Discrete Fourier Transform

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Discrete Fourier Transform

DFT (Definition)

Given a sequence $x = [x_0, x_1, \dots, x_{N-1}]$

Prende una collezione di campioni equispaziati e produce una sequenza di coefficienti come combinazioni lineari di sinusoidi con frequenze diverse

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N} \qquad k = 0, \dots, N-1 \qquad (\star)$$

- Notations : $X = \mathcal{F}(x)$ or $X = \mathcal{F}x$ F è l'operatore della trasformata di Fourier
- $(\star)$ can be evaluated for all k : periodic extension with period equal to N

$$\begin{aligned} X_{k+N} &= \sum_{n=0}^{N-1} x_n e^{-j2\pi n(k+N)/N} \\ &= \sum_{n=0}^{N-1} x_n e^{-j2\pi nk/N} \underbrace{e^{-j2\pi n}}_{1} = X_k \end{aligned}$$

- other sequence of indices are sometimes used
  - N even : $[-N/2, N/2-1]$
  - N odd : $[-(N-1)/2, (N-1)/2]$

Inverse Discrete Fourier Transform

IDFT (Definition)

Given $X = [X_0, \ldots, X_{N-1}]$

Definizione della trasformata inversa

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N} \qquad n = 0, \ldots, N-1$$

- Notations : $\mathcal{F}^{-1}(X)$ or $\mathcal{F}^{-1}X$
- Discrete analog of the formula for the coefficients of a Fourier series
- Example :

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(\mathcal{F}\delta)_k = 1 \qquad \forall k \in \{0, \ldots, N-1\}$$

$$(\mathcal{F}^{-1}\delta)_n = 1/N \qquad \forall n \in \{0, \ldots, N-1\}$$

Discrete Fourier Transform

DFT : linear trasform $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$

- Fourier basis: $\{\phi_n^{(k)}\}_{k=0,\dots,N-1}$ orthonormal basis of $\mathbb{C}^N$

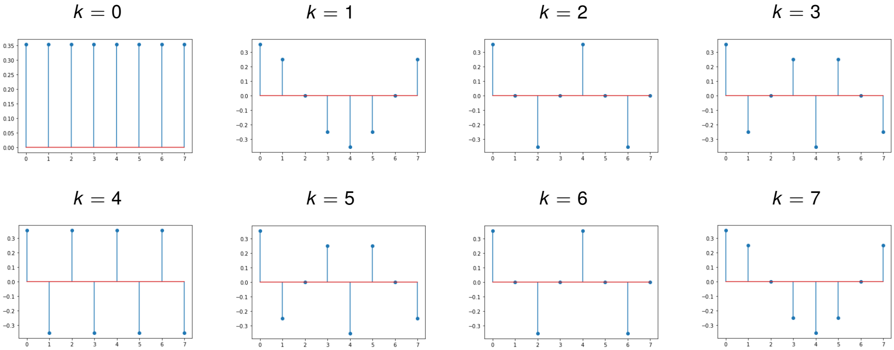
$$\phi_n^{(k)} = \frac{1}{\sqrt{N}} e^{j2\pi kn/N} = \frac{1}{\sqrt{N}} \left(\cos\left(2\pi kn/N\right) + j\sin\left(2\pi kn/N\right)\right)$$

$$\langle \phi_k, \phi_{k'} \rangle = \delta_{k-k'}$$

- Parseval identity : X_k DFT coefficients of $x \in \mathbb{C}^N$ then

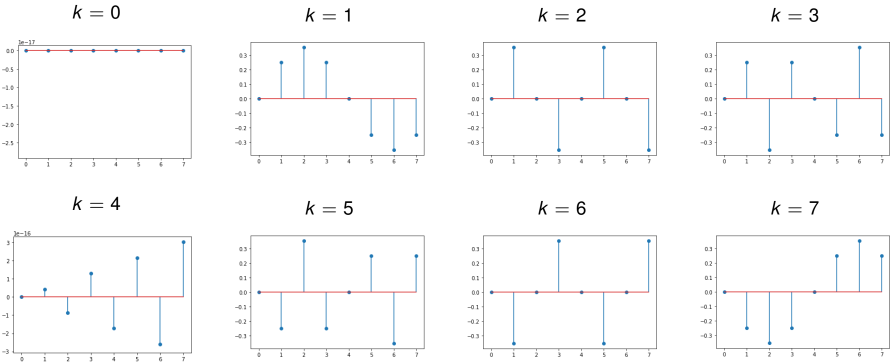
$$N \sum_{n=0}^{N-1} |x_n|^2 = \sum_{k=0}^{N-1} |X_k|^2$$

$N = 8$ – Real part



- digital frequency : k/N
- maximal digital frequency : $N/2$

$N = 8$ – Imaginary part



Discrete Fourier Transform

Properties: $x, y \in \mathbb{C}^N$

- Linearity :

$$\mathcal{F}(ax + by)_k = a\mathcal{F}(x)_k + b\mathcal{F}(y)_k$$

- Time and frequency shifting

$$\mathcal{F}(\{x_{n\pm n_0}\})_k = \mathcal{F}(x)_k e^{\pm j2\pi n_0 k/N}$$

$$\mathcal{F}(\{x_n e^{\mp j2\pi n k_0/N}\})_k = \mathcal{F}(x)_{k\pm k_0}$$

- Convolution theorem

$$\mathcal{F}(x * y)_k = \mathcal{F}(x)_k \mathcal{F}(y)_k$$

Spectral analysis with Python

Modules for spectral analysis in SciPy library : **fftpack** and **signal**

- Importing modules : `from scipy import fftpack as f`
- Call functions : `f.fft, f.ifft`

Function	**Description**
`fft`, `ifft`	General FFT and inverse FFT of a real- or complex-valued signal. The resulting frequency spectrum is complex valued.
`rfft`, `irfft`	FFT and inverse FFT of a real-valued signal.
`dct`, `idct`	The discrete cosine transform (DCT) and its inverse.
`dst`, `idst`	The discrete sine transform (DST) and its inverse.
`fft2`, `ifft2`, `fftn`, `ifftn`	The two-dimensional and the N-dimensional FFT for complex-valued signals and their inverses.
`fftshift`, `ifftshift`, `rfftshift`, `irfftshift`	Shift the frequency bins in the result vector produced by `fft` and `rfft`, respectively, so that the spectrum is arranged such that the zero-frequency component is in the middle of the array.
`fftfreq`	Calculate the frequencies corresponding to the FFT bins in the result returned by `fft`.