ECON 251

Discussion Section

Week 1 Solutions

1. Let $E(X) = 1$, $E(Y) = -3$, $Var(X) = 2$, $Var(Y) = 1$, and $Cov(X, Y) = -1$. Define a new random variable $Z = 2X + Y$. Compute the expected value and variance of Z. What is the covariance of Z with X?

$$E(Z) = E(2X + Y) = 2 \cdot E(X) + E(Y) = 2(1) + (-3) = -1$$

$$Cov(Z, X) = Cov(2X + Y, X) = Cov(2X, X) + Cov(Y, X)$$
$$= 2 \cdot Var(X) + Cov(X, Y) = 2(2) + (-1) = 3$$

2. Let $f_{XY}(x, y)$ be the joint probability distribution function of the random variables (X, Y). Compute the marginal distributions $f_X(x)$ and $f_Y(y)$, conditional distribution $f_{X|Y}(x|y)$, and the conditional expectation function $E(Y|X)$ given the values

$f_{XY}(x,y)$	$y = 0$	$y = 1$	$y = 2$	$y = 3$	$f_X(x)$
$x = 1$	1/8	2/8	1/8	0	$\frac{1}{8}+\frac{2}{8}+\frac{1}{8}=\frac{1}{2}$
$x = 0$	0	1/8	2/8	1/8	$\frac{1}{2}$
$f_Y(y)$	$\frac{1}{8}$	$\frac{2}{8}+\frac{1}{8}=\frac{3}{8}$	$\frac{1}{8}+\frac{2}{8}=\frac{3}{8}$	$\frac{1}{8}$	

$$f_X(x) = \sum_{Y_i} f_{XY}(x, Y_i)$$

$$f_Y(y) = \sum_{X_i} f_{XY}(X_i, y)$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$E(X|Y=y) = 1 \cdot f_{X|Y}(1|y) + 0 \cdot f_{X|Y}(0|y) = f_{X|Y}(1|y) = P(X=1|Y=y)$$

$$= \frac{f_{XY}(1,y)}{f_X(1)} = \frac{f_{XY}(1,y)}{1/2} = \begin{cases} \frac{1}{4} & if\ y=0 \\ \frac{1}{2} & if\ y=1 \\ \frac{1}{4} & if\ y=2 \end{cases}$$

3. Prove that for a sample of size N from some random variable X, the sum of deviations for the mean is always equal to zero. That is, $\sum_{i=1}^{N}(X_i - \overline{X}) = 0$ where $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$. (How would you prove the same holds at the population level as well using expectations?)

$$\sum_{i=1}^{N}(X_i - \overline{X}) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \overline{X} = N\overline{X} - N\overline{X} = 0$$

Note that $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i \Rightarrow N\overline{X} = \sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} \overline{X} = N\overline{X}$ since $\overline{X}$ is a constant.

At the population level, we have

$$E(X - \mu_X) = E(X) - E(\mu_X) = \mu_X - \mu_X = 0$$

Note that $E(X) = \mu_X$ by definition and $E(\mu_X) = \mu_X$ since μ_X is a constant :)

4. Prove that $Var(X) \coloneqq E[(X-\mu_X)^2] = E(X^2) - \mu_X^2$ where $\mu_X \coloneqq E(X)$.

$$Var(X) \coloneqq E[(X-\mu_X)^2] = E[X^2 - 2X\mu_X + \mu_X^2] = E[X^2] - E[2X\mu_X] + E[\mu_X^2]$$

$$= E[X^2] - 2\mu_X \cdot E[X] + \mu_X^2 = E[X^2] - 2\mu_X\mu_X + \mu_X^2 = E[X^2] - \mu_X^2$$

or equivalently $Var(X) = E(X^2) - E(X)^2$ which is why you'll hear me say that variance is just the second moment minus the first moment squared!

5. Let Y be a standard normal random variable and define $X = Y^2$. Are X and Y independent? Are X and Y correlated? True/false: uncorrelated variables are always independent? True/false: independent variables are always uncorrelated.

   The random variables are obviously dependent… if I know Y, then I know X (by squaring) and if I know X, then I know Y (by rooting). The key to this question is that these variables are nonlinearly dependent on one another! The notion of correlation and covariance is only about linear relationships between random variables. This is why even though X and Y are dependent, we find that

$$
\begin{aligned}
Cov(X, Y) = Cov(Y^2, Y) = E(Y^2Y) - E(Y^2)E(Y) \\
= E(Y^3) - E(Y^2)E(Y) \\
= 0 - 1 \cdot 0 = 0
\end{aligned}
$$

   so they are no correlated! Note that I use the fact that for a standard normal random variable $Y \sim N(\mu_Y = 0, \sigma_Y^2 = 1)$ all the odd moments ($E(X), E(X^3), E(X^5), \dots$) are zero.

6. Recall from your micro classes that we say a consumer is "risk averse" if they prefer the expected value of a gamble (wither certainty) over the gamble itself. Suppose your utility function is given by $u(c) = \sqrt{c}$ and I offer you the following gamble: I flip a coin, if it lands heads you get \$8; tails you get \$0. Prove that you shouldn't let me flip the coin.

   Let G denote a random variable representing this gamble; it takes on a value of 8 when the coin lands heads, and a value of 0 otherwise. The expected value is thus

$$
\mu_G \coloneqq E[G] = \frac{1}{2} \cdot 8 + \frac{1}{2} \cdot 0 = 4
$$

   which confers utility $u(E[G]) = \sqrt{E[G]} = \sqrt{4} = 2$. Alternatively, if you take the gamble your *expected utility* is given by

$$E[u(G)] = \frac{1}{2} \cdot u(8) + \frac{1}{2} \cdot u(0) = \frac{1}{2}\sqrt{8} + \frac{1}{2}\sqrt{0} \approx 1.4$$

which proves that you shouldn't let me flip the coin, since $u(E[G]) > E[u(G)]$.

Note that this is an example of the general result that for nonlinear functions $u(\cdot)$ we have $E\big(u(X)\big) \neq u(E(X))$.