Algorithmic Game Theory

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Exercise Set 12

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Exercise 1

Let us consider TCP games on general networks, where each edge is a channel of some capacity.
Each player in the game sends a certain rate s_i along a predetermined path, let us call π_i the path of the player i. The utility of a generic player is the rate r_i at which the traffic arrives at destination.
Our task is to prove that the result proved for a single channel extends also to this game.
Let us call c_i the i-th channel and let us define the following functions:

- f: given an edge e_i, $f(e_i)$ returns 1 iff the edge c_i belongs to the path of the player i. Otherwise, it returns 0
- C: given an edge e_i, $C(e_i)$ returns the capacity of this edge

It is evident that, if all channels use the same Strict Priority Queuing policy, the channel which assigns the lowest capacity to a certain player define the quantity of traffic of that player which arrive at destination, in particular:

$$\begin{gathered} r_1 \leftarrow min(s_1, \min_{e\in\pi_1} C(e)) \\ r_2 \leftarrow min(s_2, \min_{e\in\pi_2}[C(e) - f_1(e)\cdot r_1]) \\ \vdots \\ r_n \leftarrow min(s_n, \min_{e\in\pi_n}[C(e) - \textstyle\sum_{j<n} f_j(e)\cdot r_j]) \end{gathered}$$

We now show that the base game is NBR-solvable with clear outcome. The elimination sequence follows the priority order of Strict Priority Queuing: set

$$\begin{gathered} s_1^* \leftarrow min(M_1, \min_{e\in\pi_1} C(e)) \\ s_2^* \leftarrow min(M_2, \min_{e\in\pi_2}[C(e) - f_1(e)\cdot s_1^*]) \\ \vdots \\ s_n^* \leftarrow min(M_n, \min_{e\in\pi_n}[C(e) - \textstyle\sum_{j<n} f_j(e)\cdot s_j^*]) \end{gathered}$$

and each player i eliminates all strategies different from s_i^* in the order above

$$E_i = \{s_i \neq s_i^*\}$$

To see that this sequence defines an NBR-solvable with clear outcome game we observe the following:

1. Define subgame G_i as the subgame where all players before i have already eliminated their strategies $E_1, ..., E_{i-1}$. So G_1 is the original game
2. In the subgame G_i the highest rate available to i is $\min_{e\in\pi_i}[C(e) - \sum_{j<i} f_j(e)\cdot s_j^*])$. Since i wants to send at most M_i, strategy s_i^* guarantees i the highest possible payoff in this subgame,

$$min(M_i, \min_{e\in\pi_i}[C(e) - \sum_{j<i} f_j(e)\cdot s_j^*])$$

3. Sending with rate smaller than s_i^* results in worse payoff for i, and sending with higher rate is not better than s_i^*. Here we use a simple tie-braking rule, namely

   Prefer smaller sending rate over higher sending rate

   Then the two cases in Definition 9 correspond to $s_i < s_i^*$ and $s_i > s_i^*$ respectively and we only keep the strategy s_i^*.

Hence, we have shown that if all channels use the same Strict Priority Queuing policy, then the result proved for a single channel extends.

Exercise 2

Let us consider the VCG mechanism for sponsored search given in the lecture notes. Our task is to show that this mechanism is a VCG mechanism. In order to do so we need to prove:

1. That the solution provided by the greedy algorithm is optimal, in the sense that it maximizes the social welfare, defined as $SW = \sum_{i=1}^{n} \alpha_i \cdot b_i$
2. That the payments respect the VCG form, that is

$$P_i(c) = Q_i(b_{-i}) - \sum_{j \neq i} c_j(A(b))$$

   Where c_j is the marginal contribution to the social welfare of the player j $(\alpha_j \cdot b_j)$

Claim 1: A is optimal

First we have to prove that the allocation A provided by the VCG mechanism for sponsored search auctions is optimal. We will prove this for induction on the length of the subsequence where the bids that are ordered.

Base case

Let us consider the set of all possible allocations between the bids and the alphas. For all the allocations where a b_i such that $i \neq 1$ is allocated to α_1, we can find a new allocation by switching b_i with b_1, such that

$$\Delta SW = (\alpha_1 b_1 + \alpha_m b_i) - (\alpha_1 b_i + \alpha_m b_1) = (b_1 - b_i) \cdot (\alpha_1 - \alpha_m) \geq 0$$

Where m is the index of the α given to b_1 in all these other allocations. Consequently, all the original permutations we were considering have a SW not higher than the one after switching. Since we just want to find one optimal allocation, we can discard all permutations which do not match α_1 to b_1. We need to observe that in case some alphas or bids have the same value we will use a fixed tie breaking rule to keep the order.

Induction step

Suppose now that we have already discarded all the allocations where the bids from 1 to $i - 1$ were not ordered. Repeating the argument of the base case we can now discard all the solutions were α_i is not matched with b_i, and we will still have the optimal solution in the set of the not discarded allocations.

We can thus conclude that the choice of our algorithm is among the ones which maximize the social welfare, and thus is an optimal one.

Claim 2: the payments are in the VCG form

In order to show that we will manipulate the expression for the payments given in the lecture notes:

$$P_i(b) = \sum_{l=i}^{k} (\alpha_l - \alpha_{l+1}) b_{l+1} = \sum_{l=i}^{k} \alpha_l b_{l+1} - \sum_{l=i}^{k} \alpha_{l+1} b_{l+1}$$

Since $\alpha_{k+1}, ..., \alpha_n = 0$, we can rewrite the sums in the following way:

$$P_i(b) = \sum_{l=i}^{k} \alpha_l b_{l+1} - \sum_{l=i+1}^{n} \alpha_l b_l$$

Now we add and we subtract the same term $\sum_{l=1}^{i-1} \alpha_l b_l$ and we obtain:

$$P_i(b) = \sum_{l=i}^{k} \alpha_l b_{l+1} - \sum_{l=i+1}^{n} \alpha_l b_l + \sum_{l=1}^{i-1} \alpha_l b_l - \sum_{l=1}^{i-1} \alpha_l b_l$$

Rearranging the terms:

$$P_i(b) = \sum_{l=i}^{k} \alpha_l b_{l+1} + \sum_{l=1}^{i-1} \alpha_l b_l - \sum_{l \neq i} \alpha_l b_l$$

In the expression the first two terms do not depend on b_i, so we can define:

$$Q_i(b_{-i}) = \sum_{l=i}^{k} \alpha_l b_{l+1} + \sum_{l=1}^{i-1} \alpha_l b_l$$

The third term is exactly the marginal contribution that we wanted to obtain the VCG payment form.

Exercise 3

In this exercise we want to show that the VCG mechanism for sponsored search satisfies the following two conditions:

1. **Envy-freeness**

$$\alpha_s v_s - P_s^{VCG}(v) \geq \alpha_t v_s - P_t^{VCG}(v) \qquad \text{for } t \in \{s-1, s+1\}$$

2. **Voluntary participation**

Part 1: proof of envy-freeness

Case 1: t = s - 1

We have to show that

$$\alpha_s v_s - P_s^{VCG}(v) \geq \alpha_{s-1} v_s - P_{s-1}^{VCG}(v) \tag{1}$$

We know that $P_{s-1}^{VCG}(v)$ is defined as

$$P_{s-1}^{VCG}(v) = (\alpha_{s-1} - \alpha_s)v_s + P_s^{VCG}$$

Thus, we can rewrite (1) in the following way:

$$\alpha_s v_s - P_s^{VCG}(v) \geq \alpha_{s-1} v_s - (\alpha_{s-1} - \alpha_s)v_s - P_s^{VCG}$$

Rearranging the terms we obtain:

$$(\alpha_{s-1} - \alpha_s)v_s \geq (\alpha_{s-1} - \alpha_s)v_s$$

That always holds.

Case 1: t = s + 1

We have to show that

$$\alpha_s v_s - P_s^{VCG}(v) \geq \alpha_{s+1} v_s - P_{s+1}^{VCG}(v) \tag{2}$$

We know that $P_s^{VCG}(v)$ is defined as

$$P_s^{VCG}(v) = (\alpha_s - \alpha_{s+1})v_{s+1} + P_{s+1}^{VCG}$$

Thus, we can rewrite (2) in the following way:

$$\alpha_s v_s - (\alpha_s - \alpha_{s+1})v_{s+1} - P_{s+1}^{VCG}(v) \geq \alpha_{s+1} v_s - P_{s+1}^{VCG}(v) \tag{3}$$
$$(\alpha_s - \alpha_{s+1})v_s \geq (\alpha_s - \alpha_{s+1})v_{s+1} \tag{4}$$

Where if $\alpha_s = \alpha_{s+1}$, we obtain $0 = 0$ and the condition is trivially satisfied.
Otherwise, it must hold that $\alpha_s > \alpha_{s+1}$ so we just need to prove:

$$v_s \geq v_{s+1}$$

That is always true since the bids are ordered.

Part 2: proof of voluntary participation

We need to prove that truth-telling bidders have non-negative utilities, thus that:

$$\alpha_i v_i - P_i^{VCG} \geq 0$$

for all the players i. It is thus sufficient to prove that

$$\alpha_i v_i \geq P_i^{VCG} \tag{5}$$

We know that:

$$P_i^{VCG}(v) = \sum_{l=i}^{k} (\alpha_l - \alpha_{l+1}) \cdot v_{l+1}$$

Hence, we can rewrite (1) as follows:

$$\alpha_i v_i \geq \sum_{l=i}^{k} (\alpha_l - \alpha_{l+1}) \cdot v_{l+1}$$

We can now observe that bids are ordered, thus it must always hold $v_i \geq v_{l+1}$ for all $l \in [i, k]$.

$$\sum_{l=i}^{k} (\alpha_l - \alpha_{l+1}) \cdot v_{l+1} \leq \sum_{l=i}^{k} (\alpha_l - \alpha_{l+1}) \cdot v_i = (\alpha_i - \alpha_{i+1} + \alpha_{i+1} + ... + \alpha_k) v_i = \alpha_i v_i$$

We have obtained that it must always hold $\alpha_i v_i \geq \sum_{l=i}^{k} (\alpha_l - \alpha_{l+1}) \cdot v_{l+1}$, that is exactly what we wanted to prove.

Exercise 4

In this exercise we want to show that it is possible to construct a bid vector b^{VCG} such that

1. $P_s^{VCG}(v) = P_s^{GSP}(b^{VCG})$
2. b^{VCG} is a symmetric pure Nash equilibrium.

Part 1: definition of the bid vector b to satisfy the first condition

We know that

$$P_s^{VCG}(v) = (\alpha_s - \alpha_{s+1})v_{s+1} + P_{s+1}^{VCG}$$

and

$$P_s^{GSP}(b^{VSG}) = \alpha_s b_{s+1}^{VCG}$$

For brevity from now on we will refer to b^{VSG} simply as b.
Let's start by defining the form of the vector b.
In order to satisfy the first condition we can define

$$b_s = \frac{P_{s-1}^{VCG}(v)}{\alpha_{s-1}}$$

for $s \leq k+1$, since $\alpha_i = 0$ for all $i > k$.
The other values of b can just be null.

Part 2: proof that b^{VCG} is a SNE

Let us now analyze the second condition:
From the characterization of SNE it must hold that:

$$\begin{cases} \alpha_s(v_s - p_s) \geq \alpha_{s-1}(v_s - p_{s-1}) \\ \alpha_s(v_s - p_s) \geq \alpha_{s+1}(v_s - p_{s+1}) \end{cases}$$

Thus, since in GSP we define $p_i = \alpha_i \cdot b_{i+1}$, we can rewrite the conditions as follows:

$$\begin{cases} \alpha_s(v_s - b_{s+1}) \geq \alpha_{s-1}(v_s - b_s) \\ \alpha_s(v_s - b_{s+1}) \geq \alpha_{s+1}(v_s - b_{s+2}) \end{cases}$$

Let's first prove that the first condition holds for the vector b we have previously defined.
It is trivial to show that this hold for all $s \geq k+2$, since $\alpha_s = \alpha_{s+1} = 0$, so the equality $\alpha_s(v_s - b_{s+1}) = \alpha_{s-1}(v_s - b_s)$ holds.
Also, if we consider the board case in which $s = k+1$, we should verify that:

$$\alpha_{k+1}(v_{k+1} - b_{k+2}) \geq \alpha_k(v_{k+1} - b_{k+1})$$

But $\alpha_{k+1} = 0$ and $b_{k+1} = \frac{P_k^{VCG}(v)}{\alpha_k} = \frac{\alpha_k v_{k+1}}{\alpha_k} = v_{k+1}$, thus we obtain:

$$0 \geq \alpha_k(v_{k+1} - v_{k+1}) = 0$$

that always holds.
Let's now consider the general case where $s \geq k$.
We can rewrite the inequality as follows:

$$\begin{aligned} \alpha_s(v_s - b_{s+1}) &\geq \alpha_{s-1}(v_s - b_s) \\ \alpha_s(v_s - \frac{P_s^{VCG}(v)}{\alpha_s}) &\geq \alpha_{s-1}(v_s - \frac{P_{s-1}^{VCG}(v)}{\alpha_{s-1}}) \\ \alpha_s v_s - P_s^{VCG}(v) &\geq \alpha_{s-1}v_s - (\alpha_{s-1} - \alpha_s)v_s - P_s^{VCG} \\ \alpha_s v_s &\geq \alpha_{s-1}v_s - (\alpha_{s-1} - \alpha_s)v_s \\ 0 &= 0 \end{aligned}$$

So the first condition always holds.
We now will check the second condition.
It is again trivial to show that for all $s \geq k+1$ the condition is valid since we always have $0 = 0$, as all α_s such that $s \geq k+1$ are null. Let's analyze the corner case in which $s = k$.

$$\begin{aligned} \alpha_k(v_k - b_{k+1}) &\geq \alpha_{k+1}(v_k - b_{k+2}) \\ \alpha_k(v_k - v_{k+1}) &\geq 0 \end{aligned}$$

That since $v_k \geq v_{k+1}$ (the bids in the VCG mechanism are ordered), always holds.
Let's now analyze the general case where $s < k$. In these cases it must hold that:

$$\begin{aligned} \alpha_s(v_s - b_{s+1}) &\geq \alpha_{s+1}(v_s - b_{s+2}) \\ \alpha_s v_s - (\alpha_s - \alpha_{s+1})v_{s+1} + P_{s+1}^{VCG}(v) &\geq \alpha_{s+1}v_s - P_{s+1}^{VCG}(v) \\ \alpha_s v_s - (\alpha_s - \alpha_{s+1})v_{s+1} &\geq \alpha_{s+1}v_s \\ v_s(\alpha_s - \alpha_{s+1}) &\geq v_{s+1}(\alpha_s - \alpha_{s+1}) \end{aligned}$$

If $\alpha_s = \alpha_{s+1}$ we obtain $0 = 0$, so the condition is trivially met.
Otherwise, if $\alpha_s \neq \alpha_{s+1}$, it has to hold that $\alpha_s > \alpha_{s+1}$, thus we can divide both the terms for $\alpha_s - \alpha_{s+1}$, without changing the verse of the inequality and we obtain:

$$v_s \geq v_{s+1}$$

that is always true since the bids are ordered.

We have then proved that it is possible to construct a bid vector b^{VCG} such that $P_s^{VCG}(v) = P_s^{GSP}(b^{VCG})$ and b^{VCG} is a symmetric pure Nash equilibrium.

Exercise 5

Let us consider the Interns-Hospitals (Stable Matching) problem with two interns (1 and 2) and two hospitals (A and B) where each hospital has its own preference over interns.

Our task is to prove that there is a mechanism that is **obviously strategyproof** for the interns.
In order to do that we will describe such a mechanism.
Our mechanism works in the following way:

1. The first player to chose his preference is the top choice of the hospital preferred by the first player. Then the other player will choose
2. At step i, intern i checks which hospitals are not yet taken, and he proposes to his most preferred one in this set.
3. Each hospital accepts only the most preferred intern that proposing to it (and this hospital is then considered taken).

We can notice that this mechanism **depends on $\prec_\mathbf{A}$ and $\prec_\mathbf{B}$**, as we decide the order in which players choose looking to hospitals' preferences.

Now, our task is to prove that the following mechanism returns a stable matching and that it is obviously strategyproof for the interns.

Claim 1: the mechanism returns a stable matching

We have to prove that the matching obtained is a **stable matching**, thus that it does not contain any blocking pair.
For the sake of contradiction consider a intern i not matched to a certain hospital h. There are two possible reasons for i not being matched to h. Either i is matched to another hospital h' which he preferred over h, or h preferred the other intern over i.
In the first case, the pair (i, h) is not blocking, because i is matched to his first preference.
In the second case, h is matched to his first preference, thus again the pair (i, h) is not blocking.

Claim 2: the mechanism is obviously strategyproof for the interns

When the first player (intern) (let us call this player p_1, the other will be p_2) has to make a choice, we are in a history h with available actions (non-taken hospitals)

$$A(h) = \{A, B\}$$

Given p_1 preference ($\prec_1$) we let $u_{p_1}^{max}(h)$ be the utility of p_1 if matched with the most preferred hospital in $A(h)$. Observe that:

1. The strategy s_1^* guarantees p_1 being matched to $j_{p_1}^*$ if the other intern prefers the other hospital or if the hospital $j_{p_1}^*$ prefer i other the other intern. This is because:
   - if the players have different preferences , the other player will propose to the other hospital, so $j_{p_1}^*$ will accept the only offer received
   - if the players have common preferences, he will be accepted even if the other player proposes to the same hospital, since he is the top preference of that hospital
2. Any strategy s_1' which deviates from s_1^* at some h will only give p_1 a worse utility (he would be matched to a worse hospital in A(h)).

That is for any $s_{p_2}^-$

$$u_i(s_{p_1}^*, s_{p_2}^- | h) = u_{p_1}^{max}(h)$$

while for any $s_{p_2}^+$

$$u_i(s_{p_1}^*, s_{p_2}^+ | h) < u_{p_1}^{max}(h)$$

This shows that the strategy $s_{p_1}^*$ in Step 1 of our mechanism is obviously dominant for the first player who chooses.

When the second player (intern) makes a choice, we are in a history h' with available actions (non-taken hospitals)

$$A(h') = \{x\}$$

where x is the remaining hospital that could be either A or B. Given the preference of p_2, we let $u_{p_2}^{max}(h)$ be the utility of p_2 if matched with the most preferred hospital $j_{p_2}^*$ in $A(h')$. We observe that:

1. The strategy s_2^* which makes him proposing to such $j_{p_2}^*$ guarantees him being matched with $j_{p_2}^*$ no matter the strategies of the other players because he is the last one remaining.
2. Any strategy s_2' which deviates from s_2^* at some h will only give p_2 a worse utility (he would remain unmatched).

That is for any $s_{p_1}^-$

$$u_{p_2}(s_{p_2}^*, s_{p_1}^- | h') = u_{p_2}^{max}(h')$$

while for any $s_{p_2}^+$

$$u_{p_2}(s_{p_2}^*, s_{p_1}^+ | h') < u_{p_2}^{max}(h')$$

This shows that the strategy $s_{p_2}^*$ in Step 2 of our mechanism is obviously dominant for the second player who chooses.

Hence, we have implemented our mechanism as an extensive form game with obviously dominant strategies and we can therefore cocnlude that our mechanism is obviously strategyproof.