Convexity properties of stochastic programming models

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CHANCE CONSTRAINED PROGRAMMING

Consider the general, joint chance constraint

$$\mathbb{P}\left(\{\omega \mid \mathbf{g}(\mathbf{x}, \boldsymbol{\xi}(\omega)) \leq \mathbf{0}\}\right) \geq 1 - \alpha$$

where $\mathbf{g}$ is a vector-valued function.

A point $\hat{\mathbf{x}}$ is feasible if the set

$$S(\hat{\mathbf{x}}) = \{\omega \mid \mathbf{g}(\hat{\mathbf{x}}, \boldsymbol{\xi}(\omega)) \leq \mathbf{0}\} \tag{1}$$

has probability measure of at least $1 - \alpha$.

Let $\mathcal{F}$ be the field of all events and $\mathcal{G} \subset \mathcal{F}$ the collection of events with probability measure at least $1 - \alpha$.

Then $\hat{\mathbf{x}}$ is feasible if there exists at least one event $G \in \mathcal{G}$ such that $\mathbf{g}(\hat{\mathbf{x}}, \boldsymbol{\xi}(\omega)) \leq \mathbf{0}$, $\forall \omega \in G$:

$$\hat{\mathbf{x}} \in \bigcap_{\omega \in G} \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, \boldsymbol{\xi}(\omega)) \leq \mathbf{0}\}.$$

Chance constrained programming

Then, the feasible set $\mathcal{X}$ is the union of all such vectors:

$$\mathcal{X} = \bigcup_{G \in \mathcal{G}} \bigcap_{\omega \in G} \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, \boldsymbol{\xi}(\omega)) \leq \mathbf{0}\}$$

Even if each set $\{\mathbf{x} \mid \mathbf{g}(\mathbf{x}, \boldsymbol{\xi}(\omega)) \leq \mathbf{0}\}$ is convex for all $\boldsymbol{\xi}$, we should not expect convexity in general, since the union of convex sets is not guaranteed to be convex.

Convexity of the overall problem depends on the type of distribution (continuous vs. discrete) and its CDF, the kind of constraints (joint or not), and the functions in $\mathbf{g}$ (linear, convex/concave, or arbitrary).

CCP: AN EASY (NONTRIVIAL) CASE

Let us consider a random linear constraint $\mathbf{a}^\mathsf{T}\mathbf{x} \le b$, where $\mathbf{a} \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$, and require $\mathbb{P}\{\mathbf{a}^\mathsf{T}\mathbf{x} \le b\} \ge \eta$.

For a given vector $\mathbf{x}$, we have $\mathbf{a}^\mathsf{T}\mathbf{x} \sim \mathrm{N}(\nu, \sigma^2)$, where $\nu = \boldsymbol{\mu}^\mathsf{T}\mathbf{x}$ and $\sigma^2 = \mathbf{x}^\mathsf{T}\Sigma\mathbf{x}$.

Using the standard normal CDF $\Phi(z)$, we rewrite the constraint as:

$$\mathbb{P}\left\{\frac{\mathbf{a}^\mathsf{T}\mathbf{x} - \nu}{\sigma} \le \frac{b - \nu}{\sigma}\right\} = \Phi\left(\frac{b - \nu}{\sigma}\right) \ge \eta \quad \Longleftrightarrow \quad \frac{b - \nu}{\sigma} \ge \Phi^{-1}(\eta)$$

Assuming $\eta > 0.5$, so that $\Phi^{-1}(\eta) > 0$, this is a second-order cone constraint:

$$\boldsymbol{\mu}^\mathsf{T}\mathbf{x} + \Phi^{-1}(\eta)\, \|\Sigma^{1/2}\mathbf{x}\|_2 \le b,$$

where $\Sigma^{1/2}$ is the square root of the covariance matrix.

Hence, an LP with *disjoint* chance constraints of this kind is a convex SOCP.

Convexity of the recourse function

Let us consider the recourse function $\mathcal{Q}(\mathbf{x}) \equiv \mathbb{E}\left[Q(\mathbf{x}, \xi)\right]$ in the case of deterministic recourse cost:

$$Q(\mathbf{x}, \xi) \equiv \min_{\mathbf{y}} \left\{ \mathbf{q}^{\mathsf{T}}\mathbf{y} \mid \mathbf{W}\mathbf{y} = \mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}, \mathbf{y} \geq \mathbf{0} \right\}$$

The effective domain of $\mathcal{Q}(\mathbf{x})$ consists of vectors $\mathbf{x}$ that are feasible for the first-stage constraints and such that $\mathcal{Q}(\mathbf{x}) < +\infty$, i.e., the second stage problem is (almost surely) feasible.

Note: we assume that, barring second-stage infeasibility, the above expectation is always defined, i.e., random variables are not heavy-tailed.

By LP duality, we have $Q(\mathbf{x}, \xi) = \max_{\boldsymbol{\pi}} \left\{ [\mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}]^{\mathsf{T}}\boldsymbol{\pi} \mid \mathbf{W}^{\mathsf{T}}\boldsymbol{\pi} \leq \mathbf{q} \right\}$.

Let us denote the feasible set of the dual by $\Pi = \{\boldsymbol{\pi} \mid \mathbf{W}^{\mathsf{T}}\boldsymbol{\pi} \leq \mathbf{q}\}$. Note that, since $\mathbf{q}$ is deterministic, that this set is nonrandom.

Convexity of the recourse function

Let us consider $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{D}$, $\lambda \in [0, 1]$, and $\mathbf{x}_\lambda = \lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$.

$$
\begin{aligned}
\mathcal{Q}(\mathbf{x}_\lambda) &= \int_\Xi Q(\mathbf{x}_\lambda, \xi)\, dP \\
&= \int_\Xi \max_{\boldsymbol{\pi} \in \Pi} \left\{ [\mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}_\lambda]^\mathsf{T} \boldsymbol{\pi} \right\} dP \\
&= \int_\Xi \max_{\boldsymbol{\pi} \in \Pi} \left\{ \lambda \left[\mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}^1 \right]^\mathsf{T} \boldsymbol{\pi} + (1-\lambda) \left[\mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}^2 \right]^\mathsf{T} \boldsymbol{\pi} \right\} dP \\
&\le \lambda \int_\Xi \max_{\boldsymbol{\pi} \in \Pi} \left\{ \left[\mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}^1 \right]^\mathsf{T} \boldsymbol{\pi} \right\} dP \\
&\quad + (1-\lambda) \int_\Xi \max_{\boldsymbol{\pi} \in \Pi} \left\{ \left[\mathbf{h}(\xi) - \mathbf{T}(\xi)\mathbf{x}^2 \right]^\mathsf{T} \boldsymbol{\pi} \right\} dP \\
&= \lambda \mathcal{Q}(\mathbf{x}^1) + (1-\lambda)\mathcal{Q}(\mathbf{x}^2)
\end{aligned}
$$

CONVEXITY OF THE RECOURSE FUNCTION

The previous proof is rather simple, because the dual feasible region is nonrandom. Actually, convexity of the recourse function can be shown in a more general setting.

To cut a long story short (and cutting a few technical corners along the way...):

- The recourse function is typically continuous for continuous probability distributions.
- The recourse function is polyhedral for discrete probability distributions.

In both cases, we may rely on Kelley's cutting planes to solve the problem.

The bonus, given problem structure, is that we may find cutting planes by a decomposition strategy.