.4. It may happen that we have $i_p(L, C) \geq 3$ for the line L through p. Such a p is called an *inflection point*, or *flex*.

Example 3.5. Let $f(x,y) = y - y^3 - x^3$; note that $(0,0)$ is a smooth point on C. At $(0,0)$, $\frac{\partial f}{\partial y} = 1$ and $\frac{\partial f}{\partial x} = 0$. Let $L = (y = 0)$. It's not hard to see that $i_p(L, C) = 3$ (it's the multiplicity of 0 as a point of x^3).

3.2. **Projective Plane Curves.** We use $\mathbb{P}^2$, or $\mathbb{P}^2_k$ when we want to specify the field k, to denote the projective plane over $k = \overline{k}$. We define $\mathbb{P}^2$ in the following way: $\mathbb{P}^2 := \mathbb{A}^3 \setminus \{0\}/k^* = \mathbb{A}^3 \setminus \{0\}/\sim$, where $\sim$ is the equivalence relation given by $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if there exists some $\lambda \in k^*$ such that $\lambda(x_1, y_1, z_1) = (x_2, y_2, z_2)$. Notation for a point in $\mathbb{P}^2$: $[x : y : z]$; these are called homogeneous coordinates. Note that $[x : y : z]$ is an equivalence class, and that there are many triples of numbers that can represent this point in $\mathbb{P}^2$!

Remark 3.6. $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$, where $(x,y) \mapsto [x : y : 1]$. If $[x : y : z] \in \mathbb{P}^2$ with $z \neq 0$, then $[x : y : z] = [x/z : y/z : 1]$. Thus, if z is nonzero, $[x : y : z]$ always lives in affine space, and we may write

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \{[x : y : 0] \mid x, y \in k \text{ not both } 0\}.$$

Now, by forgetting about the third coordinate, we see that $\{[x : y : 0] \mid x, y \in k \text{ not both } 0\} = \mathbb{P}^1$. This set is sometimes denoted L_∞ and called the "line at infinity." Each point on this line corresponds to a slope—a "direction"—in $\mathbb{A}^2$. Writing $\mathbb{P}^2$ in this way involves a choice: we chose $z \neq 0$; in fact, we may define $\mathbb{A}^2_x = \{[1 : y : z]\} \subset \mathbb{P}^2$, $\mathbb{A}^2_y = \cdots$, and $A^2_z = \cdots$. Therefore, we may write

$$\mathbb{P}^2 = \mathbb{A}^2_x \cup \mathbb{A}^2_y \cup \mathbb{A}^2_z.$$

This is what will be known as an affine paving of a variety.

Definition 3.7. A *projective plane curve* is the zero locus of a nonconstant homogeneous polynomial $F = k[X, Y, Z]$. Recall that F is homogeneous of degree d if $F(\lambda x, \lambda y, \lambda z) = \lambda^d F(x, y, z)$, or, equivalently, if all the monomials in F have degree d.

Question 3.8. Does it make sense to say that a polynomial takes the value 0 at a point in $\mathbb{P}^2$? Does it make sense to say that a polynomial takes the value 5 at a point?

Because F is homogeneous, it makes sense to say that $[x : y : z]$ is a zero of F: no matter what representative of the point $[x : y : z]$ you choose, if $F(x, y, z) = 0$, we see that $F(\lambda(x, y, z)) = \lambda^d F(x, y, z) = 0$. However, any other value does not make sense, and thus we cannot think of polynomials as functions on projective space!

Homogenization ("Projective closure"): Suppose $C = Z(f(x, y)) \subset \mathbb{A}^2$ and that $\deg(f) = n$. Consider $F(x, y, z) := z^n f(x/z, y/z)$. A quick check tells us that F is a homogeneous polynomial in x, y, z of degree n. Hence, F defines a curve $\overline{C} \subset \mathbb{P}^2$. Note that $F(x, y, 1) = f(x, y)$, so $C = \overline{C} \cap \mathbb{A}^2$.

Dehomogenization: Say $F(x, y, z)$ is a homogeneous polynomial of degree n, and consider $\overline{C} = Z(F) \subset \mathbb{P}^2$. By intersecting $\overline{C}$ with $\mathbb{A}^2_x$, $\mathbb{A}^2_y$, or $\mathbb{A}^2_z$ we can get affine curves defined by $F(1, y, z)$, $F(x, 1, z)$, and $F(z, y, 1)$ respectively.

Smooth points and tangent lines to projective plane curves:

(1) Recall: let $C = Z(f) \subset \mathbb{A}^2$ and $p = (x_0, y_0)$ be a smooth point on C. Then we have the tangent line at p (see Equation (1)), which we defined earlier.

(2) Now suppose $C = Z(F) \subset \mathbb{P}^2$, $p \in C$, and $p \in \mathbb{A}^2_z$ so that $p = [x_0 : y_0 : 1]$. Suppose F is homogeneous of degree n. Let $f(x, y) = F(x, y, 1)$, and note that $\frac{\partial f}{\partial x}(x, y) = \frac{\partial F}{\partial x}[x : y : 1]$ and that $\frac{\partial F}{\partial y}(x, y) = \frac{\partial F}{\partial y}[x : y : 1]$. We say that $p \in C$ is a *singular point* of C if $\frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = 0$.

Exercise 3.9. Let F be a homogeneous polynomial of degree n. Then we have *Euler's formula* (assume char$k = 0$). Then

$$nF = x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z}.$$

Hence, $P \in C$ is singular if and only if $\frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = \frac{\partial F}{\partial z} = 0$.

If we homogenize Equation (1) at z and use Euler's formula again, we get the tangent line to C at p in $\mathbb{P}^2$, given by

$$L : \frac{\partial F}{\partial x}(x_0, y_0, 1)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, 1)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, 1)(z - 1) = 0.$$

We conclude with a preliminary statement of Bézout's Theorem:

Theorem 3.10 (Bézout's Theorem, preliminary version)**.** *Let C and D be projective plane curves. Suppose that C is smooth and that $C \not\subset D$. Then*

$$\sum_{p \in C \cap D} i_p(C, D) = \deg(C)\deg(D).$$

4. Wednesday February 1

4.1. **Cubic Curves.** In the affine case, a cubic curve is defined by a degree 3 polynomial $f \in k[x, y]$.

Fact: It is possible, by elementary manipulations (e.g., change of coordinates), to bring f to the form $y^2 = x^3 + ax^2 + bx + c$, where $a, b, c \in k$. More precisely, the curve defined by $y^2 = x^3 + ax^2 + bx + c$ is birational to $Z(f)$. The form $y^2 = x^3 + ax^2 + bx + c$ is called the *Weierstrass normal form.*

Some motivation: there are inflection points, or flexes, on cubic curves. Projectivize your curve, and declare the tangent line at the inflection point to be the line at infinity; doing so will bring the cubic to this form.

Another change of coordinates sending $x \mapsto x - a/3$ brings our curve to the form $y^2 = x^3 + px + q$ for $p, q \in k$ (this is sometimes referred to as the Weierstrass form). We'll denote the resulting curve by C_{st}, where the st denotes that the curve is in "standard form."

Lemma 4.1. *C_{st} is smooth if and only if $4p^3 + 27q^2 \neq 0$ (in characteristic 0).*

Proof. Note that $\frac{\partial f}{\partial y} = 2y = 0$ if and only if $y = 0$. Let $g(x) = x^3 + px$. Thus, if $(x_0, 0)$ is a singular point of the curve, then $g(x_0) = 0$ and $\frac{\partial g}{\partial x}(x_0) = 0$. Hence, g has a multiple root at x_0, which occurs if and only if the discriminant of g, $\Delta = 4p^3 + 27q^2$, is 0. □

If $4p^3 + 27q^2 \neq 0$ (i.e., C_{st} is smooth), we call C_{st} an elliptic curve (really, we'll think about its projective closure—the zero locus in $\mathbb{P}^2$ of $y^2z = x^3 + pxz^2 + qz^3$).

On this week's homework: every irreducible singular cubic ($e.g., y^2 = x^3, y^2 = x^3 + x^2$) is rational! But what about elliptic curves? Are these rational?

Proposition 4.2. *Elliptic curves are* not *rational. :'(*

Proof. Suppose $C_{st} = Z(y^2 = x^3 + px + q)$ is a smooth cubic and that it is rational, i.e., birational to $\mathbb{A}^1$. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^1 = k$ be given by $\varphi(x) = \frac{P(x)}{Q(x)}$. We want to determine when $\varphi(x) = z$ has exactly one solution in x, which occurs if and only if $P(x) - Q(x)z = 0$ has exactly one solution in x for general z. This implies that P and Q are linear polynomials, say $P(x) = ax + b$ and $Q(x) = cx + d$. We claim that there at most two fixed points of this map. If x is a fixed point, then $ax + b = x(cx + d)$, so this map has at most two fixed points.

For C_{st}, if we find a map from the curve to itself that has three fixed points or more, then we are done. Consider the map $\psi : C_{st} \to C_{st}$ given by $(x, y) \mapsto (x, -y)$. ψ is an involution because $\psi \circ \psi = 1$. It is not hard to verify that this indeed maps the curve to itself. In order to fix the point, $y = -y$, so $y = 0$. Thus, any fixed point has the form $(x_0, 0)$, and this contradicts our assumption that C_{st} is smooth (i.e., that discriminant $\Delta = 4p^3 + 27q^2 \neq 0$). $\square$

4.2. **Bézout's Theorem.** We can think of Bézout's theorem as a generalization of the fundamental theorem of algebra:

Proof of Bézout in the special case of intersections with lines. Let $C = L \subset \mathbb{P}^2$, and let $D \subset \mathbb{P}^2$ be a curve of degree d. Suppose that $L \not\subset D$. We want to show the following:

$$\sum_{P \in L \cap D} i_P(L, D) = d \cdot 1.$$

We can assume that $|L \cap D| < \infty$.

Exercise 4.3. Show that we can coordinates so that $L = (y = 0)$. Then $\mathbb{P}^2 \setminus \mathbb{A}^2_z = L_\infty = (z = 0)$. Note that L_∞ doesn't pass through the intersection points of L and D.

In $\mathbb{A}^2_z$, suppose D is given by $f(x, y) = 0$, where $\deg(f) = d$. Write $f = f_d + g(x)$, where f_d denotes the terms of degree d and $g(x)$ is the lower degree terms. Consider the point $[1 : 0 : 0] \in L \cap L_\infty$. By assumption, this is not an intersection point of L and D. This implies that f_d must be of the form $f_d = a_d x^d + \ldots$ where $a_d \neq 0$. Otherwise, every term would be of lower degree in x; homogenizing would give a polynomial whose degree d terms all are divisible by y or z. This would imply $[1 : 0 : 0] \in D$, which is a contradiction.

By definition, $i_P(L, D)$ is the multiplicity of P as a root of $f|_{L=(y=0)}$. Since f always contains the term $a_d x^d$, it follows that f is a polynomial of degree d in the variable x. Hence,

$$\sum i_p(L, D) = \sum \text{mult(all roots of } f|_L \text{ in } x \text{ of degree } d) = d,$$

where the last equality follows from the Fundamental Theorem of Algebra. $\square$

Everyone's favorite example:

Theorem 4.4 (Pascal's Theorem)**.** *If a hexagon is inscribed in a conic, then the three points at which the pairs of opposite sides meet lie on a straight line (see picture from class, or just look this up).*

Proof. Let our conic be denoted by C. Order the vertices of the hexagon $x_1, \ldots, x_6$ going counterclockwise. Intersect the "opposite sides" in the following way: let $\ell_1, \ldots, \ell_6$ denote the sides of the hexagon, where ℓ_i connects x_{i-1} to x_i (work mod 6); let the lines continue infinitely in both directions. Let $P = \ell_1 \cap \ell_4$, $Q = \ell_2 \cap \ell_5$, and $R = \ell_3 \cap \ell_6$. Then we claim that P, Q, and R are colinear!

For any $t \in k$, consider the cubic $C_t := (\ell_1\ell_3\ell_5 + t\ell_2\ell_4\ell_6 = 0)$. Let $\ell_1\ell_3\ell_5 + t\ell_2\ell_4\ell_6$ be denoted by f_t. It is not so difficult to see that $x_1, \ldots, x_6 \in C_t$ for all t. Fix a point $y \notin \{x_1, \ldots, x_6\}$. Then we can find some $s \in k$ such that $y \in C_s$ (just solve for t). Thus, we see that $x_1, \ldots, x_6, y \in C_s$.

Applying Bézout's Theorem, we see that if $C \not\subset C_2$, then there are at most 6 points in $C \cap C_s$, since $\deg(C) = 2$ and $\deg(C_s) = 3$. It follows that $C \subset C_s$, implying that $C_s = C \cup L$ for some line L.

Similar reasoning implies that $P, Q, R \in C_s \setminus C$; therefore $P, Q, R \in L$, as desired. $\square$

4.3. **Affine Varieties.** In this section, we'll begin a more formal introduction to algebraic geometry. Throughout let k be an algebraically closed field.

Definition 4.5. The *affine space* over k is

$$\mathbb{A}^n = \mathbb{A}^n_k = \{(a_1, \dots, a_n) \mid a_i \in k\} = k^n.$$

Remark 4.6. A polynomial $f : k[x_1, \dots, x_n]$ defines a function $f : \mathbb{A}^n \to k$. (Though this is not true in more general spaces, e.g., projective space).

Definition 4.7. Let $S \subset k[x_1, \dots, x_n]$. Then

$$Z(S) := \{x \in \mathbb{A}^n \mid f(x) = 0 \forall f \in S\}$$

is called the *zero set of* S. A subset $X \subset \mathbb{A}^n$ is called an *algebraic set* if there exists $S \subset k[x_1, \dots, x_n]$ such that $X = Z(S)$.

Example 4.8. (1) $f = xy \in k[x, y, z]$. This is the union of two planes: the plane $(x = 0)$ and the plane $(y = 0)$.

(2) $f = x^2 + y^2 - 1 \in k[x, y, z]$. Over $\mathbb{R}$, $Z(f)$ is just a cylinder of radius 1.

(3) The algebraic sets of the form $Z(f)$ for $f \in k[x_1, \dots, x_n]$ are called *hypersurfaces*.

(4) $X = Z(xy, x^2 + y^2 - 1) \subset \mathbb{A}^3$. This is the intersection of the previous two examples, which is just four vertical lines.

(5) Some algebraic sets (not necessarily hypersurfaces) can be *parameterized*. For example, consider the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ given by $t \mapsto (t, t^2, t^3)$. $C := \operatorname{im} \varphi$ is called the *twisted cubic* in $\mathbb{A}^3$, and it's very famous. As an algebraic set, $C = Z(y - x^2, z - x^3)$.

Definition 4.9. (Algebra Reminder) A ring R (commutative with unit from now on, unless specified otherwise), is called *Noetherian* if every ideal in R is finitely generated. Equivalently, R satisfies the ascending chain condition (ACC) on ideals, i.e., every ascending chain of ideals

$$I_1 \subset I_2 \subset \cdots \subset I_k$$

stabilizes so that $I_k = I_{k+1} = \cdots$ for some k.

Exercise 4.10. Prove this equivalence if you haven't seen it before.

Theorem 4.11 (Hilbert's Basis Theorem)**.** *Let R be a Noetherian ring. Then $R[x]$ is Noetherian.*

Thus, $k[x_1, \dots, x_n]$ is Noetherian.

5. Monday February 6

5.1. **Affine Algebraic Sets and the Zariski Topology.** Let $S \subset k[x_1, \dots, x_n]$ be a subset. Consider *the ideal generated by* S, denoted $\mathcal{I}(S) \leq k[x_1, \dots, x_n]$. Recall that

$$\mathcal{I}(S) = \left\{ \sum_{i=1}^{m} g_i f_i \mid g_i \in k[x_1, \dots, x_n], f_i \in S \right\}.$$

Lemma 5.1. *With notation as in the above, $Z(S) = Z(\mathcal{I}(S))$.*

Proof. "$\supseteq$" is clear, since $S \subset \mathcal{I}(S)$.

"$\subseteq$": Let $x \in Z(S)$ so that for all $f_i \in S$, $f_i(x) = 0$. Take an arbitrary $h = \sum_i g_i f_i \in \mathcal{I}(S)$. Then $h(x) = 0$, implying that $x \in Z(\mathcal{I}(S))$. □

Hence, every algebraic set may be written as $Z(I)$ for some ideal $I \subset k[x_1, \dots, x_n]$.

Lemma 5.2. *(1) $\emptyset$ and $\mathbb{A}^n_k$ are algebraic.*

(2) If $S_1 \subset S_2$, then $Z(S_2) \subset Z(S_1)$.

(3) If $(S_i)_{i \in I}$ is a family of subsets, then $\bigcap_{i \in I} Z(S_i) = Z(\bigcup_{i \in I} S_i)$.

(4) If S_1 and S_2 are subsets, then $Z(S_1) \cup Z(S_2) = Z(S_1 \cdot S_2)$.

By the preceding lemma, we can safely replace subsets with ideals in the above.

Proof. (1)-(3) are left as exercises for the reader.

We'll prove (4). "$\subseteq$": Let $x \in Z(S_1) \cup Z(S_2)$. Then for all $f_1 \in S_1, f_2 \in S_2$, $f_1(x) = 0$ or $f_2(x) = 0$. Hence $f_1(x)f_2(x) = 0$, so $x \in Z(S_1 \cdot S_2)$. For the reverse inclusion, suppose that $x \in Z(S_1) \cup Z(S_2)$. It follows that there exists $f_1 \in S_1$ and $f_2 \in S_2$ such that $f_1(x) \neq 0$, $f_2(x) \neq 0$, implying that $f_1 f_2(x) \neq 0$. Thus, $x \notin Z(S_1 \cdot S_2)$. □

Therefore, $\emptyset$, $\mathbb{A}^n_k$, and intersections and finite unions of algebraic sets are algebraic sets! In other words, there exists a topology on $\mathbb{A}^n$ where the closed sets are the algebraic sets. this is called the *Zariski topology.*

To a differential geometer, this might seem like a "bad" topology. For example, the open sets in the Zariski topology are huge. In fact any two nonempty open subsets of $\mathbb{A}^n$ intersect. Thus, our space is not Hausdorff, which is actually sort of terribly from a purely topological perspective. However, it is refined by the classical topology.

Example 5.3. In $\mathbb{A}^1$, the closed sets are $\emptyset, \mathbb{A}^1$, and finite sets of points. Recall that $k[x]$ is a PID, implying that every ideal is of the form (f) for $f \in k[x]$. By what we proved earlier, the zero locus of any subset of $k[x]$ is the zero locus of an ideal of $k[x]$, which just looks like (f). Now,

$$f = c(x - a_1) \cdots (x - a_n),$$

since we are working in an algebraically closed field, and $Z(f) = \{a_1, \dots, a_n\}$.

In $\mathbb{A}^2$, the closed sets are $\emptyset, \mathbb{A}^2$, plane curves union with finite sets of points (consider intersecting two plane curves—this will give you a finite set of points).

The Zariski topology on any $X \subset \mathbb{A}^n$ is the induced topology on X from $\mathbb{A}^n$.

Definition 5.4. A topological space X is called *irreducible* if there are no proper closed subsets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$. Otherwise, X is called *reducible.*

In the usual topology, this is somewhat of a pointless definition. Note that $\mathbb{C} = \{z \mid |z| \leq 1\} \cup \{z \mid |z| \geq 1\}$.

Definition 5.5. An irreducible affine algebraic set is called an *affine variety.*

Example 5.6. (1) Let $f \in k[x_1, \dots, x_n]$ be irreducible. Then the hypersurface $Z(f) \subset \mathbb{A}^n$ is an affine variety.

(2) Let $X = Z(xz, yz) \subset \mathbb{A}^3$. Note that we may write $X = (z = 0) \cup (x = y = 0)$, i.e., as the union of the plane $(z = 0)$ and the z-axis. This is reducible and therefore not a variety.

(3) (Challenge exercise) Let X be the twisted cubic, which, recall, is defined by the map $\mathbb{A}^1 \to \mathbb{A}^3$, $t \mapsto (t, t^2, t^3)$. Show that this is an affine variety. (Later, we'll formalize the notion of morphisms of algebraic subsets. This will allow us to show that the map in question is an isomorphism, which will tell us immediately that the twisted cubic is irreducible.)

Definition 5.7. A topological space is *Noetherian* if for all descending chain of closed subsets

$$X \supset X_1 \supset X_2 \supset \cdots$$

is stationary (i.e., there exists k_0 such that for all $k \geq k_0$ $X_k = X_{k+1}$).

We'll soon see that algebraic sets are Noetherian topological spaces.

Proposition 5.8. *Every Noetherian topological space can be written as a finite union*

$$X = X_1 \cup \cdots \cup X_r$$

of irreducible closed subsets. If we assume that $X_i \not\subset X_j$ for every $i \neq j$, then the decomposition in unique up to reordering.

Proof. Suppose that the conclusion is false. Then we can write $X = X_1 \cup X_1'$ so that at least one of X_1, X_1' (say X_1) isn't irreducible. It follows that $X_1 = X_2 \cup X_2'$, where one of X_2, X_2' isn't irreducible. Continuing in this manner, we get a descending chain that doesn't stabilize,

$$X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots,$$

which is a contradiction.

For uniqueness, suppose that

$$X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_s',$$

which satisfy the desired properties (from the statement of the theorem). Note that

$$X_1 \subset X_1' \cup \cdots \cup X_s',$$

implying that

$$X_1 = \bigcup_{j=1}^{s} X_1 \cap X_j'.$$

Because X_1 is irreducible, it follows that one of the X_j''s contains X_1. Otherwise, $X_1 \cap X_j'$ is properly contained in X_1 for all j, implying that X_1 is not reducible. By symmetry, there exists some i such that $X_j' \subset X_i'$. Thus, we have that there exist i, j such that

$$X_1 \subset X_j' \subset X_i,$$

forcing $X_1 = X_i = X'_j$. Reindex so that $j = 1$ and throw X_1 and X'_1 away. Look at $Z = \overline{X \setminus X_1}$, and recall that

$$X_2 \cup \cdots \cup X_r = Z = X'_2 \cup \cdots \cup X'_s.$$

The argument is finished by induction on r.[2] $\square$

Note that the above proposition allows us to write every algebraic subset as a union of affine varieties, provided that our original algebraic subset of $\mathbb{A}^n$ is Noetherian.

Definition 5.9. Let $\emptyset \neq X$ be an irreducible topological space. The dimension of X is the largest integer n such that there exists a chain of irreduicble closed subsets

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X.$$

Example 5.10. Consider $X = \mathbb{A}^n \supset \mathbb{A}^{n-1} \supset \mathbb{A}^{n-2} \supset \cdots \supset \mathbb{A}^1 \supset \mathbb{A}^0 \supset \emptyset$. This implies $\dim \mathbb{A}^n \geq n$. "If there is any justice in this world, then we should have $\dim \mathbb{A}^n = n \ldots$ can you show that?" - Prof. Popa

This is a trick question. Showing that $\dim \mathbb{A}^n = n$ directly from Definition 5.9 would probably win you an award; to show this, we'll need some powerful notions from commutative algebra. In particular, we'll need to define Krull dimension, transcendence degree, etc. and to show that $\dim \mathbb{A}^n = \text{trdeg} k(x_1, \ldots, x_n) = n$.

Definition 5.11. Let $X \subset \mathbb{A}^n$ be an arbitrary set. The *vanishing ideal* of X, denoted $\mathcal{I}(X)$, is

$$\mathcal{I}(X) = \{f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \ \forall x \in X\}.$$

Prove that $\mathcal{I}(X)$ is an ideal.

Restricting to algebraic sets we have a correspondence:

$$\{\text{algebraic sets in } \mathbb{A}^n\} \longleftrightarrow \{\text{ideals in } k[x_1, \ldots, x_n]\}.$$

The direction $\leftarrow$ is given by $Z(-)$ (taking the zero locus of a set of polynomials); the direction $\rightarrow$ is given by $\mathcal{I}(-)$.

Question: To what extent are these operations inverse to each other? For example, x^2 and x in $k[x]$ have the same vanishing locus.

The obvious obstruction is this: if f^r vanishes at x for some $r > 0$, then f vanishes at x. Thus, we'll want to introduce the idea of taking square roots.

Example 5.12. (1) Let $J = (x^2) \subset k[x, y]$. Then $Z(J)$ is the y-axis, and $\mathcal{I}(Z(J)) = (x)$. Therefore, the correspondence introduced above is not a bijection.

(2) Let $J = (y, y^2 - x^2 - x^3) \subset k[x, y]$. It is not difficult to see that $Z(J)$ consists of two points. Now, note that $(y, y^2 - x^2 - x^3) = (y, x^2 + x^3) = (y, x^2(x+1))$, and that $\mathcal{I}(Z(J)) = (y, x(x+1))$.

Definition 5.13. Suppose $I \leq R$ is an ideal in R, a commutative ring. The *radical of* I is

$$\sqrt{I} = \text{rad}(I) := \{f \in R \mid \exists r > 0 \text{ such that } f^r \in I\}$$

[2]Work out the details of this argument as an exercise.

6. Wednesday February 8

6.1. More on Algebraic Varieties and Vanishing Ideals.

Remark 6.1. Suppose X is an algebraic set. Then $\mathcal{I}(X)$ is a *radical ideal*, i.e., $\operatorname{rad}(I) = \sqrt{I} = I$. In other words, if $f^r \in I$ for some nonzero r, then $f \in I$.

Lemma 6.2. *Let X be an affine algebraic set. Then X is an affine variety if and only if $\mathcal{I}(X)$ is prime. (Recall I prime means that if $fg \in I$, one of f or g is in I.)*

Proof. " $\Longleftarrow$ ": Suppose $X = X_1 \cup X_2$ for proper closed subsets. Then $\mathcal{I}(X) \subsetneq \mathcal{I}(X_1), \mathcal{I}(X_2)$; hence there exists $f_i \in \mathcal{I}(X_i) \setminus \mathcal{I}(X)$ for $i \in \{1, 2\}$. If $x \in X$, then $f_1(x) = 0$ or $f_2(x) = 0$, implying that $f_1 f_2(x) = 0$. Hence, $f_1 f_2 \in \mathcal{I}(X)$, which contradicts the primality of $\mathcal{I}(X)$.

" $\Longrightarrow$ ": Suppose for a contradiction that $\mathcal{I}(X)$ is not prime. Then there exist $f, g \notin \mathcal{I}(X)$ such that $fg \in \mathcal{I}(X)$. Hence, $X \subset Z(fg) = Z(f) \cup Z(g)$, and we may write

$$X = (X \cap Z(f)) \cup (X \cap Z(g)).$$

Let $X_1 = X \cap Z(f)$ and $X_2 = X \cap Z(g)$. We see that $X \neq X_1$ and $X \neq X_2$ because $X \not\subset Z(f)$ and $X \not\subset Z(g)$. □

Remark 6.3. Recall that an ideal I is prime if and only if R/I is a (integral) domain. The proof is left as an exercise for the reader. Usually, the easiest way to check whether an ideal is prime or not is to take the quotient and look at the resulting ring. In general, it will probably be easiest to show that the quotient ring is an integral domain.

Exercise 6.4. Over $\mathbb{R}$, check that $f = x^6 + y^6$ is reducible. Show that $f = y^2 + x^2(x-1)^2$ is irreducible.

6.2. Hilbert's Nullstellensatz.

Theorem 6.5 (Hilbert's Nullstellensatz, Variant I, or "Weak Nullstellensatz")**.** *Let k be an algebraically closed field. Consider the polynomial ring $k[x_1, \ldots, x_n]$. Then the maximal ideals in the ring are necessarily of the form*

$$\mathfrak{M} = (x_1 - a_1, \ldots, x_n - a_n).$$

In particular, every ideal $I \subsetneq k[x_1, \ldots, x_n]$ has $Z(I) \neq 0$: by Zorn's Lemma, every ideal I is contained in a maximal ideal $\mathfrak{M}$, so $Z(\mathfrak{M}) \subset Z(I)$, and the above then tells us that $Z(I)$ is nonempty.

Remark 6.6. In German, Nullstellen means root, or zero-point, and Satz means theorem. So Nullstellensatz translates roughly to root-theorem. Last year, Fabian taught us a result which he called the "Nichtnullstellensatz," or the "nonroot theorem." Nichtnullstellensatz is a pretty cool word.

Proposition 6.7. (*1*) *Suppose X is an algebraic set in $\mathbb{A}^n$. Then $Z(\mathcal{I}(X)) = X$.*

(*2*) *(Hilbert's Nullstellensatz, Variant II) For every $I \subset k[x_1, \ldots, x_n]$, $\mathcal{I}(Z(I)) = \operatorname{rad}(I) = \sqrt{I}$.*

Proof. (1). "$\supset$" is clear. "$\subset$": Say $X = Z(J)$. Then $J \subset \mathcal{I}(Z(J)) = \mathcal{I}(X)$. Taking the zero locus of these ideals gives us $Z(\mathcal{I}(X)) \subset Z(J) = X$.

(2). The inclusion $\operatorname{rad}(I) \subset \mathcal{I}(Z(I))$ is clear. For the reverse inclusion, we'll need Theorem 6.5, the Weak Nullstellensatz. Because $k[x_1, \ldots, x_n]$ is Noetherian, we know that $I = (f_1, \ldots, f_r)$ for

$f_i \in k[x_1, \dots, x_n]$. Take $g \in \mathcal{I}(Z(I))$. Now, we do something called the "Rabinowitch Trick." Consider a new ideal $J = (f_1, \dots, f_r, x_{n+1}g - 1) \subset k[x_1, \dots, x_{n+1}]$. Consider $p \in Z(J)$. Note that $f_1(p) = \cdots = f_r(p) = 0$. Since $g \in \mathcal{I}(Z(I))$, $g(p) = 0$. We also have that $(x_{n+1}g - 1)(p) = 0$ if and only if $-1 = 0$. Therefore, p does not exist, and $Z(J) = \emptyset$. The Weak Nullstellensatz (Theorem 6.5) forces $J = k[x_1, \dots, x_{n+1}]$, implying $1 \in J$. Hence, there exist $h_1, \dots, h_{r+1} \in k[x_1, \dots, x_{n+1}]$ such that $\sum_{i=1}^r h_i f_i + h_{r+1}(x_{n+1}g - 1) = 1$. This is an identity in the variables $x_1, \dots, x_{n+1}$. In particular, it holds if we set $x_{n+1} = 1/g(x_1, \dots, x_n)$. This implies that

$$\sum_{i=1}^{r} h_i(x_1, \dots, x_n, 1/g(x_1, \dots, x_n)) f_i(x_1, \dots, x_n) = 1.$$

Multiply both sides of the above by a sufficiently large power of g to clear the denominators, which gives us

$$g^r = \sum h_i' f_i,$$

where the h_i''s are polynomials in $x_1, \dots, x_n$. Therefore, $g^r \in I$, which is exactly what we wanted to show. $\square$

Corollary 6.8. *There is a one-to-one correspondence:*

$$\{\text{algebraic sets in an affine space } \mathbb{A}^n\} \longleftrightarrow \{\text{radical idealsin } k[x_1, \dots, x_n]\}$$

$$\{\text{affine varieties in } \mathbb{A}^2\} \longleftrightarrow \{\text{prime ideals in } k[x_1, \dots, x_n]\}$$

$$\{\text{points in } \mathbb{A}^n\} \longleftrightarrow \{\text{maximal ideals in } k[x_1, \dots, x_n]\}.$$

The left arrows correspond to applying $\mathcal{I}(-)$; the right correspond to applying $Z(-)$. The sets are ordered by inclusion going up the page (so the largest sets are at the top).

6.3. **Coordinate Rings.**

Definition 6.9. Let $X \subset \mathbb{A}^n$ be an affine variety. Then $A(X) := k[x_1, \dots, x_n]/I(X)$ is the affine coordinate algebra (or ring) of X. Note that $A(X)$ is a finitely generated k-algebra and an integral domain. If X is just an algebraic set, we can still consider $A(X)$, but note that in this case $A(X)$ may not be an integral domain.

Suppose $X = X_1 \cup \cdots \cup X_r$ is the decomposition into X into irreducible components. Recall that X_i is irreducible if and only if $\mathcal{I}(X_i) = \mathfrak{p}_i$ is a prime ideal in $k[x_1, \dots, x_n]$. Then,

$$\mathcal{I}(X) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r,$$

and $\mathfrak{p}_i$ are the minimal primes in $k[x_1, \dots, x_n]$ containing $\mathcal{I}(X)$ (i.e., by the Correspondence Theorem, the $\mathfrak{p}_i$ are the minimal primes in $A(X)$).

$$\mathcal{I}(X) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$$

is called the *primary decomposition of* $I(X)$ (read more about this in a commutative algebra book, such as Atiyah-MacDonald).

6.4. **Affine Theory of Dimension.**

Definition 6.10. Let R be a ring. The (Krull) dimension of R is

$$\dim R := \sup\{\text{ht}(\mathfrak{p}) \mid \mathfrak{p} \subset R \text{ prime ideal}\}.$$

Here $\text{ht}(\mathfrak{p})$ is the supremum over the length k of any chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k = \mathfrak{p}.$$

Remark 6.11. It is possible that $\dim R = \infty$ even for Noetherian rings!

By carefully examining Definition 5.9 and Definition 6.10, we see that $\dim(X) = \dim(A(X))$. Thus, we've reduced the problem of determining the dimension of algebraic subsets of $\mathbb{A}^n$ to finding the Krull dimension coordinate rings! This is made easier by the following theorem:

Theorem 6.12 (Main Theorem of Dimension Theory)**.** [3] *Let B be an integral, finitely-generated k-algebra, where k is an arbitrary field. Then*

(*1*) $\dim B = \operatorname{trdeg}_k \mathcal{Q}(B)$*, where* $\mathcal{Q}(B) = \{a/b \mid a, b \in B, b \neq 0\}$ *denotes the field of fractions of* B*.*

(*2*) *For every prime* $\mathfrak{p} \subset B$*,* $\operatorname{ht}\mathfrak{p} + \dim B/\mathfrak{p} = \dim B$*.*

Example 6.13. If $B = k[x_1, \ldots, x_n]$, then $\mathcal{Q}(B) = k(x_1, \ldots, x_n)$, which clearly has transcendence degree n over k. Thus, $\dim \mathbb{A}^n = \operatorname{trdeg}_k \mathcal{Q}(B) = n$.

In general, $A(X) = k[x_1, \ldots, x_n]/I(X) = k[\overline{x_1}, \ldots, \overline{x_n}]$. Some of the variables will be algebraically independent, so $\mathcal{Q}(A(X)) = k(x_1, \ldots, x_n) = k(x_1, \ldots, x_r)(x_{r+1}, \ldots, x_n)$. The extension $k \subset k(x_1, \ldots, x_r)$ is purely transcendental, whereas $k(x_1, \ldots, x_r) \subset k(x_1, \ldots, x_n)$ will be algebraic.

For example, consider $Z(y^2 - x^3)$. We have $k \subset k(x) \subset k(x, y)$, where $y^2 = x^3$ in the last extension. Thus, the transcendence degree is 1, and $\dim(Z(y^2 - x^3)) = 1$.

For X an algebraic subset of $\mathbb{A}^n$, we let $\operatorname{codim}(X) = n - \dim(X)$.

We won't prove Theorem 6.12, but feel free to look up a proof on your own!

7. Monday February 13

7.1. **A Commutative Algebra Interlude.**

A good reference for what follows is H. Matsumura's *Commutative Ring Theory*. Other standard references are Atiyah-MacDonald and Eisenbud's book.

Theorem 7.1 (Krull's Principal Ideal Theorem)**.** *Let R be a Noetherian ring, and let $f \in R$ such that*

(*1*) *f is not a zero divisor;*

(*2*) *f is not a unit.*

Then every minimal prime $\mathfrak{p}$ over (f) has $\operatorname{ht}(\mathfrak{p}) = 1$*.*

The geometric version of the above is as follows:

Theorem 7.2 (Krull's Principal Ideal Theorem, Geometric Boogaloo)**.** *Let $X \subset \mathbb{A}^n$ is an algebraic set and $f \in k[x_1, \ldots, x_n]$ such that $Z(f)$ has the following significance:*

(*1*) *$Z(f)$ does not contain any component of X;*

(*2*) *$Z(f)$ does not miss X.*

Then every component of $X \cap Z(f)$ has dimension $\dim(X) - 1$*.*

Corollary 7.3. *Let $X \subset \mathbb{A}^n$ be an affine variety. Then* $\dim(X) = n - 1$ *if and only if* $X = Z(f)$*, where $f \in k[x_1, \ldots, x_n]$ is an irreducible polynomial.*

[3] This name might be somewhat disputed; e.g., Atiyah-MacDonald use this name to refer to a different result.

Proof. It is equivalent to show that a prime ideal $\mathfrak{p} \subset k[x_1, \ldots, x_n]$ has $\mathrm{ht}(\mathfrak{p}) = 1$ if and only if $\mathfrak{p} = (f)$ for some irreducible polynomial f. Recall Theorem 6.12, which tells us that

$$\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R).$$

Thus,

$$\mathrm{ht}(\mathfrak{p}) + \dim(A(X)) = n.$$

The forward implication is easy: let $0 \notin f \in \mathfrak{p}$. Without loss of generality, we may assume f is irreducible. Otherwise, factor f into irreducibles (recall we are in a UFD); the primality of $\mathfrak{p}$ guarantees one of the irreducible factors is in $\mathfrak{p}$. Then we have $0 \subset (f) \subset \mathfrak{p}$, implying $\mathfrak{p} = (f)$, since $\mathrm{ht}(\mathfrak{p}) = 1$.

The converse follows immediately from Krull's Principal ideal theorem (Theorem 7.1). $\square$

Note that the above does not hold when we allow our space to be cut out by multiple equations (look up complete intersection rings).

7.2. **More Commutative Algebra.**

Definition 7.4. Let R, S be two rings $R \subset S$. Then:

(1) S is finitely generated over R (i.e., *module-finite* over R) if there exist $s_1, \ldots, s_n \in S$ which generate S as an R-module. I.e., for all $s \in S$ there exist $r_i \in R$ such that

$$s = r_1 s_1 + \cdots + r_n s_n.$$

(2) S is finitely generated as an R-algebra[4] (*ring-finite* over R) if there exist $s_1, \ldots, s_n \in S$ such that $S = R[s_1, \ldots, s_n]$ (subring generated by R and $s_1, \ldots, s_n$). In this case, there exists a surjection $R[x_1, \ldots, x_n] \twoheadrightarrow R[s_1, \ldots, s_n] = S$ that sends $R \mapsto R$ and $x_i \mapsto s_i$. The kernel of this surjection is the relations between the s_i's.

Example 7.5. $R[x_1, \ldots, x_n]$ is ring finite over R, but not finite over R, since $1, x, x^2, x^3, \ldots$ are independent over R.

Exercise 7.6. Show that module-finiteness implies ring-finiteness.

Exercise 7.7. (Homework). Both module-finiteness and ring-finiteness are preserved under compositions of inclusions $R \subset S \subset T$.

Definition 7.8. If $R \subset S$ and $s \in S$, we say that s is *integral* over R if there exists a monic polynomial $f \in R[x]$ such that $f(s)$. We say S is integral over R if every element of S is integral over R.

Example 7.9. $\mathbb{Q}[\sqrt{2}]$ is integral over $\mathbb{Q}$, since $\sqrt{2}$ is a root of $x^2 - 2$. But $\mathbb{Q}[\pi]$ is not integral over $\mathbb{Q}$.

Example 7.10. $\mathbb{Q}[\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \ldots]$ is an integral extension of $\mathbb{Q}$, but is not ring-finite (so not module finite either).

Proposition 7.11. *Let $R \subset S$, where S is an integral domain. Let $s \in S$. Then the following are equivalent:*

(*1*) *s integral over R;*

[4]Justify to yourself why S is an R-algebra.

(2) $R[s]$ *is module-finite over* R*;*

(3) *There exists a subring* $R[s] \subset R' \subset S$ *which is module-finite over* R*.*

Proof. (1) $\implies$ (2): Suppose $f = x^n + r_{n-1}x^{n-1} + \cdots + r_0 \in R[x]$ such that $f(s) = 0$. Thus,

$$s^n = -r_{n-1}s^{n-1} - \cdots - r_1 s - r_0$$

lies in the R-submodule generated by $1, s, \ldots, s^{n-1}$. It follows that all higher powers are generated by these elements.

(2) $\implies$ (3): Clear: take $R' = R[s]$.

(3) $\implies$ (1): Suppose R' is generated over R by $v_1, \ldots, v_n \in R'$. Consider sv_i. For each i, $sv_i = \sum_{j=1}^n a_{ij}v_j$. We can write things using matrices: let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in M_n(R),$$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

so that

$$Av = (sI_n)v.$$

Hence,

$$(A - sI_n)v = 0$$

for $v \neq 0$, which is the case if and only if $\det(A - sI_n) = 0$. However, this is a monic polynomial in s with coefficients in R, implying that s is integral over R. □

Corollary 7.12. *The set of elements in* S *that are integral over* R *is a subring* $\overline{R} \subset S$ *containing* R*. (*$\overline{R}$ *is the* integral closure *of* R *in* S*).*

Proof. Say $a, b \in S$ are integral over R. We know that $R[a]$ is module-finite over R by the above proposition. Now, b is integral over $R[a]$. Again, the proposition tells us that $R[a, b]$ is module-finite over $R[a]$. Hence, $R[a, b]$ is module-finite over R. Now, taking $s \in \{a + b, a - b, ab\}$, we see that $R[s] \subset R' = R[a, b] \subset S$. By the proposition, $R[a, b]$ module-finite over R implies that s is integral over R. □

Corollary 7.13. *Say* $R \subset S$ *is ring-finite. Then,* $R \subset S$ *is module-finite if and only if* S *is integral over* R*.*

Proof. "$\implies$": Let $s \in S$. Then $R[s] \subset S$ is module finite over R. Hence, s is integral.

"$\impliedby$": Let $S = R[s_1, \ldots, s_n]$, where s_i is integral over R for all i. Suppose $n = 1$. Then $R[s_1]$ is module-finite over R by the proposition. For the inductive step, suppose $R[s_1, \ldots, s_k]$ is module-finite over R. Because s_{k+1} is integral over $R[s_1, \ldots, s_k]$, it follows that $R[s_1, \ldots, s_{k+1}]$ is module finite over $R[s_1, \ldots, s_k]$, which is module-finite over R by the inductive hypothesis. □

In the case of fields, let $K \subset L$ be a field extension, and let $s_1, \ldots, s_n \in L$. Then $K(s_1, \ldots, s_n)$ is a subfield of L generated by K and $s_1, \ldots, s_n$ (the field of fractions of $K[s_1, \ldots, s_n]$).

Definition 7.14. Let K, L be as above.

(1) L is a *finitely generated field extension* of K if $L = K(s_1, \ldots, s_n)$ for some $s_1, \ldots, s_n \in L$.

(2) L is an *algebraic extension* over K if all elements in L are algebraic over K (satisfy monic polynomial equation). As before, we can consider the algebraic closure of K in L, $K \subset \overline{K} \subset L$.

Theorem 7.15. *Let $K \subset L$ be a field extension. If L is ring-finite over K, then L is module-finite over K (if and only if algebraic over K).*

We'll prove this later, and we'll use this theorem to prove the Weak Nullstellensatz (Theorem 6.5).

Proof of Theorem 6.5, Weak Nullstellensatz. Recall that we want to show the following: If $k = \overline{k}$ and $\mathfrak{M} \subset k[X_1, \ldots, X_n]$ is a maximal ideal, then $\mathfrak{M} = (X_1 - a_1, \ldots, X_n - a_n)$ with $a_i \in K$.

Let $R = k[x_1, \ldots, x_n]$. We have the following map of fields:

$$k \hookrightarrow R/\mathfrak{M} = L.$$

Let $R/\mathfrak{M} = k[x_1, \ldots, x_n]$, where $x_i = \overline{X_i}$. Note that $k[x_1, \ldots, x_n]$ is ring-finite over K. By the above theorem, L is algebraic over k; k algebraically closed implies that $k = L$. Thus, $\overline{X_i} = \overline{a_i} \in R/\mathfrak{M}$ for some $a_i \in k$. This occurs if and only if $X_i - a_i \in \mathfrak{M}$. Therefore, $(X_1 - a_1, \ldots, X_n - a_n) \subset \mathfrak{M}$. The fact that $(X_1 - a_1, \ldots, X_n - a_n)$ is already a maximal ideal forces equality, which is exactly what we wanted to show. ☐

8. Wednesday February 15

8.1. **Even More Commutative Algebra.** Recall that last time, we stated the following theorem:

Theorem 8.1. *Let $K \subset L$ be a field extension. If L is ring-finite over K, then L is module-finite over K (if and only if algebraic over K).*

We'll need the following proposition:

Proposition 8.2. *Let x be a variable and k a field.*

(1) $k(x)$ is a finitely generated field extension that is not *ring-finite over k.*

(2) $k[x] = \overline{k[x]} \subset k(x)$. I.e., $k[x]$ is integrally closed in its field of fractions.

Proof. (1) is left as an exercise to the reader. Hint: clear denominators.

(2): Say $s \in k(x)$ is integral over $k[x]$. Then $s = P/Q$ for $P, Q \in k[x]$ relatively prime. Because s is integral over $k[x]$, there are polynomials $f_i \in k[x]$ such that

$$s^n + f_{n-1}s^{n-1} + \cdots + f_0 = 0.$$

Hence,

$$P^n + f_{n-1}P^{n-1}Q + \cdots + f_0 Q^n = 0,$$

and moving the rightmost $(n-1)$-terms to the RHS, we see that this implies $Q|P^n$, contradicting our assumption that P and Q are relatively prime. Therefore, $Q = 1$ and $s \in k[x]$, as desired. ☐

Proof of Theorem. By assumption, we may write $L = K[s_1, \ldots, s_n]$ for $s_i \in L$. We'll do induction on n. When $n = 1$, we have the following exact sequence of rings

$$0 \longrightarrow I = \ker\varphi \longrightarrow k[x] \xrightarrow{\varphi} k[s] = L,$$

where φ is the map taking $x \mapsto s$ and thus $P(s) \mapsto P(s)$. Now, the exactness of the above tells us that $L = k[x]/I$, which implies that I is a maximal ideal. It follows that $I = (f)$ for some monic, irreducible polynomial f. Then, $f(s) = 0$ implies s is algebraic over K.

For the inductive step, suppose the result holds for $n-1$ elements. Now, $K(s_1) \hookrightarrow L = K(s_1)[s_2, \ldots, s_n]$ is algebraic by induction.

Case 1: if s_1 is algebraic over K, then

$$K \hookrightarrow K(s_1) \hookrightarrow L = K(s_1)[s_2, \ldots, s_n]$$

is an algebraic extension.

Case 2: s_1 is not algebraic over K. It follows that $K(s_1) \cong K(x)$. For all $i \in \{2, \ldots, n\}$, there exist $f_{i,j} \in K(s_1)$ such that

$$s_i^{n_i} + f_{i,n-1} s_i^{n_i - 1} + \ldots + f_{i,0} = 0.$$

Now, there exists some $f \in K[s]$ that will clear all denominators, i.e., f "is a multiple of everything in sight." Multiply both sides of the above by f^{n_i}, which gives us the following polynomial relation whose coefficients are in $K[s_1]$:

$$(fs_i)^{n_i} + f f_{i,n-1} (fs_i)^{n_i - 1} + \cdots = 0.$$

Therefore, fs_i are integral over $K[s_1]$ for all $i \in \{2, \ldots, n\}$. Let $L = K[s_1, \ldots, s_n]$. If $t \in L$, then there exists $N > 0$ such that $f^N \cdot t \in k[s_1][fs_2, \ldots, fs_n]$, and recall that $fs_2, \ldots, fs_n$ are integral over $K[s_1]$. So $f^N t$ is integral over $K[s_1]$. Now take $t = \frac{1}{g}$, where $g \in k[s_1]$ is relatively prime to f. By the above, there exists an N such that f^N/g is integral over $k[s_1]$. By the above proposition, $K[s_1]$ is integrally closed in $K(s_1)$ (recall that we may treat s_1 as a formal variable because it is transcendental over K). It follows that $f^N/g \in k[s_1]$, which contradicts our assumption that $\gcd(f, g) = 1$. □

8.2. **Functions and Morphisms.** Suppose $X \subset \mathbb{A}^n$ is an affine variety. Consider its coordinate ring

$$k[x_1, \ldots, x_n]/I(X) = A(X) = \{\text{polynomial functions } f : X \to k\}.$$

We call the above the *ring of regular functions* on X.

Definition 8.3. Let $x \in X$. Define

$$\mathcal{O}_{X,x} = \{\varphi = f/g \mid f, g \in A(x), g(x) \neq 0\}.$$

This is called the *local ring* of $x \in X$. Note that any such quotient f/g is in $K(X) := \mathcal{Q}(A(X)) = \mathrm{Frac}(A(X))$, which is called the *function field of* X.

A helpful exercise is to verify the following:

(1) $\mathcal{O}_{X,x}$ is a ring is the ring of rational functions that are *regular* (defined) at x.[5]

(2) Let $U \subset X$ be open. We define the *ring of rational functions regular* on U, denoted $\mathcal{O}_X(U)$, as

$$\mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_{X,x}.$$

We can think of these as the algebro-geometric functions on our open set U.

[5] If you take scheme theory, you will see that $\mathcal{O}_{X,x}$ is something called the stalk of the sheaf of rings $\mathcal{O}_X$ on X. If you're interested, you should look all of these things up! In particular, note that the definition of $\mathcal{O}_{X,x}$ is as the colimit of the rings $\mathcal{O}_X(U)$ over all opens U containing x. Equivalently, the stalk at x can be thought of as equivalence classes of germs of functions defined on x.

Example 8.4. It is NOT the case that a regular function on U can be written everywhere as f/g for f, g polynomials. Consider

$$U = \{(x, y, z, t) \in X \mid y \neq 0 \text{ or } t \neq 0\} \subset X = (xt - yz = 0) \subset \mathbb{A}^4.$$

Note that $\varphi = x/y$ is defined only for $y \neq 0$, z/t is defined only for $t \neq 0$, and that $\varphi = x/y = z/t$ in $\mathcal{O}_X(U)$.

Definition 8.5. With notation as in the above, $x \in X$ is a *pole* of $\varphi \in K(x)$ if φ is not defined (regular) at x.

Example 8.6. Consider $X = (xy - z^2 = 0) \subset \mathbb{A}^3$. Consider $\varphi = x/z = z/y$, and note that $(a, 0, 0)$ is a pole for φ.

Just as the set of singular points on an algebraic plane curve is closed, we will see that the set of poles of a rational function on X is likewise closed.

Lemma 8.7. *The set of poles of a rational function on X is a closed subset of X.*

Proof. We have $X \subset \mathbb{A}^n$ and $\varphi \in K(X)$. Consider the following ideal in $A(X)$:

$$I_\varphi := \{h \in A(X) \mid h\varphi \in A(X)\}.$$

As an exercise, verify that this is an ideal. Claim: the set of poles of φ is equal to $Z(I_\varphi)$. What will be helpful here is the correspondence ~~theorem~~ observation: the ideals in R/I are the ideals in R containing I. Note that x is not a pole if and only if $\varphi = f/g$ for g such that $g(x) \neq 0$. Hence, $g \in I_\varphi$, $g(x) \neq 0$ if and only if $x \notin Z(I_\varphi)$. $\square$

Lemma 8.8. $\mathcal{O}_X(X) = \bigcap_{x \in X} \mathcal{O}_{X,x} = A(X) \subset K(X)$.

Proof. $\supset$ is clear.

"$\subset$": Suppose $\varphi \in \bigcap_{x \in X} \mathcal{O}_{X,x}$. In other words, φ has no poles! Lemma 8.7 tells us that this is the case if and only if $Z(I_\varphi) = \emptyset$. By the Nullstellensatz, this can be the case if and only if $1 \in I_\varphi$, which is the case if and only if $\varphi \in A(X)$. $\square$

Now, note that there is an evaluation map $ev : \mathcal{O}_{X,x} \to k$, where $\varphi = f/g \mapsto \varphi(x) = f(x)/g(x)$, where $g(x) \neq 0$. Verify that this map is well-defined (i.e., that another representative of φ gets mapped to the same element of k). Let $\mathfrak{M}_x = \ker(ev_x) = \{\varphi = f/g \mid g(x) \neq 0, f(x) = 0\}$. Since ev is a surjection onto k, we see that $\mathfrak{M}_x$ is a maximal ideal of $\mathcal{O}_{X,x}$. Not only is $\mathfrak{M}_x$ a maximal ideal, but note that every element of $\mathcal{O}_{X,x} \setminus \mathfrak{M}_x$ is invertible. In other words, $(\mathcal{O}_{X,x}, \mathfrak{M}_x)$ is a *local ring.*

Definition 8.9. R is a *local ring* if, equivalently:

(1) There exists only one maximal ideal $m \leq R$;

(2) There exists a maximal ideal $m \leq R$ such that for all $x \in R \setminus m$ is invertible.

As an exercise, prove the above are equivalent definitions.

Exercise 8.10. Prove that $\mathcal{O}_{X,x}$ is a Noetherian ring.

9. Wednesday February 22

9.1. **Review.** Recall that for every $x \in X$ we associate the Noetherian, local ring $\mathcal{O}_{X,x}$, which has maximal ideal $\mathfrak{M}_x = \{f \mid f(x) = 0\}$.

Example 9.1. (1) $k[x_1, \dots, x_n]$ is *not* local, since $(x_1 - a_1, \dots, x_n - a_n)$ are all maximal;

(2) $k[[x_1, \dots, x_n]]$ is local, with maximal ideal $(x_1, \dots, x_n)$. Recall that any power series with nonzero constant coefficient is always invertible. If you haven't seen this before, prove this as an exercise.

(3) $R = k[x]_{(x)} = \{f/g \mid f, g \in k[x], x \not| g\} \supset \mathfrak{M}_x = \{f/g \mid x|f\}$.

9.2. **Regular Maps (or Morphisms).**

Definition 9.2. Suppose $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine algebraic sets. A function $f : X \to Y$ is called *regular*, or a *morphism*, if there exist $p_1, \dots, p_m \in k[x_1, \dots, x_n]$ such that $f(x) = (p_1(x), \dots, p_m(x))$ for all $x \in X$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \\ \mathbb{A}^n \arrow[r] & \mathbb{A}^m \end{tikzcd}$$

Example 9.3. (1) Consider $f : \mathbb{A}^1 \to \mathbb{A}^2$ given by $t \mapsto (t, t^2)$. This is a morphism from $X = \mathbb{A}^1$ to $Y = Z(y - x^2)$. Consider $g : Y \to \mathbb{A}^1$ given by $(x, y) \mapsto x$. Note that g is also a morphism and that f and g are inverse to each other. If there exists such a g, we call f an isomorphism.

(2) Consider the map $f : \mathbb{A}^1 \to \mathbb{A}^2$ given by $t \mapsto (t^2, t^3)$. Note that $\mathrm{im}(f) = Y := Z(y^2 - x^3)$ and that f is a morphism. Moreover, we also see that f is bijective, but it is *not* an isomorphism. Heuristically, this is because Y is singular (there is a cusp at $(0, 0)$), whereas $\mathbb{A}^1$ is not. More rigorously, we see that f is inverted by $g : Y \to \mathbb{A}^1$ given by $(x, y) \mapsto y/x$, which is not defined at 0.

Remark 9.4. A regular function on X is the same as a morphism $X \to \mathbb{A}^1$.

Let $f : X \to Y$ be a morphism. Then we may define $f^* : A(Y) \to A(X)$ by $g \mapsto g \circ f$.

$$X \xrightarrow{f} Y \xrightarrow{g} \mathbb{A}^1$$

Example 9.5. (1) Consider $p : \mathbb{A}^n \to \mathbb{A}^m$ (with $n > m$) given by projection, so $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_m)$. The induced map $k[x_1, \dots, x_m] \to k[x_1, \dots, x_n]$ on coordinate rings is given by $x_i \mapsto x_i$.

(2) The map from $\mathbb{A}^1 \to Z(y - x^2)$ given by $t \mapsto (t, t^2)$ induces the map $f^* : k[x, y]/(y - x^2) \to k[T]$ taking $\overline{x} \mapsto T$.

(3) If we consider the map from $\mathbb{A}^1 \to Z(y^2 - x^3)$ given by $t \mapsto (t^2, t^3)$, this induces the map $f^* : k[x, y]/(y^2 - x^3) \to k[T]$ taking $\overline{x} \mapsto T^2$ and $\overline{y} \mapsto T^3$. Note that this is not an isomorphism of rings! (There are no linear terms in the image.)

Remark 9.6. $(f \circ h)^* = h^* \circ f^*$ for morphisms $f : X \to Y$ and $h : Y \to Z$.

Proposition 9.7. *Suppose $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are algebraic sets. Then there is a one-to-one correspondence between morphisms $f : X \to Y$ and k-algebra homomorphisms $A(Y) \to A(X)$ given by $f \mapsto f^*$.*

Remark 9.8. For those who know category theory: we are establishing a contravariant functor between the category of affine algebraic sets and k-algebras.

Proof of Proposition. We need to construct an inverse for the map taking $f \mapsto f^*$. Let $\varphi : A(Y) = k[y_1, \dots, y_m]/\mathcal{I}(Y) \to A(X) = k[x_1, \dots, x_n]/\mathcal{I}(X)$ be a homomorphism of k-algebras.

Define $f_i : X \to \mathbb{A}^1$ by $f_i = \varphi(y_i)$. This gives a map $f : X \to \mathbb{A}^m$ given by $f = (f_1, \dots, f_m)$. We want to show that $f(X) \subset Y$. For all $x \in X$ and $g \in I(Y)$, $g(f(x)) = g(\varphi(y_1)(x), \dots, \varphi(y_n)(x)) = \varphi(g)(x) = 0$ since $g \in \mathcal{I}(Y)$. Hence, $f(X) = Z(I(Y)) = Y$.

To finish, we need to prove that $\varphi = f^*$. In other words, we need to show that for all $h \in A(Y)$, $f^*(h)(x) = \varphi(h)(x)$ for all $x \in X$. But note that, by definition, $f^*(h) = h \circ f = \varphi(h)$, so we are done. □

Corollary 9.9. *A morphism $f : X \to Y$ is an isomorphism if and only if $f^* : A(Y) \to A(X)$ is an isomorphism of k-algebras.*

Remark 9.10. This is another way to see the line is not isomorphic to the cusp! Recall that the coordinate rings are not isomorphic.

Lemma 9.11. *Let $f : X \to Y$ be a morphism. Then f is continuous (with respect to the Zariski topology).*

Proof. We need to show that if $Z \subset Y$ is an algebraic set, then $f^{-1}(Z) \subset X$ is an algebraic set. Say $Z = Z(g_1, \dots, g_r)$, $g_i \in A(Y)$. Note that $f^{-1}(Z) = \bigcap_i f^{-1}(Z(g_i))$. Now, $f^{-1}(Z(g_i)) = \{x \in X \mid g_i(f(x)) = 0\} = \{x \in X \mid f^*(g_i)(x) = 0\}$. So

$$\bigcap Z(f^*(g_i)) = Z(f^* g_1, \dots, f^* g_r)$$

is indeed algebraic. □

We've seen, for affine sets X and Y, that a continuous map $f : X \to Y$ is a morphism if and only if for all $g \in A(Y) = \mathcal{O}_Y(Y)$, we have $f^*(g) \in A(X) = \mathcal{O}_X(X)$.

Proposition 9.12. *Let $f : X \to Y$ be a continuous map. Then the following are equivalent:*

(1) *f is a morphism*

(2) *for every $U \subset Y$ open, $f^*(\mathcal{O}_Y(U)) \subset \mathcal{O}_X(f^{-1}(U))$.*

(3) *for all $x \in X$ and $\varphi \in \mathcal{O}_{Y,f(x)}$, $f^*(\varphi) \in \mathcal{O}_{X,x}$*

Remark 9.13. The huge advantage of this proposition is that we can check things locally. Rather than relying on checking things globally, by checking things locally we'll make our lives much easier. Especially when you study schemes. In scheme theory, you learn the motto "a morphism of schemes locally looks like morphisms of affine schemes." In other words, whenever we are working with schemes, we'll always just want to choose affine covers of our domain and target schemes and then define morphsims of affine schemes (think: morphisms of rings) on each affine (think: ring) and then check that these morphisms glue.

Proof. (ii) $\implies$ (i) $\implies$ (ii) when $U = X$.

(iii) $\implies$ (ii) is clear, because $\mathcal{O}_X(U) = \bigcap_{x\in U} \mathcal{O}_{X,x}$.

(i) $\implies$ (iii): let $\varphi \in \mathcal{O}_{Y,f(x)} \subset K(Y) = Q(A(Y))$. In other words, $\varphi = g/h$ for $g, h \in A(Y)$ such that $h(f(x)) \neq 0$. Note that $f^*(h)(x) = f(h(x))$. Note that $\mathcal{O}_{X,x} \ni f^*\varphi = f^*g/f^*h \in A(X)$. □

In the future, we'll see that we can use these new definitions for morphisms between any varieties. (E.g., open subsets of affine varieties, projective varieties, open subsets in projective varieites (aka quasi-projective)).

Definition 9.14. Let $f : X \to Y$ be a continuous map between open subsets. If X and Y are affine varieties, then f is a morphism if either (ii) or (iii) holds.

Lemma 9.15. *Let $f : X \to Y$ be a morphism of affine varieties. Then:*

(1) $Z \subset f(X)$ such that $f^{-1}(Z)$ is irreducible, then Z is irreducible;

(2) $\overline{f(X)}$ is irreducible;

(3)

Remark 9.16. Note that the image of a morphism is not necessarily closed. Consider the projection of the hyperbola $xy = 1$ to the x axis. We'll see that in a map of projective varieties, $f(X)$ is always closed.

Proof. Say $Z = Z_1 \cup Z_2$. Then $f^{-1}(Z) = f^{-1}(Z_1) \cup f^{-1}(Z_2)$. It follows that $f^{-1}(Z) = f^{-1}(Z_i)$ for some i; applying f to both sides tells us that $Z = Z_i$, implying Z is irreducible. □

Proposition 9.17. *If $f : X \to Y$ is a surjective morphism, then f^* is injective.*

Proof. Let $g \in A(Y)$. Suppose $f^*(g) = 0$. Then, for every point in x, $g(f(x)) = 0$, but since f is surjective, g must be 0 on Y. Therefore, f^* is injective. □

Remark 9.18. The converse is false! (We will show later that f^* is injective if and only if f is *dominant*.) Consider the example of the projection of $yx = 1$ to the x-axis. Then $f^* : k[T] \to k[X,Y]/(XY - 1) = k[X, 1/X]$ is an injective map, since it takes $T \mapsto X$.

Note that surjectivity is not a great property to study, since the image of a morphism is not necessarily closed. Thus, we make the following definition:

Definition 9.19. $f : X \to Y$ is called *dominant* if $\overline{f(X)} = Y$.

Exercise 9.20. $f : X \to Y$ is dominant if and only if f^* is injective.

Proof. HW. □

10. Monday February 27

10.1. **More on Morphisms.**

Proposition 10.1. *If $f : X \to Y$ is a morphism of affine varieties, then $f^* : A(Y) \to A(X)$ is surjective if and only if f is an isomorphism onto its image.*

Proof. $\Longleftarrow$: We have the following diagram:

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[dr, "f"'] & Z = \mathrm{im}(f) \arrow[d] \\ & Y. \end{tikzcd}$$

Then we also get

$$\begin{tikzcd} A(X) & A(Z) \arrow[l, "\sim"] \\ & A(Y) \arrow[u, two heads] \arrow[ul, "f^*"] \end{tikzcd}$$

which commutes. The commutativity of the above implies that f^* is surjective.

$\Longrightarrow$: Let Z denote $\overline{\mathrm{im}(f)}$. This tells us that the map $f : X \to Z$ is dominant, and Z injects into Y. This tells us that the induced map $A(Z) \to A(X)$ is injective.

$$\begin{tikzcd} A(X) & A(Z) \arrow[l, hook', "\sim"] \\ & A(Y) \arrow[u, two heads] \arrow[ul, two heads, "f^*"] \end{tikzcd}$$

It follows that the map $A(Z) \to A(X)$ is surjective, implying $A(Z) \cong A(X)$. Therefore $f : X \to Z$ is an isomorphism. $\square$

10.2. **Projective Varieties.**

Definition 10.2. The *projective space* $\mathbb{P}^n$ over a field is the set of lines through the origin in $\mathbb{A}^{n+1}$. Equivalently, it is the quotient $(\mathbb{A}^{n+1} \setminus \{0\})/k^*$ via the action $\lambda(x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n) \sim (x_0, \ldots, x_n)$ for every $\lambda \in k^*$.

Remark 10.3. Each point in $\mathbb{P}^n$ can be denoted by $[x_0 : \ldots : x_n]$; these are called the homogeneous coordinates on $\mathbb{P}^n$.

The "value" x_i is not well-defined, but the quotient x_i/x_j is.

Remark 10.4. Over $\mathbb{C}$, $\mathbb{P}^n$ is compact in the classical topology. We have a surjective map $S^{2n+1} \twoheadrightarrow \mathbb{P}^n$ (justify to yourself why this should imply compactness).

Note that $\mathbb{P}^n$ is compact for any k, but this statement doesn't really make sense yet...

Definition 10.5. A polynomial $f \in k[x_0, \dots, x_n]$ is *homogeneous* of degree d if $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$ for all $\lambda \in k$.

Equivalently: f is a sum of monomials of degree d in the x_i's. An ideal I of $k[x_0, \dots, x_n]$ is *homogeneous* if it can be generated by homogeneous polynomials.

Exercise 10.6. I is homogeneous if and only if for all $f \in I$, if we write $f = \sum_d f_d$, where f_d denotes the sum of monomials of degree d, then $f_d \in I$ for all d.

Definition 10.7. (1) Let $I \subset k[x_0, \dots, x_n]$ be a homogeneous ideal. Let $Z(I) = \{[x_0 : \dots : x_n] \in \mathbb{P}^n \mid f(x_0, \dots, x_n) = 0 \ \forall f \in I\}$. We call this the *zero set* of I. These are the algebraic sets in $\mathbb{P}^n$.

(2) If $X \subset \mathbb{P}^n$ is any subset, we call

$$I(X) = \langle f \in k[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \rangle$$

the vanishing ideal of X (the brackets notation in the above means the ideal generated by the elements in the brackets).

(3) $S(X) := k[x_0, \dots, x_n]/I(X)$ is defined to be the homogeneous coordinate ring of X. This is a graded ring, and can be rewritten as

$$S(X) = \bigoplus_d (k[x_0, \dots, x_n]/I(X))_d$$

Remark 10.8. A *graded ring* is a ring $S = \bigoplus_{d \in \mathbb{N}} S_d$, where S_d is an additive subgroup of $(S, +)$ and the direct sum decomposition is a decomposition of $(S, +)$ as an abelian group, and where we have the property that $S_k S_\ell \subset S_{k+\ell}$.

Note that the polynomial ring in n variables is a graded ring, where the degree of the polynomials is our grading. Prove that the quotient of any graded ring is also graded.

Proposition 10.9. *Exactly as in the affine case:*

(1) Let $I_1 \subset I_2 \subset k[x_0, \dots, x_n]$ be homogeneous ideals. Then $Z(I_2) \subset Z(I_1)$.

(2) For $\{I_i\}_{i \in I}$ a family of homogeneous polynomials, $\bigcap_{i \in I} Z(I_i) = Z(\bigcup_i I_i)$.

(3) For I_1, I_2 homogeneous ideals, $Z(I_1) \cup Z(I_2) = Z(I_1 \cdot I_2)$.

Definition 10.10. The Zariski topology on $\mathbb{P}^n$ is generated by the projective algebraic sets in $\mathbb{P}^n$, where these sets are taken to be a basis of closed sets.

(1) For all $X \subset \mathbb{P}^n$, the Zariski topology on X is the induced topology from $\mathbb{P}^n$.

(2) A *projective variety* is an irreducible closed subset of $\mathbb{P}^n$.

(3) The notion of dimension is the same as in $\mathbb{A}^n$ (Homework: $\dim S(X) = \dim X + 1$).

Example 10.11. Here are some basic examples:

(1) $Z(F) \subset \mathbb{P}^n$ for $F \in k[x_0, \dots, x_n]$, homogeneous of degree d, is called a *hypersurface* of degree d.

(2) The image of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ given by

$$[x_0 : x_1] \mapsto [x_0^n : x_0^{n-1} x_1 : \dots : x_1^n]$$

is called the *rational normal curve* of degree n in $\mathbb{P}^n$. For $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, this is called the *twisted cubic.*

(3) Intersections of quadrics: $Y_0Y_3 = Y_1Y_2$, $Y_1^2 = y_0Y_2$, and $Y_2^2 = Y_1Y_3$. Note that these equations are the 2×2 minors of the following matrix

$$\begin{bmatrix} Y_0 & Y_1 & Y_2 \\ Y_1 & Y_2 & Y_3 \end{bmatrix}.$$

This is an example of something called the *Veronese embedding.*

However, working out some other examples, we run into a problem:

$$Z((x_0, \ldots, x_n)) = \emptyset.$$

It turns out that this is the only problem, and we can fix it!

Definition 10.12. An affine algebraic set $Y \subset \mathbb{A}^{n+1}$ is called a *cone* if for all $\lambda \in k$, $(x_0, \ldots, x_n) \in Y$ if and only if $(\lambda x_0, \ldots, \lambda x_n) \in Y$.

If $X \subset \mathbb{P}^n$ is an algebraic set, we define

$$C(X) = \{(x_0, \ldots, x_n) \in \mathbb{A}^{n+1} \mid (x_0 : \ldots : x_n) \in X\} \cup \{0\}$$

to be the *affine cone* over X. (There are some nice pictures you can draw, but unfortunately my tikz skills are extremely limited. Look these up if you're interested, or refer to someone's class notes.)

Note that if I is an ideal such that $X = Z(I)$, then $C(X) = Z(I)$, where in the latter case we are "forgetting" I is homogeneous. Thus, we see that every projective variety is intimately related to an affine variety.

Proposition 10.13 (Projective Nullstellensatz)**.** *We have the following basic properties:*

(1) *If $X_1 \subset X_2 \subset \mathbb{P}^n$ are algebraic sets, then $I(X_2) \subset I(X_1)$.*

(2) *For any algebraic $X \subset \mathbb{P}^n$, we have $Z(\mathcal{I}(X)) = X$.*

(3) *For any homogeneous ideal $J \subset k[x_0, \ldots, x_n]$ such that $Z(J) \neq \emptyset$, we have $I(Z(J)) = \operatorname{rad}(J)$.*

(4) *For any homogeneous ideal J such that $Z(J) = \emptyset$, then we have either $J = (1)$ or $\operatorname{rad}(J) = (x_0, \ldots, x_n)$. Equivalently $Z(J) = \emptyset$ if and only if $(x_0, \ldots, x_n)^r \subset J$ for some r.*

Remark 10.14. The ideal $(x_0, \ldots, x_n)$ is called the *irrelevant ideal.* This name comes from the fact that the corresponding points are irrelevant to geometry in $\mathbb{P}^n$.

Proof. (1),(2) are all as in the affine case. So is the $\supset$ direction of (3).

For the $\subset$ case of (3), we'll reduce to the affine case. Because $Z(J) \neq \emptyset$, it follows that $C(Z(J)) \neq \emptyset$, since both sets are given by the same ideal. Then apply the affine Nullstellensatz, which tells us that $I(Z(J)) = \operatorname{rad}(J)$.

(4): If $Z(J) = \emptyset$, then in $\mathbb{A}^{n+1}$ (forgetting J is homogeneous) we have that either $Z(J) = \emptyset$, or $Z(J) = \emptyset$. So either: $J = (1)$, or $\operatorname{rad}(J) = (x_0, \ldots, x_n)$ (by the Nullstellensatz). The minutia are left to the reader. $\square$

11. Wednesday March 1

11.1. **Projective Nullstellensatz.**

Recall that there exists a one-to-one inclusion reversing correspondence between

$$\{\text{algebraic sets in } \mathbb{P}^n\} \longleftrightarrow \{\text{homog. radical ideals in } k[x_1, \ldots, x_n],\ \text{different from } (x_0, \ldots, x_n)\}$$

The left arrow is given by $Z(-)$ and the right by $\mathcal{I}(-)$. There exist similar correspondences between irreducible algebraic sets and homogeneous primes, points and maximal ideals, etc. just as in the affine case.

Example 11.1. Here are some examples:

(1) The *Veronese embedding*: for $n, d > 0$ consider the mapping $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$, and where
$$[x_0 : \ldots : x_n] \mapsto [x_0^d : x_0^{d-1}x_1 : \ldots : x_n^d].$$
Ordering the monomials lexicographically and denoting them P_i, we have the following map of rings $\varphi : k[Y_0, \ldots, Y_N] \to k[x_0, \ldots, x_n]$, where $Y_i \mapsto P_i$. Note that $\operatorname{im} \nu_d$ is cut out by the polynomials in $\ker \varphi = I$. When $n = 1$ and $d = 3$, we have a closed embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $[x_0 : x_1] \mapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]$.

(2) Preview: $\nu_n(\mathbb{P}^1) \hookrightarrow \mathbb{P}^n$ is an example of a *rational normal curve.* $\nu_2(\mathbb{P}^2) \hookrightarrow \mathbb{P}^5$ is called a *Veronese surface.* These are examples of varieties of *minimal degree.* See Harris' book; in particular Lecture I.

(3) We know $\mathbb{P}^n$ is a projective variety, but what about $\mathbb{P}^n \times \mathbb{P}^m$? This is why we consider the *Segre map.* Let $n, m > 0$; consider the mapping $\varphi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$, where $N = nm + n + m = (n+1)(m+1) - 1$, given by
$$([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) \mapsto [\ldots : x_i y_j : \ldots]_{i,j}.$$
Again order the monomials lexicographically and consider the map of rings
$$k[\ldots, Z_{ij}, \ldots] \to k[x_0, \ldots, x_n, y_0, \ldots, y_m]$$
given by $Z_{ij} \mapsto x_i y_j$. This realizes $\mathbb{P}^n \times \mathbb{P}^m$ as a projective variety.

Take $Q : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, which takes
$$([x_0, x_1], [y_0, y_1]) \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1].$$
This is a hypersurface given by the equation $Y_0 Y_3 - Y_1 Y_2 = 0$ and is an example of what is called a *quadric surface* in $\mathbb{P}^3$. One can show that every quadric in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Recall here that the Zariski topology is *not* the product topology.

Exercise 11.2. There are exactly $\binom{n+d}{n}$ monomials of degree d in $x_0, \ldots, x_n$. Equivalently, you could ask the dimension of $\mathrm{Sym}^d V$, the dth symmetric algebra over V, an $(n+1)$-dimensional vector space.

11.2. **Functions, Morphisms, and more.**

Definition 11.3. (1) Let X be a projective variety and $S(X) = k[x_0, \ldots, x_n]/I(X) = \bigoplus S(X)_d$, where $S_d(X) = \{\overline{f} \mid f \in k[x_0, \ldots, x_n] \mid \deg(f) = d\}$. The *field of rational functions* is $K(X) = \{f/g \mid f, g \in S(X)_d, g \neq 0\}$.

(2) For all $x \in X$, we define $\mathcal{O}_{X,x} = \{f/g \in K(X) \mid g(x) \neq 0\}$.

(3) For all $U \subset X$ open, $\mathcal{O}_X(U) = \bigcap_{x \in X} \mathcal{O}_{X,x}$.

We can talk about morphisms between morphisms between (open subsets of) projective varieties (an open subset of a projective variety is called a quasi-projective variety).

Definition 11.4. A continuous map $f : X \to Y$ is a morphism if one of the following equivalent definitions holds:

(1) For all $U \subset Y$ open, $f^*(\mathcal{O}_Y(U)) \subset \mathcal{O}_X(f^{-1}(U))$, where $f^*(\varphi) = \varphi \circ f$.

(2) For all $x \in X$ and $\varphi \in \mathcal{O}_{Y,f(x)}$, we have that $f^*(\varphi) \in \mathcal{O}_{X,x}$.

(3) There exists an open cover $\{U_i\}_i$ of Y such that (1) holds for all U_i. Note that we'll usually check things are morphsims in the following way: choose an open (affine) cover of our source and target projective varieties, define morphisms between the opens in this cover, and then check that the morphisms all agree on overlaps.

For $\mathbb{P}^n$, we have the standard affine open cover:
$$\mathbb{P}^n = \bigcup_i U_i = \bigcup_i \{[x_0 : \ldots : x_n] \mid x_i \neq 0\}.$$
Note that each U_i in the cover is isomorphic to $\mathbb{A}^n = \{(y_0, \ldots, 1, \ldots, y_n) \mid y_0, \ldots, \hat{y_i}, \ldots, y_n \in k\}$ (the 1 is in the ith spot) via the mapping $[x_0 : \ldots : x_n] \mapsto (x_1/x_i, \ldots, x_i/x_i, \ldots, x_n/x_i)$. For any closed $X \subset \mathbb{P}^n$, we may write $X = \bigcup_{i=1}^n X_i$, where $X_i = X \cap U_i \subset \mathbb{A}^n$.

Suppose $X = Z(F_1, \ldots, F_r)$ for F_i a homogeneous polynomial for all i. Say $i = 0$. Then $f_j(x_1, \ldots, x_n) = F_j(1, x_1, \ldots, x_n)$ for all $j \in \{1, \ldots, r\}$. Consider $Y = Z(f_1, \ldots, f_r) \subset \mathbb{A}^n$. We have the following map
$$\varphi : X_0 = X \cap U_0 \to Y$$
given by
$$[x_0 : \ldots : x_n] \mapsto (x_1/x_0, \ldots, x_n/x_0).$$
Note that φ has an inverse φ^{-1}, which takes $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \ldots : x_n]$.

There exist better "global" statements:

Lemma 11.5. *Let $X \subset \mathbb{P}^n$ be a projective variety, and let $F_0, \ldots, F_m \in k[x_0, \ldots, x_n]$ be homogeneous polynomials of the same degree such that for every $x \in X$, there exists some j such that $F_j(x) \neq 0$. Then the mapping $f : X \to \mathbb{P}^m$ given by*
$$x \mapsto [F_0(x) : \ldots : F_m(x)]$$
is a morphism.

Proof. f is well-defined set theoretically because: (1) not all $F_j(x) = 0$, and (2) homogeneity of same degree of the F_j's.

Consider the standard open cover of $\mathbb{P}^m$, which we denote by $V_i := (y_i \neq 0)$ for $i = 0, \ldots, m$. Let $U_i = f^{-1}(V_i)$. By restricting f, we get maps $f|_{U_i} : U_i \to V_i$ for all i. By definition, for $x \in U_i$, we have $F_i(x) \neq 0$, and so the map $f|_{U_i}$ is given by $x \mapsto [F_j(x)/F_i(x)]_{j=0,\ldots,\hat{i},\ldots,n}$ and is thus regular. □

Definition 11.6. A polynomial $F \in k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is called *bihomogeneous* of *bi-degree* (d, e) if for all $\lambda, \mu \in k$:
$$F(\lambda x_0, \ldots, \lambda x_n, \mu y_0, \ldots, \mu y_m) = \lambda^d \mu^e F(x_0, \ldots, x_n, y_0, \ldots, y_m).$$
Such a polynomial defines a subset of $\mathbb{P}^n \times \mathbb{P}^m$.

Proposition 11.7. *The closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ are exactly the zero loci of collections of bihomogeneous polynomials.*

Proof. The closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ are of the form $\varphi_{n,m}^{-1}(Z)$, where $Z \subset \mathbb{P}^N$ is closed and $\varphi_{n,m}$ is the Segre map. Now, Z itself is the zero locus of some homogeneous polynomials $Z = Z(F_1, \dots, F_r)$. Recall that the F_i's are polynomials in $k[\dots, Z_{ij}, \dots]$. In the X, Y variables, we get $F_k(X_i Y_j)$, which is bihomogeneous of bidegree (d, d), where $d = \deg(F_k)$. The following remark finishes the proof:

Remark 11.8. Suppose G is a bihomogeneous polynomial of bidegree (d, e). Then $Z(G)$ is the same as the zero locus of $X_i^e Y_j^d G$ for all i, j.

The proof of the remark is left as an exercise to the reader. $\square$

Example 11.9. Consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, where recall that

$$([x_0 : x_1], [y_0 : y_1]) \mapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1).$$

This is defined by $Y_0 Y_3 - Y_1 Y_2 = 0$ in $\mathbb{P}^3$. Let $Q = \varphi_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$ denote the resulting quadric surface.

Also recall the twisted cubic $\nu_3 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. $C := \nu_3(\mathbb{P}^1)$ is given by $Y_0 Y_3 - Y_1 Y_2 = 0$, $Y_1^2 - Y_0 Y_2 = 0$, $Y_2^2 - Y_1 Y_3 = 0$. It follows that this rational normal curve C lies on the quadric surface Q. If we let $Q' = (Y_1^2 - Y - 0 Y_2 = 0)$ and $Q'' = (Y_2^2 - Y_1 Y_3 = 0)$, we see that $C = Q \cap Q' \cap Q''$. Writing everything in terms of the original variables x and y, we get $(x_0 y_1)^2 = (x_0 y_0)(x_1 y_0)$ and $(x_1 y_0)^2 (x_0 y_1)(x_1 y_1)$. But these are in fact the same equation: $x_1 y_0^2 = x_0 y_1^2$, which is bihomogeneous of bidegree $(1, 2)$. This is the equation for the twisted cubic realized as a curve lying in $\mathbb{P}^1 \times \mathbb{P}^1$.

12. Monday March 6

12.1. Completeness of Projective Varieties.

Lemma 12.1. *If X is a projective variety, then the diagonal $\Delta = \Delta_X \subset X \times X$ is closed, where $\Delta_x = \{(x, x) \mid x \in X\}$.*

Proof. Note that $\Delta_X = \Delta_{\mathbb{P}^n} \cap (X \times X)$ $(X \subset \mathbb{P}^n)$. Thus, it suffices to show that $\Delta_{\mathbb{P}^n}$ is closed. Note that

$$\Delta_{\mathbb{P}^n} = \{([x_0 : \dots : x_n], [y_0 : \dots : y_n]) \mid [x_0 : \dots : x_n] = [y_0 : \dots : y_n]\},$$

which is true if and only if

$$\operatorname{rank} \begin{bmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{bmatrix} \leq 1,$$

which itself is true if and only if $x_i y_j = x_j y_i$ for all i, j (note that these are bihomogeneous polynomials of degree (1,1)). $\square$

Definition 12.2. An algebraic variety Y is called complete if for all algebraic varieties X we have that $p_2 : Y \times X \to X$ is closed (i.e., images of closed sets are closed).

Theorem 12.3. *Every projective variety is complete.*

We'll postpone the proof of the above until after the break. Instead, we'll take it for granted and think about applications.

Not every variety is complete:

Example 12.4. Let $X = (xy = 1) \subset \mathbb{A}^2$. Consider projection $p : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto x$. We see that $p(X) = \mathbb{A}^1 - \{0\}$, which is not closed in $\mathbb{A}^1$. So $\mathbb{A}^1$ is not complete.

Corollary 12.5. *If X is a projective variety, then every morphism $\varphi : X \to \mathbb{P}^n$ is closed.*

Proof. Consider the map $\gamma_f : X \hookrightarrow X \times \mathbb{P}^n$ under which $x \mapsto (x, f(x))$. Then consider the projection onto the second coordinate. We see that $p_2 \circ \Gamma_f = f$. Apply Theorem 12.3. □

Corollary 12.6. *Every regular function on a projective variety is constant.*

Proof. Recall that a regular function is a morphism $X \to \mathbb{A}^1$; $\mathbb{A}^1$ embeds into $\mathbb{P}^1$. By the above Corollary, we see that $f(X) \subset \mathbb{P}^1$ is closed. We also know that X irreducible implies $f(X)$ is irreduicble. Therefore, $f(X)$ must be a singleton. □

Corollary 12.7. *The only projective subvarieties of affine varieties are the points.*

Proof. If X is an affine variety, then $A(X)$ is the "algebra of functions on X" If X is not a point, then $A(X) \neq k$. □

Corollary 12.8. *Let $X \subset \mathbb{P}^n$ be a projective variety of positive dimension (i.e., X is not a point). Then if $H \subset \mathbb{P}^n$ is any hypersurface, $X \cap H \neq \emptyset$.*

Proof. Assume there exists $H = Z(F) \subset \mathbb{P}^n$ (F homogeneous of degree d) such that $H \cap X$. For all G homogeneous of degree d, it follows that G/F is a regular function on X ($F(x) \neq 0$ for all $x \in X$). This implies G/F is constant on X: recall the Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ takes X to a point, which is a contradiction, since the Veronese map is an injection. □

12.2. **Applications to Varieties with Group Structure.**

Theorem 12.9 (Rigidity Lemma I)**.** *Let $\varphi : V \times W \to Z$ be a regular map of (quasi-projective) varieties such that there exist $v_0 \in V$, $w_0 \in W$, and $z_0 \in Z$ such that $\varphi(\{v_0\} \times W) = \varphi(V \times \{w_0\}) = z_0$. Also suppose V is complete. Then $\varphi(V \times W) = \{z_0\}$.*

Proof. By completeness, $p_2 : V \times W \to W$ is closed by completeness. Let U be an affine open neighborhood of z_0 inside Z. Define $T = p_2(\varphi^{-1}(Z \setminus U))$. Note that $Z \setminus U$ is closed, φ is a morphism, so $\varphi^{-1}(Z \setminus U)$ is closed, and therefore T is closed. Now, consider $W \setminus T = \{w \in W \mid \varphi(V \times w) \subset U\}$, which clearly contains w_0 (and is thus nonempty). Let $w \in W \setminus T$ be some arbitrary point in this set. Now, $\varphi(V \times \{w\}) \subset U$, which is affine, but V is complete! By one of the above corollaries, $\varphi(V \times \{w\}) = \{*\}$ for all $w \in W \setminus T$ (by completeness). It follows that $\varphi(V \times \{w\}) = \{z_0\}$. Hence φ is constant ($= z_0$) on $V \times (W \setminus T)$, which is open and therefore dense in $V \times W$. Therefore φ must be constant. □

Definition 12.10. An *algebraic group* is a quasi-projective variety G with a group structure, $(G, \cdot)$, such that the morphism $G \times G \to G$, which sends $(g, g') \mapsto g \cdot g'$, and the inverse map $G \to G$ taking $g \mapsto g^{-1}$ are both morphisms of varieties.

Example 12.11. Here are some examples:

(1) $(k, +)$, which is denoted $\mathbb{G}_a$.

(2) $(k^*, \cdot)$, which is denoted $\mathbb{G}_m = GL_1(k)$.

(3) $(GL_n(k), \cdot\,)$, which is an open subset of affine space and an example of an affine algebraic group.

Note: going forward, assume that all varieties we mention are quasi-projective.

Definition 12.12. An abelian variety is a projective algebraic group.

Corollary 12.13. *Every morphism (as varieties) $\alpha : A \to B$ between abelian varieties is the composition of a group homomorphism and a translation.*

Proof. After a translation, we may assume that $\alpha(0) = 0$. Define $\varphi : A \times A \to B$, $\varphi(a, a') = \alpha(a + a') - \alpha(a) - \alpha(a')$; note that this is a morphism of varieties. We see that $\varphi(\{0\} \times A) = \varphi(A \times \{0\}) = \varphi(A \times \{0\}) = \{0\}$. By the Rigidity Lemma, $\varphi \equiv 0$, which implies α is a homomorphism. □

Corollary 12.14. *The group law on an abelian variety is commutative.*

Proof. Consider the inverse map $i : A \to A$, $a \mapsto -a$. By the definition of an algebraic group i is a morphism. Since $i(0) = 0$, i is a group homomorphism. Hence $i(ab) = (ab)^{-1} = a^{-1}b^{-1}$, which is the case if and only if $ab = ba$. □

Consider the map $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$. Under φ the horizontal lines get sent to the lines through zero except for the y-axis. The vertical lines not equal to the y-axis are sent to vertical lines, except for the y-axis, which is sent to $(0, 0)$. Hence, the image of φ is the complement of the y-axis union with the origin.

Definition 12.15. A *locally closed* subset of a topological space is the intersection of a closed set and an open set. A finite union of locally closed subset is called a *constructible* set.

Exercise 12.16. Show that $\operatorname{im}\varphi$ (with φ defined as in the above paragraph) is not locally closed. But, $\operatorname{im}\varphi$ is constructible.

Theorem 12.17 (Chevalley's Theorem)**.** *If $f : X \to Y$ is a morphism of quasi-projective varieties, then $f(X)$ is a constructible set.*

13. Wednesday March 8

13.1. **Rational Maps.**

Example 13.1. Let V be a vector space over k, and suppose we can decompose V as $V = U \oplus W$. We have a projection map $p_w : V \to W$. Noncanonically, we can write $V = k^{n+1}$, and we can think of the projectivization $\mathbb{P}(V) = \mathbb{P}^n$ of this space, i.e., the set of lines through the origin in V. We have that $\mathbb{P}(W), \mathbb{P}(U) \subset \mathbb{P}(V)$. Let $p : \mathbb{P}(V) \setminus \mathbb{P}(U) \to \mathbb{P}(W)$ be the map induced by p_w. As an exercise, prove that p is a morphism. This is called "projection with center $\mathbb{P}(V)$."

If $\dim(U) = 1$, then $\mathbb{P}(U)$ is a point, which we denot by u. Then $p : \mathbb{P}^n \setminus \{u\} \to \mathbb{P}^{n-1}$ takes x to the intersection of the line through u and x with $\mathbb{P}^{n-1}$. I.e., $p(x) = \overline{ux} \cap \mathbb{P}^{n-1}$.

Exercise 13.2. Give a similar description for arbitrary U.

The above are all examples of what are called *rational maps.*

Definition 13.3. (1) Let X and Y be (quasiprojective) varieties. Consider pairs (u, U), where U is an open subset of X and $u : U \to Y$ is a morphism. We say $(u, U) \sim (v, V)$ if u and v coincide on $U \cap V$.

(2) A *rational map* $u : X \dashrightarrow Y$ is an equivalence class of such pairs. We say u is defined at $x \in X$ if it has a representative (u, U) such that $x \in U$. The domain of definition of u is the set of points where u is defined. Note that this is an open subset of X.

(3) A *rational function* is a rational map from $X \dashrightarrow \mathbb{A}^1 = k$.

Proposition 13.4. *Let X be a quasi-projective variety. Then the rational functions on X form a field $K(X)$ which is a k-extension. If $\emptyset \neq U \subset X$ is open, then $K(U) = K(X)$. If X is affine then $K(X) = Q(A(X))$ (i.e., the old definition).*

Proof. We have $U \subset X$. Let $v : V \to k$. Because any two open sets intersect, we see that $v|_{V \cap U} : V \cap U \to k$ is equivalent to $v : V \to k$, implying that $K(X) \subset K(U)$.

To prove the second part of the proposition, we let $f/g \in Q(A(X))$. We associate a rational function $f/g : X \setminus Z(g) \to k$. For the other direction, start with $u : U \to k$ a regular function. Letting $Y = X \setminus U \subset X$, we see that Y is closed. Since X is affine, there exists a function h on X such that $h \equiv 0$ on Y. Thus, $X_h := X \setminus Z(h) = \{x \in X \mid h(x) \neq 0\}$ is an open subset of X. Now, $A(X_h) = A(X)_h$, and define $u = \frac{g}{h^p}$ for $p \geq 0$ on $V \subset U$. Hence, we've associated to u the element $g/h^p \in Q(A(X))$; we leave it as an exercise to the reader to check that the mappings are inverse to each other. $\square$

Definition 13.5. (1) A morphism $\varphi : X \to Y$ is called dominant if $\overline{\varphi(X)} = Y$.

(2) A rational map $m : X \dashrightarrow Y$ is dominant if it has a regular representative which is dominant.

Remark 13.6. Why are rational maps useful? For example, we see that in general, rational maps cannot be composed.

If $\mu : X \dashrightarrow Y$ is dominant, we can compose it with another rational map $v : Y \dashrightarrow T$. In particular, we can compose with rational functions $\varphi : Y \dashrightarrow k$. This induces a field homomorphism $\mu^* : K(Y) \hookrightarrow K(X)$, implying $K(X)$ is a field extension of $K(Y)$.

Proposition 13.7. *Let X and Y be quasi-projective varieties.*

(1) The correspondence $u \mapsto u^*$ gives a one-to-one mapping between

$$\{\text{dominant rational maps } u : X \dashrightarrow Y\} \longleftrightarrow \{k \text{ extensions } K(Y) \hookrightarrow K(X)\}.$$

(2) u^* is an isomorphism if and only if u induces an isomorphism nonempty open sets $U \subset X$ and $V \subset Y$. Such u are called birational map *and we say X and Y are* birational*, or* birationally isomorphic.

This notion of birationality is useful: knowing that two varieties are the same on an open set is really helpful—because open sets are so big, we know that the varieties are *almost* the same. Thus, when people try to classify varieties, they do so up to birational equivalence. We see immediately that $\mathbb{P}^1 \times \mathbb{P}^1$ is birationally equivalent to $\mathbb{P}^2$, since both spaces contain a copy of $\mathbb{A}^2$.

Proof. Let $i : K(Y) \hookrightarrow K(X)$ be a field extension over k. We want a rational map $u : X \dashrightarrow Y$. Assume from the beginning that X and Y are affine, contained in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively. So, $K(X) = Q(A(X))$ and $K(Y) = Q(A(Y))$. Suppose $A(Y)$ has generators $\{y_j\}$ as a k-algebra.

Let $i(y_j) = a_j/b_j \in K(X)$, so that $a_j, b_j \in A(X)$. We obtain an induced map of k-algebras $i : A(Y) \hookrightarrow A(X)_{b_1 \cdots b_n}$. By the main theorem on affine varieties, this is the same as a dominant morphism $u : X \setminus Z(b_1 \cdots b_n) \to Y$ (it's dominant because the map is injective). $\square$

Definition 13.8. If a morphism $u : X \to Y$ is a birational map, we call it a birational morphism.

Definition 13.9. A variety X is *rational* if it is birational to $\mathbb{P}^n$. By the equivalence we just established, this is equivalent to asking whether $K(X)$ is a purely transcendental extension of k.

A variety X is *unirational* if there exists a dominant rational map $\mu : \mathbb{P}^n \dashrightarrow X$. Equivalently $K(X)$ is a subextension of k inside a purely transcendental one.

This begs the following question: Is every unirational variety actually rational?

Theorem 13.10 (Lüroth, Theorem 2.4)**.** *The answer is yes for curves (i.e.,* $\operatorname{trdeg}_k = 1$*).*

Theorem 13.11 (Castelnuovo)**.** *The answer is yes for surfaces (i.e.,* $\operatorname{trdeg}_k = 2$*).*

Unfortunately, however, the answer in general is no :(In fact, there exist unirational varieties that are not rational when dimension is greater than or equal to 3. The first examples of this are due to Clemens and Griffiths, who showed that the cubic 3-fold, $X^3 \subset \mathbb{P}^4$, is not rational but unirational; similarly Iskovskikh-Mann showed this for the quartic 3-fold $X^4 \subset \mathbb{P}^4$. However this is unknown for $X^3 \subset \mathbb{P}^5$. However, by now, every hypersurface $X^n \subset \mathbb{P}^n$ is known not to be rational.

13.2. **Blow-ups.** We start by blowing up points in $\mathbb{P}^n$. Let $x_0 \in \mathbb{P}^n$, and let $H \subset \mathbb{P}^n$ be a hyperplane such that $x_0 \notin H$. Recall that we can project onto H using the lines through x_0. Choose coordinates such that $x_0 = [0 : \ldots : 0 : 1]$. Let $H = Z(x_n)$, and note that p our projection takes $[x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_n]$. Consider the graph of p: $\mathbb{P}^n \times \mathbb{P}^{n-1} = \mathbb{P}^n \times H \supset \Gamma_p = \{(x, y) \mid x \neq x_0, [x_0 : \ldots : x_{n-1}] = [y_0 : \ldots : y_{n-1}]\}$. Take $\overline{\Gamma_p} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$. This is a closed subset of a product of projective spaces; therefore the equations of $\overline{\Gamma_p}$ are $X_iY_j = X_jY_i$ for $0 \le i, j \le n-1$. Denote $\overline{\Gamma_p}$ by $\tilde{\mathbb{P}^n}$, and we call this the *blow up* of $\mathbb{P}^n$ at x_0.

The first projection $\pi : \tilde{\mathbb{P}^n} \to \mathbb{P}^n$ is the blow-up map. Let's consider its fibers: for $x \neq x_0$, $\pi^{-1}(x) = \{x\}$; if $x = x_0$, then the first $n-1$ coordinates of x are 0, so $\pi^{-1}(x) = H = \mathbb{P}^{n-1}$.

14. Monday March 20

Disclaimer: without pictures, I'm $\approx 90\%$ certain that the following lecture notes are basically useless. So perhaps they will be a good reference if you need to remember definitions, but for actually learning about blow-ups, I'll refer you to Prof. Popa's lecture.

14.1. **Blow-ups.** We'll first want to blow-up points in $\mathbb{P}^n$. Fix the point $x_0 = [0 : 0 : \ldots : 1]$. We also want to choose a hyperplane H not containing x_0; the most natural of which is $H = Z(x_n)$. Now, we have the projection map $p : \mathbb{P}^n \setminus \{x_0\} \to H \cong \mathbb{P}^{n-1}$, whiere $[x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_{n-1}]$. Define the blowup $Bl_{x_0}(\mathbb{P}^n) = \tilde{\mathbb{P}^n} := \overline{\Gamma_p} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$, i.e., the graph of the projection map p. This is given by the equations $X_iY_j = X_jY_i$ for $0 \le i, j \le n-1$. Note that $\tilde{\mathbb{P}^n}$ has two projections—one onto the first factor, which we denote by $\pi : \tilde{\mathbb{P}^n} \to \mathbb{P}^n$, is called the *blow-up map*.

So what are the fibers of this projection? Over $x \neq x_0$, $\pi^{-1}(x) = x$. On the other hand, over x_0, $\pi^{-1}(x_0) = H = \mathbb{P}^{n-1}$. There is a nice picture of this, which you should look up! (Prof. Popa drew this in his Zoom lecture.)

Remark 14.1. π induces a birational morphism, more specifically, an isomorphism $\pi : \tilde{\mathbb{P}^n} \setminus H \to \mathbb{P}^n \setminus \{x_0\}$. In other words, π takes the point x_0 and replaces it in $\tilde{\mathbb{P}^n}$ by a copy of H (everywhere else it's one-to-one).

Another interpretation of blow-ups: We can think of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ as a copy of $\mathbb{P}^n$ times the set of "lines through $x_0 \in \mathbb{P}^n$." Thus, the blow-up $\tilde{\mathbb{P}^n} = \{(x, \ell) \mid x \in \ell\} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$; this is called an *incidence correspondence*. If we project onto $\mathbb{P}^n$, then we are looking for fibers over points, so for a point different from x_0, we have the line passing through x and x_0. For x_0, we have all the lines passing through x_0, which is just H. For the other projection, denoted q, (to $\mathbb{P}^{n-1}$, the space of lines), we have each $[\ell] \in \mathbb{P}^{n-1}$, $q^{-1}([\ell]) = \ell = \mathbb{P}^1$. This is called the "tautological bundle," and this second projection map q is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$.

Exercise 14.2. Say $\{U_i\}_i$ is the standard open affine cover of $\mathbb{P}^{n-1} = H$, $i = 0, \dots, n-1$. Show that $q^{-1}(U_i) \cong U_i \times \mathbb{P}^1$.

Definition 14.3. Let $X \subset \mathbb{P}^n$ be a subvariety, and suppose $x_0 \in X$. We define the *blowup of X at x_0* to be $\tilde{X} := \overline{\pi^{-1}(X \setminus \{x_0\})} \subset \tilde{\mathbb{P}^n}$.

Remark 14.4. The birational morphism $\pi : \tilde{X} \to X$ is independent of the choice of embedding $X \subset \mathbb{P}^n$ (we won't prove this right now, or possibly at all).

Local Version: Consider $0 \in \mathbb{A}^n$. We construct the following projection $\pi : Bl_0(\mathbb{A}^n) = \tilde{\mathbb{A}^n} \to \mathbb{A}^n$ by dehomogenizing the previous construction with respect to X_n. (It shouldn't be too difficult to see that blow-ups rely only on the local data near x_0.) We have the following diagram:

$$\begin{tikzcd} \tilde{\mathbb{A}^n} \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d, "\pi"] \\ & \mathbb{A}^n, \end{tikzcd}$$

where $\tilde{\mathbb{A}^n}$ is defined in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ by the equations:

$$X_i Y_j = X_j Y_i \tag{2}$$

$0 \leq i, j \leq n-1$.

As before, if $x \neq 0$ in $\mathbb{A}^n$, then $\pi^{-1}(x)$ is a single point. So $\pi : \tilde{\mathbb{A}^n} \setminus \pi^{-1}(0) \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism with inverse $\psi : \mathbb{A}^n \setminus \{0\} \to \tilde{\mathbb{A}^n} \setminus \pi^{-1}(0)$, where

$$(x_0, \dots, x_{n-1}) \mapsto ((x_0, \dots, x_{n-1}), (x_0 : \dots : x_{n-1})).$$

We also see that $\pi^{-1}(0) = \mathbb{P}^{n-1} \ni (0, [y_0 : \dots : y_{n-1}])$. Again, we may think of $\pi^{-1}(0) = \mathbb{P}^{n-1}$ as the set of lines through the origin in $\mathbb{A}^n$. As before, there is a nice picture that will help with building intuition; you should look this up in Prof. Popa's lecture (it probably also exists elsewhere; it's the one that looks like DNA :)).

Terminology: the codimension 1 subset $\pi^{-1}(0)$ is called the *exceptional divisor* (or exceptional locus) of π.

Remark 14.5. Let L be a line through the origin. Then $\pi^{-1}(L \setminus \{0\})$ intersects $\pi^{-1}(0)$ exactly in the point corresponding to L. Therefore, $\tilde{\mathbb{A}^n} \setminus \pi^{-1}(0) \cong \mathbb{A}^n \setminus \{0\}$ is dense in $\tilde{\mathbb{A}^n}$, implying $\tilde{\mathbb{A}^n}$ is irreducible (the same argument applies for $\tilde{\mathbb{P}^n}$).

To make computations easier, there is a standard way to cover blow-ups by affine coordinate charts. Cover the exceptional divisor $\mathbb{P}^{n-1}$ by the standard open affines isomorphic to $\mathbb{A}^{n-1}$. We'll do this for $\tilde{\mathbb{A}^2} \subset \mathbb{A}^2 \times \mathbb{P}^1$. On $\mathbb{P}^1$: $U_0 = (Y_0 \neq 0) \cong \mathbb{A}^1$, and $U_1 = (Y_1 \neq 0) \cong \mathbb{A}^1$. For $Y_0 \neq 0$, the coordinates on $\mathbb{A}^2 \times \mathbb{A}^1$ are $x_0, x_1, u = y_1/y_0$, since $[y_0 : y_1] = [1 : y_1/y_0]$. Now, $\tilde{\mathbb{A}^2}$ is defined by $x_0 y_1 = x_1 y_0$, so in this coordinate chart, the equation becomes $x_1 = x_0 u$. So $U_0 \cap \tilde{\mathbb{A}^2} \subset \mathbb{A}^2 \times \mathbb{A}^1$; we have two coordinates (x, u). The blow-up map π is then a map from $\mathbb{A}^2 \to \mathbb{A}^2$ taking $(x, u) \mapsto (x, xu)$; recall that this is the kind of map we studied when we saw Chevalley's Theorem (the image is not locally closed; again, there are some nice pictures you should look at...). On the other chart, $Y_1 \neq 0$, the coordinates on $\mathbb{A}^2 \times \mathbb{A}^1$ are $x_0, x_1, v = y_0, y_1$, and $\tilde{\mathbb{A}^2}$ is given by $x_0 = v x_1$. The map $\pi : U_1 \cap \tilde{\mathbb{A}^2} \to \mathbb{A}^2$ is $(v, y) \mapsto (vy, y)$, which is just like the previous case.

Remark 14.6. Here it is just a coincidence that the charts on $\tilde{\mathbb{A}^2}$ are isomorphic to $\mathbb{A}^2$.

Definition 14.7. (1) Suppose $Y \subset \mathbb{A}^n$ is a closed subset. We have the blow-up map $\pi : \tilde{\mathbb{A}^n} \to \mathbb{A}^n$. The *total transform* of Y is $\pi^{-1}(Y)$. The *proper transform* of Y is $\tilde{Y} := \overline{\pi^{-1}(Y \setminus \{0\})}$. (See pictures!)

(2) If Y is a subvariety of $\mathbb{A}^n$ such that $0 \in Y$, then the blow-up of Y at 0 is precisely $\pi|_{\tilde{Y}} : \tilde{Y} \to Y$.

14.2. **Concrete Calculations.**

Example 14.8. $Y = (y = x^2) \subset \mathbb{A}^2$. By our previous discussion, we have the following charts: $U_0 = (Y_0 \neq 0)$ and $U_1 = (Y_1 \neq 0)$. In $U_0 \subset \mathbb{A}^2 \times \mathbb{A}^1$, where $((x, y), u)$ are the coordinates, we have $y = ux$. Pulling back the equation of the curve, we have

$$\pi^*(y - x^2) = ux - x^2 = x(u - x).$$

The pullback is the exceptional divisor $(x = 0)$ and another curve (meeting it transversely). In the second chart $U_1 = (Y_1 \neq 0)$, the equation is $x = vy$. Pulling back our original equation, we have $\pi^*(y - x^2) = y - v^2 y^2 = y(1 - v^2 y)$. Again, $(y = 0)$ is the exceptional divisor, and $(1 - v^2 y)$ is some other curve. See Prof. Popa's lecture to see how to put these together! (Apologies, pictures are really helpful for this stuff, but I am unable to draw them in LaTeX.)

Example 14.9. Consider $Y = (y^2 = x^2 + x^3) \subset \mathbb{A}^2$. In the chart $U_0 = (y_0 \neq 0)$, we have $y = ux$, so

$$\pi^*(y - x^2 - x^3) = u^2 x^2 - x^2 - x^3 = x^2(u^2 - 1 - x).$$

Again we have the exceptional divisor $(x^2 = 0)$; in fact this "appears twice." There is, again, another curve which intersects the exceptional divisor in two places, at $u = \pm 1$. In the other chart, $U_1 = (Y_1 \neq 0)$, the equation is $x = vy$, so $\pi^*(y^2 - x^2 - x^3) = y^2 - v^2 y^2 - v^3 y^3 = y^2(1 - v^2 - v^3 y)$. Again, y^2 gives us the exceptional locus with multiplicity 2, and we have another curve intersecting it at $v = \pm 1$.

Overall: On $U_0 \cap U_1$, $u = 1/v$ (so intersection is the same). We see that the node at 0 gets "pulled apart," and the two points in which the other curve intersects the exceptional divisor correspond to the two different tangent directions at the node.

15. Wednesday March 22

16. Monday March 27

16.1. **Geometrically Finite Morphisms.**

Definition 16.1. $f : X \to Y$, a morphism of (quasi-projective) varieties, is called *generically finite* if there exists open $\emptyset \neq U \subset Y$ such that $f^{-1}(y)$ is finite for all $u \in U$.

Example 16.2. Recall that blow-ups are generically finite!

Example 16.3. Let $X \subset \mathbb{P}^n$ is a projective variety, and let $x_0 \in \mathbb{P}^n \setminus X$. Let $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a hyperplane. We have the projection map $p : \mathbb{P}^n \setminus \{x_0\} \to H$; consider its restriction $p|_X : X \to H$ (there is a nice picture for this; Prof. Popa drew it in class).

Pick coordinates such that $x_0 = [1 : 0 : \ldots : 0]$, $H = (x_0 = 0)$. Let $\overline{x_0 x}$ be the line passing through x and x_0. Note that $p(x) = \overline{x_0 x} \cap H$. Now, $\overline{x_0 x}$ is given parametrically by

$$[\lambda + \mu x_0 : \mu x_1 : \ldots : \mu x_n]$$

for $\lambda, \mu \in k$. So $p([x_0 : \ldots : x_n]) = [x_1 : \ldots : x_n]$.

Claim: all of the fibers of $p|_X$ are *finite*. Let $y \in p(X)$, and suppose $p(x) = y$. We know that $p|_X^{-1}(y) \subset \overline{x_0 x} = \mathbb{P}^1$, which is a closed set by the continuity of $p|_X$. But $x_0 \notin p|_X^{-1}(y)$, implying that $p|_X^{-1}(y)$ cannot be all of $\mathbb{P}^1$. But the only closed subsets of $\mathbb{P}^1$ that are not all of $\mathbb{P}^1$ are finite.

By projectivity, $p(X)$ is closed and is therefore a variety in $\mathbb{P}^{n-1}$. If $p(X) \subsetneq \mathbb{P}^{n-1}$, we can repeat the process. Eventually, this gives a surjective morphism with finite fibers $f : X \to \mathbb{P}^m$ (by composing all of these projections together). If $X \subset \mathbb{P}^n$, we can even take f to be $[x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_m]$.

Exercise 16.4. Interpret this as a unique projection from a higher dimensional linear subspace.

Algebraic interpretation of the notion of "degree":

Theorem 16.5. *Let $f : X \to Y$ be a dominant, generically finite morphism of varieties. Then $K(Y) \hookrightarrow K(X)$ is a* finite *field extension. In particular,* $\operatorname{trdeg}_k K(Y) = \operatorname{trdeg}_k K(X)$*. The degree $[K(X) : K(Y)]$ is called the degree of f (denoted* $\deg(f)$*). If* $\operatorname{char} k = 0$*, then* $\deg(f)$ *is the same as $|f^{-1}(y)|$ for $y \in U \subset Y$ a nonempty open.*

Proof. We may assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine (exercise: prove this!). We can also factor f as

$$X \xhookrightarrow{\Gamma_f} X \times Y \xrightarrow{p_2} Y$$

$f = p_2 \circ \Gamma_f$. We can decompose this into n projections, one variable at a time. So we may assume $X \subset \mathbb{A}^{m+1}$, $Y \subset \mathbb{A}^m$, where the projection is given by $[x_0 : \ldots : x_m] \mapsto [x_1 : \ldots : x_m]$. We have the map of coordinate rings

$$A(Y) = k[x_1, \ldots, x_m]/I(Y) \hookrightarrow A(X) = k[x_0, \ldots, x_m]/I(X) = A(Y)[x_0].$$

Thus, $K(X) = K(Y)(x_0)$. We want to show that $K(X)$ is finite over $K(Y)$; this reduces to showing that x_0 is algebraic over $K(Y)$.

For a contradiction, assume that x_0 is transcendental over $K(Y)$. Take $F \in I(X)$. Write $F = F_d(x_1, \ldots, x_m)x_0^d + \ldots + F_0(x_1, \ldots, x_m)$. Note that x_0 transcendental over $K(Y)$ implies $F_i \equiv 0$ on Y. But this implies the following: if $(x_1, \ldots, x_m) \in Y$, then $(x_0, x_1, \ldots, x_m) \in X$ for all $x_0 \in k$. Therefore $f^{-1}(y) = \mathbb{A}^1$, which is a contradiction! Thus, $K(Y) \subset K(X)$ is algebraic; therefore is a finite extension.

We call $\deg(f) = [K(X) : K(Y)]$. Assume now that $\mathrm{char}(k) = 0$. Let F (write this as in the above) be the minimal polynomial of x_0 over $K(Y)$. Clear denominators so that the coefficients are in $A(Y)$. Note that $d = \deg(f)$. Take $\Delta(x_1, \ldots, x_m) = c\mathrm{Res}(f, f')$ to be the discriminant of f. We have that $\Delta \neq 0$ on Y (F irreducible with coefficients in $K(Y)$ and $\mathrm{char}(k) = 0$). This implies $(\Delta = 0)$ and $(F_d = 0)$ are proper closed subsets of Y. It follows that we have exactly d points in the fiber, for $y \in U = Y \setminus ((\Delta = 0) \cup (F_d = 0))$. □

Remark 16.6. The same proof works if $K(Y) \hookrightarrow K(X)$ is a *separable* extension in characteristic $p > 0$. (I.e., $f' \not\equiv 0$ if f is the minimal polynomial of any $x_0 \in K(X)$).

Definition 16.7. Suppose $X \subset \mathbb{P}^n$ is a projective variety. Take the surjective projection map $f : X \to \mathbb{P}^m$ discussed earlier, where $m = \dim(X)$. This is a generically finite map. Define the *degree of* X (in $\mathbb{P}^n$) to be $\deg(X) := \deg(f)$.

Remark 16.8. A priori, it seems like this depends on the choice of projection! We'll show later that in fact, this construction is independent of our choice of projection. Let's assume this for now.

Remark 16.9. $\deg(X)$ does depend on the embedding in $\mathbb{P}^n$. Note that a line, which is isomorphic to $\mathbb{P}^1$, has degree 1 in $\mathbb{P}^2$, but the degree of a conic (also isomorphic to $\mathbb{P}^2$) is 2 in $\mathbb{P}^2$.

Remark 16.10. In general, $\deg(X)$ is the number of points in the intersection of X with a general $(n-m)$-dimensional linear subspace in $\mathbb{P}^n$, which is equal to the number of points of X intersected with $n-m$ general hyperplanes in $\mathbb{P}^n$. Note that *general* always means "in a nonempty Zariski open set."

Example 16.11. If $X = Z(F) \subset \mathbb{P}^n$ is a hypersurface, then $\deg(X) = \deg(F)$.

Example 16.12. Take the conic $X = Z(xy - z^2) \subset \mathbb{P}^2$. Consider the projection $\mathbb{P}^2 \to \mathbb{P}^1$ given by $(x : y : z) \mapsto (x : y)$. If the characteristic of k is not 2, then the fibers consist of two points, except over $(0 : 1)$ and $(1 : 0)$.

If the characteristic is 2, then all fibers consist of 1 point.

17. Wednesday March 29

Theorem 17.1. *Let $X, Y \subset \mathbb{P}^n$ be quasi-projective varieties. Then*

(1) *Every irreducible component of $X \cap Y$ has dimension greater than or equal to* $\dim X + \dim Y - n$ *if the intersection is nonempty.*

(2) *If X and Y are closed and* $\dim X + \dim Y \geq n$*, then* $X \cap Y \neq \emptyset$.

Proof. Consider the cone construction $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, where $C^0(X) := \pi^{-1}(X)$. As an exercise prove that $C^0(X)$ is quasi-projective and irreducible if X is, and $\dim C^0(X) = \dim X + 1$. Let $C(X) = \overline{C^0(X)} \subset \mathbb{A}^{n+1}$. In particular, if X is closed, then $C(X)$ is the affine cone over X.

We'll show (1): First note that $C^0(X \cap Y) = C^0(X) \cap X^0(Y)$. Then, $C^0(X) \cap C^0(Y) = C^0(X) \times C^0(Y) \cap \Delta_{\mathbb{A}^{2n+2}}$. Note that the diagonal $\Delta_{\mathbb{A}^{2n+2}}$ is cut out by $n+1$ linear equations. Therefore,

$$= \dim(X \cap Y) + 1 = \dim(C^0(X) \cap X^0(Y)) \geq \dim C^0(X) + \dim C^0(Y) - n - 1 = \dim X + 1 + \dim Y + 1 - n - 1 = \dim X + d$$

For (2): The same reasoning as in (1) shows that each component of $C(X) \cap C(Y)$ has dimension greater than or equal to $\dim X + \dim Y - n + 1 \geq 1$ (by assumption $\dim X + \dim Y - n \geq 0$). So

$C(X) \cap C(Y) = C(X \cap Y) = \overline{C^0(X \cap Y)}$ has to contain nonzero points. Therefore, $C^0(X \cap Y) \neq \emptyset$. ☐

17.1. **Dimension of the Fibers of a Morphism.**

Definition 17.2. Let X be a variety, and let $\varphi : X \to \mathbb{Z}$ be any function. Then φ is said to be *upper semicontinuous* if for all $k \in \mathbb{Z}$, the set $\{x \in X \mid \varphi(x) \geq k\}$ is closed (similarly we can define *lower semicontinuous* functions).

Consequently, there exists a nonempty open set on which φ attains its *minimal* value.

Theorem 17.3. *Let X be a variety and $f : X \to \mathbb{P}^n$ a morphism. For each $x \in X$, let $X_x = f^{-1}(f(x))$. Then the function $\varphi : X \to \mathbb{N}$ given by $\varphi(x) = \dim_x X_x$ is upper semi-continuous. Moreover, $\dim X = \dim \overline{f(X)} + \varphi_0$, where $\varphi_0 := \min_{x \in X} \varphi(x)$. (See Harris Theorem 11.12; this should be in Shafarevich.)*

Notation 17.4. *For Z some algebraic set containing x, $\dim_x(Z)$ is the maximum dimension of the components passing through x.*

Remark 17.5. This is really a statement about any $f : X \twoheadrightarrow Y$ a map between quasi-projective varieties.

Corollary 17.6. *Let $f : X \to \mathbb{P}^n$ be a morphism, X a quasi-projective variety. Denote $Y = \overline{f(X)}$. Then*

(1) *For all $y \in f(X)$, every irreducible component has dimension greater than or equal to $\dim X - \dim Y$.*

(2) *There exists $\emptyset \neq U \subset Y$ open such that for all $y \in U$, $\dim f^{-1}(y) = \dim X - \dim Y$.*

Proof. We'll prove this in the case that X and Y are projective varieties. In this case, $f : X \to Y$ is surjective. For all $r \in \mathbb{N}$, $X(r) := \{x \in X \mid \dim_x X_x \geq r\}$. Fix $y \in Y$, and let X' be an irreducible component in $f^{-1}(y)$. Fix $x \in X'$ that is not in any other component. Then, $\dim X' = \dim_x X_x \geq \varphi_0 = \dim X - \dim Y$ (by Theorem 17.3). (1) follows.

For (2), we show the following for all irreducible components $Z \subset X(\varphi_0 + 1)$, $f(Z) \subsetneq Y$. Let $x \in Z \backslash \{\text{other components of } X(\varphi_0{+}1)\}$. Therefore, there exists an irreducible component $X' \subset X_x$ passing through x such that $\dim X' \geq \varphi_0 + 1$. By our choice of x, it follows that $X' \subset Z$. Therefore, for $f|_Z : Z \to f(Z)$, the function φ from THeorem 17.3 has minimum value $\varphi_0 + 1$. Applying the theorem to $f|_Z$, we ee that $\dim \overline{f(Z)} \leq \dim Z + \varphi_0 + 1 \leq \dim X - \varphi_0 - 2 = \dim Y - 2$. ☐

Corollary 17.7. *The image of every dominant morphism morphism contains a nonempty open set.*

Corollary 17.8 (Chevalley's Theorem). *Let $f : X \to Y$ be a morphism of quasi-projective varieties. Then the image of every constructible set is constructible. (Recall: constructible sets are finite unions of locally closed subsets.)*

Proof. Induction on the dimension of Y. If $\dim Y = 0$, then Y is a singleton and the result is obvious. For the inductive step, we may assume without loss of generality that X is irreducible and f is dominant. We will try to show that $f(X)$ is constructible. By Corollary 17.7, there exists a nonempty open $U \subset f(X)$. Let $Z = Y \setminus U$; note that Z is closed and properly contained Y. Now, $f(X) = U \cup f(f^{-1}(Z))$. Because $Z \subsetneq Y$, we have $\dim Z < \dim Y$. Therefore, $f(f^{-1}(Z))$ is constructible by the inductive hypothesis. Therefore $f(X)$ is constructible. ☐

17.2. **Grassmannians.**

Definition 17.9. The *Grassmannian* $G(k,n)$ is the set of k-dimensional linear subspaces of K^n. Equivalently (and probably, preferably, since the following does not require choosing a basis), let V be an n-dimensional K-vector space; then $G(k,V)$ is the set of k-dimensional linear subspaces of V. We also will denote $G(k,n)$ by $\mathbb{G}(k-1,n-1)$, where the blackboard bold $\mathbb{G}$ signals that we are working in projective space. I.e., $\mathbb{G}(k-1,n-1)$ is the set of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$. Similarly, $G(k,V) = \mathbb{G}(k-1,\mathbb{P}(V))$.

The following is a fundamental fact about Grassmannians: $G(k,r)$ is a projective variety of dimension $k(n-k)$.

Example 17.10. $G(1,n) \cong \mathbb{P}^{n-1}$.

We'll realize $G(k,n)$ as a closed subvariety of the projective variety. In particular, $G(k,V) \hookrightarrow \mathbb{P}(\wedge^k V)$; this is called the *Plücker* embedding.

18. Monday April 3

18.1. **The Exterior Algebra.** Assume char$k \neq 2$. Let V be an n-dimensional vector space over k; let $e_1,\dots,e_n$ be a basis for V. The exterior algebra of V, denoted E, is defined in the following way:

$$E = \bigwedge V = \bigwedge^0 V \oplus \bigwedge^1 V \oplus \ldots \oplus \bigwedge^n V,$$

where

$$\bigwedge^k V = \bigotimes^k V / (v_1 \otimes \cdots \otimes v_k - \operatorname{sgm}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}).$$

In other words, $\bigwedge V = \bigotimes V / \langle x \otimes y + y \otimes x \rangle$, where $\langle x \otimes y + y \otimes x \rangle$ denotes the relations generated by $x \otimes y + y \otimes x$ for all $x, y \in V$. Therefore, for all $x, y \in V$, $x \wedge y = -y \wedge x$. Note that E is a graded ring, with ring structure given by $\wedge : E \times E \to E$ taking $(v,w) \mapsto v \wedge w$, and because the characteristic of k is not 2, we see that $x \wedge x = 0$ for all $x \in E$.

$\bigwedge V$ satisfies the following properties:

(1) for all $\lambda_1, \lambda_2 \in k$ and $v_1, v_2, w \in E$,

$$(\lambda_1 v_1 + \lambda_2 v_2) \wedge w = \lambda_1 (v_1 \wedge w) + \lambda_2 (v_2 \wedge w);$$

(2) $v_1 \wedge v_2 = -v_2 \wedge v_1$ (making the multiplication *skew-commutative)*

(3) $\wedge$ *is associative.*

Every element in $\bigwedge^k V$ can be written as

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1,\ldots,i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Note that $a_{i_1,\ldots,i_k} \in k$ and that the $e_{i_1} \wedge \cdots \wedge e_{i_k}$ are linearly independent over k. The set $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \leq i_1 < \ldots < i_k \leq n}$ form a basis for $\bigwedge^k V$, so $\dim \bigwedge^k V = \binom{n}{k}$. In particular, $\bigwedge^n V \cong k e_1 \wedge \cdots \wedge e_n$.

18.2. **Properties of the Exterior Product.** Suppose $\varphi : V \to V$ is a linear transformation. If $e_1, \ldots, e_n$ is a basis for V, then φ can be identified with a matrix $A \in M_n(k)$. This induces a map $\bigwedge^k \varphi : \bigwedge^k V \to \bigwedge^k V$, and in particular a map $\bigwedge^n \varphi : \bigwedge^n V \cong k \to \bigwedge^n V \cong k$.

Exercise 18.1. $\bigwedge^n \varphi$ corresponds to multiplication by $\det(A)$.

Corollary 18.2. $\mathrm{rank}(\varphi) < n$ *if and only if* $\bigwedge^n \varphi = 0$.

Corollary 18.3. *Suppose* $v_1, \ldots, v_k$ *are linearly independent, and suppose* $v_1', \ldots, v_k' \in W := \langle v_1, \ldots, v_k \rangle$. *Write* $v_i' = \sum_j a_{ij} v_j$. *Then* $v_1' \wedge \cdots \wedge v_k' = \det(a_{ij})_{i,j} v_1 \wedge \cdots \wedge v_k$.

Definition 18.4. We make the following definitions:

(1) $\omega \in \bigwedge^k V$ is *completely decomposable* (a "pure wedge") if $\omega = v_1 \wedge \cdots \wedge v_k$ for $v_1, \ldots, v_k \in V$.

(2) $\omega \in \bigwedge^k V$ is partially decomposable if there exists $v \in V$ and $u \in \bigwedge^{k-1} V$ such that $\omega = v \wedge u$.

Proposition 18.5. *Let* $\omega \in \bigwedge^k V$. *Then*

(1) ω *partially decomposable implies* $\omega \wedge \omega = 0$*;*

(2) ω *partially decomposable, then the linear transformation* $\alpha_\omega : V \to \bigwedge^{k+1} V$ *taking* $v \mapsto v \wedge \omega$ *has nonzero kernel;*

(3) If $v_1, \ldots, v_m$ *forms a basis for* $\ker \alpha_\omega$*, then* $\omega = v_1 \wedge \cdots \wedge v_m \wedge \eta$ *for some* $\eta \in \bigwedge^{k-m} V$*;*

(4) ω *is completely decomposable if and only if* $\dim \ker \alpha_\omega = k$.

Proof. (1): Write $\omega = v \wedge u$ for $v \in V$. Then $\omega \wedge \omega = v \wedge u \wedge v \wedge u = \pm v \wedge v \wedge u \wedge u = 0$.

(2),(4): These are special cases of (3).

(3): Complete $v_1, \ldots, v_m$ to a basis of V, say $v_1, \ldots, v_m, v_{m+1}, \ldots, v_n$. Write

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \cdots i_k} v_{i_1} \wedge \cdots \wedge v_{i_k},$$

and consider $v_j \wedge \omega$. It is not hard to see that

$$v_j \wedge \omega = \sum a_{i_1 \cdots i_k} v_j \wedge v_{i_1} \wedge \cdots \wedge v_{i_k},$$

so $a_{i_1 \cdots i_k} = 0$ if $j - i_p$ for some p. Otherwise, $v_j \wedge v_{i_1} \wedge \cdots \wedge v_{i_k}$ is a basis vector for $\bigwedge^{k+1} V$. Hence, $v_j \wedge \omega = 0$ if and only if $a_{i_1 \cdots i_k} = 0$ for every $\{i_1, \ldots, i_k\} \not\ni j$. Because $v_1, \ldots, v_m$ are a basis for $\ker \alpha_\omega$, we know $v_1 \wedge \omega = \cdots = v_m \wedge \omega = 0$, implying that $a_{i_1 \cdots i_k} = 0$ if $\{1, \ldots, m\} \not\subset \{i_1, \ldots, i_k\}$. ☐

18.3. **Applications to Grassmannians.** For an n-dimensional vector space V over k, recall that $G(k, V)$ denotes the set of k-dimensional linear subspaces of V. Alternatively, $\mathbb{G}(k-1, \mathbb{P}(V))$ denotes the set of $(k-1)$-planes in $\mathbb{P}(V)$.

Lemma 18.6. *There exists an injection from* $i : G(k, V) \hookrightarrow \mathbb{P}(\bigwedge^k V)$. *Given* $W = \langle v_1, \ldots, v_k \rangle \in G(k, V)$*, this map takes* $W \mapsto [v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k V) = \mathbb{P}^{\binom{n}{k}-1}$.

Proof. The lemma follows from showing that the map is well-defined and injective; it's proof is left as homework for the reader. ☐

Lemma 18.7. *Moreover, i realizes $G(k,V)$ as a closed subset in $\mathbb{P}(\bigwedge^k V)$; we call i the* Plücker embedding.

Proof. By Proposition 18.5, $[\omega] \in \operatorname{im}(i)$ if and only if $\alpha_\omega : V \to \bigwedge^{k+1} V$ has rank less than or equal to $(n-k)$. Now, $\alpha_\omega : V \to \bigwedge^{k+1} V$ has rank less than or equal to $(n-k)$ if and only if all the minors of α_ω of type $(n-k+1) \times (n-k+1)$ are 0. This gives (polynomial) equations describing $\operatorname{im}(i) \subset \mathbb{P}(\bigwedge^k V)$.

More specifically, consider the following argument: Fix a basis $e_1, \ldots, e_n \in V$. The coordinates on $\mathbb{P}(\bigwedge^k V)$ are $[\ldots : x_{i_1 \cdots i_k} : \ldots]_{1 \le i_1 < \cdots < i_k \le n}$. Write $\omega \in \bigwedge^k V$ as $\omega = v_1 \wedge \cdots \wedge v_k$. Then $\ker \alpha_\omega = \langle v_1, \ldots, v_k \rangle$, where $v_i = \sum_j x_{ij} e_j$. Now, recall that α_ω takes $e_1 \mapsto e_1 \wedge \omega = e_1 \wedge v_1 \wedge \cdots \wedge v_k = e_1 \wedge (\sum x_{1j} e_j) \wedge \cdots \wedge (\sum x_{kj} e_j) = \sum x_{1i_1} \cdots x_{ki_k} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}$. Similar calculations can be performed for the other e_i's. Each $x_I = x_{1i_1} \cdots x_{ki_k}$ correspond to $e_{i_1} \wedge \cdots \wedge e_{i_k}$. The minors of α_ω give the equations for the Grassmannian in the x_I. □

Example 18.8. Say $k = 2$. Then $i : G(2,V) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$. We claim that $\operatorname{im}(i)$ is given by the equations $\omega \wedge \omega = 0$ (this is a quadratic relation in the x_I's). These are called the *Plücker relations*. The proof is left as an exercise for the reader.

The first nontrivial Grassmannian is when $k = 2$ and $n = 4$. We have $G(2,4) = \mathbb{G}(1,3) \hookrightarrow \mathbb{P}(\bigwedge^2 k^4) = \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$ (recall that $G(2,3) \cong G(1,3) \cong \mathbb{P}^2$). The dimension of $G(k,n)$ is $k(n-k)$, so $\dim G(2,4) = 4$. Therefore, $G(2,4)$ is a hypersurface in $\mathbb{P}^5$, i.e., a quadric. Let $e_1, \ldots, e_4$ be a basis for k^4. We have that

$$\omega = \sum_{1 \le i < j \le 4} x_{ij} e_i \wedge e_j = x_{12}(e_1 \wedge e_2) + \ldots + x_{34}(e_3 \wedge e_4),$$

so $x_{12}, \ldots, x_{34}$ are our coordinates on $\mathbb{P}^5$. We also see that $\omega \wedge \omega = 0$ if and only if $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$. This is the equation of $G(2,4)$ in $\mathbb{P}^5$.

Remark 18.9. In fact, every $G(k,n)$ is defined by quadrics. (See Griffiths-Harris for more on Grassmannians).

19. Wednesday April 5

20. Monday April 10

Example 20.1. Example with quadrics deforming into two planes that intersect in a line.

Example 20.2. Universal family of k-planes: Recall that $\mathbb{G}(k,n)$ parameterizes the k-planes in $\mathbb{P}^n$. Consider the following incidence correspondence:

$$\Sigma = \{(W,x) \mid x \in W\} \subset \mathbb{G}(k,n) \times \mathbb{P}^n.$$

We have projections p_1 and p_2 to $\mathbb{G}(k,n)$ and $\mathbb{P}^n$ respectively. We call Σ the universal k-plane: this is because $p_1^{-1}(W) = W$. We have that $p_1^{-1}(W) = W$ is irreducible of dimension k. We see that Σ is irreducible of dimension $k + (k+1)(n-k)$. Studying the other projection p_2 is on the homework.

Example 20.3. Varieties of incident planes: Suppose $X \subset \mathbb{P}^n$ is a projective variety and that $k < n - \dim X$. Let

$$I_k(X) = \{W \mid W \cap X \neq \emptyset\} \subset \mathbb{G}(k,n).$$

Homework exercise: $I_k(X)$ is irreduicble of dimension $\dim(X) + k(n-k)$. Hint: You should use the universal family of k-planes from the previous example.

Example 20.4. Fix a line $\ell_0 = \mathbb{P}^1 \subset \mathbb{P}^2$. Consider

$$I_1(\ell_0) = \{\ell \mid \ell \cap \ell_0 \neq \emptyset\} \subset \mathbb{G}(1,3),$$

the lines in $\mathbb{P}^3$ intersecting ℓ_0. Form the following incidence correspondence:

$$Z = \{(\ell, x) \mid x \in \ell\} \subset \mathbb{G}(1,3) \times \ell_0.$$

The first projection map takes $Z \to I_1(\ell_0)$. We will show that $I_1(\ell_0)$ is irreducible of dimension 3. The fiber of p_1 over any $\ell \neq \ell_0$ is just $p_1^{-1}(\ell) = \{*\} = \ell \cap \ell_0$, while $p_1^{-1}(\ell_0) = \ell_0 \cong \mathbb{P}^1$. Is this the blow-up of the point ℓ_0? While it looks like it might be, the answer is no, since the fiber over ℓ_0 would need to be $\mathbb{P}^2$. In actuality, p_1 is a birational map whose exc. locus is not a hypersurface (so not a blow-up); this is an example of what is called a *small contraction.* This gives rise to a fancier version of a birational map called a FLOP.

20.1. **Smoothness, tangent spaces, and tangent cones.** Preliminary situation: let $X \subset \mathbb{A}^n$ be an affine variety with ideal $\mathcal{I}(X)$. Define the *Zariski tangent space* to X at $p \in X$ as the linear subspace of $\mathbb{A}^n$ given by the equations

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(p) X_i = 0$$

for all $f \in \mathcal{I}(X)$. Exercise: It suffices to take a set of generators for $I(X)$. What we visualize is

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(p)(X_i - p_i) = 0$$

for $p = (p_1, \ldots, p_n)$.

The drawback of this definition is that it depends on the equations of the variety, i.e., it depends on the way X embeds in $\mathbb{A}^n$. Also, we ultimately want to be able to make such a definition for other types of varieties (e.g., projective). Thus, we consider the following approach, which is motivated by differential geometry. Suppose $X \subset \mathbb{A}^n$ is affine, and let $A(X)$ be its coordinate ring $k[x_1, \ldots, x_n]/\mathcal{I}(X)$. A *derivation of X at p* is a map $D : A(X) \to k$ such that

(1) D is k-linear;

(2) D satisfies the Leibniz rule: $D(fg) = f(p)D(g) + g(p)D(f)$ for all $f, g \in A(X)$.

From now on, we'll think of things in the following way:

$$\{\text{tangent space of } X \text{ at } p\} \longleftrightarrow \{\text{derivations of } X \text{ at } p\}.$$

If $X \subset \mathbb{A}^n$ has coordinates $x_1, \ldots, x_n$, then all derivations are of the form

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

for $_i \in k$. So $D : A(X) \to k$ will take $g \mapsto \sum_i a_i \frac{\partial g}{\partial x_i}$. This is because $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$, and $D(x_1^2) = x_1 D(x_1) + x_1 D(x_1) = 2x_1 D(x_1)$ (set $D(x_i) = a_i$).

Thinking of tangent vectors as derivations, we'll now define things generally for any quasi-projective variety and show that this general definition coincides with our previous ones. If X is any quasi-projective variety and $x \in X$, recall that we have

$$\mathcal{O}_{X,x} = \{(U, f) \mid x \in U, f \text{ regular on } U\}/\sim,$$

where $(U, f) \sim (V, g)$ if $f = g$ on $U \cap V$. We have a maximal ideal $\mathfrak{M}_x = \{f \mid f(x) = 0\}$. Because we are working over an algebraically closed field k, we have $k \hookrightarrow \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{M}_x = k(x) \cong k$. (If X is affine, then $\mathcal{O}_{X,x} = A(X)_{I_x}$, and $\dim \mathcal{O}_{X,x} = \dim X$.)

Definition 20.5. The *Zariski tangent space* of X at x is

$$T_xX := (\mathfrak{M}_x/\mathfrak{M}_x^2)^*.$$

Note that $\mathfrak{M}_x/\mathfrak{M}_x^2$ is an $\mathcal{O}_{X,x}/\mathfrak{M}_x$-module. But $\mathcal{O}_{X,x}/\mathfrak{M}_x = k$, so $\mathfrak{M}_x/\mathfrak{M}_x^2$ is a k-vector space. Thus, considering $(\mathfrak{M}_x/\mathfrak{M}_x^2)^*$ is valid! We can also just define $\mathfrak{M}_x/\mathfrak{M}_x^2$ to be the *cotangent space* at x.

Remark 20.6. This is equivalent to the definition using derivations. Let $D : A(X) \to k$ be a derivation. Take $\mathfrak{M}_x \subset A(X)$ to be the maximal ideal corresponding to x. We can restrict our derivation to $\mathfrak{M}_x$, $D|_{\mathfrak{M}_x} : \mathfrak{M}_x \to k$. We claim that $D(\mathfrak{M}_x^2) = 0$: $D(fg) = f(x)D(g) + g(x)D(f) = 0$ for $f, g \in \mathfrak{M}_x$. Therefore, $D|_{\mathfrak{M}_x}$ factors through $\mathfrak{M}_x/\mathfrak{M}_x^2$. We get a linear form $\mathfrak{M}_x/\mathfrak{M}_x^2 \to k$, which is our corresponding element of T_xX.

Conversely, say $\theta \in (\mathfrak{M}_x/\mathfrak{M}_x^2)^*$. We get a derivation in the following way: set $D(f) = \theta(f - f(x))$. Prove that this is a derivation as an exercise, and show that the assignments described in this paragraph and the one above are inverse to each other.

21. Wednesday April 12

21.1. **More on tangent spaces.** Recall from multivariable calculus that if $f_1, \dots, f_n$ are polynomials in n variables, then the derivative of the function $(f_1, \dots, f_n) : k^n \to k^n$ at x is given by the matrix

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{ij} : k^n \to k^n.$$

We see that T_xX is equal to the kernel of this matrix.

Proposition 21.1. *Let X be a variety. Then the function $x \mapsto \dim_k T_xX$ is upper semicontinuous. In other words, for all $r \geq 0$, $X(r) = \{x \in X \mid \dim T_xX \geq r\}$ is closed.*

Proof. We can think locally, so assume that $X \subset \mathbb{A}^n$ is affine, and suppose $\mathcal{I}(X) = (f_1, \dots, f_k)$. Because

$$\dim T_xX = n - \operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}\right)_{ij},$$

we see that $\dim T_xX \geq r$ if and only if $\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{ij} \leq n - r$. But the rank of a matrix being greater than or equal to $n - r$ is a closed condition! (Hint if this isn't immediately obvious: minors!) ☐

Proposition 21.2. *For all $x \in X$, $\dim T_xX \geq X$, and we have equality for general x (i.e., $x \in U \neq$ open).*

Proof. We've seen before that there exists an isomorphism between an open dense subset in X and an open subset of a hypersurface in $\mathbb{A}^{n+1}$. Suppose the proposition is true for hypersurfaces in $\mathbb{A}^{n+1}$. Therefore, there exists an open set $V \subset X$ such that $V \subset X(n)$ (see Proposition 21.1). By the above proposition, $V \subset X(n)$, which is closed, while V is open. This forces $X(n) = X$; we have reduced the problem to showing the proposition on hypersurfaces.

Thus, we may assume $X = Z(f)$ for $f \in k[x_1, \dots, x_n]$. Recall that T_xX is defined by $\sum_{j=1}^{n+1} \frac{\partial f}{\partial j}(x) = 0$. Therefore, $\dim T_xX$ is either n or $n+1$, and we need to worry about the case where the dimension is $n+1$. This is the case where all of the partials are identically 0. If this is the case, then $f | \frac{\partial f}{\partial x_j}$ for all j. Since the partials have smaller degree, this can only be the case if $\frac{\partial f}{\partial x_j} = 0$ for all j. This implies that the characteristic of k is $p > 0$ and that $f = g^p$ for some g. However, this is impossible, since f was assumed to be irreducible (X is a variety). □

Definition 21.3. A point $x \in X$ is *smooth* if $\dim T_x X = \dim X$. *Regular* and *nonsingular* are also commonly used adjectives that mean the same thing. Otherwise, x is called *singular*.

By Proposition 21.2, if X is a variety (i.e., irreducible), then $X_{sing} = \mathrm{Sing}(X) = \{x \in X \mid x \text{ singular}\}$ is a *proper closed* subset of X, while $X_{reg} = \{x \in X \mid x \text{ smooth}\} \subset X$ is a dense open set.

X is called *smooth* if $X = X_{reg}$.

Example 21.4. Here are some examples/exercises:

(1) Show that $T_{(x,y)} X \times Y = T_x X \oplus T_y Y$. It follows that $\mathrm{Sing}(X \times Y) = (\mathrm{Sing} X \times Y) \cup (X \times \mathrm{Sing} Y)$.

(2) If $X = (f = 0) \subset \mathbb{A}^n$ is a hypersurface, then $\mathrm{Sing}(X) = \{x \in X \mid f(x) = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0\}$.

(3) Let $X \subset \mathbb{A}^n$, and suppose $\mathcal{I}(X) = (f_1, \ldots, f_r)$. We see that $x \in X$ is smooth if and only if $\mathrm{rank}(= \frac{\partial f_i}{\partial x_j}(x)) = n - \dim X$. Recall from calculus that $(= \frac{\partial f_i}{\partial x_j})_{ij}$ is called the *Jacobian matrix*.

(4) Suppose $X = Z(F) \subset \mathbb{P}^n$ for a homogeneous $F \in k[x_0, \ldots, x_n]$. Say $x = [1 : x_1 : \ldots : x_n]$, and now dehomogenize with respect to x_0. Let $U_0 \subset X$ be the corresponding open affine subset. Then $I(X \cap U_0)$ is given by dehomogenizing F with respect to X_0. Use (2) to do the tangent space computation. Say $\deg(F) = d$. Then we have the *Euler formula*:
$$dF = \sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}.$$
Hence the singular locus is given by $F(x) = \frac{\partial F}{\partial x_0}(x) = \cdots = \frac{\partial F}{\partial x_n}(x) = 0$. If the characteristic of the field we are working over does not divide d, it is enough to consider
$$F(x) = \frac{\partial F}{\partial x_0}(x) = \cdots = \frac{\partial F}{\partial x_n}(x) = 0,$$
and we can basically skip the dehomogenization process discussed above.

(5) The following is an exercise: if $X \subset \mathbb{P}^n$ is a projective variety, with $\mathcal{I}(X) = (F_1, \ldots, F_r)$. Then $x \in X$ is smooth if and only if $\mathrm{rank}(\frac{\partial F_i}{\partial x_j}(x)) = n - \dim X$.

(6) For an algebraic group G, and X is a G-homogeneous space (i.e., there is a transitive action of G on X), then X is smooth. To show this, it suffices to find one smooth point, say $x \in X$. Then any neighborhood of a point is isomorphic to a neighborhood of x by the transitive action. For example, $G(k, n)$ has an action by $PGL(n)$ which makes it into a homogeneous space.

21.2. **The differential of a regular map.**

Definition 21.5. Let $f : X \to Y$ be a morphism of quasi-projective varieties. Let $x \in X$, and let $y = f(x) \in Y$. We'll return to our definition of $T_x X$ as $T_x X = (\mathfrak{M}_x / \mathfrak{M}_x^2)^*$. As is usual in differential geometry/analysis, we want a differential, a map $df_x : T_x X \to T_y Y$. Recall that because f is a morphism, there exists an induced map $f^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, which is a morphism of local rings. Therefore, $f^*(\mathfrak{M}_y) \subset \mathfrak{M}_x$. Because f^* is a ring homomorphism, we have that $f^*(\mathfrak{M}_y^2) \subset \mathfrak{M}_x^2$. It follows that f^* induces a map of k-vector spaces
$$f^* : \mathfrak{M}_y / \mathfrak{M}_y^2 \to \mathfrak{M}_x / \mathfrak{M}_x^2.$$

This is the cotangent map of the morphism. To get df_x, dualize this map! This gives us our differential $df_x : T_xX \to T_yY$.

Remark 21.6. The above construction behaves well with respect to composition: if $g : Y \to Z$ is another regular map taking $y \mapsto z$, then $d_x(g \circ f) = d_y g \circ d_x f$.

Example 21.7. The following is an advertisement for Math 232. Consider the map $f : \mathbb{A}^3 \to \mathbb{A}^2$ taking $(x, y, z) \mapsto (z, x^2z + y^2)$ (assume that the characteristic of k is not 2). It is not so difficult to see that

$$df_{(x,y,z)} = \begin{bmatrix} 0 & 0 & 1 \\ 2xz & 2y & x^2 \end{bmatrix} : k^3 \to k^2.$$

This gives a surjective map (i.e., f is a submersion at the point) if and only if either $y \neq 0$ or $xz \neq 0$. If $xz = y = 0$, then the differential has rank 1. The fiber over (a, b) is $Z(x^2a + y^2 - b)$. If $a \neq 0$ and $b \neq 0$, the fiber is singular (it's a sum of squares and there will be a node). If $a = b = 0$, then the rank of the differential is 1. However, the fiber is $Z(y^2)$, a line, which is smooth. We see that $T_xF_0 \subset \ker df_{(x,0,0)}$, where F_0 is the fiber over $(0, 0)$. In fact this is a strict inequality. Because the rank of the differential is 1, the kernel is 2 dimensional. But since T_xF_0 is 1-dimensional, we run into issues. The problem stems from the fact that the fiber is $(y^2 = 0)$. The fiber of this map is not, in fact, a line. Rather, it is a double line, and we'll need to take this into account in our tangent space computation. If you work just with varieties, you'll see some statements that seem odd at first (but will later be fixed by schemes).

Exercise 21.8. Show that for any morphism $f : X \to Y$, the function taking $x \mapsto \dim_k(\ker df_x)$ is upper semicontinuous.

22. Monday April 17

22.1. Tangent Cones.

Remark 22.1. (1) Think of Taylor expansions as trying to linearize X at x.

(2) $TC_pX \subset T_pX$. T_pX is defined by the terms f_1 for all $f \in I(X)$ (when nonzero they are initial).

Example 22.2. Consider the following examples of tangent cones:

(1) $X = (y - x^2 + x) \subset \mathbb{A}^2$. Then $TC_0X = (y + x) = T_0X$.

(2) $X = (y^2 = x^3 + x^2) \subset \mathbb{A}^2$. We see that $TC_0X = (y^2 = x^2) = (y - x)(y + x) \subset T_0X = \mathbb{A}^2$.

(3) $X = (y^2 = x^3) \subset \mathbb{A}^2$. Then $TC_0X = (y^0 = 0) \subset T_0X = \mathbb{A}^2$; this is the "double line."

Geometric Interpretation: $\mathbb{P}TC_xX$ is the exceptional divisor of the blow-up of X at x.

Example 22.3. Consider a curve $C \subset \mathbb{A}^2$, given by $f(x_0, x_1) = \sum a_{ij}x_o^i x_1^j = a_{00} + a_{10}x_0 + a_{01}x^1 + \cdots$. Say $a_{00} = 0$, i.e., $0 \in C$. The tangent line at 0 is given by $a_{10}x_0 + a_{01}x_1$, if $(a_{10}, a_{01}) \neq (0, 0)$. Suppose the coordinates on $\mathbb{P}^1$ are given by $[y_0 : y_1]$. By direct calculation, the proper transform $\widetilde{C}$ of C is given by

$$a_{10}y_0 + a_{01}y_1 + a_{11}y_0x_1 + a_{20}x_0y_0 + a_{02}x_1y_1 + \cdots = 0.$$

Over the origin, $(x_0, x_1) = (0, 0)$, we get $a_{1,0}y_0 + a_{01}y_1$. The computation is the same when we have more branches.

Exercise 22.4. Let $P = \{0\} \in X \subset \mathbb{A}^n$. Consider the blow-up $\pi : \tilde{X} \to X$, which, recall, is the restriction of the blow-up map $\mu : \tilde{\mathbb{A}^n} = Bl_0(\mathbb{A}^n) \to \mathbb{A}^n$. This has exceptional divisor E (recall this is $\mathbb{P}^{n-1} \cap \tilde{X}$). Show that $E \cong \mathbb{P}TC_0X$.

Example 22.5. Let $X = (x^2 + y^2 + z^2) \subset \mathbb{Z}^3$. This is just the cone in $\mathbb{A}^3$; it is not difficult to see that TC_0X is just the cone itself. Blow-up $\tilde{\mathbb{A}^3}$; let $\mu : \tilde{\mathbb{A}^3} \to \mathbb{A}^3$ be the blow-up map. Its exceptional divisor is just a copy of $\mathbb{P}^2$, denote this by $\mathrm{Exc}(\mu) = \mathbb{P}^2$. Now take the proper transform of the cone. The cone turns into something that looks like a cylinder, which is the blow-up of X, $\tilde{X}$. Let $\pi : \tilde{X} \to X$; we see that $E = \mathrm{Exc}(\mu) \cap \tilde{X}$. Then $f = f_2 = f^{in}$, so $TC_0(X) = Z(f^{in}) = Z(f)$, so $\mathbb{P}TC_0(X)$ is a smooth conic in $\mathbb{P}^2$.

Remark 22.6. If the projectivized tangent cone is smooth, then the singularity is called *ordinary*.

Corollary 22.7. $\dim TC_xX = \dim_x X$ *(which is the dimension of X if X is irreducible).*

Proof. We know that

$$\dim \tilde{X} = \dim X = \dim \mathbb{P}TC_xX + 1 = \dim E + 1 = \dim TC_xX,$$

as desired. □

Corollary 22.8. *$x \in X$ is a smooth point if and only if $TC_xX = T_xX$.*

Intrinsic Interpretation: Let $x \in X$ be a point in a variety and let $\mathfrak{M}_x \subset \mathcal{O}_{X,x}$. We have a filtration:

$$\mathcal{O}_{X,x} \supset \mathfrak{M}_x \supset \mathfrak{M}_x^2 \supset \cdots,$$

which gives us an associative graded ring $R = gr(\mathcal{O}_{X,x}) = \bigoplus_{i=0\infty} \mathfrak{M}_x^i/\mathfrak{M}_x^{i+1}$. We may write this as

$$R = gr(\mathcal{O}_{X,x}) = \mathcal{O}_{X,x}/\mathfrak{M}_x \oplus \mathfrak{M}_x/\mathfrak{M}_x^2 \oplus \ldots \cong k \oplus T_x^*X \oplus;$$

call the summands R_0, R_1, R_2, etc., respectively. We see that R is generated by R_1 as an algebra.

Exercise 22.9. If $X \subset \mathbb{A}^n$ is affine and $x = 0$, the mapping $k[x_1, \ldots, x_n]/\mathcal{I}(X)^{in} \to R$ given by $\overline{x_i} \mapsto \hat{x_i}$ is a k-algebra isomorphism (where the tilde and hat denote the classes of x_i in their respective rings).

You can interpret TC_xX as having affine coordinate ring R, which extends to $x \in X$ for any variety X.

Let's consider $\mathfrak{M}_x/\mathfrak{M}_x^2$ again. Recall that $k \cong \mathcal{O}_{X,x}/fm_x$ as vector spaces and that $\mathfrak{M}_x/\mathfrak{M}_x^2 \cong (T_xX)^*$. Form the kth symmetric algebra $\mathrm{Sym}^k(\mathfrak{M}_x/\mathfrak{M}_x^2)$. Recall that for an arbitrary vector space C, $\mathrm{Sym}^* V = \bigoplus_{k \geq 0} \mathrm{Sym}^k V$, so if $x_1, \ldots, x_n$ is a basis for V, then $\mathrm{Sym}^* V \cong k[x_1, \ldots, x_n]$, and $\mathrm{Sym}^k V$ is the space of homogeneous degree-k polynomials. We may think of $k[x_1, \ldots, x_n]$ as the functions on $\mathbb{A}^n$. Show as an exercise that there is a surjection

$$\mathrm{Sym}^k(\mathfrak{M}_x/\mathfrak{M}_x^2) \twoheadrightarrow \mathfrak{M}_x^k/\mathfrak{M}_x^{k+1}.$$

Now,

$$A = \mathrm{Sym}^*(\mathfrak{M}_x/\mathfrak{M}_x^2) = \bigoplus_{i \geq 0} \mathrm{Sym}^i(\mathfrak{M}_x/\mathfrak{M}_x^2) \hookrightarrow \bigoplus_{i \geq 0} \mathfrak{M}_x^i/\mathfrak{M}_x^{i+1} = R.$$

We know that $\mathrm{Sym}^*(\mathfrak{M}_x/\mathfrak{M}_x^2)$ is the coordinate algebra of $T_xX = (\mathfrak{M}_x/\mathfrak{M}_x^2)^*$; this corresponds to the fact that $TC_xX \subset T_xX$ (taking coordinate rings is a contravariant functor).