Instructional Module

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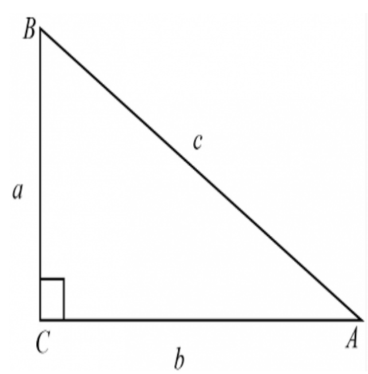
Derivatives of Trigonometric Functions

Introduction

This module will briefly discuss Derivatives of Transcendental Functions particularly Trigonometric Functions. The principles for Trigonometric Functions rely heavily on the principles of Pythagorean Theorem. In the right triangle $\triangle ABC$ shown below,$\angle BCA$ or simply $\angle C$ measures 90° (hence making it a right triangle). The legs of the triangle ABC are the line segments b and a, the lines on either side of $\angle C$, also called *legs.* Meanwhile, the *hypotenuse* of the triangle is the line segment c which is opposite $\angle C$.



The Pythagorean theorem states that the sum of the squares of the measures of the legs a and b of $\triangle ABC$ is equal to the square of the measure of the hypotenuse c that is $a^2 + b^2 = c^2$. By laying the given right triangle $\triangle ABC$ in a unit circle as shown below, the formula $a^2 + b^2 = c^2$ becomes $a^2 + b^2 = 1$. Recall that the radius of a unit circle is equal to 1 unit, which in the triangle shown in the figure above, corresponds to the hypotenuse c.

Trigonometric functions, also known as Circular Functions, can be defined as the functions of an angle of a triangle. Trig functions identify the relationships between the angles and sides of a triangle. The basic trigonometric functions are sine, cosine, tangent, cotangent, secant and cosecant.

The Trigonometric Functions

Solving for the values of the legs a and b of the triangle above leads us to the use of the SOHCAHTOA formulas, which stemmed from the three trigonometric functions sine (sin), cosine (cos), and tangent (tan).

The Sine Function

The *sine* of an angle is the ratio between the length of the opposite side to the length of the hypotenuse. From the above diagram, the value of the sine of $\angle BAC$ will be:

- **Sin A** $= \frac{Opposite}{Hypotenuse} = \frac{CB}{BA}$

The Cosine Function

The *cosine* of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse. From the above diagram, the cos function will be derived as follows:

- **Cos A** $= \frac{Adjacent}{Hypotenuse} = \frac{CA}{BA}$

The Tangent Function

The *tangent function* is the ratio of the length of the opposite side to that of the adjacent side. From the diagram taken above, the tan function will be the following:

- **Tan A** $= \frac{Opposite}{Adjacent} = \frac{CB}{CA}$

The tangent can also be expressed in terms of sin and cos as follows:

- **Tan A** $= \frac{SinA}{CosA} = \frac{CB}{CA}$

Other Trigonometric Formulas

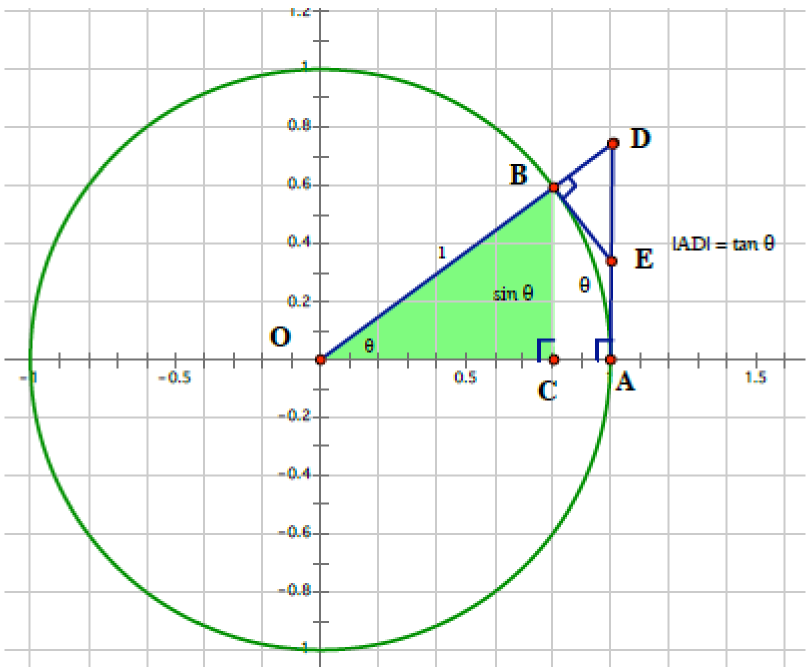
Aside from the initial three functions, there are other trigonometric formulas as shown in the following:

- **Sec A** $= \frac{1}{CosA} = \frac{Hypotenuse}{Adjacent} = \frac{BA}{CA}$
- **Cosec A** $= \frac{1}{SinA} = \frac{Hypotenuse}{Opposite} = \frac{BA}{CB}$
- **cot a** $= \frac{1}{TanA} = \frac{Adjacent}{Opposite} = \frac{CA}{CB}$

Recall that sin$x$ and cos$x$ are defined and continuous everywhere whereas the functions tan$x$, sec$x$, csc$x$, and cot$x$ are continuous on their domains (all values of x where the denominator is non-zero).

Before we dive into the derivatives of trigonometric functions, let us first discuss their limits.

Consider the unit circle given by the figure below.



Derivatives of Trigonometric Functions

Now that we have reviewed trigonometric functions, we are now ready for the derivation of trig functions.

For conformity, we will be using the Leibnitz Notation or Differential Notation throughout this module.

The Leibnitz notation is given by:

$$\frac{df}{dx}$$

or

$$\frac{d}{dx}f(x).$$

Recall that

$$\frac{d}{dx}sin(x) = cos(x)$$

and

$$\frac{d}{dx}cos(x) = -sin(x).$$

To remember which derivative contains the negative sign, recall the graphs of the sine and cosine functions. At *x=0*, *sin(x)* is increasing, and *cos(x)* is positive, thus the derivative is a positive *cos(x)*. Meanwhile, just after *x=0*, *cos(x)* is decreasing, and *sin(x)* is positive, so the derivative must be a negative *sin(x)*.

Knowledge of the derivatives of sine and cosine allows us to find the derivatives of all the other trigonometric functions using the quotient rule.Recall the following identities:

$$tan(x) = \frac{sin(x)}{cos(x)}$$

$$cot(x) = \frac{cos(x)}{sin(x)}$$

$$sec(x) = \frac{1}{cos(x)}$$

$$csc(x) = \frac{1}{sin(x)}$$

Using the above knowledge, let us find the derivative of each function.

Example 1: Find the derivative of $csc(x)$.

Recall that $csc(x) = \frac{1}{sin(x)}$.

Now let us apply the quotient rule.

$$\frac{d}{dx}csc(x) = \frac{sin(x) * 0 - 1 * cos(x)}{sin(x)^2} = -\frac{cos(x)}{sin(x)} * \frac{1}{sin(x)} = -cot(x)csc(x)$$

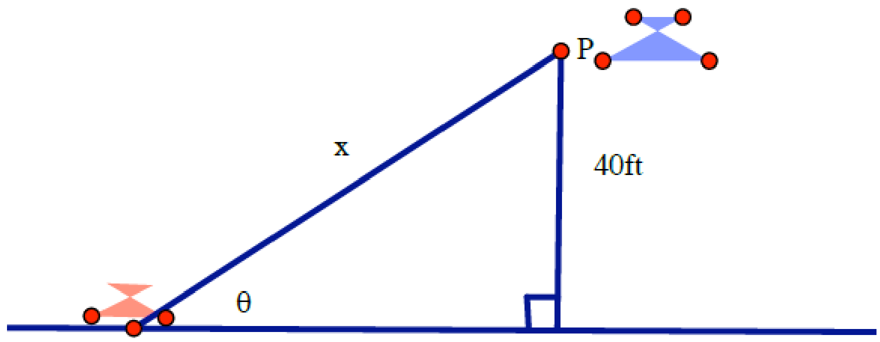
Example 2: Find the derivative of $e^x * cos(x) + x^2$.

Solution: To find this derivative, we will utilize the sum and product rules.

$$\frac{d}{dx}(e^x * cos(x) + x^2) = e^x * cos(x) + e^x * (-sin(x)) + 2x = e^x(cos(x) - sin(x)) + 2x$$

The derivative of trigonometric functions can also be used to solve real life problems. Consider the following problem and its solution.

Example 3: A police car is parked 40 feet from the road at the point P in the diagram below. Your vehicle is approaching on the road as in the diagram below and the police are pointing a radar gun at your car. Let x denote the distance from your car to the police car and let θ be the angle between the line of sight of the radar gun and the road. How fast is x changing with respect to θ when $\theta = \frac{\pi}{4}$?



Solution: From the illustration, we can write:

$$\frac{40}{x} = sin(\theta).$$

Therefore,

$$\frac{40}{sin(\theta)} = x$$

and

$$\frac{dx}{d\theta} = 40 * \frac{-cos(\theta)}{sin^2(\theta)}.$$

When $\theta = \frac{\pi}{4}$,

$\frac{dx}{d\theta} = 40 * \frac{-cos(\frac{\pi}{4})}{sin^2(\frac{\pi}{4})} = 40 * \frac{-1/\sqrt{2}}{1/2} = -40\sqrt{2}$ feet per radian.