Analytical Solution of the Transverse Field Ising Model

Eliott Rosenberg

November 8, 2022

Here, we will find the ground state energy of the transverse field Ising model. The Hamiltonian is

$$H = -J\left(\sum_i Z_i Z_{i+1} + g\sum_i X_i\right). \tag{1}$$

We will solve this by doing a Jordan-Wigner transformation. Breaking convention slightly, define the Majorana fermions:

$$\begin{aligned} \chi_j &= Z_i \prod_{j<i} X_j \\ \tilde{\chi}_j &= Y_i \prod_{j<i} X_j, \end{aligned} \tag{2}$$

which satisfy $\{\chi_i, \chi_j\} = 2\delta_{ij}$, $\{\tilde{\chi}_i, \chi_j\} = 2\delta_{ij}$, and $\{\tilde{\chi}_i, \tilde{\chi}_j\} = 2\delta_{ij}$, where the curly brackets are anti-commutators. We can write the Hamiltonain in terms of these as

$$H = -iJ\left(\sum_i \tilde{\chi}_i \chi_{i+1} - g\sum_i \tilde{\chi}_i \chi_i\right). \tag{3}$$

Next, we want to find a new set of Majorana fermions in terms of which the Hamiltonian becomes that of free fermions. Note that an orthogonal transformation of Majorana fermions preserves the anticommutation relations. That is, if we define new Majorana fermions $\psi_i = O_{ij}\chi_j$ (with implicit summation over repeated indices), then

$$\begin{aligned} \{\psi_i, \psi_j\} &= O_{ik} O_{jl} \{\chi_k, \chi_l\} \\ &= 2(OO^T)_{ij} \\ &= 2\delta_{ij}, \end{aligned} \tag{4}$$

so the anti-commutation relations are preserved under orthogonal transformations.

Now, we can write the Hamiltonian as

$$H = -i\tilde{\chi}_i \mathcal{H}_{ij} \chi_j, \tag{5}$$

where

$$\mathcal{H}_{ij} = J\delta_{j,i+1} - Jg\delta_{ij}. \tag{6}$$

We want to find a new set of Majorana fermions ψ_i and $\tilde{\psi}_i$ that diagonalize H, i.e. eliminate the hopping term. Let's define

$$\begin{aligned} \chi_i &= O_{ij}\psi_j \\ \tilde{\chi}_i &= \tilde{O}_{ij}\tilde{\psi}_j. \end{aligned} \tag{7}$$

(This is different from our previous O and ψ.) In terms of these, the Hamiltonian becomes

$$\begin{aligned} H &= -i\tilde{O}_{ik}\tilde{\psi}_k \mathcal{H}_{ij} O_{jl}\psi_l \\ &= -i\tilde{\psi}_i (\tilde{O}^T \mathcal{H} O)_{ij} \psi_j. \end{aligned} \tag{8}$$

Next, suppose that we have found the singular value decomposition of $\mathcal{H}$ and can write it as

$$\mathcal{H} = \tilde{O}\Sigma O^T, \tag{9}$$

where Σ is diagonal. (We will denote the diagonal elements of Σ as λ_i.) Then we can write

$$H = -i\sum_i \lambda_i \tilde{\psi}_i \psi_i. \tag{10}$$

To relate this back to more familiar free fermions, define the fermionic creation and annhilation operators:

$$\begin{aligned} a_i &= \frac{1}{2}\left(\psi_i + i\tilde{\psi}_i\right) \\ a_i^\dagger &= \frac{1}{2}\left(\psi_i - i\tilde{\psi}_i\right), \end{aligned} \tag{11}$$

which satisfy the canonical anticommutation relation $\{a_i, a_i^\dagger\} = \delta_{ij}$. The fermionic number operator for species ψ_i is

$$a_i^\dagger a_i = \frac{1}{2}\left(1 - i\tilde{\psi}_i\psi_i\right), \tag{12}$$

so we can finally write the Hamiltonian as

$$H = -\sum_i \lambda_i \left(1 - 2a_i^\dagger a_i\right). \tag{13}$$

The ground state energy is therefore

$$E_0 = -\sum_i \lambda_i. \tag{14}$$

Now, suppose we want the expectation values of the individual ZZ and X terms. To find them, we will write these operators in the ψ basis. We have

$$\begin{aligned} Z_i Z_{i+1} &= i\tilde{\chi}_i \chi_{i+1} \\ &= i\tilde{O}_{ij} O_{i+1,k} \tilde{\psi}_j \psi_k \\ &= \tilde{O}_{ij} O_{i+1,k} (a_j - a_j^\dagger)(a_k + a_k^\dagger) \\ \langle 0|Z_i Z_{i+1}|0\rangle &= \sum_j \tilde{O}_{ij} O_{i+1,j} \\ &= (\tilde{O}O^T)_{i,i+1}. \end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned} X_i &= -i\tilde{\chi}_i \chi_i \\ \langle 0|X_i|0\rangle &= -(\tilde{O}O^T)_{ii}. \end{aligned} \tag{16}$$