AUDIOVISUAL COMMUNICATIONS LCAV

Mathematical Foundations of Signal Processing

Module 5: Multirate Sequences and Systems

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Multirate Systems

Why Multirate?

Multirate is nothing but the combination of the downsampling, upsampling, and filtering operators, where the result of such a combination is strongly dependent on the order of execution of the operators.

A classical example of a multirate system is the changing of the sampling rate: Many everyday devices implement a change of sampling rate.

TV

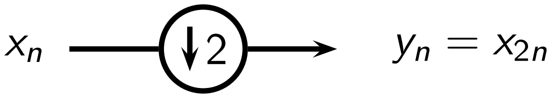
Mobile screen

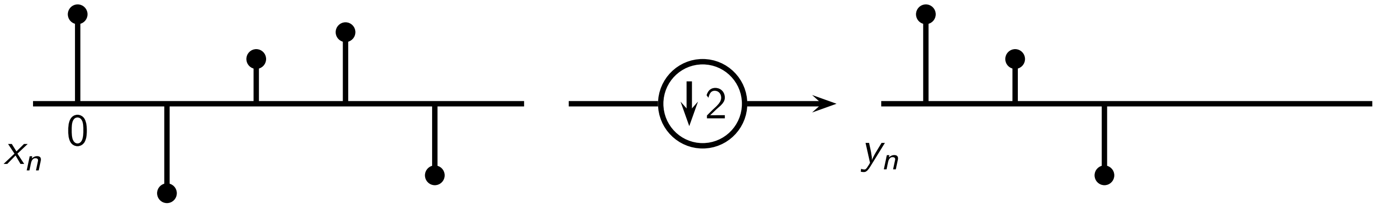
- CDs: 44.1kHz
- DVD: 48 kHz

Audio processing

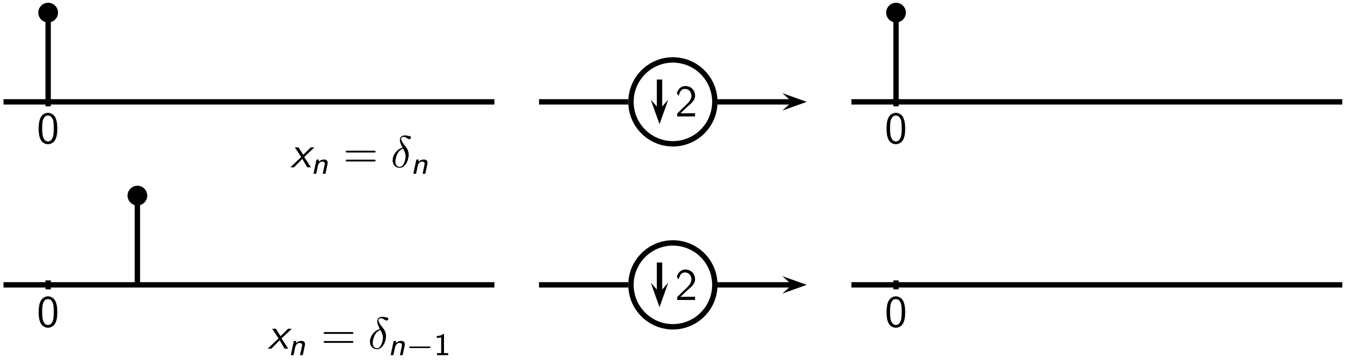
Downsampling

$N = 2$





Downsampling: Origin of time



- What happens if $x_n = \delta_{n-2}$?
- Downsampling is shift variant **but** periodically shift variant
- Downsampling is lossy → con perdita

Downsampling: Important Formulas

Some important equalities related to downsampling

By 2:

- Time domain

$$y_n = x_{2n}$$

- z-domain

$$Y(z) \;=\; \frac{1}{2}\left[X(z^{1/2}) + X(-z^{1/2})\right]$$

- Fourier domain

$$Y(e^{j\omega}) \;=\; \frac{1}{2}\left[X(e^{j\omega/2}) + X(e^{j(\omega-2\pi)/2})\right]$$

Downsampling: Important Formulas

By 2 (proof):

- z-domain

$$Y(z) = \sum_{k\in\mathbb{Z}} x_{2k} z^{-k} = \sum_{p=2k,k\in\mathbb{Z}} x_p z^{-\frac{p}{2}}$$

To keep only even terms we can use the fact that

$$z^k + (-z)^k = \begin{cases} 2z^k & k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$Y(z) = \frac{1}{2}\sum_{k\in\mathbb{Z}} x_k (z^{\frac{1}{2}})^{-k} + \frac{1}{2}\sum_{k\in\mathbb{Z}} x_k (-z^{\frac{1}{2}})^{-k} = \frac{1}{2}[X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})]$$

- Fourier domain

We simply have $z = e^{j\omega}$ and $e^{j(\omega-2\pi)/2} = e^{j\omega/2}e^{j2\pi/2} = -e^{j\omega/2}$

Downsampling: Important Formulas

By N:

- Time domain

$$y_n = x_{Nn}$$

- z-domain

$$Y(z) = \frac{1}{N}\sum_{n=0}^{N-1} X(W_N^n z^{1/N})\,, \text{ where } W_N^n = e^{-j(2\pi/N)n}$$

- Fourier domain

$$Y(e^{j\omega}) = \frac{1}{N}\sum_{n=0}^{N-1} X(e^{j(\omega-2\pi n)/N})$$

Downsampling: Important Formulas

By N (proof):

- z-domain

$$Y(z) = \sum_{k\in\mathbb{Z}} x_{Nk} z^{-k} = \sum_{m=kN, k\in\mathbb{Z}} x_m z^{-\frac{m}{N}}$$

Using the identity for roots of unity: $\frac{1}{N}\sum_{n=0}^{N-1} W_N^{-nm} = \begin{cases} 1 & \text{when } m = kN \\ 0 & \text{otherwise} \end{cases}$

We obtain:

$$Y(z) = \frac{1}{N}\sum_{m=kN} x_m \left(\sum_{n=0}^{N-1} W_N^{-nm} z^{-\frac{m}{N}}\right) = \frac{1}{N}\sum_{m\in\mathbb{Z}}\sum_{n=0}^{N-1} x_m \left(W_N^n z^{\frac{1}{N}}\right)^{-m}$$

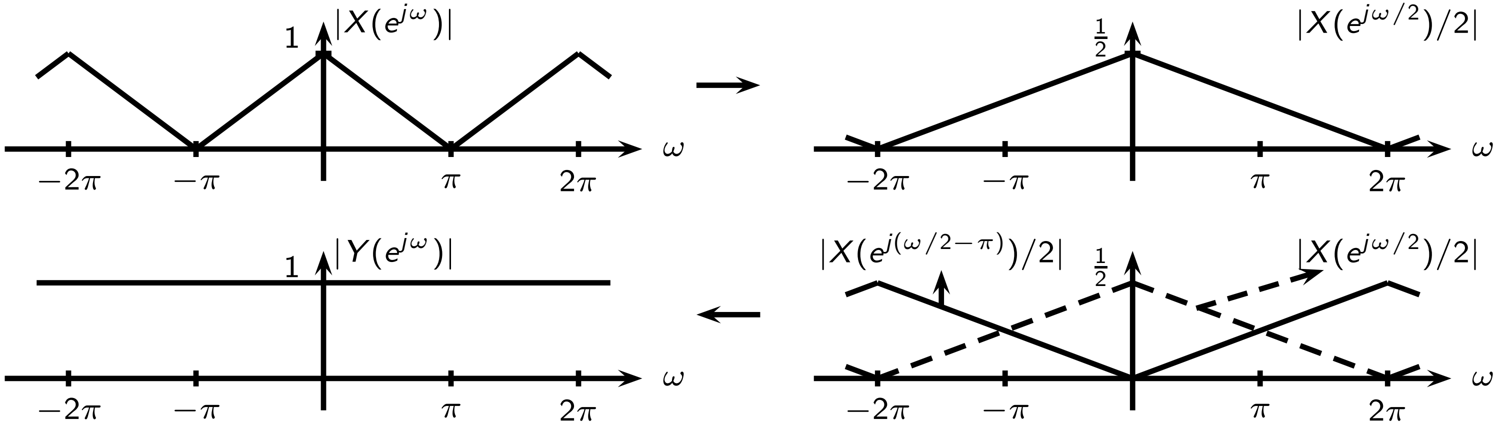
$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m\in\mathbb{Z}} x_m \left(W_N^n z^{\frac{1}{N}}\right)^{-m} = \frac{1}{N}\sum_{n=0}^{N-1} X\left(W_N^n z^{\frac{1}{N}}\right)$$

- Fourier domain

We simply have $z = e^{j\omega}$

Downsampling: Spectrum

By 2:



- Shrink in time → Expand in frequency
- Loss of information in signal reflected by overlapping spectra (aliasing)

By N:

- $(N-1)$ shifted copies overlap with the central one and create aliasing
- Copies are N times wider than the initial spectrum

Downsampling: Operators

By 2:

- Start with identity, take out every other row: keep even rows only (remember: origin of time is important)

$$
\begin{bmatrix} \vdots \\ y_{-1} \\ \boxed{y_0} \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}
=
\underbrace{\begin{bmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}}_{D_2}
\begin{bmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix}
=
\begin{bmatrix} \vdots \\ x_{-2} \\ \boxed{x_0} \\ x_2 \\ x_4 \\ \vdots \end{bmatrix}
$$

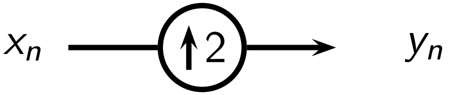
$$
y = D_2\, x
$$

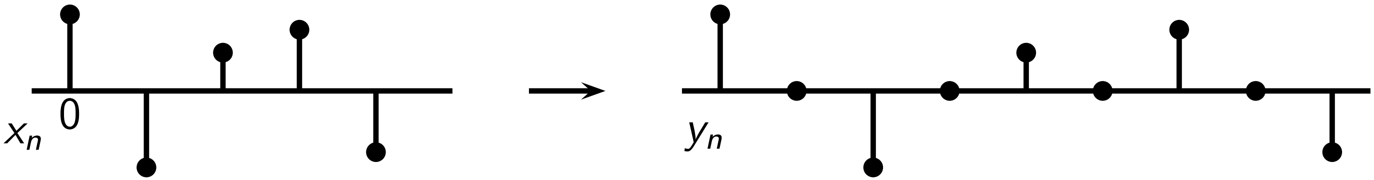
By N:

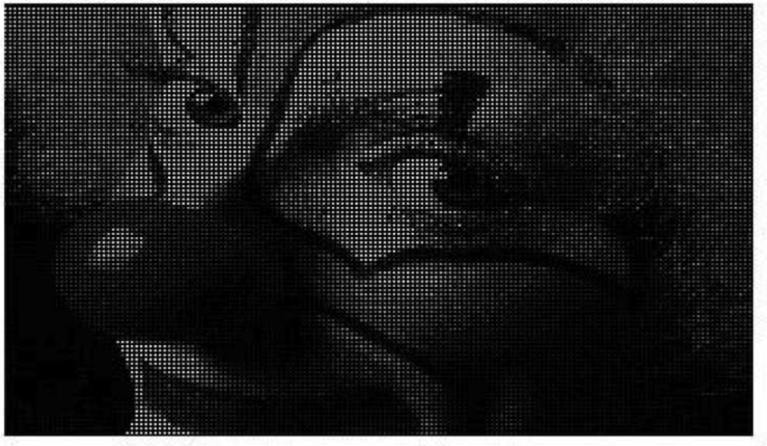
- D_N: Remove $(N-1)$ rows from identity (keep the origin)

Upsampling

- By 2:







- By N: Introduce $N-1$ zeros in-between every two samples

Upsampling: Formulas

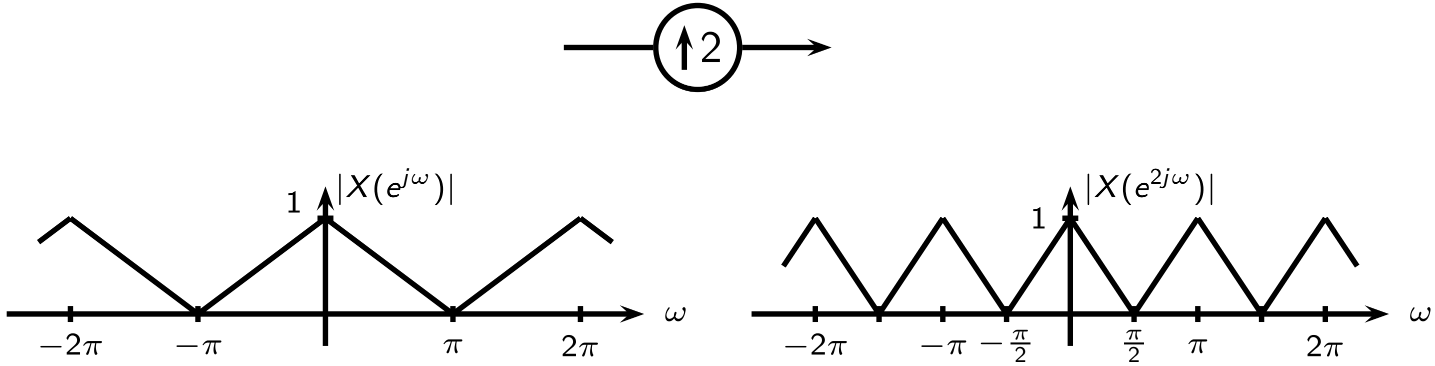
By 2:

- Time domain : $y_n = \begin{cases} x_{\frac{n}{2}} & n = 2k \\ 0 & \text{otherwise} \end{cases}$
- z-domain : $Y(z) \; = \; X(z^2)$
- Fourier domain : $Y(e^{j\omega}) \; = \; X(e^{j2\omega})$

By N:

- Time domain : $y_n = \begin{cases} x_{\frac{n}{N}} & n = Nk \\ 0 & \text{otherwise} \end{cases}$
- z-domain : $Y(z) \; = \; X(z^N)$
- Fourier domain : $Y(e^{j\omega}) \; = \; X(e^{jN\omega})$

Upsampling: Spectrum



- Expand in time → shrink in frequency
- Spectrum shrinks by N
- Lossy or not ?

Upsampling: Operator

By 2:

- Introduce an all-zero row in between every pair of rows of the identity matrix

$$
\begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \\ \boxed{y_0} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \end{bmatrix}
=
\underbrace{\begin{bmatrix}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \boxed{1} & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}}_{U_2}
\begin{bmatrix} \vdots \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}
=
\begin{bmatrix} \vdots \\ x_{-1} \\ 0 \\ \boxed{x_0} \\ 0 \\ x_1 \\ 0 \\ x_2 \\ \vdots \end{bmatrix}
$$

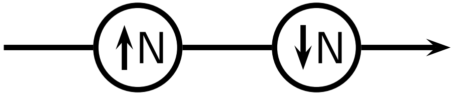
$$
y = U_2 \, x
$$

By N:

- U_N: Introduce (N-1) zero rows in the identity

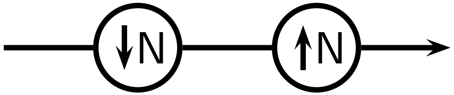
Are downsampling and upsampling friends?

- $U_N = D_N^T$
- $D_N U_N = I$



no loss

- $U_N D_N \neq I$



lossy

- $P = U_2 D_2$

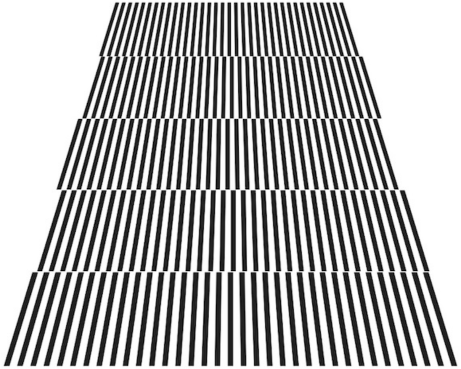
  is an orthogonal projection onto subspace of all even indexed samples

Filtering, downsampling, & upsampling

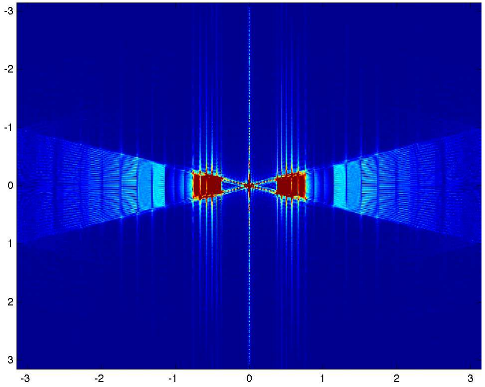
Filtering & downsampling

Should we filter before or after downsampling? Let's consider the following setup:

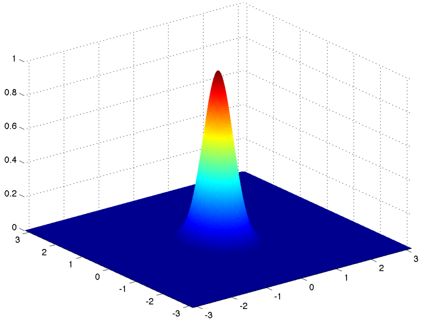
Original



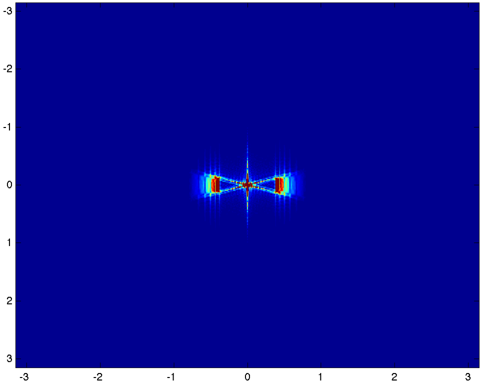
Spectrum original



Low pass filter



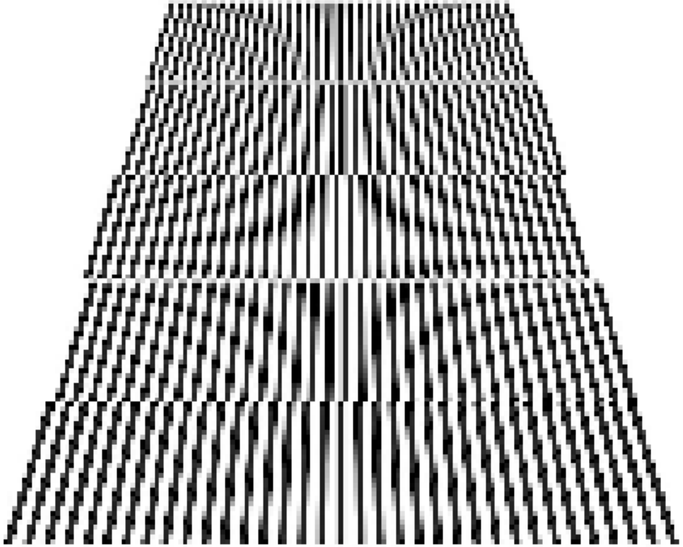
Spectrum original filtered

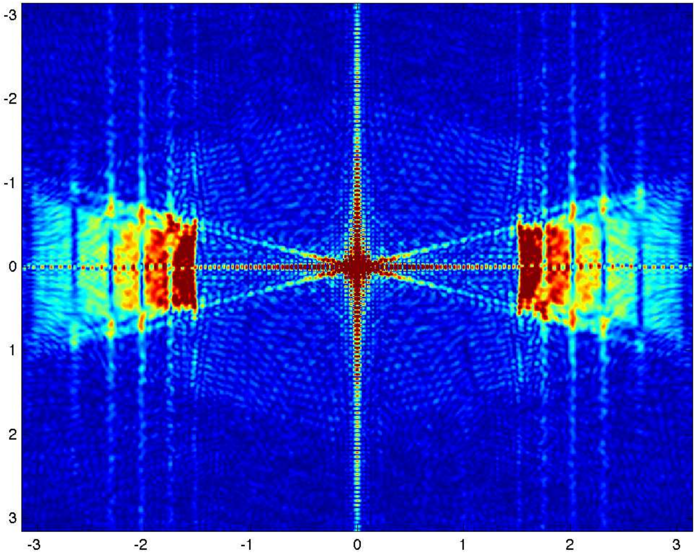


Filtering & downsampling

Original downsampled by 4

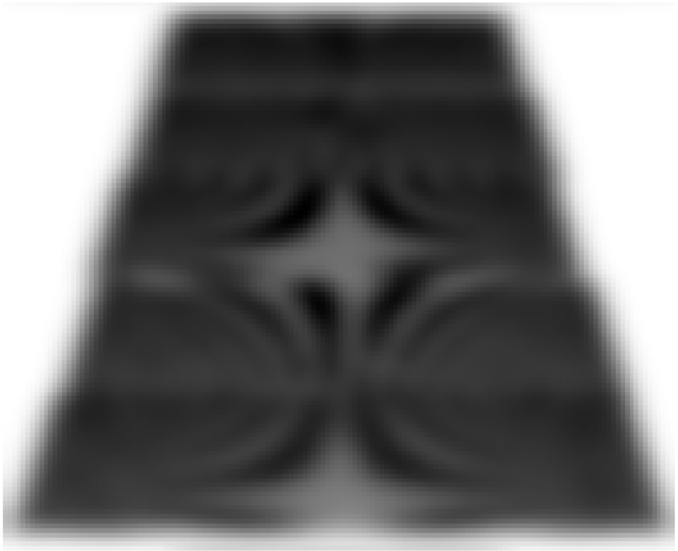
ALIASING.

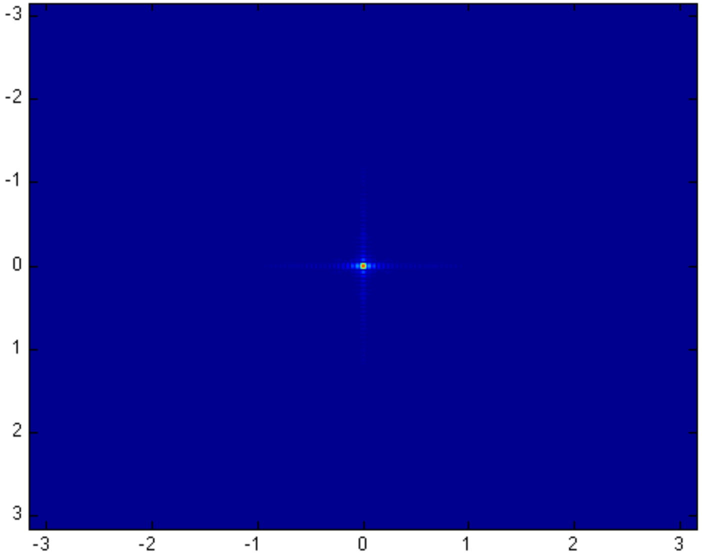




Filtering & downsampling

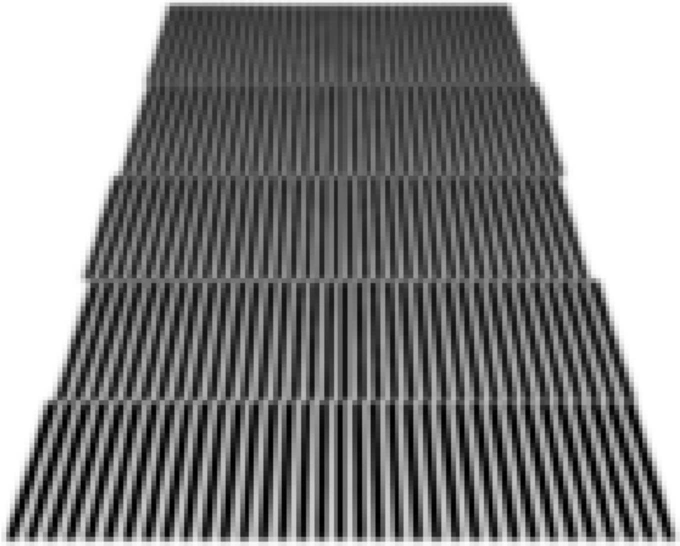
Filtering after downsampling by 4

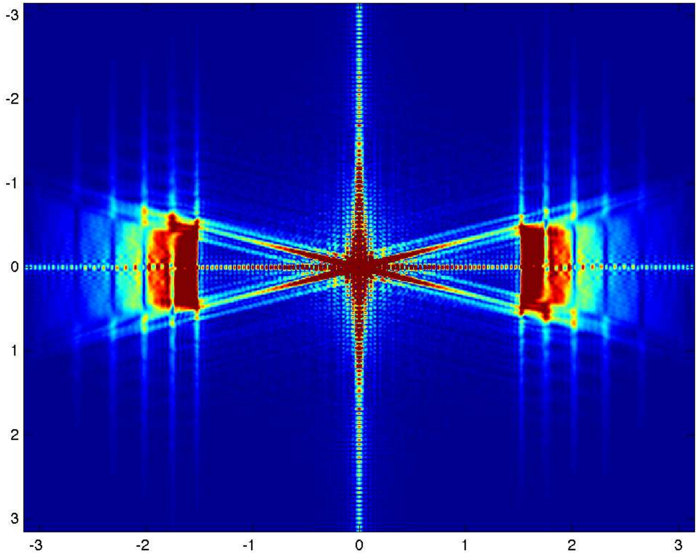




Filtering & downsampling

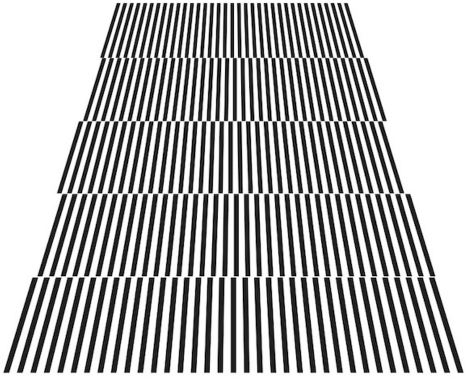
Filtering before downsampling by 4



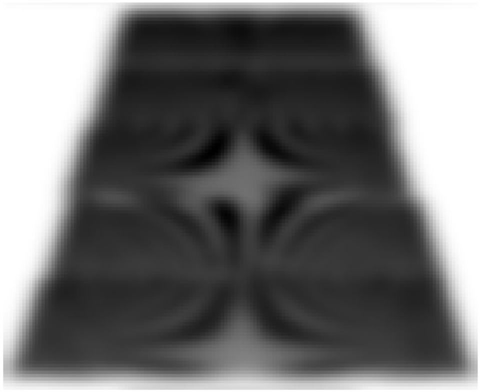


Filtering & downsampling

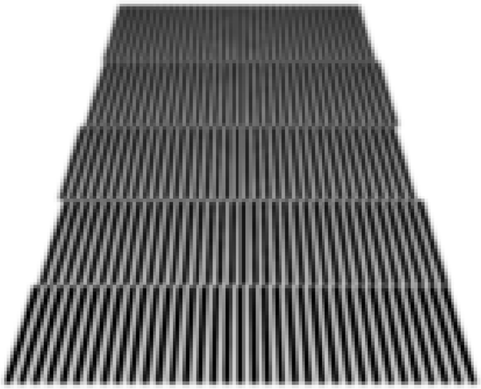
Filtering before or after downsampling?



Original



After



Before

Filtering & downsampling

Filtering before downsampling

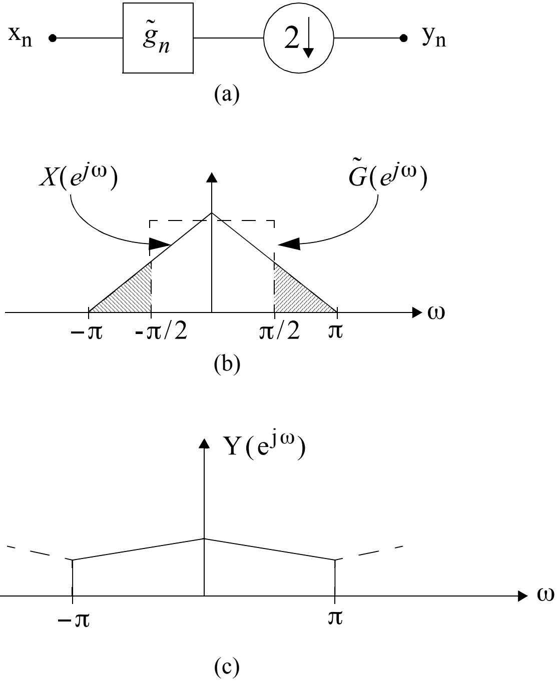
Let's see what happens in mathematical terms (consider a downsampling by 2).

$$y = \begin{bmatrix} \vdots \\ y_{-1} \\ \boxed{y_0} \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \widetilde{g}_1 & \widetilde{g}_0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \widetilde{g}_3 & \widetilde{g}_2 & \widetilde{g}_1 & \boxed{\widetilde{g}_0} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \widetilde{g}_3 & \widetilde{g}_2 & \widetilde{g}_1 & \widetilde{g}_0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \widetilde{g}_3 & \widetilde{g}_2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}}_{D_2 \widetilde{G}} \begin{bmatrix} \vdots \\ x_{-3} \\ x_{-2} \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = D_2 \widetilde{G} \, x$$

Filtering & downsampling

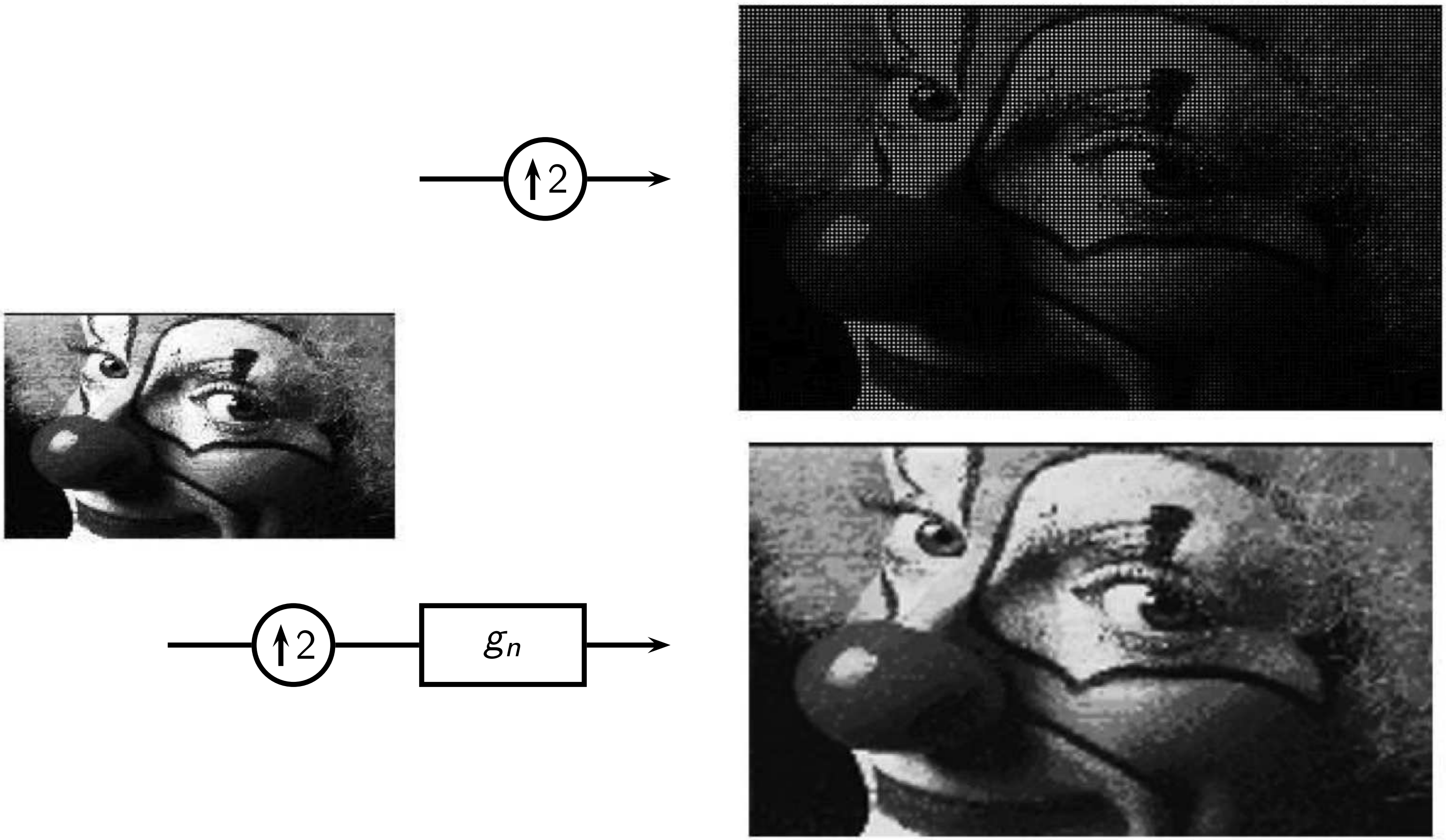
Filtering before downsampling

Low pass filtering enables to reduce bandwidth to $[-\pi/2, \pi/2]$ to avoid aliasing whaten downsampling



Filtering & upsampling

Filtering after upsampling



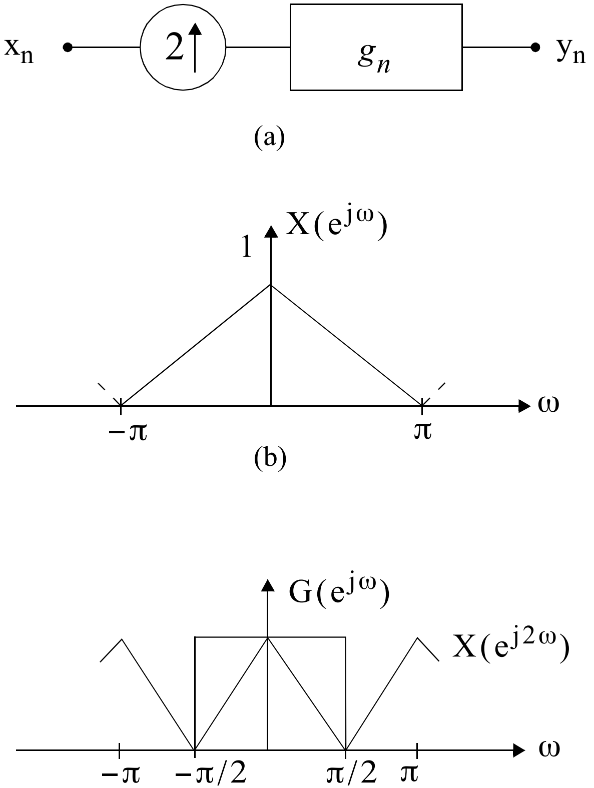
Filtering & upsampling

Filtering after upsampling

Filtering after upsampling enables to keep the bandwidth in $[-\pi/2, \pi/2]$

It amounts to interpolating the sequence y_n so as to fill in the zeros

$$y = \begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \\ \boxed{y_0} \\ y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & g_2 & g_0 & 0 & 0 & \cdots \\ \cdots & g_3 & g_1 & 0 & 0 & \cdots \\ \cdots & 0 & g_2 & \boxed{g_0} & 0 & \cdots \\ \cdots & 0 & g_3 & g_1 & 0 & \cdots \\ \cdots & 0 & 0 & g_2 & g_0 & \cdots \\ \cdots & 0 & 0 & g_3 & g_1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}}_{GU_2} \begin{bmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ \boxed{x_0} \\ x_1 \\ \vdots \end{bmatrix}$$



(a)

(b)

(c)

Downsampling, upsampling & filtering

When $\widetilde{g}_n^* = g_{-n}$, $D_2\widetilde{G}$ and GU_2 are adjoint

$$
\begin{aligned}
(D_2\widetilde{G})^* &= \begin{bmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \widetilde{g}^*_{-2} & \widetilde{g}^*_{0} & \widetilde{g}^*_{2} & \widetilde{g}^*_{4} & \widetilde{g}^*_{6} & \cdots \\
\cdots & \widetilde{g}^*_{-3} & \widetilde{g}^*_{-1} & \widetilde{g}^*_{1} & \widetilde{g}^*_{3} & \widetilde{g}^*_{5} & \cdots \\
\cdots & \widetilde{g}^*_{-4} & \widetilde{g}^*_{-2} & \boxed{\widetilde{g}^*_{0}} & \widetilde{g}^*_{2} & \widetilde{g}^*_{4} & \cdots \\
\cdots & \widetilde{g}^*_{-5} & \widetilde{g}^*_{-3} & \widetilde{g}^*_{-1} & \widetilde{g}^*_{1} & \widetilde{g}^*_{3} & \cdots \\
\cdots & \widetilde{g}^*_{-6} & \widetilde{g}^*_{-4} & \widetilde{g}^*_{-2} & \widetilde{g}^*_{0} & \widetilde{g}^*_{2} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix} \\
&= \begin{bmatrix}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & g_{2} & g_{0} & g_{-2} & g_{-4} & g_{-6} & \cdots \\
\cdots & g_{3} & g_{1} & g_{-1} & g_{-3} & g_{-5} & \cdots \\
\cdots & g_{4} & g_{2} & \boxed{g_{0}} & g_{-2} & g_{-4} & \cdots \\
\cdots & g_{5} & g_{3} & g_{1} & g_{-1} & g_{-3} & \cdots \\
\cdots & g_{6} & g_{4} & g_{2} & g_{0} & g_{-2} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix} = GU_2
\end{aligned}
$$

Multirate Identities

Filters with impulse responses orthogonal to even shifts

Consider a filter with impulse response orthogonal to its even shifts, *that is,*

$$\langle g_n, g_{n-2k} \rangle_n = \delta_k \; ,$$

which corresponds to the deterministic autocorrelation of g downsampled by 2.

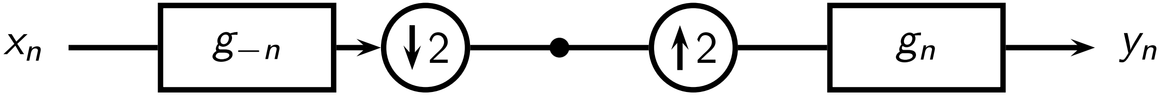
We have

$$D_2 G^* G U_2 = I \, .$$

In particular

$$G U_2 D_2 G^* = P \, ,$$

is an orthogonal projection onto the subspace spanned by g_n and its even shifts.



Filters with impulse responses orthogonal to even shifts

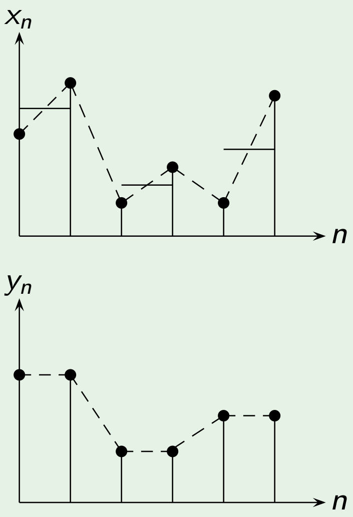
Example (Haar smoothing operator)

- Filter: two point average (normalized to unit norm), or Haar lowpass

$$g_n = \frac{1}{\sqrt{2}}[\delta_n + \delta_{n-1}]$$

- Then

$$\begin{aligned}
P &= GU_2 \cdot D_2 G^T \\
&= \frac{1}{2}\begin{bmatrix} \ddots & \vdots & \vdots & \ddots \\ \cdots & 1 & 0 & \cdots \\ \cdots & 1 & 0 & \cdots \\ \cdots & 0 & 1 & \cdots \\ \cdots & 0 & 1 & \cdots \\ \ddots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & 1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & 1 & 1 & 0 & 0 & \cdots \\ \cdots & 1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\end{aligned}$$



Filters with impulse responses orthogonal to even shifts

Quadrature mirror formula

The orthogonality to even translates $\langle g_n, g_{n-2k} \rangle_n = \delta_k$ gives

$$\frac{1}{2}\left[G(z^{1/2})G(z^{-1/2}) + G(-z^{1/2})G(-z^{-1/2})\right] = 1\,,$$

that in the frequency domain reads

$$|G(e^{j\omega})|^2 + |G(e^{j(\omega+\pi)})|^2 = 2,$$

the latter being called the *quadrature mirror formula* or *power complementarity*

Filters with impulse responses orthogonal to even shifts

Quadrature mirror formula: derivation

The orthogonality to even translates $\langle g_n, g_{n-2k}\rangle_n = \delta_k$ can be seen as a deterministic autocorrelation of g

$$a_k = \langle g_n, g_{n-k}\rangle_n \, ,$$

downsampled by 2, that is $a_{2k} = \delta_k$.

Assuming now a real g, we have

$$A(z) = G(z)G(z^{-1}) \, .$$

Keeping only the even terms can be accomplished by adding $A(z)$ and $A(-z)$ and dividing by 2:

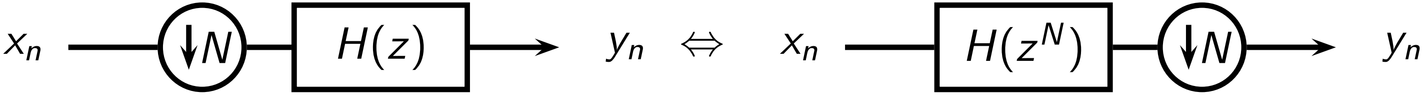
$$\frac{1}{2}(A(z) + A(-z)) \;=\; \frac{1}{2}(G(z)G(z^{-1}) + G(-z)G(-z^{-1})) \;=\; 1 \, ,$$

which on the unit circle leads to

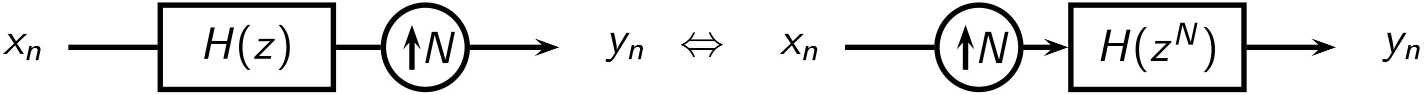
$$|G(e^{j\omega})|^2 + |G(e^{j(\omega+\pi)})|^2 = 2$$

Interchange of Multirate Operations and Filtering

- Interchange of filtering and downsampling

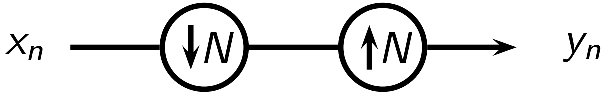


- Interchange of filtering and upsampling

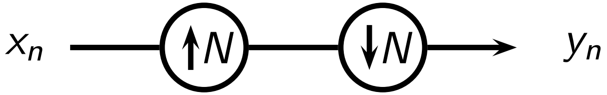


Commutativity of Multirate Operations

- Commutativity

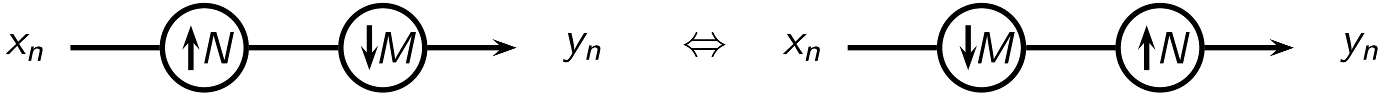


$U_N D_N = P$ projection



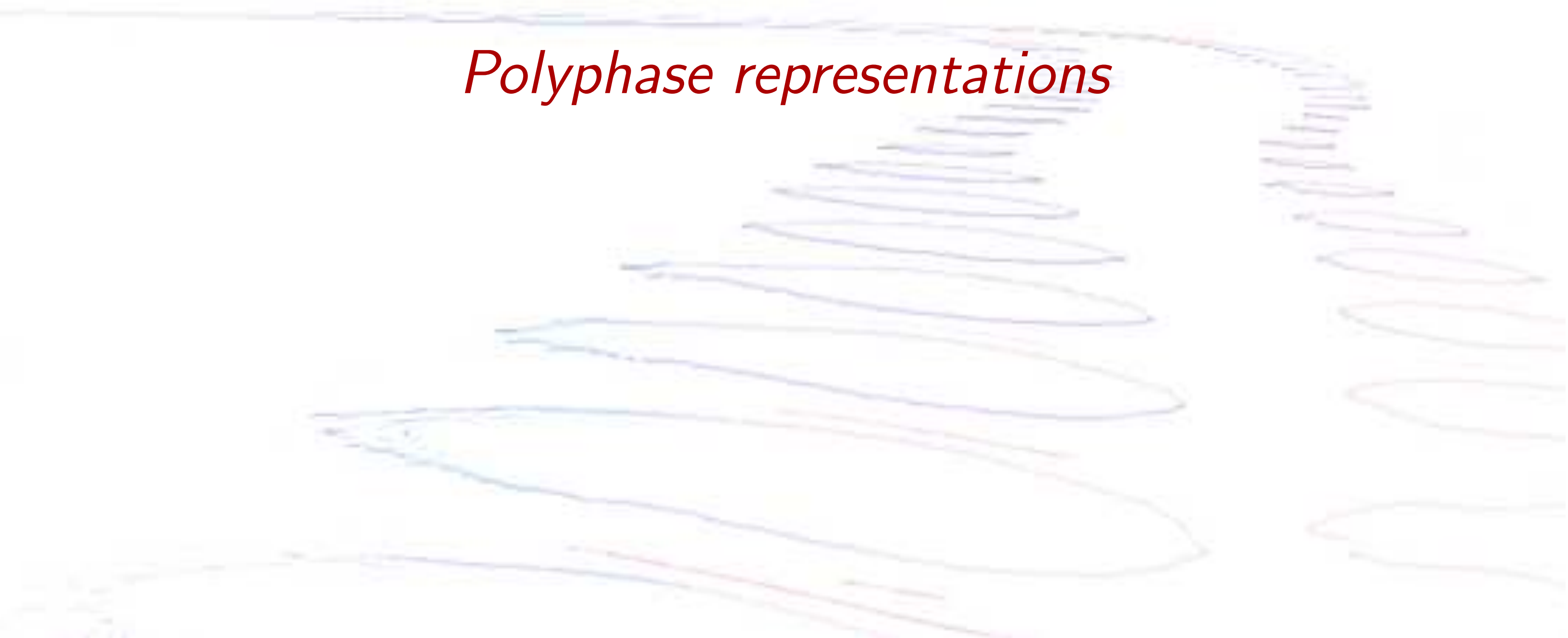
$D_N U_N = I$ identity

Upsampling and downsampling for the same integer do not commute.



Upsampling and downsampling for the different integers N and M, respectively, do commute when $gcd(N, M) = 1$.

Polyphase representations



Polyphase representation

Multirate processing brings a major twist to signal processing: shift invariance is replaced by periodic shift variance, represented by block-Toeplitz matrices.

The polyphase representation is a tool dealing with such periodic shift variance. It is a key method to transform single-input single-output linear periodically shift varying systems into multiple-input multiple-output linear shift invariant systems.

Polyphase representation: $N = 2$

Forward and inverse polyphase transforms

- Forward polyphase transform

  Decompose the signal into odd and even parts

  $$X(z) = \sum_{n\in\mathbb{Z}} x_n z^{-n} = \sum_{n\in\mathbb{Z}} x_{2n} z^{-2n} + \sum_{n\in\mathbb{Z}} x_{2n+1} z^{-2n-1} = X_0(z^2) + z^{-1} X_1(z^2)\,,$$

  where

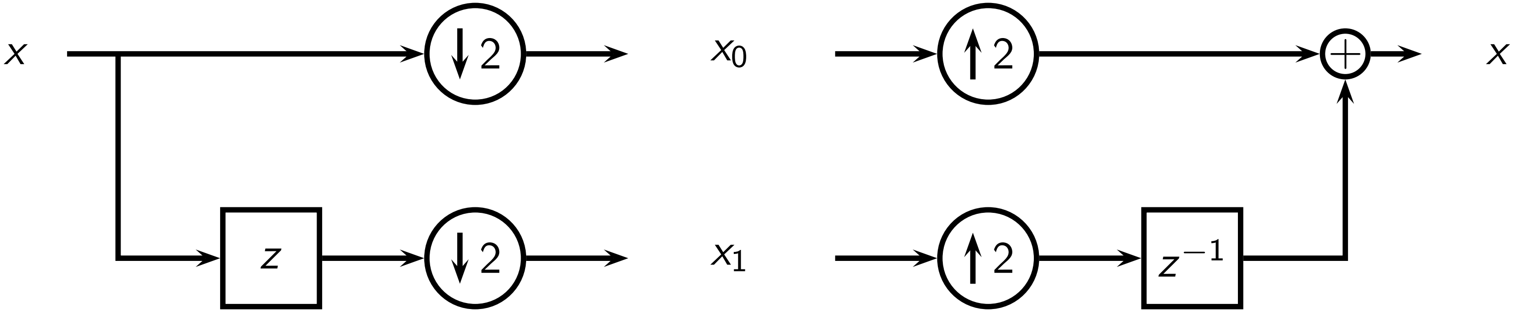
  $$X_0(z) = \sum_{n\in\mathbb{Z}} x_{2n} z^{-n}, X_1(z) = \sum_{n\in\mathbb{Z}} x_{2n+1} z^{-n}\,.$$

- Inverse polyphase transform

  Take polyphase components and reconstruct X

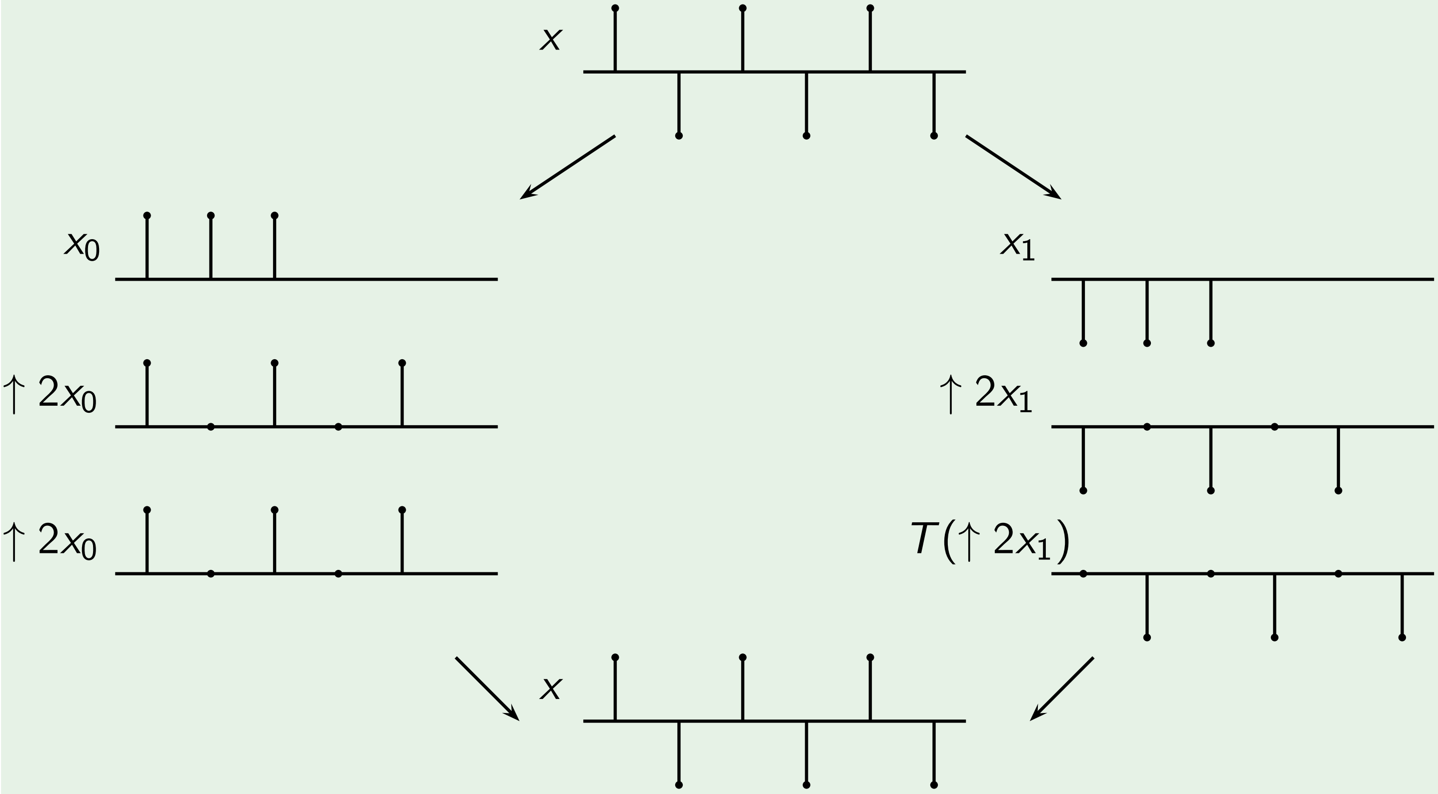
Polyphase representation: $N = 2$

Forward and inverse polyphase transforms



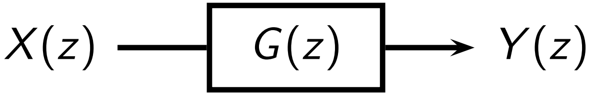
Polyphase representation: $N = 2$

Example

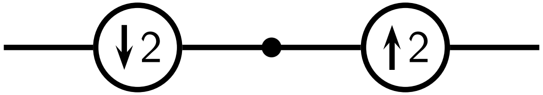


Filtering in Polyphase Domain

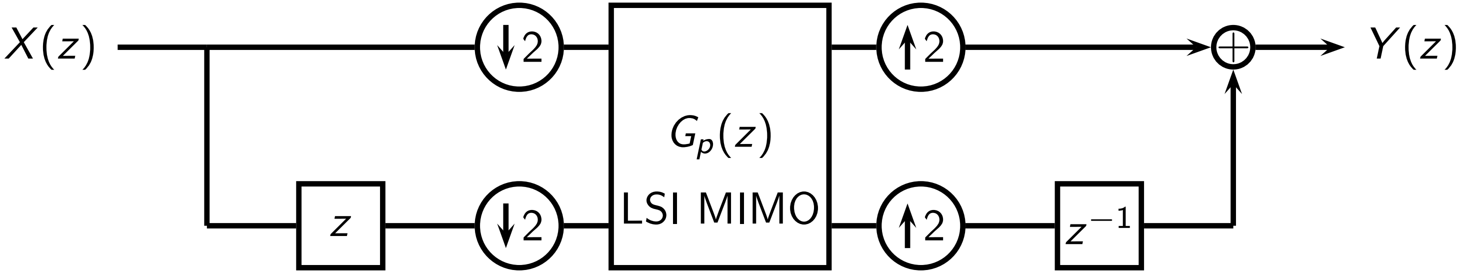
- Linear Shit-Invariant - LSI SISO systems



- Multirate leads to Linear Periodically Shift-Varying - LPSV SISO systems



- Polyphase takes SISO linear periodically shift varying system to MIMO linear shift invariant system



Filtering in Polyphase Domain

Given a filter (LSI system) with impulse response g_n, we have

$$g_{0,n} = g_{2n} \qquad G_0(z) = \sum_{n\in\mathbb{Z}} g_{2n} z^{-n}\,,$$

$$g_{1,n} = g_{2n+1} \qquad G_1(z) = \sum_{n\in\mathbb{Z}} g_{2n+1} z^{-n}\,,$$

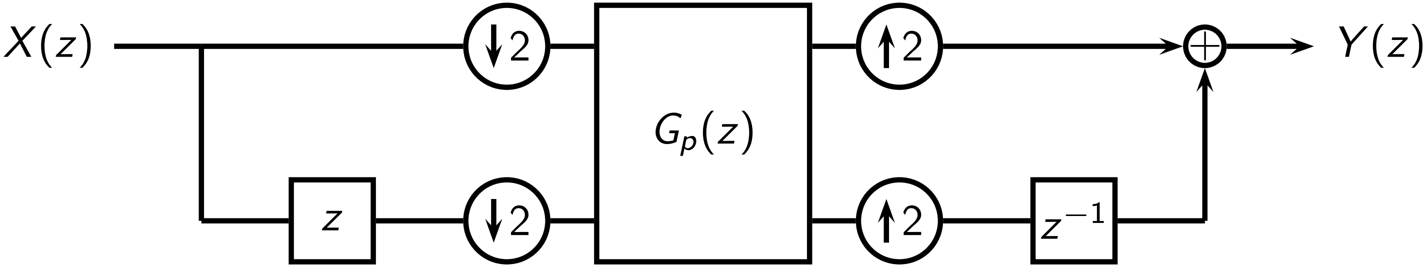
obtaining

$$G(z) = G_0(z^2) + z^{-1} G_1(z^2)\,.$$

Such a decomposition can be described vie the polyphase representation, mapping $G(z)$ into

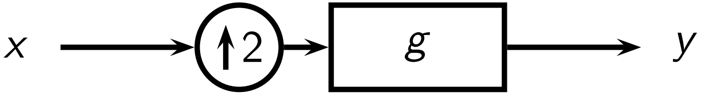
$$G_p(z) = \begin{bmatrix} G_0(z) & z^{-1}G_1(z) \\ G_1(z) & G_0(z) \end{bmatrix}$$

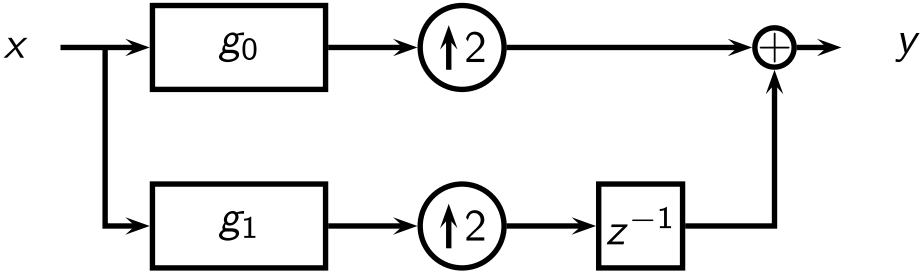
Filtering in Polyphase Domain



$$
\begin{aligned}
Y(z) &= \begin{bmatrix} 1 & z^{-1} \end{bmatrix} G_p(z^2) \begin{bmatrix} X_0(z^2) \\ X_1(z^2) \end{bmatrix} \\
&= \begin{bmatrix} 1 & z^{-1} \end{bmatrix} \begin{bmatrix} G_0(z^2) & z^{-2}G_1(z^2) \\ G_1(z^2) & G_0(z^2) \end{bmatrix} \begin{bmatrix} X_0(z^2) \\ X_1(z^2) \end{bmatrix} \\
&= \begin{bmatrix} 1 & z^{-1} \end{bmatrix} \begin{bmatrix} G_0(z^2)X_0(z^2) + z^{-2}G_1(z^2)X_1(z^2) \\ G_1(z^2)X_0(z^2) + G_0(z^2)X_1(z^2) \end{bmatrix} \\
&= G_0(z^2)X_0(z^2) + z^{-2}G_1(z^2)X_1(z^2) + z^{-1}G_1(z^2)X_0(z^2) + z^{-1}G_0(z^2)X_1(z^2) \\
&= [G_0(z^2) + z^{-1}G_1(z^2)][X_0(z^2) + z^{-1}X_1(z^2)] = G(z)X(z)\,.
\end{aligned}
$$

Polyphase representation of upsampling + filtering



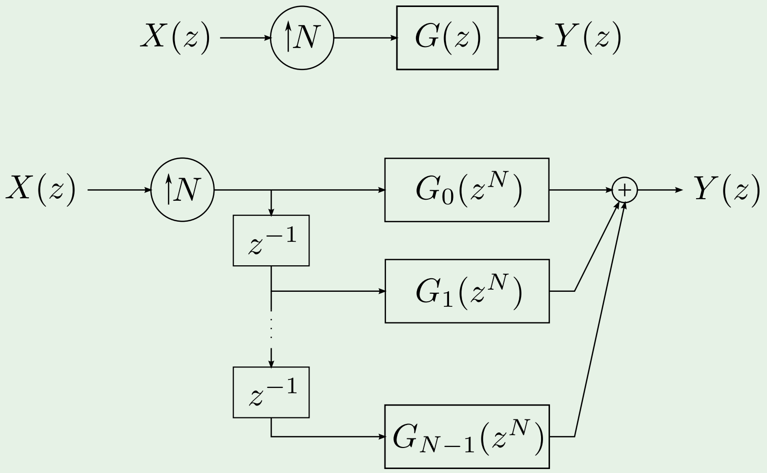


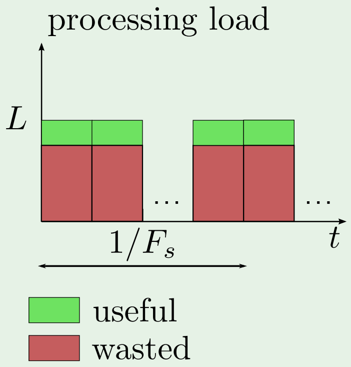
$$Y(z) \;=\; G_0(z^2)X(z^2) + z^{-1}G_1(z^2)X(z^2) \;=\; Y_0(z^2) + z^{-1}Y_1(z^2)$$

Polyphase representation of filtering + upsampling by *N*

Polyphase implementation of upsampling FIR filters

- Upsampling followed by filtering results in many multiplications by zero ($N - 1$ out of N filter inputs are zero)

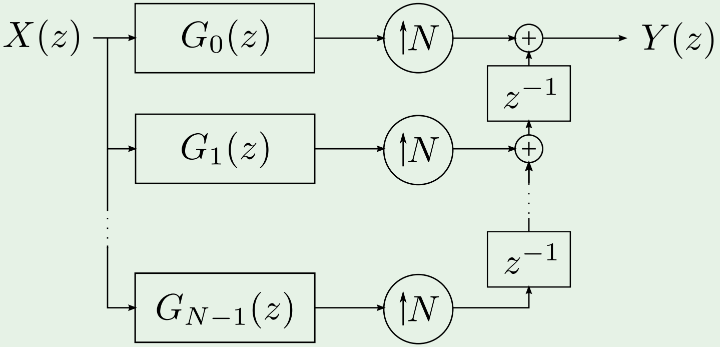


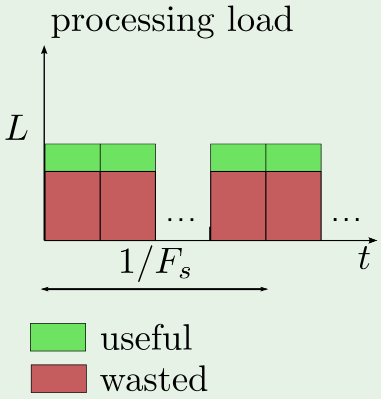


Polyphase representation of filtering + upsampling by *N*

Polyphase implementation of upsampling FIR filters

- Interchanging filtering and downsampling moves the processing to the lower rate F_s (as opposed to NF_s)
- Naive implementation
  - Compute the outputs of all polyphase filters
  - Add delayed upsampled outputs of all polyphase filters
  - Processing load decreases, but still adding many zeros (every output sampled obtained from only one non-zero usamplers' outputs)

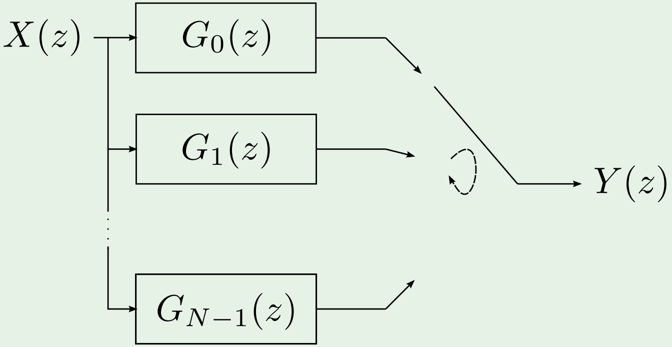


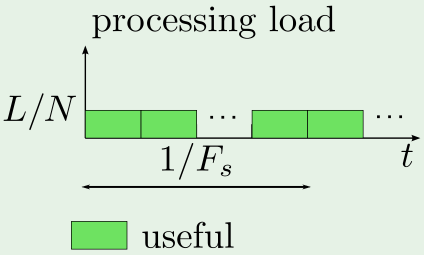


Polyphase representation of filtering + upsampling by N

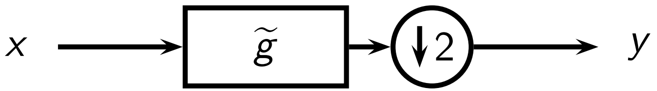
Polyphase implementation of upsampling FIR filters

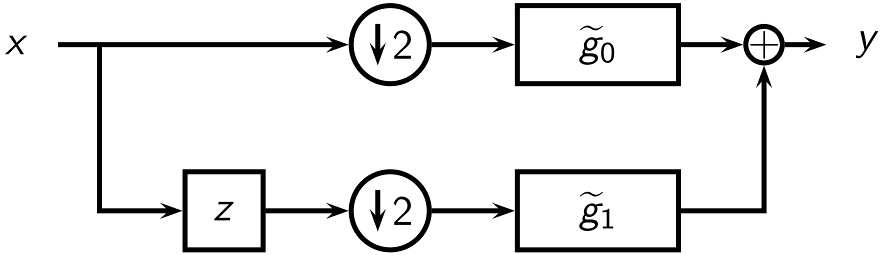
- For every output sample, only one polyphase component is computed
- Filter delay lines loaded with a new input sample every N output samples
- The processing load decreased by a factor of N





Polyphase representation of filtering + downsampling



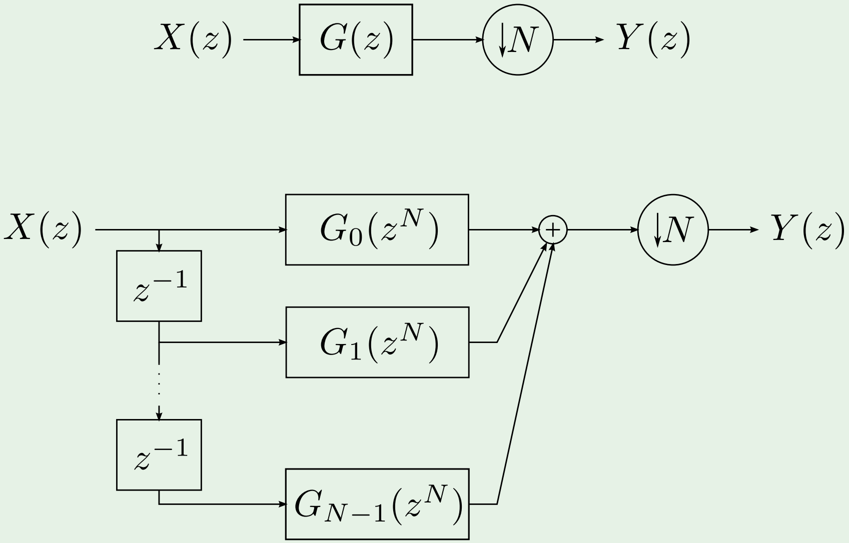


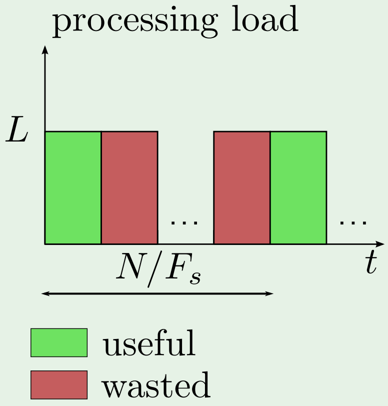
$$\begin{aligned}
\widetilde{g}_{0,n} &= \widetilde{g}_{2n} & \overset{\text{ZT}}{\longleftrightarrow} \quad & \widetilde{G}_0(z) = \sum_{n\in\mathbb{Z}} \widetilde{g}_{2n} z^{-n}, \\
\widetilde{g}_{1,n} &= \widetilde{g}_{2n-1} & \overset{\text{ZT}}{\longleftrightarrow} \quad & \widetilde{G}_1(z) = \sum_{n\in\mathbb{Z}} \widetilde{g}_{2n-1} z^{-n}, \\
& & & \widetilde{G}(z) = \widetilde{G}_0(z^2) + z\widetilde{G}_1(z^2).
\end{aligned}$$

Polyphase representation of filtering + downsampling by N

Polyphase implementation of downsampling FIR filters

- Filtering followed by downsampling discards $N-1$ every N samples and wastes processing time

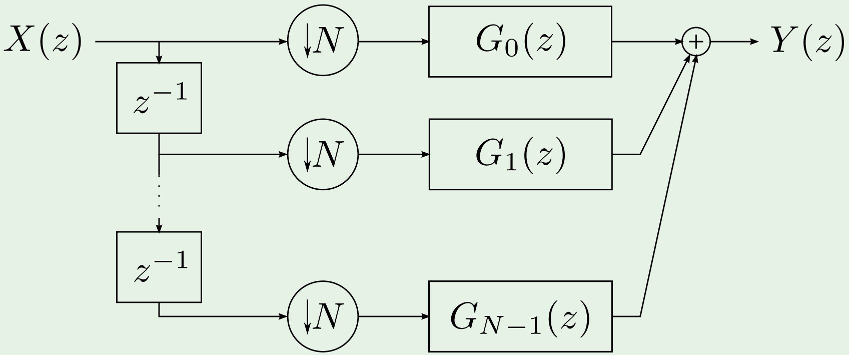


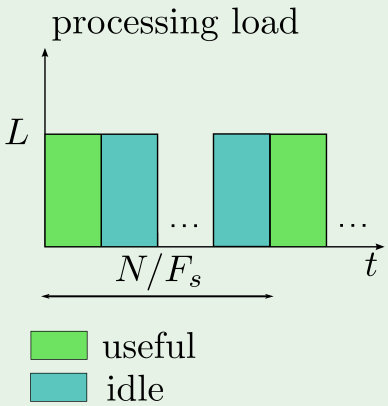


Polyphase representation of filtering + downsampling by N

Polyphase implementation of downsampling FIR filters

- Interchanging filtering and downsampling moves the processing to the lower rate F_s/N
- Naive implementation
  - Pre-load filters
  - Compute the output sample after all the polyphase filters have been loaded (every N input samples)
  - Processing load decreased, but done in the same bursts every N samples

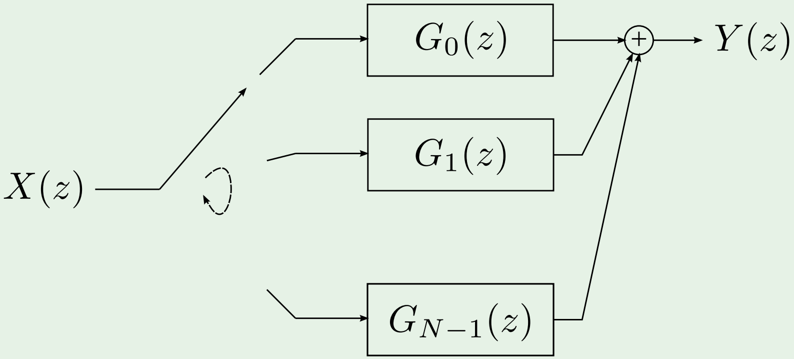


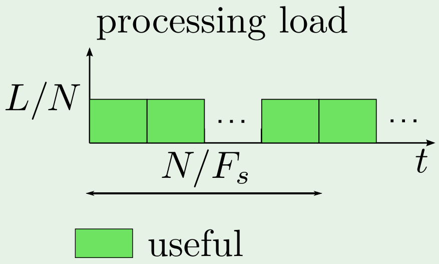


Polyphase representation of filtering + downsampling by N

Polyphase implementation of downsampling FIR filters

- Commutator runs at F_s
- One polyphase component computed per input sample and stored in accumulator
- After completing the last polyphase component, the output is emitted and accumulator reset
- One reduces the processing load by a factor of N without having idle cycles





Polyphase representation: any N

- Take indices modulo N
- N polyphase components
- $X(z) = \sum_{i=0}^{N-1} z^{-i} X_i(z^N)$ with

$$X_i(z) = \sum_{n \in \mathbb{Z}} x_{Nn+i} z^{-n} \qquad i = 0, \ldots, N-1$$