Approximate dynamic programming and reinforcement learning for discrete states

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References

These slides are taken from my book:

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Other references:

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Approximate dynamic programming (**ADP**) is the collective name for a wide array of numerical DP methods that, unlike the standard methods do not ensure that an optimal policy will be found.

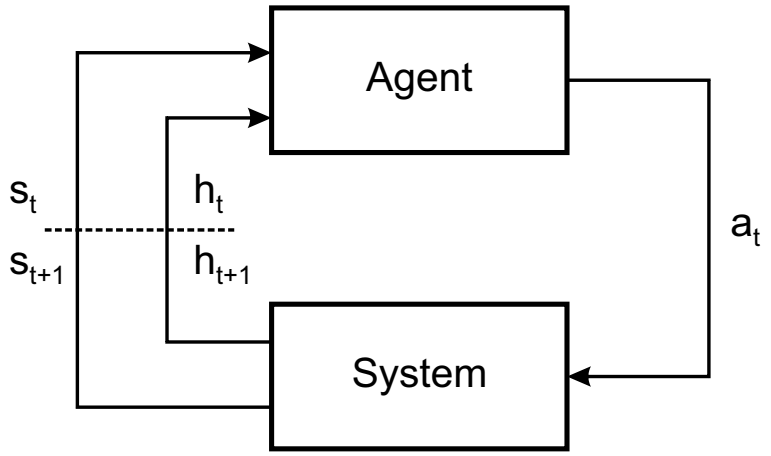
In exchange for their approximate nature, ADP methods aim at easing the overwhelming curses of DP.

Here, we consider approximate methods for finite MDPs in the infinite-horizon setting. Model-free versions of both value iteration and policy iteration are collectively labeled under the umbrella of reinforcement learning (**RL**).

The basic scheme of RL is shown in the figure.

An agent interacts with a system in order to learn a suitable control policy. The agent has no model of the system, but can observe at time t the current state s_t and try an action a_t.

The agent will then observe the immediate contribution h_{t+1} and the next state s_{t+1}.



A good policy must balance short- and long-term objectives, which are measured by the immediate contribution and by the value of the next state.

A key point in model-free RL is that even if we have perfect knowledge of the value function $V(s)$ for each state, we cannot really assess the suitability of an action, if we lack any information about the possible transitions to a next state.

A possible remedy is to introduce state–action values, in the form of Q-factors $Q(s, a)$.

If we replace state values with state–action values, we may adopt sampling strategies to learn a decision policy.

Here we assume that the size of the state–action space is not too large and that a tabular representation of Q-factors is feasible.

Quite often, this is not the case, and a compact representation is needed.

In probability theory, we are often interested in the expected value of a function $h(\cdot)$ of a random variable X or a vector $\mathbf{X}$ of random variables:

$$\theta = \mathsf{E}\big[h(\mathbf{X})\big] = \int_{\mathcal{X}} h(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x},$$

where $\mathcal{X}$ is the support of the distribution.

Since evaluating high-dimensional integrals is no piece of cake, we may resort to Monte Carlo sampling:

$$\widehat{\theta} = \frac{1}{m} \sum_{k=1}^{m} h\big(\mathbf{X}^{(k)}\big)$$

provides an unbiased estimate of the true value of θ.

In MDPs, however, we would like to estimate state values $V(i)$ or Q-factors $Q(i,a)$. In this setting, statistical learning must face two sources of complication:

- the need to balance exploration and exploitation;
- the difficulty in estimating a non-stationary target.

Suppose that we are engaged in learning the set of Q-factors $Q(i,a)$ by sampling, and that we are currently at state i. Which action should we choose?

One possibility would be to rely on the current estimates of Q-factors and pick the most promising action. However, this makes sense when the values of each action at that state have been assessed with sufficient accuracy.

However, this will probably not be the case when we have just started our learning process. There could be an action that does not look good because its immediate contribution has been poorly assessed; furthermore, the action may lead to a state which is a good one, but we just do not know it yet.

These considerations raise the fundamental issue of balancing *exploitation* and *exploration*.

Let us simplify the matter and consider a static learning problem. We have to choose one among a finite set $\mathcal{A}$ of alternative courses of action. Each action $a \in \mathcal{A}$ results in a random reward $R(a)$.

If we knew the expected value $\nu(a)$ of the random reward for each action, under a risk neutrality assumption, we would just rank actions according to their expected rewards and choose the most promising alternative.

However, imagine that we are only able to obtain noisy estimates $\widehat{\nu}(a)$ by sampling.

According to a pure exploitation strategy, we would select the action maximizing $\widehat{\nu}(\cdot)$, but this may be an unwise strategy.

As an example, let us consider a simple multi-armed bandit problem.

The bandit is a slot machine with different arms, associated with different probability distributions of payoff. Let us consider three normal distributions and sample 10 payoff observations for each one, repeating the experiment 10 times:

```
rng default  % for repeatability
for k=1:10
    fprintf(1, 'X1=%.2f   X2=%.2f   X3=%.2f\n', ...
        mean(normrnd(20,5,10,1)), mean(normrnd(25,10,10,1)),...
        mean(normrnd(30,15,10,1)))
end
```

The expected value is largest for the third arm, $\mu_3 = 30$; however this also features the largest standard deviation, $\sigma_3 = 15$.

```
X1=23.12   X2=32.05   X3=34.90
X1=17.79   X2=27.06   X3=28.36
X1=20.64   X2=22.35   X3=34.54
X1=18.77   X2=26.96   X3=24.69
X1=21.39   X2=22.91   X3=19.78
X1=20.19   X2=25.72   X3=24.76
X1=20.12   X2=27.59   X3=30.11
X1=19.36   X2=24.90   X3=21.59
X1=20.23   X2=26.62   X3=25.80
X1=20.31   X2=27.51   X3=20.13
```

It is quite likely that we will miss the optimal choice, with such a small sample size.

On the opposite end of the spectrum, the alternative of pure exploration consists of a random selection of alternatives, based on uniform probabilities. Clearly, we need to find some sensible compromise between these extremes.

- **The static ϵ-greedy approach.** In the static version, we fix a probability ϵ. Then, with probability $1-\epsilon$ we select the most promising action, and with probability ϵ we select a random action.

- **The dynamic ϵ-greedy approach.** A possible refinement is to change ϵ along the way. One possibility would be to set, at iteration k,

$$\epsilon^{(k)} = \frac{c}{d+k}, \qquad \text{or} \qquad \epsilon^{(k)} = d + \frac{c}{k}.$$

- **Boltzmann exploration.** This approach relies on a soft-max function to define the probability $\epsilon(a)$ of choosing action $a \in \mathcal{A}$:

$$\epsilon(a) = \frac{\exp\left(\rho\,\widehat{\nu}(a)\right)}{\sum_{a' \in \mathcal{A}} \exp\left(\rho\,\widehat{\nu}(a')\right)}.$$

  When $\rho = 0$, we have pure exploration, whereas the limit for $\rho \to +\infty$ is pure exploitation.

We observe that we are effectively resorting to random policies.

Even though the system itself might be stationary, the policy that we follow to make decisions is not, as we are learning along the way.

Consider the familiar sample mean used to estimate the expected value $\theta = \mathrm{E}[X]$ of a scalar random variable X. In the sample mean, all of the collected observations have the same weight.

Let $\widehat{\theta}^{(m)}$ be the estimate of θ after collecting m observations:

$$\begin{aligned}\widehat{\theta}^{(m)} &= \frac{1}{m}\sum_{k=1}^{m} X^{(k)} = \frac{1}{m}\left(X^{(m)} + \sum_{k=1}^{m-1} X^{(k)}\right) = \frac{1}{m}\left(X^{(m)} + (m-1)\widehat{\theta}^{(m-1)}\right) \\ &= \frac{1}{m}X^{(m)} + \frac{m-1}{m}\cdot\widehat{\theta}^{(m-1)} = \widehat{\theta}^{(m-1)} + \frac{1}{m}\left(X^{(m)} - \widehat{\theta}^{(m-1)}\right).\end{aligned}$$

When we obtain a new observation $X^{(m)}$, we apply a correction that is proportional to the forecasting error. to the old estimate $\widehat{\theta}^{(m-1)}$ The amount of correction is smoothed by $1/m$, and tends to vanish as m grows.

In a non-stationary setting, we may wish to keep the amount of correction constant:

$$\widehat{\theta}^{(m)} = \widehat{\theta}^{(m-1)} + \alpha\left(X^{(m)} - \widehat{\theta}^{(m-1)}\right) = \alpha X^{(m)} + (1-\alpha)\widehat{\theta}^{(m-1)}, \tag{1}$$

where the coefficient $\alpha \in (0,1)$ specifies the weight of the new information with respect to the old one.

This approach is known as **exponential smoothing**:

$$\begin{aligned}\widehat{\theta}^{(m)} &= \alpha X^{(m)} + (1-\alpha)\widehat{\theta}^{(m-1)} \\ &= \alpha X^{(m)} + \alpha(1-\alpha)X^{(m-1)} + (1-\alpha)^2\,\widehat{\theta}^{(m-2)} \\ &= \sum_{k=0}^{m-1} \alpha(1-\alpha)^k X^{(m-k)} + (1-\alpha)^m\,\widehat{\theta}^{(0)},\end{aligned}$$

where $\widehat{\theta}^{(0)}$ is an initial estimate. Unlike the standard sample mean, the weights of older observations $X^{(k)}$ are exponentially decreasing.

The coefficient α is known as **learning rate**, or smoothing coefficient, or even forgetting factor.

The larger the learning rate α, the larger the amount of correction in Eq. (1), resulting in more responsive and nervous updates; if, on the contrary, we keep the learning rate small, we introduce a larger amount of inertia and noise filtering.

If the non-stationarity is only due to learning while applying a decision policy, and not to changes in the system dynamics, we may also consider a decreasing sequence of learning rates (step sizes) $\alpha^{(k)}$, to be applied at step k. Sensible and commonly used conditions are

$$\sum_{k=1}^{\infty} \alpha^{(k)} = \infty, \qquad \sum_{k=1}^{\infty} \left[\alpha^{(k)}\right]^2 < \infty.$$

We have introduced the operator $\mathcal{T}_\mu$, whose definition is repeated here:

$$[\mathcal{T}_\mu \widetilde{\mathbf{V}}](i) = \sum_{j \in \mathcal{S}} \pi(i, \mu(i), j) \left\{ h(i, \mu(i), j) + \gamma \widetilde{V}\big(j\big) \right\}, \qquad i \in \mathcal{S}, \tag{2}$$

where μ is a stationary policy and $\widetilde{\mathbf{V}}$ is a value function.

The value $\mathbf{V}_\mu$ of μ may be obtained by finding the fixed point of $\mathcal{T}_\mu$, i.e., by solving the fixed-point equation $\mathcal{T}_\mu \mathbf{V}_\mu = \mathbf{V}_\mu$.

Let us rewrite the fixed-point equation in terms of Q-factors $Q_\mu(s, a)$ associated with a stationary policy μ:

$$Q_\mu\big(i, \mu(i)\big) = \mathsf{E}\Big[h\big(i, \mu(i), j\big) + \gamma Q_\mu\big(j, \mu(j)\big) \Big]. \tag{3}$$

Here the expectation is taken with respect to the next state j, and the action $a = \mu(i)$ is defined by the policy that we want to evaluate.

Let us abstract a bit and consider a fixed-point equation

$$\mathbf{y} = H\mathbf{y},$$

where $\mathbf{y}$ is a vector in some linear space and H is an operator.

One possibility to solve the equation is plain fixed-point iteration,

$$\mathbf{y}^{(k)} = H\mathbf{y}^{(k-1)},$$

whose convergence is not guaranteed in general.

Even if the operator H is a contraction, we cannot evaluate it exactly, since we do not know the probability distribution needed to evaluate the expectation in Eq. (3).

Let us rewrite the fixed-point equation as follows:

$$\mathbf{y} = (1-\alpha)\mathbf{y} + \alpha\mathbf{y} = (1-\alpha)\mathbf{y} + \alpha H\mathbf{y} = \mathbf{y} + \alpha(H\mathbf{y} - \mathbf{y}).$$

Then, we may consider using this equation to define an iterative scheme:

$$\mathbf{y}^{(k)} = \mathbf{y}^{(k-1)} + \alpha(H\mathbf{y}^{(k-1)} - \mathbf{y}^{(k-1)}). \tag{4}$$

We may apply exponential smoothing, replacing the expectation with random observations, which is exactly what we do in statistical learning.

Say that we are at state $s^{(k)} = i$, after observing k state transitions starting from an initial state $s^{(0)}$. Then, we apply action $a^{(k)} = \mu(i)$ and, as a result, we will observe:

- the next state $s^{(k+1)} = j$;
- the immediate contribution $h\big(i, \mu(i), j\big)$, or just $f\big(i, \mu(i)\big)$ if it is deterministic.

The adaptation of the scheme of Eq. (4) to this setting leads to the following iterative scheme to learn the value of a given stationary policy:

$$\Delta^{(k)} = \Big[h\big(i,\mu(i),j\big) + \gamma \widehat{Q}_{\mu}^{(k-1)}\big(j,\mu(j)\big)\Big] - \widehat{Q}_{\mu}^{(k-1)}\big(i,\mu(i)\big) \tag{5}$$

$$\widehat{Q}_{\mu}^{(k)}\big(i,\mu(i)\big) = \widehat{Q}_{\mu}^{(k-1)}\big(i,\mu(i)\big) + \alpha\Delta^{(k)}, \tag{6}$$

where $i = s^{(k)}$ and $j = s^{(k+1)}$. The quantity $\Delta^{(k)}$ in Eq. (5) is an example of a **temporal difference**.

The temporal difference plays the role of the term $H\mathbf{y}^{(k-1)} - \mathbf{y}^{(k-1)}$ in Eq. (4), where the role of operator H is played by the operator $\mathcal{T}_{\mu}$.

The expected value of the temporal difference, if we could use the true Q-factors, would be zero because of Eq. (3). Rather than the exact expected value of a random variable, we have to settle for an observation of the random variable itself.

A non-zero value suggests that we should correct the current estimate. Since corrections are noisy, they are smoothed by the learning rate α.

Since use estimates to obtain another estimate, the term **bootstrapping** is used to refer to this kind of scheme.

The above algorithm is known as **SARSA**, which is the acronym for the sequence (State, Action, Reward, State, Action). In fact, we observe the current state i, select the action $\mu(i)$, observe a new state j and an immediate reward $h\big(i,\mu(i),j\big)$, and the action $\mu(j)$ at the new state, chosen according to the stationary policy μ that we are evaluating.

SARSA is an **on-policy** learning approach, as we behave according to a given policy to learn the value of that policy. This is what we are supposed to do in policy iteration.

We need two further ingredients:

1. A suitable amount of exploration, which may be obtained by introducing ϵ-greedy exploration. With a probability $\epsilon^{(k)}$ we deviate from the incumbent policy and select a random action.

2. With SARSA, we will learn the value of a policy only in the limit, after a possibly large amount of steps. Clearly, we have to stop prematurely and perform an improvement step of the incumbent policy before the exact assessment of its value.

   This approach is known as **generalized** or **optimistic policy iteration**. The term "optimistic" is due to the fact that we optimistically assume that we were able to assess the true value of the current stationary policy.

   This may have a positive effect on the speed of learning, but a detrimental one on convergence.

When we stop learning and have an approximation $\widehat{Q}_\mu(i,a)$ of the Q-factors, we try improving the incumbent policy $\mu(\cdot)$ by a greedy approach:

$$\tilde{\mu}(i) \in \operatorname*{arg\,opt}_{a \in \mathcal{A}(i)} \widehat{Q}_\mu(i,a), \qquad i \in \mathcal{S}.$$

Unlike exact iteration, there is no guarantee that the new policy $\tilde{\mu}(\cdot)$ is really improved, as the evaluation of the incumbent policy $\mu(\cdot)$ need not be exact.

Q-learning is an alternative model-free reinforcement learning approach to cope with finite MDPs.

Let us recall the essential equations for DP in terms of optimal Q-factors:

$$Q(i,a) = \sum_{j \in \mathcal{S}} \pi(i,a,j) \left[h(i,a,j) + \gamma \operatorname*{opt}_{a' \in \mathcal{A}(j)} Q(j,a') \right], \qquad i \in \mathcal{S},\ a \in \mathcal{A}(i)$$

$$V(i) = \operatorname*{opt}_{a \in \mathcal{A}(i)} Q(i,a).$$

Note that, when dealing with SARSA, we have used equations for the Q-factors associated with a stationary policy μ, whereas here we deal with the *optimal* factors.

We have to learn about $h(i,a,j)$ and $\pi(i,a,j)$ by experimentation and sampling. At iteration k, when we are about to choose action $a^{(k)}$, we have a current set of estimates of Q-factors

$$\widehat{Q}^{(k-1)}(i,a).$$

The superscript $(k-1)$ points out that these estimates were updated at the previous step, after observing the effect of action $a^{(k-1)}$.

Now we are at state $s^{(k)} = i$ and, based on current estimates, we select the next action by considering what looks best:

$$a^{(k)} \in \operatorname*{arg\,opt}_{a \in \mathcal{A}(i)} \widehat{Q}^{(k-1)}(i,a). \tag{7}$$

It is worth observing that in Q-learning we are applying the same off-policy logic the is used in value iteration.

Unlike policy iteration, we are not evaluating a given policy μ. The policy is implicit in the Q-factors and, since we continuously update them, we keep changing the policy; in some sense, we use a policy to learn about another one.

After applying the selected action we will observe:

- the next state $s^{(k+1)} = j$;
- the immediate contribution $h\big(i, a^{(k)}, j\big)$, or just $f\big(i, a^{(k)}\big)$ if it is deterministic.

Then, we can use this information to update the estimate of $Q\big(s^{(k)}, a^{(k)}\big)$. We have obtained a new observation of a random variable that expresses the tradeoff between the short-term objective (immediate contribution) and the long-term objective (value of the next state), when we are at state $s^{(k)} = i$ and select action $a^{(k)}$:

$$\widetilde{q} = h\big(i, a^{(k)}, s^{(k+1)}\big) + \gamma \operatorname*{opt}_{a' \in \mathcal{A}(s^{(k+1)})} \widehat{Q}^{(k-1)}\big(s^{(k+1)}, a'\big). \tag{8}$$

Note that this includes a random observation of the next state $s^{(k+1)} = j$, as well as its value when following the policy implicit in the Q-factor estimates at step $k-1$:

$$\widehat{V}^{(k-1)}\big(s^{(k+1)}\big) = \operatorname*{opt}_{a' \in \mathcal{A}(s^{(k+1)})} \widehat{Q}^{(k-1)}\big(s^{(k+1)}, a'\big).$$

Then, we update the Q-factor for the state–action pair $\left(s^{(k)}, a^{(k)}\right)$, by using the same logic of exponential smoothing:

$$\widehat{Q}^{(k)}\left(s^{(k)}, a^{(k)}\right) = \alpha\widetilde{q} + (1-\alpha)\widehat{Q}^{(k-1)}\left(s^{(k)}, a^{(k)}\right). \tag{9}$$

Here we assume a constant smoothing coefficient α, but we could use a diminishing one. We may also express the same idea in a temporal difference style:

$$\Delta^{(k)} = \left[h\left(s^{(k)}, a^{(k)}, j\right) + \gamma \operatorname*{opt}_{a' \in \mathcal{A}(j)} \widehat{Q}^{(k-1)}\left(j, a'\right)\right] - \widehat{Q}^{(k-1)}\left(s^{(k)}, a^{(k)}\right) \tag{10}$$

$$\widehat{Q}^{(k)}(s,a) = \begin{cases} \widehat{Q}^{(k-1)}(s,a) + \alpha\Delta^{(k)} & \text{if } s = s^{(k)} \text{ and } a = a^{(k)}, \\ \widehat{Q}^{(k-1)}(s,a) & \text{otherwise,} \end{cases} \tag{11}$$

where $j = s^{(k+1)}$ is the next observed state.

The difference between SARSA and Q-learning may be appreciated by comparing Eqs. (5) and (10). These expressions are both corrections to be applied to current state–action value estimates, but the former case is on-policy and refers to an incumbent stationary policy μ; on the contrary, the latter case is off-policy, as we keep changing the policy.

MATLAB implementation of Q-learning.

```
function [QFactors, policy, visits] = QLearning(systemObj,objSense, ...
    discount,numSteps,startQFactors,alpha,epsFlag,epsilon)
if nargin < 7, epsFlag = false; end
QFactors = startQFactors;
visits = 0*startQFactors; % initialize to zero, using shape of Q table
currentState = systemObj.currentState;
for k = 1:numSteps
    if epsFlag
        if rand > epsilon
            [~, action] = findBest(currentState);
        else
            action = datasample(systemObj.feasibleActions{currentState},1);
        end
    else
        [~, action] = findBest(currentState);
    end
    visits(currentState, action) = visits(currentState, action) + 1;
    [contribution, newState] = systemObj.getNext(action);
    qtilde = contribution + discount*findBest(newState);
    QFactors(currentState,action) = alpha*qtilde + ...
                                (1-alpha)*QFactors(currentState,action);
    currentState = newState;
end % for
```

```
% now find optimal policy
[numStates,~] = size(QFactors);
policy = zeros(numStates,1);
for i = 1:numStates
    [~, policy(i)] = findBest(i);
end

% nested function
    function [value, action] = findBest(state)
        actionList = systemObj.feasibleActions{state};
        qf = QFactors(state,actionList);
        if strcmpi(objSense, 'max')
            [value, idx] = max(qf);
        else
            [value, idx] = min(qf);
        end
        action = actionList(idx);
    end

end
```

A possible MATLAB implementation of Q-learning is shown in Fig. **??**. The function `QLearning` produces a matrix `QFactors` of Q-factors, a vector `policy` describing the stationary policy, and a matrix `visits` counting the number of times that a state–action pair has occured. The function needs the following input arguments:

- `systemObj`, an object of class `systemClass` (to be described below) implementing the simulation of the dynamic system we want to control;
- `objSense`, a flag setting the direction of optimization, `min` or `max`;
- `discount`, the discount factor;
- `numSteps`, the number of simulation steps to learn the policy;
- `startQFactors`, a matrix containing the initial Q-factors;
- `alpha`, the learning rate.

This code is also able to implement a simple (static) ϵ-greedy action selection. If the `epsFlag` flag is set to true, a random action is selected with probability `epsilon`. We use the `datasample` MATLAB function for the random selection of an action; here, the function selects at random an element of a vector containing the feasible actions in the current state. The nested function `findBest` picks the optimal action, taking care of the selected optimization sense.

MATLAB definition of a generic system class

Since Q-learning is a model-free approach, we want to keep the Q-learning algorithm well separated from the simulation model of the system. The interface with the system is provided here by the MATLAB class `systemClass`.

```
classdef systemClass < handle
    properties
        numStates
        numActions
        feasibleActions
        transArray
        payoffArray
        currentState
    end
    methods
        % Constructor
        function obj = systemClass(transArray, payoffArray, ...
                        feasibleActions, initialState)
            obj.transArray = transArray;
            obj.payoffArray = payoffArray;
            obj.feasibleActions = feasibleActions;
            obj.currentState = initialState;
            [obj.numStates, obj.numActions] = size(payoffArray);
        end
```

```
        function obj = setState(obj, state)
            obj.currentState = state;
        end

        function [contribution, nextState] = getNext(obj,action)
            contribution = obj.payoffArray(obj.currentState, action);
            nextState = find(1 == ...
                 mnrnd(1,obj.transArray(obj.currentState,:,action)));
            obj.currentState = nextState;
        end
    end % methods
end
```

NOTE: In a serious object-oriented implementation, this should be an *abstract* class defining only the interface between the Q-learner and the system.

The class properties include: `feasibleActions`, a cell array containing, for each state, the list of feasible actions; `transArray`, a three-dimensional array of transition probability matrix (actions correspond to the third index); `payoffArray`, a matrix giving, for each state–action pair, the immediate contribution.

The essential method of this class is `getNext`, which provides the immediate contribution and the next state for a selected action, based on the `currentState`.

Note that we associate the contribution with the pair, which implicitly assumes that it is either deterministic or that we know its expected value. In order to implement alternative systems, we could just instantiate a lower level class overriding `getNext`.

A numerical example

Let us apply Q-learning to the recurrent optimal stoping problem, where we find, assuming a discount factor of $\gamma = 0.8$:

$$V(1) = 9.67742, \quad V(2) = 17.7419, \quad V(3) = 27.7419, \quad V(4) = 37.7419.$$

The corresponding optimal policy is to always reset the state and collect the immediate reward.

In order to apply Q-learning, we need to instantiate a specific `systemObj` object of class `systemClass` and run the `QLearning` function, choosing the initial factors as well.

The setting of parameters is the same that we have used for value and policy iteration, and the initial values of the Q-factors are set to 50 for the wait action, and to the immediate reward for the reset action.

```
probs = [0.7; 0.8; 0.9; 0];
payoff = [0; 10; 20; 30];
discount = 0.8;
payoffArray = [zeros(4,1), payoff];
% build transition matrices
transReset = [ones(4,1), zeros(4, 3)];
auxProbs = probs(1:(end-1)); % the last one is not relevant
transWait = diag([auxProbs;1]) + diag(1-auxProbs,1);
transArray(:,:,1) = transWait;
transArray(:,:,2) = transReset;
feasibleActions = {1, [1;2], [1:2], 2};
initialState = 1;
systemObj = systemClass(transArray,payoffArray,feasibleActions,initialState);

rng('default')
numSteps = 1000;
alpha = 0.1;
startQFactors = [ repmat(50,4,1), payoff(:)];
[QFactors, policy] = QLearning(systemObj,'max',discount,numSteps, ...
    startQFactors,alpha);
display(startQFactors);
display(QFactors);
disp(['policy  ', num2str(policy')]);
```

We set the smoothing coefficient to $\alpha = 0.1$ and run 1000 iterations, obtaining the following output:

```
startQFactors =
    50     0
    50    10
    50    20
    50    30
QFactors =
   11.0476         0
    9.8825   17.6721
   19.8226   26.9803
   50.0000   47.7861
policy  1  2  2  2
visits =
   608     0
   168   132
    61    22
     0     9
```

If we compare the Q-factors corresponding to the optimal decisions against the state values obtained by value iteration, we find them in fairly good agreement, with the exception of the last state, which is visited quite rarely.

What really matters, however, is that the state–action value estimates are good enough to induce the optimal policy.

This may look reassuring, but what if we change the initialization a bit?

In the second run, we initialize the factors for the wait action ($a = 1$) to 900:

```
startQFactors =
   900     0
   900    10
   900    20
   900    30

QFactors =
   10.9569         0
   14.0696   10.0000
   19.9080   25.5919
  900.0000   58.8755

policy  1  1  2  2
```

With this bad initialization, we assign a very large value to wait, and after 1000 iterations we fail to find the optimal policy.

Note that the factor $Q(2, 2)$ for reset is not changed, as this action is never applied.

Arguably, we may solve the issue by adopting a larger value of α, at least in the initial steps, in order to accelerate learning.

If we increase α to 0.5, we obtain

```
startQFactors =
   900     0
   900    10
   900    20
   900    30

QFactors =
   16.4106         0
    9.5378   22.2903
   19.6721   32.5977
  900.0000  114.9065

policy  1  2  2  2
```

Note that the poor estimates for state 4 are inconsequential.

In this case, it seems that a faster learning rate does the job. However, what if we keep $\alpha = 0.5$ and use an alternative initialization?

```
startQFactors =
     1     0
     1     0
     1     0
     1     0
QFactors =
    7.2515         0
   14.3098         0
   21.2052         0
    1.0000   38.8769
policy  1  1  1  2
visits =
   195     0
   294     0
   452     0
     0    59
```

If the Q-factors for the reset action are initialized to zero, then this action will never be tried, and we will never discover that it is, in fact, the optimal one.

Indeed, we observe from `visits` that we never apply action reset at states 2 and 3.

Changing the learning rate is of no use, as the Q-factors for wait cannot get negative.

This illustrates the need of balancing exploitation and exploration.

As a remedy, we could resort to ϵ-greedy exploration. Let us try it with $\epsilon = 0.1$:

```
numSteps = 10000;
[QFactors, policy] = QLearning(systemObj,'max',discount,numSteps, ...
    startQFactors,alpha,true,0.1);
```

This yields the following result:

```
startQFactors =
     1     0
     1     0
     1     0
     1     0
QFactors =
   10.5408         0
   16.3370   18.0537
    1.7721   26.2163
    1.0000    7.1070
policy  1  2  2  2
visits =
        7611           0
         129        2214
          19          25
           0           2
```

We observe again that the state values are not exact, but now they are good enough to find the optimal policy. However, there is a price to pay in terms of computational effort.