Optimal Decision Making: Group Project
Supplementary Information about Games and Graphs

This document provides useful background information on game theory and graph theory.

1 Game Theory

Game theory is a branch of mathematics and economics that studies strategic decision-making in competitive situations. It provides a framework for modeling and analyzing the behavior of rational agents, who aim to maximize their own gains while anticipating the actions of their opponents. Game theory is widely applicable across various disciplines, including economics, political science, psychology, and biology, as it offers insights into cooperation, conflict, and negotiation in complex environments.

1.1 Zero-Sum Games

A zero-sum game formalizes the competitive interaction between two players when the total amount of available rewards is constant—without loss of generality assumed to be zero. This means that any gain by one player comes at an equal loss to the other, resulting in a net reward of zero. Hence, the players' interests are completely opposed—one player's gain directly translates into the other player's loss.

Key concepts in game theory such as Nash equilibria, pure strategies, and mixed strategies are best explained in the context of a zero-sum game. A classical example is a penalty shot in soccer, where the striker (she) must decide whether to shoot left or right, while the goalkeeper (he) chooses whether to vault in either direction. If the goalkeeper jumps in the same direction as the shot, he blocks it, resulting in a payoff of 0 for the striker; if he jumps the other way, the striker scores and earns a payoff of +1.

		keeper: left	keeper: right
striker	left	0	+1
	right	+1	0

Figure 1: Payoff matrix of the striker

Introducing the concepts of pure and mixed strategies is essential for understanding strategic decision-making in game theory. A pure strategy involves a player consistently choosing a single action without any variation. In the penalty shot example, a pure strategy would involve the striker always shooting to the right or the goalkeeper always vaulting to the left. However, this deterministic approach can be easily exploited by an observant opponent.

Mixed strategies, on the other hand, involve players probabilistically choosing between multiple actions, adding an element of unpredictability to their decision-making. By employing a randomization device, such as a die or a random number generator, players can randomize their choices, making it difficult for their opponents to predict their actions. For instance, a striker might decide to shoot to the right with a 60% probability and to the left with a 40% probability, determined by the roll of a die or another randomization device. This randomization, combined with an optimal probability distribution, helps players maximize their expected payoff even when facing an opponent who is fully aware of their strategy. The use of mixed strategies adds a layer of sophistication to game theory analysis and is particularly relevant when discussing Nash equilibria and optimal play in competitive scenarios.

1.2 Nash Equilibria and the Minimax Theorem

If a player adopts a suboptimal strategy, we can take advantage of this. For example, if the striker always shoots to the right, the keeper could simply jump to the right to catch the ball. However, if we assume that the keeper knows our strategy and responds optimally, we must determine our optimal strategy, taking into account the opponent's knowledge and actions. This is where the concept of a Nash equilibrium comes into play. In a Nash equilibrium, each player uses a strategy that maximizes their own payoff, given the strategies chosen by the other players. In a two-player zero-sum game, the Nash equilibrium can be found using the minimax approach. Each player seeks to maximize their minimum possible gain with respect to all possible decisions of the opponent. Indeed, this is exactly what a rational opponent will do to maximize their own gain. This results in a pair of strategies, one for each player, which are optimal under the assumption that the opponent is also playing optimally. In other words, if the opponent is playing optimally given the knowledge of the game and my strategy, what is the optimal action for me? Mathematically, we can find a Nash equilibrium of the zero-sum game using the following recipe, where $\Delta_d = \{p \in \mathbb{R}^d_+ : \sum_{i=1}^d p_i = 1\}$ denotes the probability simplex in $\mathbb{R}^d$.

1. Assume that there are m possible actions for player one and n possible actions for player two. The mixed strategies $x \in \Delta_m$ and $y \in \Delta_n$ of the two players represent probability vectors for randomly selecting among the m and n actions, respectively. That is, x_i (y_j) denotes the probability of player one (two) choosing action i (j). Denote further the payoff matrix by $A \in \mathbb{R}^{m\times n}$, where A_{ij} denotes the payoff of player one if player one chooses action i and player two chooses action j. Thus, it makes sense to call player one the row-player and player two the column player.

2. Recall that the payoff of the column player equals the negative payoff of the row-player. If the row-player selects strategy x and the column player selects an optimal response y, then the row-player's (worst-case) payoff amounts to
$$f(x) = \min_{y\in\Delta_n} x^\top A y.$$

3. If the row-player anticipates that the column player plays the best response to any strategy x, she can compute her Nash strategy by solving the minimax problem
$$p^\star = \max_{x\in\Delta_m} f(x) = \max_{x\in\Delta_m} \min_{y\in\Delta_n} x^\top A y.$$

   The column player proceeds analogously. In addition, we have
$$\begin{aligned}
\max_{x\in\Delta_m} \min_{y\in\Delta_n} x^\top A\, y &= \max_{x\in\Delta_m, \alpha\in\mathbb{R}} \{\alpha : \min_{y\in\Delta_n} x^\top A\, y \geq \alpha\} \\
&= \max_{x\in\Delta_m, \alpha\in\mathbb{R}} \{\alpha : \min_{j=1,\ldots,n} (x^\top A)_j \geq \alpha\} \\
&= \max_{x\in\mathbb{R}^m, \alpha\in\mathbb{R}} \{\alpha : x^\top A \geq \alpha \mathbf{1}^\top,\ \mathbf{1}^\top x = 1,\ x \geq 0\} \\
&= \min_{y\in\mathbb{R}^n, \beta\in\mathbb{R}} \{\beta : Ay \leq \mathbf{1}\beta,\ \mathbf{1}^\top y = 1,\ y \geq 0\} \\
&= \min_{y\in\Delta_n, \beta\in\mathbb{R}} \{\beta : \max_{i=1,\ldots,m} (A\, y)_i \leq \beta\} \\
&= \min_{y\in\Delta_n, \beta\in\mathbb{R}} \{\beta : \max_{x\in\Delta_m} x^\top A\, y \leq \beta\} = \min_{y\in\Delta_n} \max_{x\in\Delta_m} x^\top A\, y,
\end{aligned}$$

   The second and sixth equalities hold because, when optimizing a linear function over the probability simplex, the optimal solution places all weights on a single coordinate. Thus, the smallest (respectively largest) entry of $x^\top A$ (respectively $A\, y$) yields the minimum (respectively maximum) value. The fourth equality then follows from strong linear programming duality. We have thus shown the classical minimax theorem, which asserts that the maximum of the row-player's minimum payoff equals the minimum of the column player's maximum payoff. We refer to $p^\star$ as the value of the zero-sum game because it denotes the payoff a player can guarantee, regardless of the opponent's strategy, when both players play optimally. Note that the only information needed to compute the Nash strategies of both players (that is, the Nash equilibrium) is the payoff matrix A. By construction, neither player can improve their guaranteed payoff by unilaterally changing their strategy. Hence, for two-player zero-sum games, computing the minimax solution directly gives us the Nash equilibrium. This result underpins many real-world problems in competitive environments, from sports tactics to security models and beyond.

2 Graph Theory

An undirected graph $G = (V, E)$ is an ordered pair that consists of a set of finitely many vertices V (also called nodes) and a set of edges E. All edges correspond to unordered pairs $\{u, v\}$ of vertices $u, v \in V$. The incidence matrix $I \in \mathbb{R}^{n \times m}$ of a graph with n vertices $v_1, \ldots, v_n$ and m edges $e_1, \ldots, e_m$ is defined through $I_{ij} = 1$ if the e_j is incident on vertex v_i (*i.e.*, if v_i is an endpoint of e_j) and $I_{ij} = 0$ otherwise. A bipartite graph is a graph whose vertices can be split into two disjoint sets such that every edge connects one vertex in the first set with one vertex in the second set. Thus, $V = X \cup Y$ with $X \cap Y = \emptyset$, with edges running only between X and Y but never within X or within Y. Bipartite graphs naturally emerge in pairing or assignment problems, which seek to assign workers to jobs, students to dorm rooms, factories to suppliers, and so on.

Let $G = (X \cup Y, E)$ be the bipartite graph shown in Figure 2 with vertex sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ and with edges $e_1 = \{x_1, y_2\}$, $e_2 = \{x_2, y_1\}$, $e_3 = \{x_2, y_2\}$, $e_4 = \{x_2, y_3\}$, and $e_5 = \{x_3, y_2\}$. Relabeling the vertices as $v_1 = x_1$, $v_2 = x_2$, $v_3 = x_3$, $v_4 = y_1$, $v_5 = y_2$ and $v_6 = y_3$, one readily verifies that the incidence matrix $I \in \mathbb{R}^{6 \times 5}$ of G is given by

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

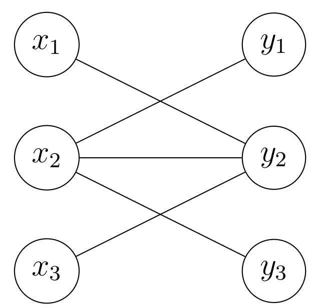

Figure 2: Example of a bipartite graph

A *matching* in a graph $G = (V, E)$ is a subset of edges $E' \subseteq E$ such that no two edges in E' share a vertex. Equivalently, each vertex in V is incident to at most one edge in E'. In the example of Figure 2, both $E'_1 = \{e_1\}$ and $E'_2 = \{e_1, e_4\}$ qualify as matchings. A *maximum matching* is one with the largest possible number of edges among all matchings in G. In our example, E'_2 constitutes a maximum matching, while E'_1 does not.

A *vertex cover* is a set of vertices such that every edge in the graph has at least one endpoint in this set. In other words, these chosen vertices "touch" all the edges. Formally, a vertex cover of $G = (V, E)$ is a set $V' \subseteq V$ of vertices with the property that every edge in E has at least one endpoint in V'. In our example, $V'_1 = \{x_1, x_2, y_2\}$ and $V'_2 = \{x_2, y_2\}$ are both valid vertex covers. A *minimum vertex cover* is one with the smallest number of vertices among all vertex covers of G. In our example, V'_2 is a minimum vertex cover because it uses the fewest possible vertices.

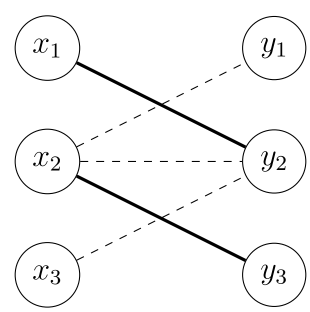

Figure 3: Visualization of $E'_2 = \{e_1, e_4\}$