Wavelets

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Uncertainty principle for finite-length sequences

Theorem (Vetterli, Sec. 7.3.4)

Let a nonzero $x \in \mathbb{C}^N$ have N_t nonzero entries and DFT X with N_f nonzero entries. Then,

$$N_t N_f \geq N$$

Proof :

1. $x \in \mathbb{C}^N$ have N_t nonzero entries $\Longrightarrow$ X cannot have N_t consecutive zero entries (interpeted mod N)
2. arrange the points of X in a circle and choose one nonzero entry to start from, this can be followed by at most $(N_t - 1)$ zeros, then

$$N_f \geq \left\lceil \frac{N}{N_t} \right\rceil \geq \frac{N}{N_t}$$

Uncertainty principle for finite-length sequences

Proof of point 1. (by contradiction)

1. $x \in \mathbb{C}^N$ have N_t nonzero entries $\Longrightarrow$ X cannot have N_t consecutive zero entries (interpeted mod N)
   - $x \in \mathbb{C}^N$, $X = \mathcal{F}(x) \in \mathbb{C}^N$, $W_N = \mathrm{e}^{-2\pi j/N}$
   - $x \in \mathbb{C}^N$ have N_t nonzero entries : $x_{i_0} = \ldots = x_{i_{N_t-1}} \neq 0$
   - define $y \in \mathbb{C}^{N_t}$: $y_\ell = x_{i_\ell} \neq 0$
   - suppose N_t consecutive zero entries in X :

     $$\exists k \in \{0, \ldots, N-1\} : X_{k+m} = 0 \quad \forall m \in \{0, \ldots, N_t - 1\}$$

   - notice

     $$0 = X_{k+m} = \sum_{n=0}^{N-1} x_n W_N^{(k+m)n} = \sum_{\ell=0}^{N_t-1} x_{i_\ell} W_N^{(k+m)i_\ell} = \sum_{\ell=0}^{N_t-1} y_\ell z_\ell^{k+m}$$

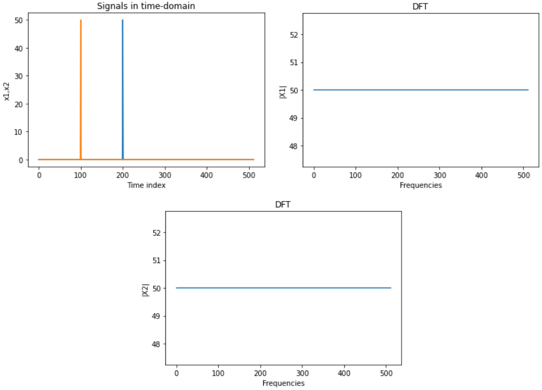
   - in vector form : $Z_{m,\ell} = z_\ell^{k+m}$, with $0 \leq m, n \leq N_t - 1$

     $$0 = Zy$$

   - y nonzero $\Longrightarrow$ Z not full rank <u>but</u> ... $Z = \widehat{Z}D$ full rank
     - $\widehat{Z}_{m,n} = z_n^m$ Vendermonde matrix (full rank)
     - $D = \mathrm{diag}(z_0^k, z_1^k, \ldots, z_{N_t-1}^k)$ (full rank)

Uncertainty principle for finite-length sequences

The equality is achieved...



DFT & DWT

Fourier analysis

- DFT : decomposition of a signal into sinusoidal basis functions of different frequencies
- DFT is invertible (no information lost) --> possiamo sempre recuperare interamente il segnale

Wavelet analysis --> permette di rappresentare bene segnali transitori

- DWT : decomposition of a signal into a set of mutually orthogonal wavelet basis functions
- Wavelet functions are spatially localized --> le ondine sono spazialmente localizzate, quinidi sono non nulle solo su una parte del segnale: hanno supporto non nullo su un sottointervallo del segnale
- functions are dilated, translated and scaled versions of a common mother wavelet
- DWT refers not just to a single transform, but rather a set of transforms (Haar, Daubechies, ...) --> quindi a una famiglia di trasformate
- DWT is invertible (no information lost) --> è possibile la ricostruzione completa del segnale

One-dimensional DWT

Esempio trasformata DWT monodimensionale

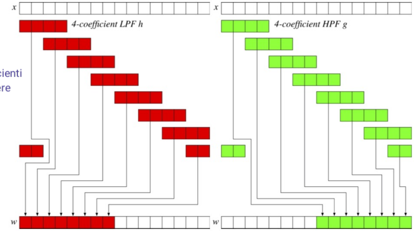
The (one-dimensional) DWT

- operates on a real-valued vector x of length 2^n, $n \in 2, 3, \ldots,$
- results in a transformed vector w of equal length.

--> La DWT trasforma il segnale in un altro segnale della stessa lunghezza

1. Applicazione filtro passa basso
2. Applicazione filtro passa alto

Il segnale viene filtrato a incrementi di due e i coefficienti risultatanti vengono messe nelle prime posizioni libere



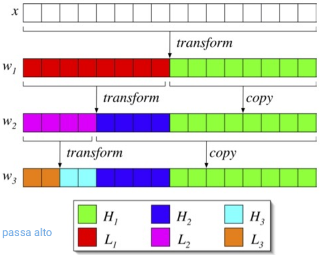
(a) (b)

- The original signal is low-pass filtered in increments of two, and the resulting coefficients are grouped as the first eight elements of the vector.
- The original signal is high-pass filtered in increments of two, and the resulting coefficients are grouped as the last eight elements of the vector.

Three-level wavelet transform

Repeat identical procedure :

Le traformate si possono fare a diversi libvelli (come abbiamo visto in teoria)



La parte arancione corrisponde a tre applicazioni consecutive del filtro passa basso

La parte azzurrina due passa basso e un passa alto

La parte blu abbiamo l'applicazione di un passa basso e di un passa alto

- values L3 are the result of three consecutive low-pass filters,
- values H3 are the result of two consecutive low-pass filtering operations followed by a high-pass filter
- values H2 are the result of a low-pass filter followed by a high-pass filter
- values H1 are the result of one high-pass filter.
- the highest frequencies will be isolated and localized in values H1 of w3, intermediate frequencies will be isolated and localized in values H2 of w3, etc
- lower frequencies, the resolution is decimated by half for each level of the wavelet transform

Filter coefficients

If h low-pass filter of length n and g high-pass filter, then they lead to basis functions with properties of compactness and orthogonality...

$$g_k = (-1)^k h_{n-k-1}, k \in \{0, \dots, n-1\}$$

Examples :

- $n = 2$:

$$h = [c_0 \quad c_1], \qquad g = [c_1 \quad -c_0]$$

(e.g. Haar : $h = [1/\sqrt{2}, \quad 1/\sqrt{2}]$

- $n = 4$:

$$h = [c_0 \quad c_1 \quad c_2 \quad c_3], \qquad g = [c_3 \quad -c_2 \quad -c_1 \quad -c_0]$$

(e.g. Daubechies-4 : $h = [\frac{1+\sqrt{3}}{4\sqrt{2}}, \quad \frac{3+\sqrt{3}}{4\sqrt{2}} \quad \frac{3-\sqrt{3}}{4\sqrt{2}} \quad \frac{1-\sqrt{3}}{4\sqrt{2}}]$

Press, William H., Brian P. Flannery, Saul A. Teukolsky, and William T. Vetterling. Numerical Recipes in C : The Art of Scientific Computing. 2nd ed., Cambridge University Press, 1992.

Inverse DWT

La DWT è invertibile e esiste un algoritmo che dai coefficienti della DWT possiamo ritornare al segnale originale

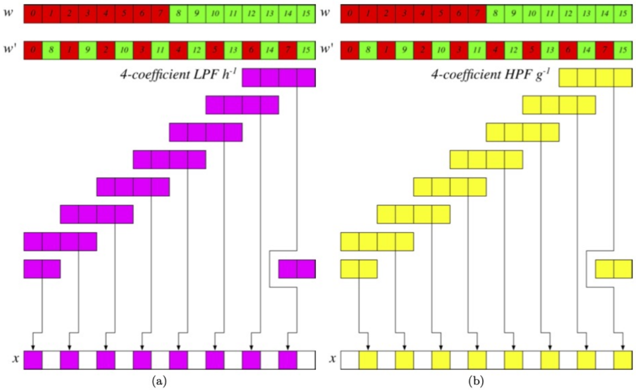
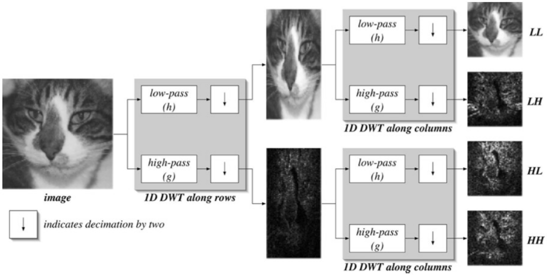


Figure 5: Illustration of the inverse DWT for a one-level DWT w of length 16. First, the low-pass and high-pass elements of w are interleaved. Then, (a) the inverse low-pass filter h^{-1} is applied in increments of two, and (b) the inverse high-pass filter g^{-1} is applied in increments of two.

DWT in two dimensions

Two-dimensional DWT through repeated application of the one-dimensional DWT :



- apply a one-level, one- dimensional DWT along the rows of the image.
- apply a one-level, one-dimensional DWT along the columns of the transformed image
- four distinct bands :
  - LL band ⇝ down-sampled (by a factor of two) version of the original image
  - LH band ⇝ preserve localized horizontal features
  - HL band ⇝ to preserve localized vertical features in the original image
  - HH band ⇝ isolate localized high-frequency point features in the image.