

HW 2-PART 2

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PART2

###Point a and b :Online Expert Aggregation on ACI (AgACI)

The **AgACI** method involves running the same **ACI algorithm** multiple times, each with different values of γ (denoted γ_k for $k \in \{1, \dots, K\}$), which generates K distinct values of $\alpha_t^{(k)}$ at each time t . These values are then combined to form the final prediction interval.

Each value of γ_k corresponds to an “expert,” and the goal is to aggregate the prediction bounds from each expert into a final set of bounds. This is done by calculating a **weighted mean** of the K intervals, with weights determined by each expert’s historical performance. Specifically, the weights are derived using the **pinball loss function** (ρ_β), which quantifies how well each expert’s prediction matches the true outcome.

Inputs required:

- the miscoverage rate α
- a grid of values for γ $\gamma_k, k \in [1, K]$
- the aggregation rule Φ (we can choose among different ones in R) and it can include the computation of the gradients of the loss (gradient trick)
- threshold values M for the lower and the upper bounds (since aggregation rules require bounded experts)

At each temporal step the algorithm performs two independent aggregations of the K-ACI intervals

$$\hat{C}_{\alpha,t}(\cdot) = [\hat{b}_{t,k}^{(l)}(\cdot), \hat{b}_{t,k}^{(u)}(\cdot)]$$

and outputs

$$\tilde{C}_t(\cdot) = [\tilde{b}_t^{(l)}(\cdot), \tilde{b}_t^{(u)}(\cdot)]$$

Weights are computed as:

$$\omega_{t,k}^{(\cdot)} = \Phi\left(\rho_{\beta^{(\cdot)}}(y_s, \hat{b}_{s,l}^{(\cdot)}(x_s)), s \in [T_0 + 1, t], l \in [1, k]\right)$$

where $\beta^{(l)}$ is set to $\frac{\alpha}{2}$ and $\beta^{(u)}$ is set to $1 - \frac{\alpha}{2}$

As aggregation rule, the authors have chosen The Bernstein Online Aggregation (BOA) procedure. The weights outputted by BOA have an exponential form. In the exponent is plugged the difference between the loss suffered by the last aggregated forecast and the current squared loss suffered by the expert.

In order to adapt this strategy to build the prediction intervals of our eGARCH(1,1)-std model, what we present below is a version inspired by, but slightly different from, that of the AgACI just described. The inputs are the same ones we use in the original Agaci. Instead of building the lower and upper bounds of

the prediction interval for each expert and then building the “aggregate” bounds, we focus directly on the α_t . Each expert produces an $\alpha_t^{(k)}$ and what we want to do is create an “aggregate” $\tilde{\alpha}_t$ given by the weighted average of the $\alpha_t^{(k)}$. The weights of each expert are calculated with a BOA aggregation and with an MSE loss function. The arguments of the MSE are a constant vector with elements equal to the miscoverage rate α and the cumulative average of the errors of each expert k :

$$\hat{e}_t^{(k)} = \frac{1}{t} \sum_{s=1}^t \text{err}_s^{(k)}.$$

In this way we give more weight to the experts who guarantee us a better coverage rate (i.e. close to α) over time.

The aggregate estimate looks like this:

$$\tilde{\alpha}_{t+1} = \sum_{g=1}^K w_t^{(g)} \alpha_{t+1}^{(g)}.$$

once we have $\tilde{\alpha}_t$, we calculate the error of the aggregate $\text{err}_t^{(\text{agg})}$ which will then be compared with the errors of the experts $\text{err}_t^{(k)}$.

```
library(quantmod)
```

```
## Caricamento del pacchetto richiesto: xts
```

```
## Caricamento del pacchetto richiesto: zoo
```

```
##  
## Caricamento pacchetto: 'zoo'
```

```
## I seguenti oggetti sono mascherati da 'package:base':  
##  
##      as.Date, as.Date.numeric
```

```
## Caricamento del pacchetto richiesto: TTR
```

```
## Registered S3 method overwritten by 'quantmod':  
##      method      from  
##      as.zoo.data.frame zoo
```

```
library(rugarch)
```

```
## Warning: il pacchetto 'rugarch' è stato creato con R versione 4.4.2
```

```
## Caricamento del pacchetto richiesto: parallel
```

```
##  
## Caricamento pacchetto: 'rugarch'
```

```
## Il seguente oggetto è mascherato da 'package:stats':  
##  
##      sigma
```

```
library(opera)
```

```
## Warning: il pacchetto 'opera' è stato creato con R versione 4.4.2
```

```
library(dplyr)
```

```
##  
## ##### Warning from 'xts' package #####  
## #  
## # The dplyr lag() function breaks how base R's lag() function is supposed to #  
## # work, which breaks lag(my_xts). Calls to lag(my_xts) that you type or #  
## # source() into this session won't work correctly. #  
## #  
## # Use stats::lag() to make sure you're not using dplyr::lag(), or you can add #  
## # conflictRules('dplyr', exclude = 'lag') to your .Rprofile to stop #  
## # dplyr from breaking base R's lag() function. #  
## #  
## # Code in packages is not affected. It's protected by R's namespace mechanism #  
## # Set `options(xts.warn_dplyr_breaks_lag = FALSE)` to suppress this warning. #  
## #  
## #####
```

```
##  
## Caricamento pacchetto: 'dplyr'
```

```
## I seguenti oggetti sono mascherati da 'package:xts':  
##  
##      first, last
```

```
## I seguenti oggetti sono mascherati da 'package:stats':  
##  
##      filter, lag
```

```
## I seguenti oggetti sono mascherati da 'package:base':  
##  
##      intersect, setdiff, setequal, union
```

```
agaciVolatilityForecasting <- function(returns, alpha = 0.05,  
                                       gammaGrid = seq(0.001, 0.01, by = 0.001),
```

```

lookback = 1250, garchP = 1, garchQ = 1,
startUp = 100, verbose = FALSE,
updateMethod = "Momentum", momentumBW = 0.9

```

```

5) {
  myT <- length(returns)
  T0 <- max(startUp, lookback)

  garchSpec <- ugarchspec(
    mean.model = list(armaOrder = c(0, 0), include.mean = FALSE),
    variance.model = list(model = "eGARCH", garchOrder = c(garchP, garchQ)),
    distribution.model = "std" # Heavy-tailed distribution
  )

  # Initialize storage variables
  K <- length(gammaGrid) # Number of "experts"
  Tlen <- myT - T0

  # Vector to store GARCH (variance) predictions at each step
  garchForecastVec <- rep(NA, Tlen)

  scores <- array(NA, dim = Tlen)
  alphaSeq <- matrix(alpha, nrow = K, ncol = Tlen)

  alphaSeq2 <- rep(alpha, Tlen) # Base alpha for the BOA part

  errSeqOC <- matrix(NA, nrow = K, ncol = Tlen) # Matrix for errors of each expert
  aggregatedalpha <- rep(NA, Tlen)
  err <- rep(NA, Tlen) # Vector for aggregated errors

  lower_limit <- 0.001
  upper_limit <- 0.999

  # 1) Interval lengths for experts (Tlen x K)
  intervalLengths <- matrix(NA, nrow = Tlen, ncol = K)

  # 2) Interval lengths for the aggregate (Tlen)
  intervalLengths2 <- rep(NA, Tlen)

  # === 1) Compute predictions using the ACI logic ===
  for (t in T0:(myT - 1)) {

    # Fit GARCH and forecast
    garchFit <- ugarchfit(garchSpec, returns[(t - lookback + 1):(t - 1)], solver = "hybrid")
    sigmaNext <- sigma(ugarchforecast(garchFit, n.ahead = 1))
    garchForecast <- sigmaNext^2 # GARCH forecast

    colIdx <- t - T0 + 1

    # Save GARCH forecast
    garchForecastVec[colIdx] <- garchForecast

    # Conformity score

```

```

scores[colIdx] <- abs(returns[t]^2 - sigmaNext^2) / sigmaNext^2

# Historical scores
historical <- scores[max(1, colIdx - lookback + 1):(colIdx - 1)]

# Compute thresholds and update alpha for each expert
for (g in seq_along(gammaGrid)) {
  gamma <- gammaGrid[g]

  if (t == T0) {
    thr <- quantile(historical, 1 - alpha)
    errSeqOC[g, colIdx] <- as.numeric(scores[colIdx] > thr)
    intervalLengths[colIdx, g] <- 2 * thr

  } else {
    thr <- quantile(historical, 1 - alphaSeq[g, colIdx - 1])
    errSeqOC[g, colIdx] <- as.numeric(scores[colIdx] > thr)
    intervalLengths[colIdx, g] <- 2 * thr

    # Update alphaSeq
    if (updateMethod == "Simple") {
      alphaSeq[g, colIdx] <- alphaSeq[g, colIdx - 1] + gamma * (alpha - errSeq
OC[g, colIdx])

    } else if (updateMethod == "Momentum") {
      w <- rev(momentumBW^(1:(colIdx - 1)))
      w <- w / sum(w)
      alphaSeq[g, colIdx] <- alphaSeq[g, colIdx - 1] + gamma * (alpha - sum(er
rSeqOC[g, colIdx] * w))
    }

    # Clipping
    alphaSeq[g, colIdx] <- min(upper_limit, max(lower_limit, alphaSeq[g, colId
x]))
  }
}

# Cumulative mean for each column
cummean_cols <- apply(errSeqOC, 2, cummean)

# === 2) Aggregation using BOA ===
aggalpha <- mixture(
  Y      = alphaSeq2,
  experts = t(cummean_cols),
  model  = "BOA",
  loss.gradient = TRUE,
  loss.type = 'square'
)

print(aggalpha$weights)

# Convert weights to a data frame for readable output

```

```

weights_df <- as.data.frame(aggalpha$weights)
colnames(weights_df) <- paste0("Expert_", 1:K) # Names for experts
rownames(weights_df) <- paste0("T", 1:nrow(weights_df)) # Names for time steps

# Print the first 10 rows as an example
print(head(weights_df, 10))

# Compute aggregate and error vector for the aggregate
for (t in T0:(myT - 1)) {
  colIdx <- t - T0 + 1

  historical <- scores[max(1, colIdx - lookback + 1):(colIdx - 1)]

  if (colIdx == 1) {
    aggregatedalpha[colIdx] <- alpha
    thr <- quantile(historical, 1 - aggregatedalpha[colIdx])
    intervalLengths2[colIdx] <- 2 * thr
    err[colIdx] <- as.numeric(scores[colIdx] > thr)

  } else {
    w_t <- aggalpha$weights[colIdx - 1, ]
    aggregatedalpha[colIdx] <- sum(w_t * alphaSeq[, colIdx])

    thr <- quantile(historical, 1 - aggregatedalpha[colIdx])
    intervalLengths2[colIdx] <- 2 * thr
    err[colIdx] <- as.numeric(scores[colIdx] > thr)
  }
}

# Statistics on interval lengths

# 1) Mean lengths for each expert
avgIntervalLengthsExperts <- colMeans(intervalLengths, na.rm = TRUE)
# 2) Mean lengths for the aggregate
avgIntervalLengthsAgg <- mean(intervalLengths2, na.rm = TRUE)

# 3) Median lengths for each expert
medIntervalLengthsExperts <- apply(intervalLengths, 2, median, na.rm = TRUE)
# 4) Median lengths for the aggregate
medIntervalLengthsAgg <- median(intervalLengths2, na.rm = TRUE)

# Return results
return(list(
  aggregatedalpha      = aggregatedalpha,
  err                  = err,
  errSeqOC             = errSeqOC,
  garchForecastVec     = garchForecastVec,
  alphaSeq             = alphaSeq,
  scores               = scores,
  weights_df           = weights_df,

  intervalLengths      = intervalLengths,
  intervalLengths2     = intervalLengths2,

```

```

# Means
avgIntervalLengthsExperts = avgIntervalLengthsExperts,
avgIntervalLengthsAgg     = avgIntervalLengthsAgg,

# Medians
medIntervalLengthsExperts = medIntervalLengthsExperts,
medIntervalLengthsAgg     = medIntervalLengthsAgg
))
}

getSymbols("^GSPC", from = "2015-01-01", to = "2023-12-31") # Retrieve S&P 500 data

```

```
## [1] "GSPC"
```

```

sp500 <- Cl(GSPC) # Extract closing prices

returns <- dailyReturn(sp500) # Compute daily returns
returns <- na.omit(returns) # Remove missing values

gammaGrid = seq(0.001, 0.013, by = 0.003) # Define the grid of gamma values
print(gammaGrid)

```

```
## [1] 0.001 0.004 0.007 0.010 0.013
```

```

# Run the AGACI volatility forecasting function
results <- agaciVolatilityForecasting(
  returns    = returns,
  alpha      = 0.01,
  gammaGrid  = gammaGrid,
  lookback   = 1250,
  garchP     = 1,
  garchQ     = 1,
  startUp    = 100,
  verbose    = TRUE,
  updateMethod = "Simple",
  momentumBW = 0.95
)

```

```

##           X1           X2           X3           X4           X5
## [1,] 0.2000000 0.2000000 0.2000000 0.2000000 0.2000000
## [2,] 0.2000000 0.2000000 0.2000000 0.2000000 0.2000000
## [3,] 0.2000000 0.2000000 0.2000000 0.2000000 0.2000000
## [4,] 0.2000000 0.2000000 0.2000000 0.2000000 0.2000000
## [5,] 0.2000000 0.2000000 0.2000000 0.2000000 0.2000000
## [6,] 0.2000000 0.2000000 0.2000000 0.2000000 0.2000000
## [7,] 0.2000000 0.2000000 0.2000000 0.2000000 0.2000000
## [8,] 0.2000000 0.2000000 0.2000000 0.2000000 0.2000000
## [9,] 0.2000000 0.2000000 0.2000000 0.2000000 0.2000000

```

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

```
## [982,] 0.4760547 0.3850668 0.04942849 0.05515734 0.03429265
## [983,] 0.4760547 0.3850668 0.04942849 0.05515734 0.03429265
## [984,] 0.4760547 0.3850668 0.04942849 0.05515734 0.03429265
## [985,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [986,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [987,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [988,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [989,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [990,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [991,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [992,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [993,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [994,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [995,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [996,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [997,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [998,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [999,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [1000,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [1001,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [1002,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [1003,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [1004,] 0.4765079 0.3858026 0.04952294 0.05472215 0.03344442
## [1005,] 0.5888997 0.3025579 0.03301926 0.04085190 0.03467123
## [1006,] 0.5888997 0.3025579 0.03301926 0.04085190 0.03467123
## [1007,] 0.5888997 0.3025579 0.03301926 0.04085190 0.03467123
## [1008,] 0.5888997 0.3025579 0.03301926 0.04085190 0.03467123
## [1009,] 0.5888997 0.3025579 0.03301926 0.04085190 0.03467123
## [1010,] 0.5750374 0.2911105 0.03193878 0.03989027 0.06202306
## [1011,] 0.5750374 0.2911105 0.03193878 0.03989027 0.06202306
## [1012,] 0.5750374 0.2911105 0.03193878 0.03989027 0.06202306
## [1013,] 0.5750374 0.2911105 0.03193878 0.03989027 0.06202306
## [1014,] 0.5750374 0.2911105 0.03193878 0.03989027 0.06202306
##      Expert_1 Expert_2 Expert_3 Expert_4 Expert_5
## T1          0.2      0.2      0.2      0.2      0.2
## T2          0.2      0.2      0.2      0.2      0.2
## T3          0.2      0.2      0.2      0.2      0.2
## T4          0.2      0.2      0.2      0.2      0.2
## T5          0.2      0.2      0.2      0.2      0.2
## T6          0.2      0.2      0.2      0.2      0.2
## T7          0.2      0.2      0.2      0.2      0.2
## T8          0.2      0.2      0.2      0.2      0.2
## T9          0.2      0.2      0.2      0.2      0.2
## T10         0.2      0.2      0.2      0.2      0.2
```

```
myT <- length(returns)
T0   <- 1250
K    <- length(gammaGrid)
print(K)
```

```
## [1] 5
```

```
date <- index(returns)[T0 : (myT - 1)]
```

```
# Print results
```

```
cat("\nMean interval lengths for each expert:\n")
```

```
##
```

```
## Mean interval lengths for each expert:
```

```
print(results$avgIntervalLengthsExperts)
```

```
## [1] 42.27133 29.23043 31.16759 26.43918 24.42173
```

```
cat("\nMean interval length for the aggregate:\n")
```

```
##
```

```
## Mean interval length for the aggregate:
```

```
print(results$avgIntervalLengthsAgg)
```

```
## [1] 29.86828
```

```
cat("\nMedian interval lengths for each expert:\n")
```

```
##
```

```
## Median interval lengths for each expert:
```

```
print(results$medIntervalLengthsExperts)
```

```
## [1] 29.26440 20.79933 22.22856 17.69180 14.40585
```

```
cat("\nMedian interval length for the aggregate:\n")
```

```
##
```

```
## Median interval length for the aggregate:
```

```
print(results$medIntervalLengthsAgg)
```

```
## [1] 21.49677
```

Mean interval lengths for each expert: [1] 42.27132 29.23036 31.16750 26.43909 24.42164

Mean interval length for the aggregate: [1] 29.86819

Median interval length for the aggregate: [1] 21.49678

[illegible]

```

0
## [852] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0
## [889] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0
## [926] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0
## [963] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0
## [1000] 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0

```

```
mean(err)
```

```
## [1] 0.01084813
```

```
# Print the mean error for each expert
for (k in 1:K) {
  print(mean(errSeqOC[k,-1]))
}
```

```
## [1] 0.011846
## [1] 0.0148075
## [1] 0.01678184
## [1] 0.017769
## [1] 0.017769
```

```
par(mfrow = c(1, 1))

# Distinct colors for each expert
colors <- rainbow(K)

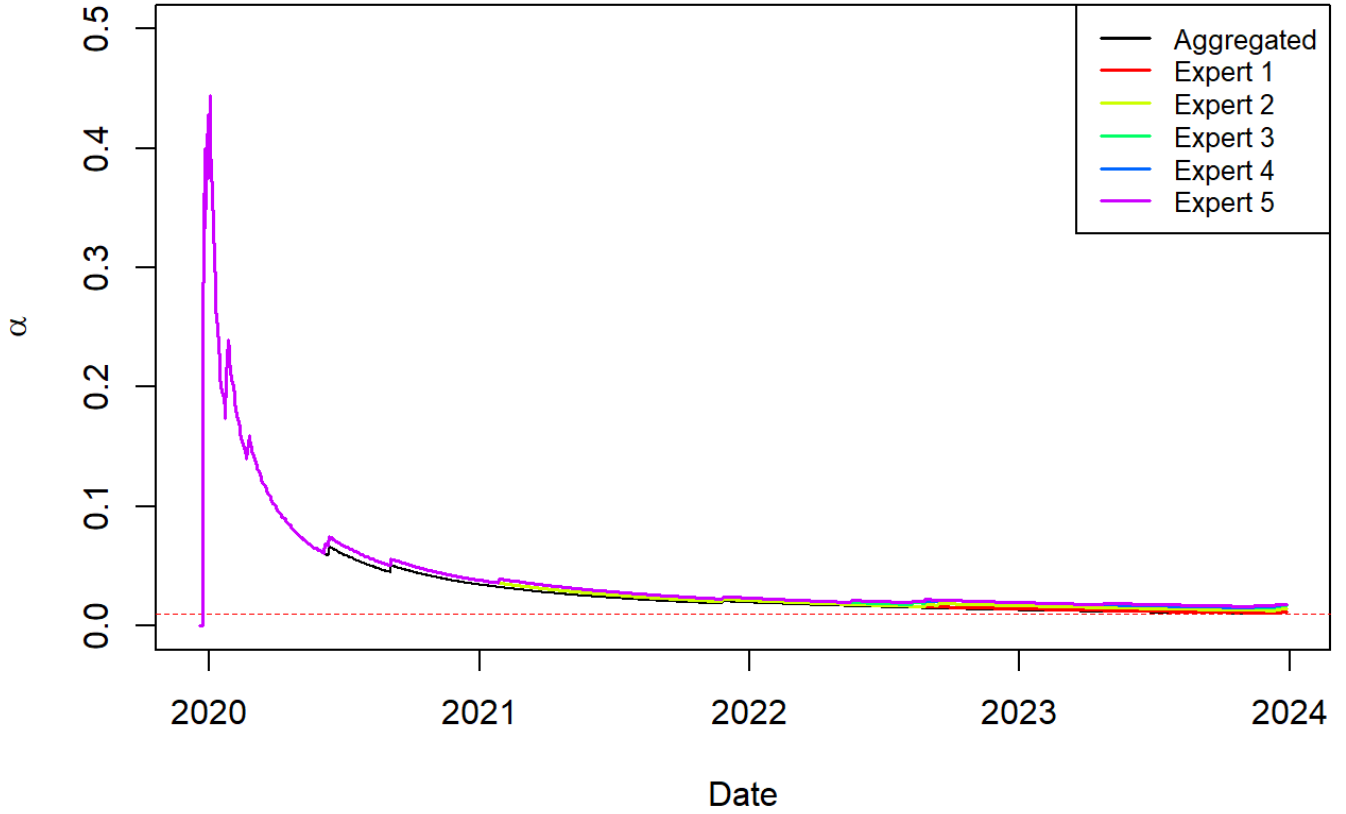
# Create an empty plot for all errors
plot(date[-1], cummean(err[-1]), type = "l", col = "black",
      ylim = c(0, 0.5),
      main = "Cumulative errors for experts and aggregate ",
      xlab = "Date", ylab = expression(alpha))

# Add a horizontal line for the threshold
abline(h = 0.01, col = "red", lwd = 0.3, lty = 2)

# Add lines for each expert with distinct colors
for (k in 1:K) {
  lines(date[-1], cummean(errSeqOC[k,-1]), col = colors[k], lwd = 1.5)
}

# Add a legend to distinguish experts
legend("topright", legend = c("Aggregated", paste("Expert", 1:K)),
      col = c("black", colors), lty = 1, lwd = 1.5, cex = 0.8)
```

Cumulative errors for experts and aggregate



Here we showed the results for $\alpha = 0.001$, but in our conclusions we consider also the case for $\alpha = 0.005$ (if you want to verify change the α in the code above).

Comments on results:

Results for $\alpha = 0.005$: The plot shows that the aggregate offers better coverage than all experts. However, the Median and Average Interval Lengths of the aggregate, although not the lowest ones in absolute terms, are still among the lowest ones. These results emphasize how the adopted strategy effectively combines the strengths of each expert, yielding a superior prediction interval compared to what a single expert can provide.

Results for $\alpha = 0.001$: the conclusions remain the same as before.

The Naive Strategy

The **naive strategy** is an adaptive method designed to dynamically select the value of γ based on the observed efficiency and validity of predictive intervals constructed in previous steps. The strategy is applied after an initial “warm-up” period and involves calculating the optimal γ from a user-defined grid of values. Below is a detailed explanation:

The selection of the optimal γ is based on past performance, particularly after a **warm-up period**. During the first T_w steps, the strategy operates arbitrarily, using a predefined value (e.g., $\gamma = 0$) instead of selecting the optimal γ . This warm-up phase allows the system to “warm-up” and gather enough data for later choosing an appropriate value for γ . Once the warm-up period ends, i.e., for each $t \geq T_0 + T_w$, the strategy begins selecting γ_{k^*} based on past performance, aiming to minimize the length of the predictive intervals while ensuring their validity. The selection of γ_{k^*} is formalized as follows:

$$k_{t+1}^* \in \arg \min_{k \in A_t} \left(t^{-1} \sum_{s=1}^t \text{length}(C_{\alpha_{s,k}}(x_s)) \right),$$

where

$$A_t = \left\{ k \in [1, K] \mid t^{-1} \sum_{s=1}^t \mathbb{I}_{y_s \in C_{\alpha_{s,k}}(x_s)} \geq 1 - \alpha \right\}$$

represents the set of candidate values for γ_k . This condition ensures the validity of the predictive intervals by maintaining the miscoverage below a threshold $(1 - \alpha)$. If $A_t = \emptyset$ instead, we take γ_{k^*} as:

$$k_{t+1}^* \in \arg \min_{k \in [1, K]} (|1 - \alpha - t^{-1} \sum_{s=1}^t \mathbb{I}_{y_s \in C_{\alpha_{s,k}}(x_s)}|),$$

```
library(quantmod)
library(PerformanceAnalytics)
```

```
## Warning: il pacchetto 'PerformanceAnalytics' è stato creato con R versione
## 4.4.2
```

```
##
## Caricamento pacchetto: 'PerformanceAnalytics'
```

```
## Il seguente oggetto è mascherato da 'package:graphics':
##
##      legend
```

```
library(rugarch)
library(dplyr)
#Download the historical data of Microsoft (MSFT) from 2015 to 2023
getSymbols("MSFT", from = "2015-01-01", to = "2023-12-31")
```

```
## [1] "MSFT"
```

```
msft <- Cl(MSFT)      # Estrai i prezzi di chiusura

# Daily returns calculation
returns <- dailyReturn(msft)
returns <- na.omit(returns) # Rimuovo eventuali valori NA

garchNaiveStrategy <- function(returns, alpha = 0.05,
                                gammaGrid = seq(0.001, 0.01, by = 0.001),
                                lookback = 1250, garchP = 1, garchQ = 1,
                                startUp = 100, warmUp = 50,
                                garchType = "sGARCH", verbose = FALSE) {
```

```

myT <- length(returns)
T0 <- max(startUp, lookback)

# Prepare the matrices and support vectors
nGamma <- length(gammaGrid)
nSteps <- myT - T0 + 1

scores      <- matrix(0, nrow = nSteps, ncol = nGamma)
errSeq      <- matrix(0, nrow = nSteps, ncol = nGamma)
alphaSeq    <- matrix(alpha, nrow = nSteps, ncol = nGamma)

# Interval length for each gamma at each step: (nSteps x nGamma)
intervalLengths <- matrix(Inf, nrow = nSteps, ncol = nGamma)

# For the selected gamma
selectedErrSeq      <- rep(0, nSteps)
selectedIntervalLengths <- rep(0, nSteps)
selectedAlphaSeq    <- rep(0, nSteps)
bestGammaIndex      <- rep(1, nSteps)

# GARCH model specification
garchSpec <- ugarchspec(
  mean.model = list(armaOrder = c(0, 0), include.mean = FALSE),
  variance.model = list(model = garchType, garchOrder = c(garchP, garchQ)),
  distribution.model = "std"
)

# =====
# Loop principale
# =====
for (t in T0:myT) {

  #GARCH model fitting
  garchFit <- ugarchfit(garchSpec, returns[(t - lookback + 1):(t - 1)], solver
= "hybrid")
  sigmaNext <- sigma(ugarchforecast(garchFit, n.ahead = 1))

  # Conformity score calculation
  # (e.g., relative difference between realized variance and forecast variance)
  for (k in seq_along(gammaGrid)) {
    gamma <- gammaGrid[k]

    # step index
    stepIdx <- t - T0 + 1

    scores[stepIdx, k] <- abs(returns[t]^2 - sigmaNext^2) / sigmaNext^2

    # Subsample of recent scores for quantile calculation
    recentScores <- scores[max(stepIdx - lookback + 1, 1):stepIdx, k]

    # Threshold estimation
    if (t == T0) {
      # Use the 'fixed' alpha value in the first step

```



```

    qValue <- quantile(recentScores, 1 - alpha, na.rm = TRUE)
  } else {
    # In the next step, we use the updated alphaSeq
    qValue <- quantile(recentScores, 1 - alphaSeq[stepIdx - 1, k], na.rm = TRUE)
  }

  # Calculation of error and interval length.
  errSeq[stepIdx, k] <- as.numeric(scores[stepIdx, k] > qValue)
  intervalLengths[stepIdx, k] <- 2 * qValue # intervallo +/- qValue

  # Update alphaSeq for the next step.
  if (stepIdx >= 2) {
    alphaSeq[stepIdx, k] <- alphaSeq[stepIdx - 1, k] + gamma * (alpha - errSeq[stepIdx, k])
    # Clipping to avoid values less than 0 or greater than 1.
    alphaSeq[stepIdx, k] <- min(1, max(0, alphaSeq[stepIdx, k]))
  }
}

# -----
# Selection of the optimal gamma
# -----
stepIdx <- t - T0 + 1

if (t < T0 + warmUp) {
  # During the warm-up: we always use gammaGrid[1], for example.
  gammaIdx <- 1
} else {
  # After the warm-up, we select the gammas that have had average errors <= alpha so far
  # Among these, we select the one with the minimum average length
  validGammas <- which(colMeans(errSeq[1:stepIdx, , drop = FALSE]) <= alpha)

  if (length(validGammas) > 0) {
    # Among the valid gammas, I take the one with the smallest average interval
    mediaLunghezze <- colMeans(intervalLengths[1:stepIdx, validGammas, drop = FALSE])
    bestIndex <- which.min(mediaLunghezze)
    gammaIdx <- validGammas[bestIndex]
  } else {
    # "If there is no gamma with an average error <= alpha, I take the gamma with the average error closest to alpha
    mediaErrori <- colMeans(errSeq[1:stepIdx, , drop = FALSE])
    gammaIdx <- which.min(abs(mediaErrori - alpha))
  }
}

bestGammaIndex[stepIdx] <- gammaIdx

# Construction of the interval with the selected gamma
selectedRecentScores <- scores[max(stepIdx - lookback + 1, 1):stepIdx, gammaIdx]
x]

```

```

if (t == T0) {
  # First step
  qValue <- quantile(selectedRecentScores, 1 - alpha, na.rm = TRUE)
  selectedAlphaSeq[stepIdx] <- alpha
} else {
  qValue <- quantile(selectedRecentScores, 1 - alphaSeq[stepIdx - 1, gammaIdx]
, na.rm = TRUE)
  selectedAlphaSeq[stepIdx] <- alphaSeq[stepIdx - 1, gammaIdx]
}

selectedErrSeq[stepIdx]          <- as.numeric(scores[stepIdx, gammaIdx] > qVa
lue)
selectedIntervalLengths[stepIdx] <- 2 * qValue
}

# =====
# Calculation of the average interval lengths
# =====
# 1) Average for each gamma (vector of length = nGamma)
avgIntervalLenPerGamma <- colMeans(intervalLengths, na.rm = TRUE)

# 2) Average interval for the selected gamma
avgIntervalLenSelected <- mean(selectedIntervalLengths, na.rm = TRUE)

# =====
# Calculation of the medians of interval lengths
# =====
# 1) Median for each gamma
medianIntervalLenPerGamma <- apply(intervalLengths, 2, median, na.rm = TRUE)

# 2) Median of the interval for the selected gamma
medianIntervalLenSelected <- median(selectedIntervalLengths, na.rm = TRUE)

# If we want to print (if verbose = TRUE):
if (verbose) {
  cat("\n=====\\n")
  cat("Average interval length for EACH gamma:\\n")
  print(avgIntervalLenPerGamma)

  cat("\\nAverage interval length for the SELECTED gamma:\\n")
  print(avgIntervalLenSelected)

  cat("\\n---\\n\\nMedian interval length for EACH gamma:\\n")
  print(medianIntervalLenPerGamma)

  cat("\\nMedian interval length for the SELECTED gamma:\\n")
  print(medianIntervalLenSelected)
  cat("=====\\n")
}

# =====
# Final result

```

```
# =====
return(list(
  alphaSeq          = alphaSeq,
  errSeq            = errSeq,
  intervalLengths   = intervalLengths,
  bestGammaIndex    = bestGammaIndex,
  selectedErrSeq     = selectedErrSeq,
  selectedIntervalLengths = selectedIntervalLengths,
  selectedAlphaSeq   = selectedAlphaSeq,

  # Medie su queste lunghezze
  avgIntervalLenPerGamma = avgIntervalLenPerGamma,
  avgIntervalLenSelected = avgIntervalLenSelected,

  # Medianas of these lengths
  medianIntervalLenPerGamma = medianIntervalLenPerGamma,
  medianIntervalLenSelected = medianIntervalLenSelected
))
}

results <- garchNaiveStrategy(
  returns = returns,
  alpha   = 0.05,
  gammaGrid = seq(0.001, 0.010, by = 0.001),
  lookback = 1250,
  garchP    = 1,
  garchQ    = 1,
  startUp   = 100,
  warmUp    = 50,
  verbose   = TRUE
)
```

```
##
## =====
## Average interval length for EACH gamma:
## [1] 6.746173 6.901056 7.202726 7.239034 7.460935 7.578465 7.649217 7.606568
## [9] 7.823446 7.872616
##
## Average interval length for the SELECTED gamma:
## [1] 7.194258
##
## ---
##
## Median interval length for EACH gamma:
## [1] 6.873996 6.938913 6.976310 6.970595 7.006243 7.123205 6.968306 6.932987
## [9] 6.962098 6.955754
##
## Median interval length for the SELECTED gamma:
## [1] 6.829778
## =====
```

```
# Visualization of the results
date <- index(returns)[1250:length(index(returns))]
cat("\n===== \n")
```

```
##
## =====
```

```
cat("Average interval length for EACH gamma:\n")
```

```
## Average interval length for EACH gamma:
```

```
print(results$avgIntervalLenPerGamma)
```

```
## [1] 6.746173 6.901056 7.202726 7.239034 7.460935 7.578465 7.649217 7.606568
## [9] 7.823446 7.872616
```

```
cat("\nAverage interval length for the SELECTED gamma:\n")
```

```
##
## Average interval length for the SELECTED gamma:
```

```
print(results$avgIntervalLenSelected)
```

```
## [1] 7.194258
```

```
cat("\n---\n\nMedian interval length for EACH gamma:\n")
```

```
##
## ---
##
## Median interval length for EACH gamma:
```

```
print(results$medianIntervalLenPerGamma)
```

```
## [1] 6.873996 6.938913 6.976310 6.970595 7.006243 7.123205 6.968306 6.932987
## [9] 6.962098 6.955754
```

```
cat("\nMedian interval length for the SELECTED gamma:\n")
```

```
##
## Median interval length for the SELECTED gamma:
```

```
print(results$medianIntervalLenSelected)
```

```
## [1] 6.829778
```

```
cat("=====\n")
```

```
## =====
```

```
# Cumulative errors (per gamma)
cumErrSeq <- apply(results$errSeq, 2, cummean)

# Colors creation for different gamma
colors <- rainbow(ncol(cumErrSeq))

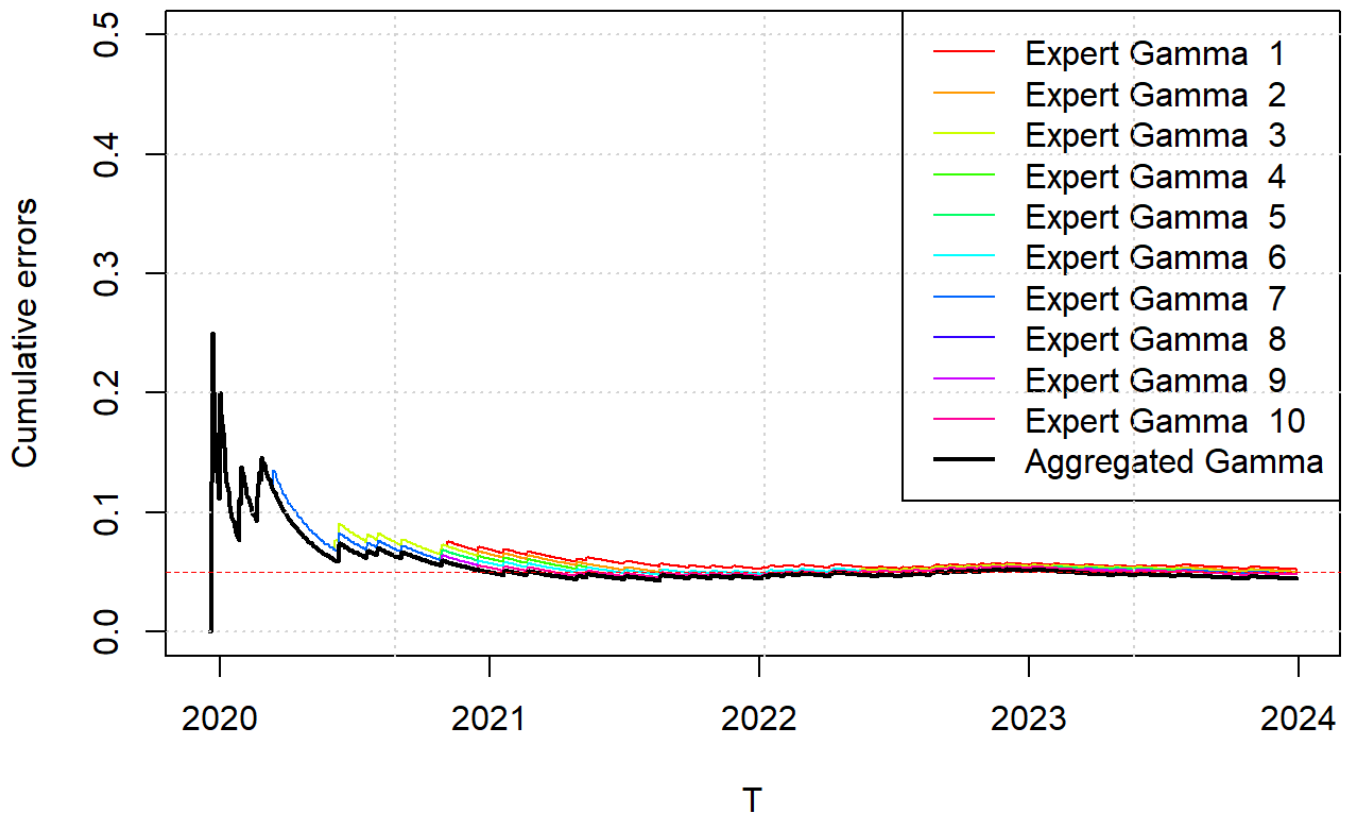
# Main graph
plot(date, cumErrSeq[, 1], type = "l", col = colors[1], ylim = c(0, 0.5),
     main = "Cumulative errors for different gammas", xlab = "T", ylab = "Cumulative errors")
for (k in 2:ncol(cumErrSeq)) {
  lines(date, cumErrSeq[, k], col = colors[k])
}

# Addition of the cumulative error curve for the aggregated gamma
lines(date, cumsum(results$selectedErrSeq) / seq_along(results$selectedErrSeq),
     col = "black", lwd = 2)

abline(h = 0.05, col = "red", lwd = 0.3, lty = 2)

# Legend creation
legend("topright",
     legend = c(paste0("Expert Gamma  ", 1:ncol(cumErrSeq)), "Aggregated Gamma"),
     col = c(colors, "black"),
     lwd = c(rep(1, ncol(cumErrSeq)), 2))
grid()
```

Cumulative errors for different gammas



Here we showed the results for $\alpha = 0.005$, but in our conclusions we consider also the case for $\alpha = 0.001$ (if you want to verify change the α in the code above).

Comments on Results:

For $\alpha = 0.005$, the plot illustrates that the aggregate provides better coverage than any individual expert. Moreover, the aggregate demonstrates a lower Median Interval Length compared to each expert. Although the Average Interval Length of the aggregate is not the absolute lowest, it still ranks among the smallest. These results highlight how the adopted strategy manages to effectively combine the strengths of each expert, providing an overall better prediction interval than that obtainable from a single expert.

However, for $\alpha = 0.001$, the performance deteriorates in accordance with what is showed in the paper "Adaptive Conformal Predictions for Time Series".