

ANALISI TEMPO-FREQUENZA E MULTISCALE

2024/25

LEZIONI : FABIO NICOLA (NANEDÌ)

ESERCITAZIONI: CHIARA RAVAZZI (LUNEDI)

(nelle prime 2 settimane lezione anche il lunedì)

TESTI (consigliati, non strettamente necessari).

[FSP] VETTERLI, KOVACEVIC, GOYAL, Foundation of signal processing

[FWSP] = Fourier and wavelet signal processing

Scaricabili gratuitamente da

[www.Fourierandwavelets.org](http://www Fourierandwavelets.org)

Altri materiali (facoltativo): testi (Thellst, Peyré) in un avviso sul portale.

ESERCITAZIONI IN PYTHON (nessuna conoscenza preliminare richiesta).

MODALITÀ DI ESAME È same ore: una domanda sulle teorie e una domanda su una esercitazione svolta nei LAIB (potete consultare appunti per la domanda sulle esercitazioni).

Analog World, Digital Computation

- You want an analysis (CAMPIONAMENTO) → Acquisizione del segnale fatto da un filtro



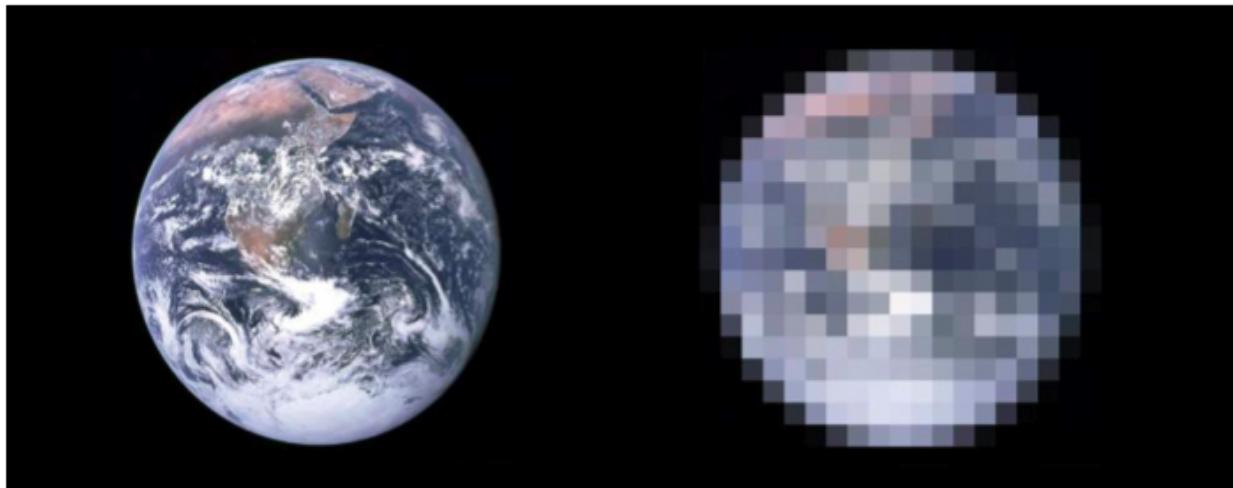
- You want a synthesis (RICOSTRUZIONE) → Per riportarlo nel mondo analogico



From Analog to Digital

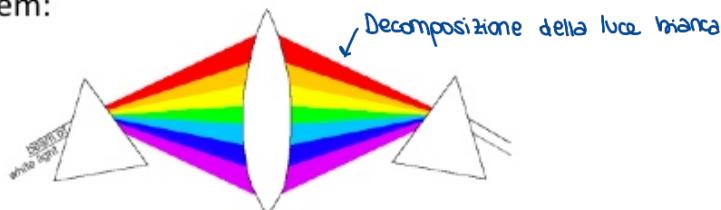
Da analogica a digitale c'è una perdita di info

PERDITA DI INFORMAZIONI
=> DI RISOLUZIONE



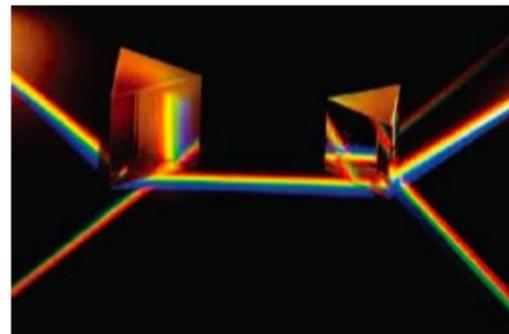
From Rainbows to Spectras

The analysis-synthesis problem:

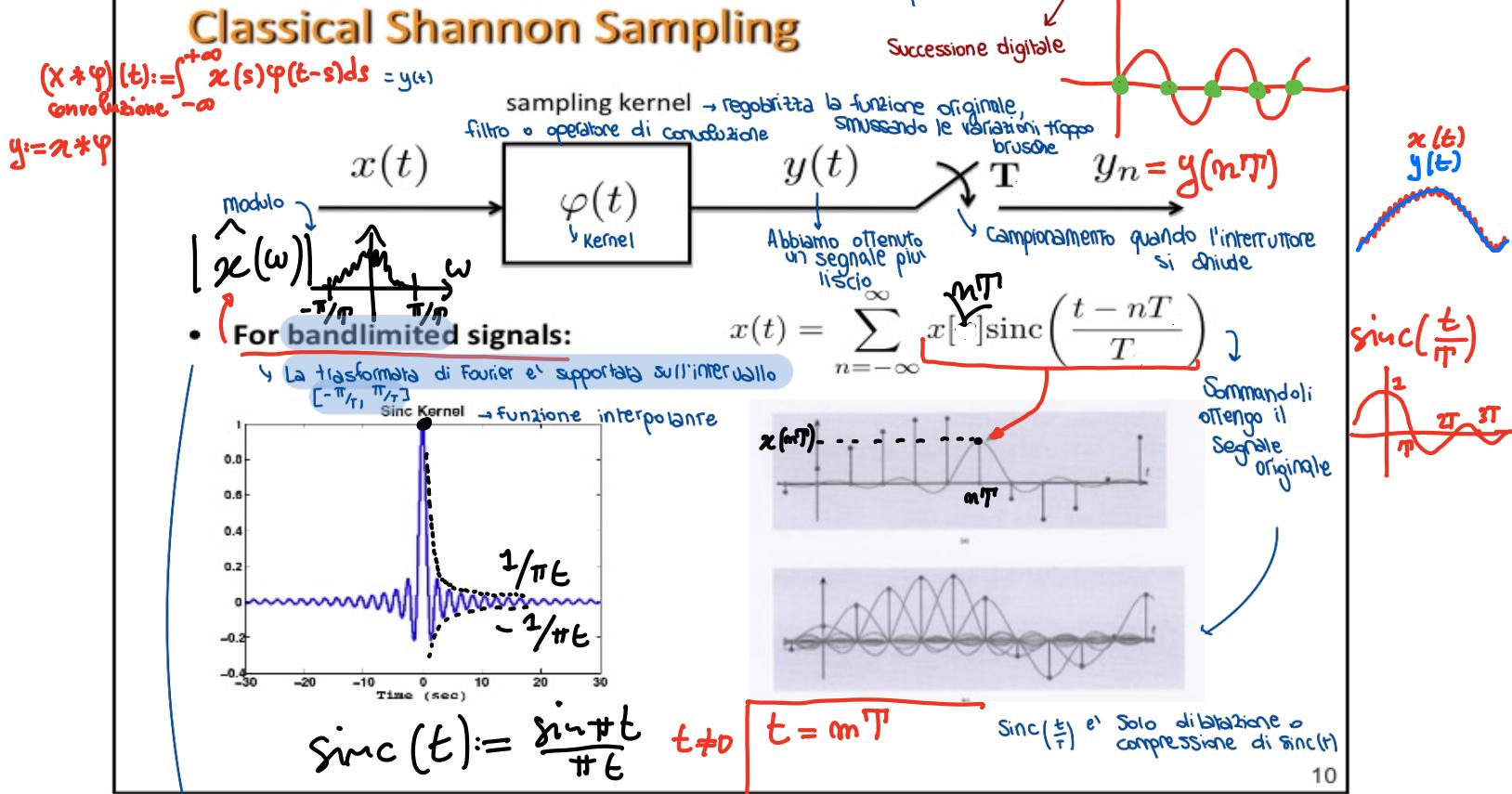


Newton's experiment

~1670
SPECTRO



In this class: we will study the analysis-synthesis problem from a signal processing point of view!

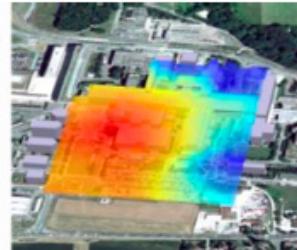


Are these real problems?

- **Sensor networks:**
 - How many sensors?
 - How to reconstruct?
- **Google maps**
 - How to register images?
 - What resolution images to give an adequate view of the world?

November 13th 2006
Air Temperature Kriging
9h00 pm local time

Air Temperature (°C)
High - 3.00
Low - 0.00
Unknown



• Segnale modellizzato in uno spazio di Hilbert come vettore

$L^2(\mathbb{R})$, ℓ^2

← Dove viviamo i segnali

RAPPRESENTAZIONE DEI SEGNALI

$\tau \in L^2(\mathbb{R})$, $\tau \in \ell^2$, $\tau \in \mathbb{C}^N$

Proprietà che vogliamo:

A More General View: Bases and Frames

• Ingredients:

- as set of vectors, or "atoms"
- an inner product, e.g.
- a series expansion

Problema del pattern recognition

Strutture elementari che si leggono direttamente dal segnale

Prodotto Scalare

\downarrow

$\{\varphi_n\}$

Elementi della base

\downarrow

$\alpha_m = \langle f, \varphi_m \rangle = \int f \varphi_m$

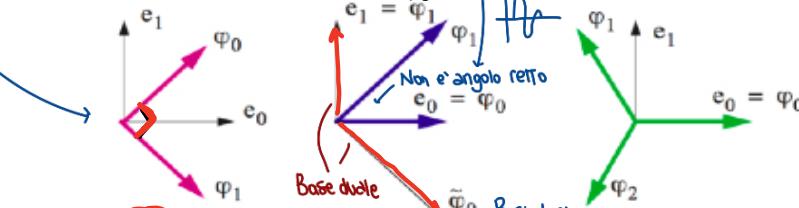
Correlazione / similitudine

$f(t) = \sum_n \langle f, \varphi_n \rangle \varphi_n(t)$

Decomposizione del segnale w.r.t una base

• Many possibilities:

- orthonormal bases (e.g. Fourier series, wavelet series)
- biorthogonal bases (DUALE)
- overcomplete systems or frames



$$\begin{aligned}\langle \varphi_0, \tilde{\varphi}_0 \rangle &= 1 \\ \langle \varphi_0, \varphi_1 \rangle &= 0 \\ \langle \varphi_1, \tilde{\varphi}_0 \rangle &= 0 \\ \langle \varphi_1, \tilde{\varphi}_1 \rangle &= 1\end{aligned}$$

Elementi di base duale es. base di span

Lunghezza ∞

Ondina

funzione con andamento sinusoidale con durata breve

$$d_k = \langle f, \varphi_k \rangle = \int_0^1 f(t) e^{-2\pi i k t} dt$$

Ha senso a volte per comporre meglio il segnale (es. dictionario)

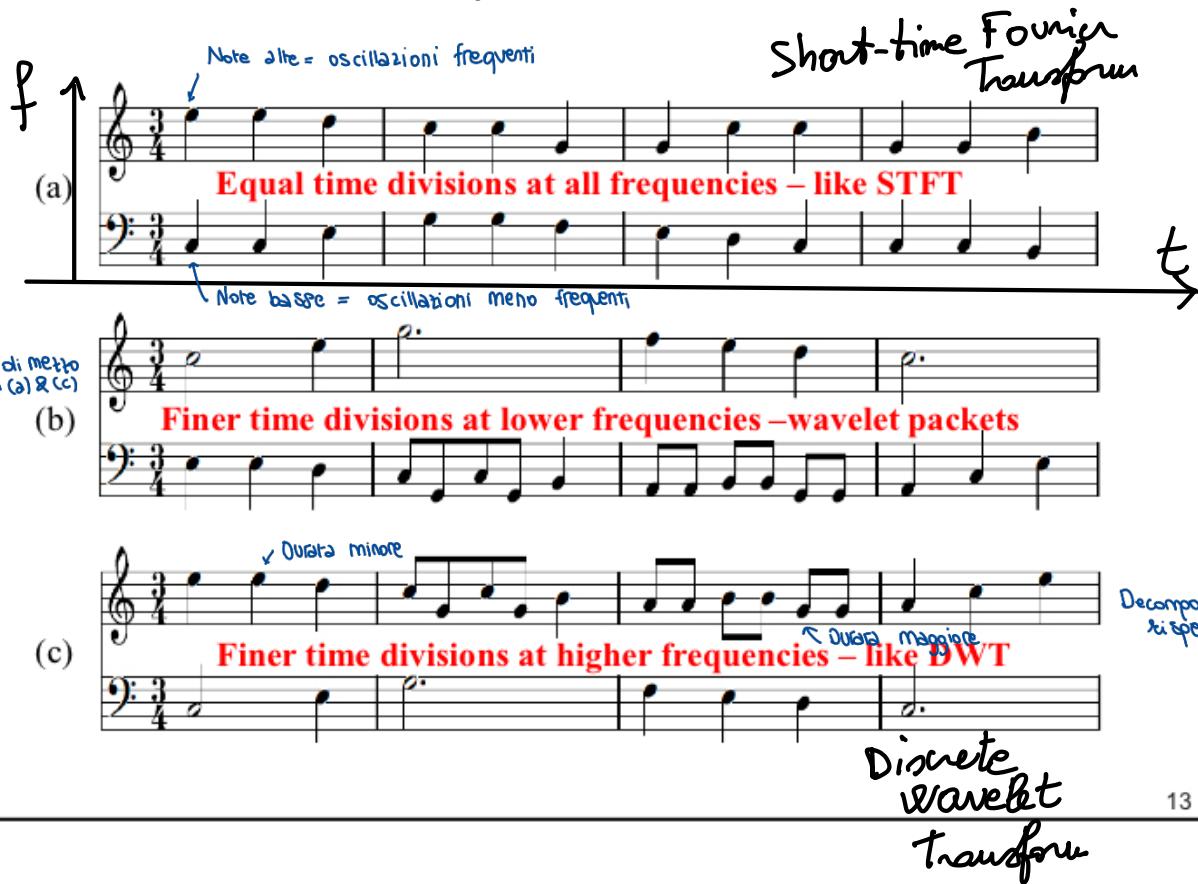
FRAMES e RIDONDANZA

el' analisi della trasformata
 $f \mapsto (d_n) \mapsto f$
el' analisi dell'antitrasformata

perché ci sono più elementi di quanti necessari per descrivere le informazioni

Relation to music (ANALOGIA)

Note tutte con la stessa durata



Matrice di base di \mathbb{C}^8 Haar Wavelet Bases

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0 \right)$$

$$(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0)$$

$$(0, 0, 0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0)$$

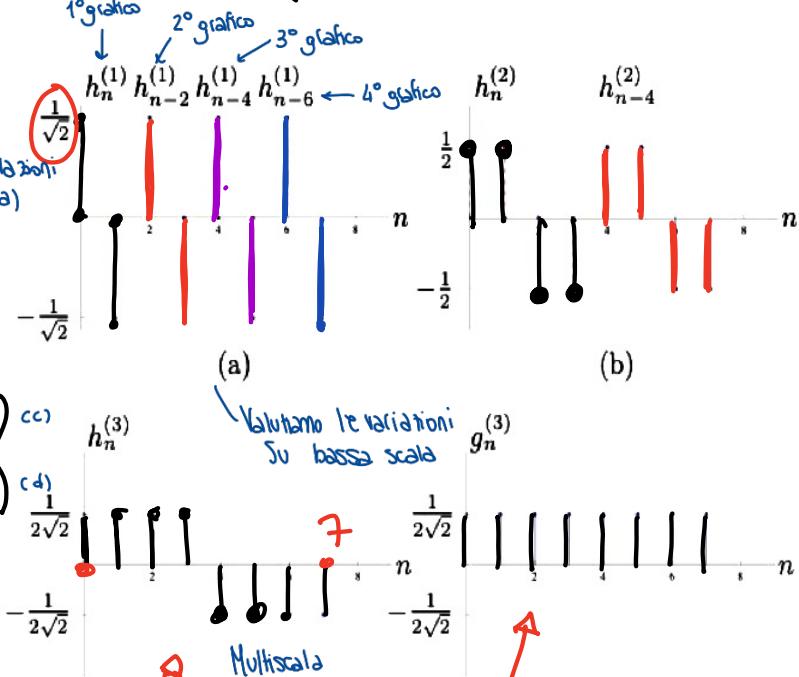
$$(0, 0, 0, 0, 0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

$$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0)$$

$$(0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$$

$$(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$$

$$(\frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}})$$



$$\langle X, \varphi \rangle = \frac{1}{2\sqrt{2}} \sum_{j=0}^7 x_j$$

Valutiamo la media

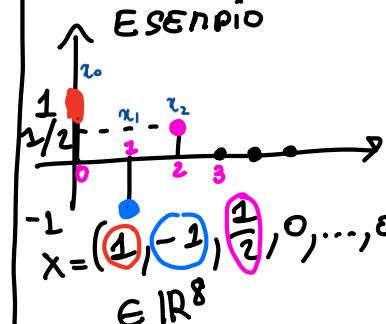
$$\langle X, \psi \rangle = \frac{1}{2\sqrt{2}} (x_0 + x_1 + x_2 + x_3 - x_4 - x_5 - x_6 - x_7)$$

Valutiamo le variazioni su ampia scala

$$X = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$$

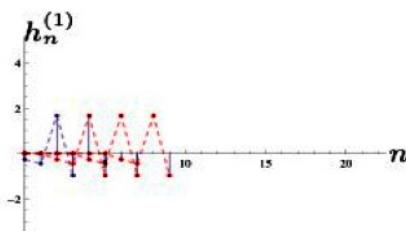
Vettori di lunghezza 8 di numeri complessi:
 $(x_0, x_1, x_2, \dots, x_7) \in \mathbb{C}^8$

Rappresentati come se fosse una successione, una serie temporale

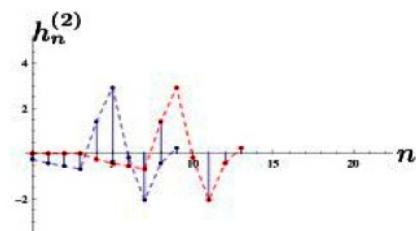


Daubechies (D2) Wavelet Bases

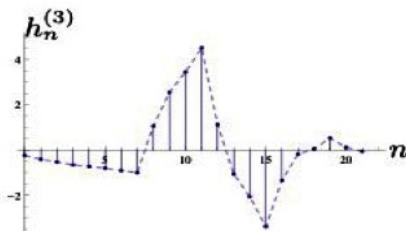
↓ Preferibile a quella di Haar perche' la trasformata di Fourier e' localizzata



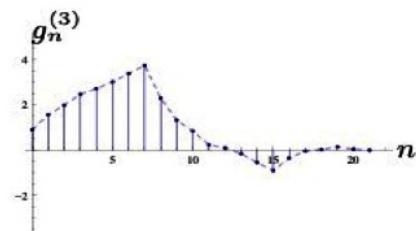
(a)



(b)



(c)



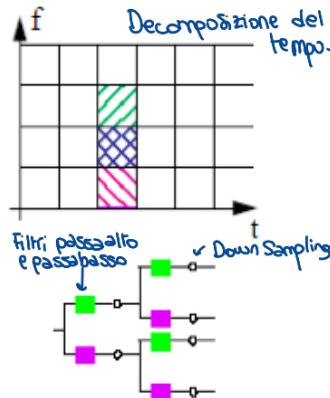
(d)

→ Operatore di convoluzione

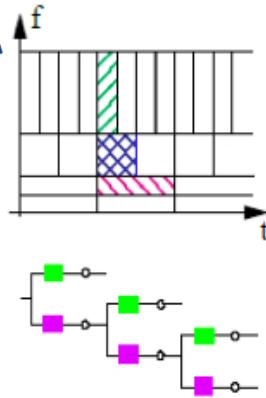
Computing such analysis using filter banks

IMPLEMENTAZIONE

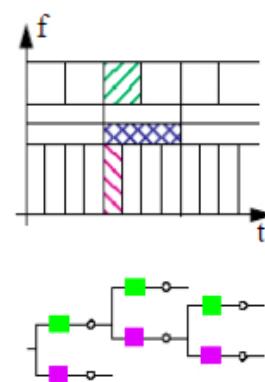
Short-time
Fourier transform



Discrete
wavelet transform



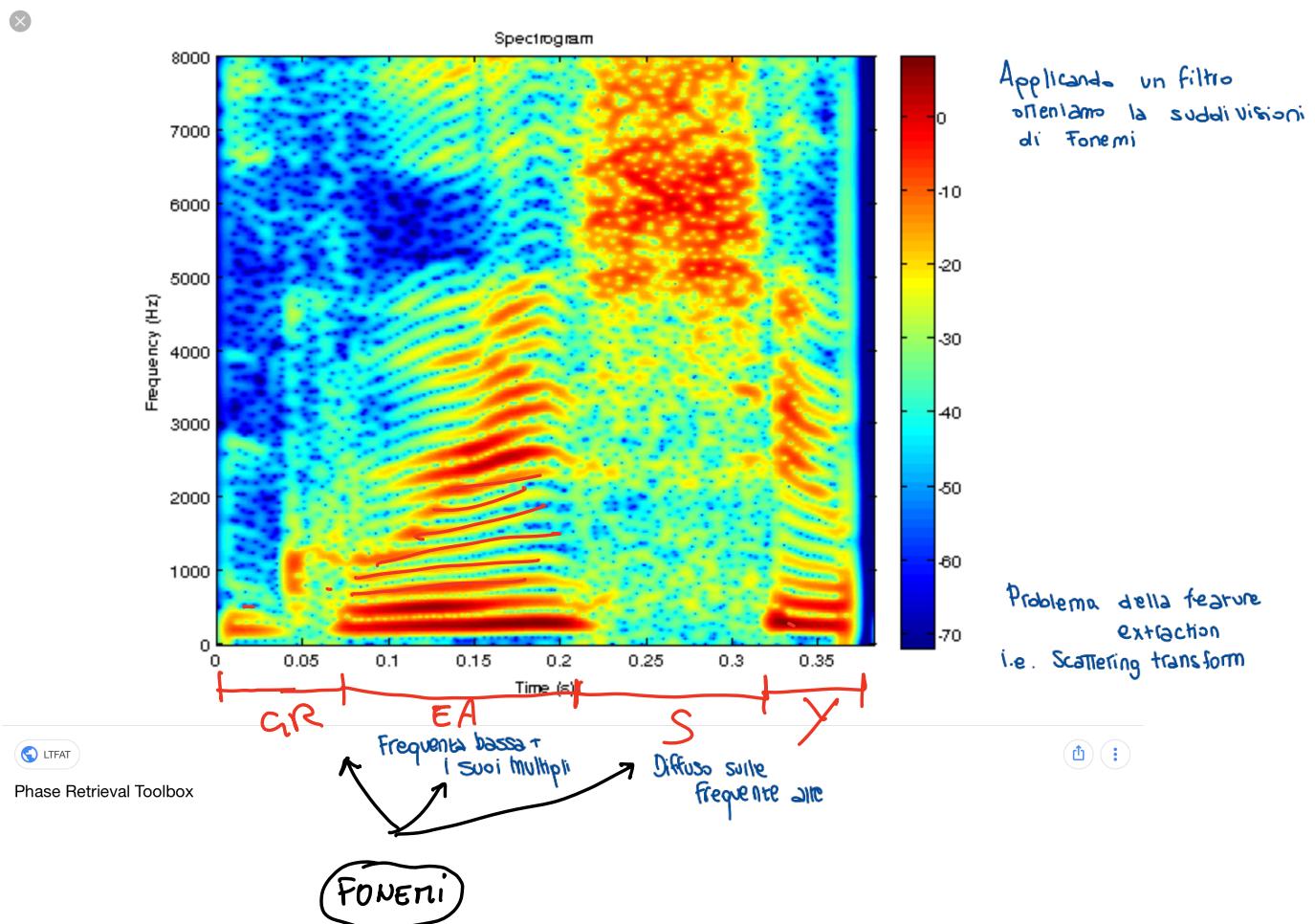
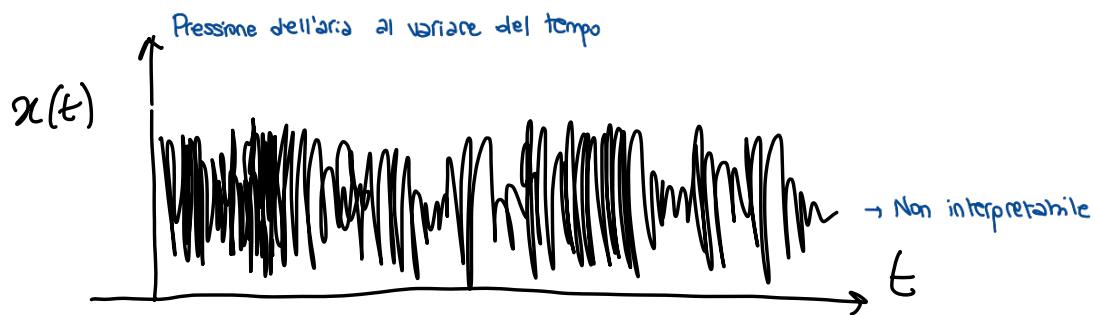
Wavelet packet
transform (1 of many)



FILTRE + DOWNSAMPLING
+ UPSAMPLING

i.e. da $x_0, x_1, x_2, x_3, x_4, x_5$ a x_0, x_2, x_4 operazioni per ridimensionare le immagini

SPETTROGRAMMA DEL SEGNALE VOCALE "GREASY"



Approximation

- The linear/nonlinear approximation method

Given an orthonormal basis $\{g_n\}$ for a space S and a signal

$$f = \sum_n \langle f, g_n \rangle \cdot g_n \quad \text{Decompositione di una funzione rispetto a una base}$$

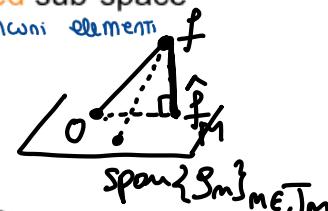
- Linear approximation is given by the projection onto a **fixed** sub-space of size M (**independent** of f!) \rightarrow Tronchiamo e teniamo solo alcuni elementi della sommatoria

$$\hat{f}_M = \sum_{n \in J_M} \langle f, g_n \rangle \cdot g_n$$

\nwarrow Proiezione ortogonale

The error (MSE) is thus

$$\varepsilon_M = \| f - \hat{f}_M \|_M^2 = \sum_{n \notin J_M} | \langle f, g_n \rangle |^2 \quad \rightarrow \begin{matrix} \text{somma degli elementi} \\ \text{che abbiamo trascurato} \end{matrix}$$



- Nonlinear approximation projects onto M vectors corresponding to largest inner products with f

- typically lowpass

$$\hat{f} - \hat{f}_n = \sum_{m \notin J_M} \langle f, g_m \rangle g_m =$$

16

\rightarrow In maniera additiva
scegli i coefficienti
migliori per approssimare
 \downarrow
applico filtri
passa basso

A seconda della base, l'approssimazione del vettore è migliore o peggiore

Non-Linear Approximation Example

- Nonlinear approximation power depends on basis

Example:



$$f \in L^2(0,1), \varphi_n(t) = e^{j2\pi n t}$$

$$c_k = \langle f, \varphi_k \rangle = \int_0^1 f(t) e^{-j2\pi k t} dt$$

Base di Fourier

$$\left\| f(t) - \sum_{|k| \leq M/2} c_k e^{j2\pi k t} \right\|_2^2 = \varepsilon_M$$

↑
unità
immaginaria

- Two different bases for $[0, 1]$:

- Fourier series $\{e^{j2\pi k t}\}_{k \in \mathbb{Z}}$
- Wavelet series: Haar wavelets

$$\varepsilon_M \sim \frac{1}{M} \rightarrow \text{n° di termini che prendiamo}$$

- Linear approximation in Fourier or wavelet bases

$$\tilde{\varepsilon}_M \sim \frac{1}{M}$$

- Nonlinear approximation in a Fourier basis

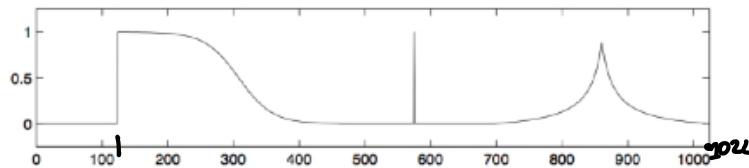
$$\tilde{\varepsilon}_M \sim \frac{1}{2^M}$$

- Nonlinear approximation in a wavelet basis

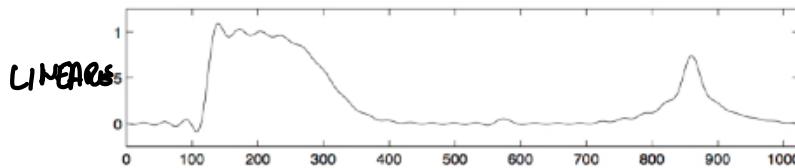
Non-linear Approximation Example

\mathbb{R}^{1024}

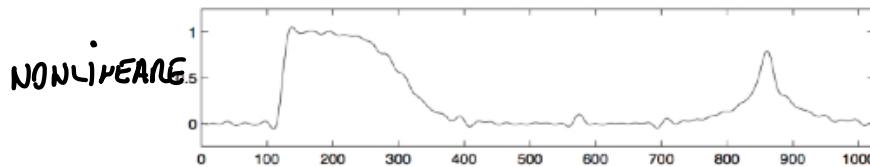
- Fourier basis: N=1024, M=64, linear versus nonlinear



1024



D = 2.7



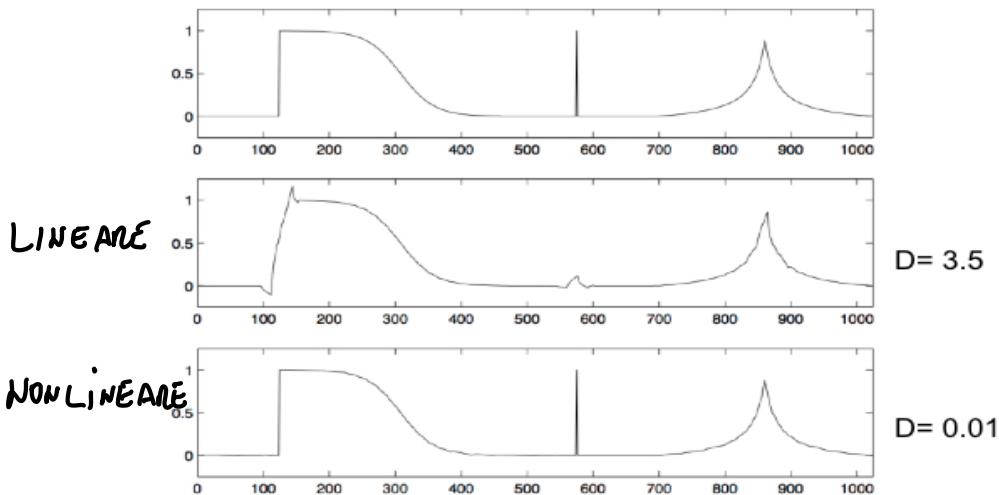
D = 2.4

- Nonlinear approximation is **not necessarily** much better!

Non-linear Approximation Example

- Wavelet basis: $N=1024$, $M=64$, $J=6$, linear versus nonlinear

Fissata una certa risoluzione



- Nonlinear approximation is vastly superior!

Representation, Approximation and Compression: Why does it matter anyway?

Parsimonious or sparse representation of information is key in → Ricerca di una base che

faccia intervenire pochi elementi

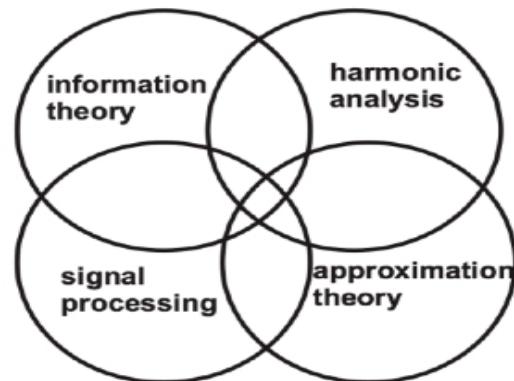
- storage and transmission
- indexing, searching, classification, watermarking = filigrana elettronica (copyright)
- denoising, enhancing, resolution change

But: it is also a fundamental question in

- information theory
- signal/image processing
- approximation theory
- vision research

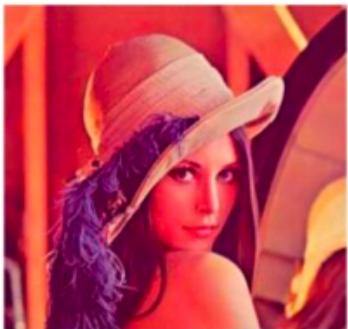
Successes of wavelets in image processing:

- compression (JPEG2000)
- denoising
- enhancement
- classification



Compressione = Salvare immagine con meno "peso" possibile

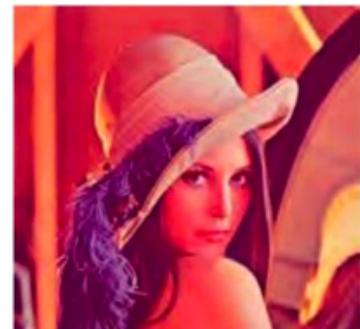
New image coding standard ... JPEG 2000



Original Lena Image (256 x 256 Pixels,
24-Bit RGB)



JPEG Compressed (Compression Ratio
43:1)



JPEG2000 Compressed (Compression
Ratio 43:1) → Applica la trasformata ondina

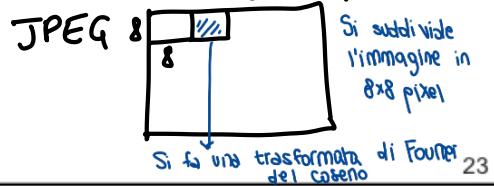
$256 \times 256 \times 24 = \dots \rightarrow 8 \text{ bit per ogni canale di colore}$

From the comparison,

- JPEG fails above 40:1 compression
- JPEG2000 survives

Note: images courtesy of www.dspworx.com

SCALA DI GRIGIO 0, ..., 255
 $256 = 2^8$
8 BIT per PIXEL

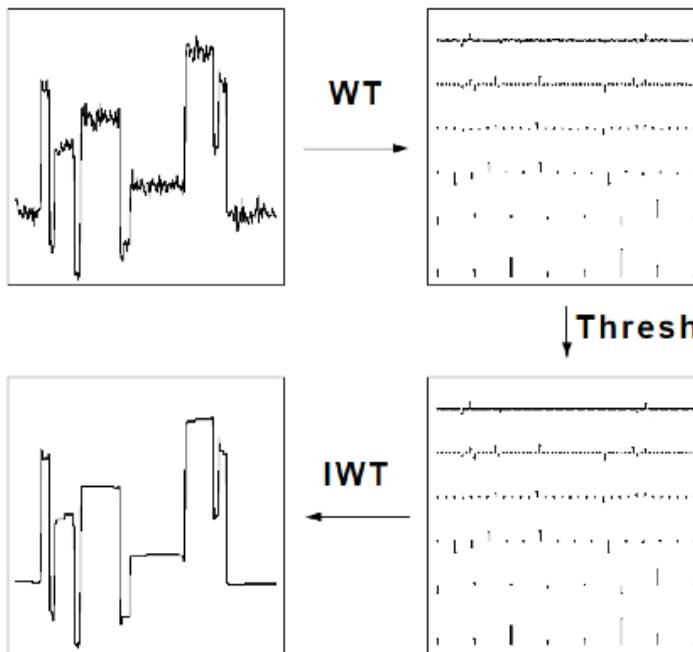


Alcuni di questi coefficienti sono trascurabili
e tengo solo quelli delle basse frequenze
Ricostruisco poi la versione originale
con la decompressione

Denoising as an Application

→ Cancellazione del rumore

La trasformata ondina e' meglio dei filtri: si perdono meno info



This is a
thresholding
method adapted to
wavelet
singularities

Denoising as an Application

Original image	Original image + i.i.d. Gaussian noise	Denoised in Fourier domain	Denoised in Wavelet domain
		Applicando Fourier	Applicando basi di ondine

