Politecnico di Torino
Financial Engineering-Exam 06-19-2025
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SURNAME and NAME

Student number

All the answers must be clearly motivated, the numerical results are not sufficient. All the answers that will be considered in the correction MUST be written here. If the student number and the name are not filled in the text will not be corrected.

Answers written with a pencil are null.

Exercise 1 (10 points)
A portfolio consists of a 4-year zero-coupon bond with face value of 10000 \$ and a 3-year coupon bond that pays a coupon on 2% ona annual basis with face value of 3000 \$. The current yield on all bonds is 5% per annum.
(a) Compute the duration of the portfolio (using continuous compounding).
(b) Compute the percentage change in the value of the portfolio in the case of a 0.2% per annum decrease in yields.
(c) Compare the result obtained in (b) with the $1st$ order and $2nd$ order approximations based on the use of duration and of convexity, and comment the results. Specifically explain what duration and convexity are.

(a) Compute the duration of the portfolio (using continuous compounding).

Zero-coupon bond:
The present value is:

$$P_Z = 10{,}000 \times e^{-0.05\times 4} = 10{,}000 \times e^{-0.2} \approx 8{,}187.31.$$

The duration of a zero-coupon bond is simply its maturity:

$$D_Z = 4.$$

Coupon bond:
Annual coupon:

$$C = 0.02 \times 3{,}000 = 60.$$

Present value:

$$P_C = 60e^{-0.05} + 60e^{-0.10} + 3{,}060e^{-0.15}.$$

Numerically:

$$P_C = 60 \times 0.9512 + 60 \times 0.9048 + 3{,}060 \times 0.8607 \approx 57.07 + 54.29 + 2{,}633.74 = 2{,}745.10.$$

Portfolio value

$$P = P_B + P_C = 10932$$

Duration of the coupon bond:

$$D_C = \frac{1}{P_C}\left[1 \times 60e^{-0.05} + 2 \times 60e^{-0.10} + 3 \times 3{,}060e^{-0.15}\right].$$

Numerically:

$$D_C = \frac{1}{2{,}745.10}[57.07 + 108.58 + 7{,}901.22] = \frac{8{,}066.87}{2{,}745.10} \approx 2.94.$$

Portfolio duration:

$$P_P = P_Z + P_C = 8{,}187.31 + 2{,}745.10 = 10{,}932.41.$$

$$D_P = \frac{P_Z}{P_P}D_Z + \frac{P_C}{P_P}D_C = \frac{8{,}187.31}{10{,}932.41} \times 4 + \frac{2{,}745.10}{10{,}932.41} \times 2.94.$$

$$D_P = 0.749 \times 4 + 0.251 \times 2.94 \approx 2.996 + 0.738 = 3.73.$$

(b) Compute the percentage change in value if yields decrease by 0.2%.

if $r = 0.48\%$ **we find (same computations as in point a)**

$$P' = 11.014$$

and

$$\frac{\Delta P}{P} = 0.75\%$$

(c) Compute the approximations

First order approximation:

$$\frac{\Delta P}{P} \approx -D_P \times \Delta y.$$

$$\Delta y = -0.002.$$

$$\frac{\Delta P}{P} \approx -3.73 \times (-0.002) = 0.00746 = 0.746\%.$$

Convexity (for second order):
Zero-coupon bond:

$$C_Z = T^2 = 16.$$

Coupon bond:

$$C_C = \frac{1}{P_C}\left[1^2 \times 60e^{-0.05} + 2^2 \times 60e^{-0.10} + 3^2 \times 3{,}060e^{-0.15}\right].$$

$$C_C = \frac{1}{2{,}745.10}[57.07 + 217.16 + 23{,}703.66] = \frac{23{,}977.89}{2{,}745.10} \approx 8.73.$$

$$C_P = \frac{P_Z}{P_P}C_Z + \frac{P_C}{P_P}C_C = 0.749 \times 16 + 0.251 \times 8.73 = 11.98 + 2.19 = 14.17.$$

Second order approximation:

$$\frac{\Delta P}{P} \approx -D_P\Delta y + \frac{1}{2}C_P(\Delta y)^2.$$

$$\frac{\Delta P}{P} \approx 0.00746 + 0.5 \times 14.17 \times (0.002)^2 = 0.00746 + 0.000028 = 0.00749.$$

So,

$$0.749\%.$$

Solution

Comment: see the theory.

Exercise 2(10 points)

A stock (indexed by A) is available on the market at the current price $S_A(0) = 8$ euros. In one year, the price may increase by 25% or decrease by 25% or stay unchanged. Another stock (indexed by B) is also available. Its current price is $S_B(0) = 12$ The possiblescenarios for B are

$$\begin{cases} S_B^+ \text{ increase by25\%,} & \text{if A goes down,} \\ S_B^0 \text{decrease by25\%,} & \text{if A unchanged,} \\ S_B^- \text{ decrease by25\%,} & \text{if A goes up.} \end{cases}$$

Another stock (indexed by B) is also available. Its current price is $S_B(0) = 12$ euros that, in one year, may increase by 25% (when the price of stock A is increased or unchanged) or decrease by 25% (when the price of stock A is increased). The risk-free interest rate on the market is 4% per year (simple compounding).

1. Consider a European put option with maturity of one year, with strike of 8 euros and written on stock A. Verify if it is possible to replicate such an option only by means of stock A and of cash invested or borrowed at risk free rate. If yes explain the replicating strategy, if no explain why.
2. Verify if it is possible to replicate the European put option above investing in stock A, stock B and cash. If yes, find a replicating portfolio and compute the cost of the replicating strategy. Comment on the result.
3. Discuss whether the market (A, B, C), where C is the risk-free asset is complete. If possible find a risk neutral measure.

Solution

We have

$$S_A(0) = 8. \quad \text{Possible outcomes:} \quad \begin{cases} S_A(u) &= 10, \\ S_A(s) &= 8, \\ S_A(d) &= 6. \end{cases}$$

The second stock B has:

$$S_B(0) = 12. \quad \text{Possible outcomes:} \quad \begin{cases} S_B^+ = 15, & \text{if A goes down,} \\ S_B^0 = 9, & \text{if A unchanged,} \\ S_B^- = 9, & \text{if A goes up.} \end{cases}$$

Risk-free return: $r = 4\%$ (simple compounding).

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1. **1)**

   The European put payoff:

$$\begin{cases} \phi(S_A) & = \max(8-10,0) = 0, \\ \phi(S_A) & = \max(8-8,0) = 0, \\ \phi(S_A) & = \max(8-6,0) = 2. \end{cases}$$

   Replicating portfolio:$P = (h_A, h_C)$ with h_A shares of A and h_C euros risk-free.

   At $t = 1$ $V^h(1) = h_A S_A(1) + h_C C(1)$

$$\begin{cases} 6h_A + 1.04h_C = 2, \\ 8h_A + 1.04h_C = 0, \\ 10h_A + 1.04h_C = 0. \end{cases}$$

   we have

$$A^c = \begin{bmatrix} 6 & 1.04 & 2 \\ 8 & 1.04 & 0 \\ 10 & 1.04 & 0 \end{bmatrix}.$$

   Since the Rank of the matrix of the coefficient is 2 and $Rank(A^c) = 3$ the system does not have a solution. We cannot replicat with only cash and S_A because the market $\mathcal{M} = (A, C)$ is not complete. The market is arbitrage free since $0.75 < 1.04 < 1.25$, but the system

$$\begin{cases} q_u + q_s 8 + q_d 6 = 8 \cdot 1.04, \\ q_u + q_s + q_d = 1, \end{cases}$$

   has infinite positive solutions.

2. **2)**

   Portfolio P=(Δ_A, Δ_B ,h_C). Δ_A units of A, Δ_B of B, h_C cash.

   At $t = 1$:

$$\begin{cases} 10\Delta_A + 9\Delta_B + 1.04h_C = 0, \\ 8\Delta_A + 9\Delta_B + 1.04h_C = 0, \\ 6\Delta_A + 15\Delta_B + 1.04h_C = 2. \end{cases}$$

   Since $Rank(A) = 12.8$ the system has a unique solution , that is

$$\Delta_A = 0, \quad \Delta_B = \frac{1}{3}, \quad B_0 \approx -2.8846.$$

Cos of the strategy t:

$$V_0 = \Delta_A S_A(0) + \Delta_B S_B(0) + h_C = 0 + \frac{1}{3} \times 12 - 2.8846 = 4 - 2.8846 \approx 1.1154.$$

So the put can be replicated with A, B and cash for about 1.12 euros.

Notice that $h_A = 0$, we repliacte only with B and cash. In fact $\mathcal{M} = (B, C)$ is complete and the Put has the same payoff of the call option $(S_B(T) - 13)^+$.

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3. **3)**

   The market has 3 states and 2 stocks:

$$M = \begin{bmatrix} 10 & 8 & 6 \\ 9 & 9 & 15 \\ 1 & 1 & 1 \end{bmatrix}.$$

   $Rank(A) = 3$, so the matrix rank is 3 $\Longrightarrow$the system has a unique solution. The solution is

   $Q = (0.74, -0.32, 0.58)$, that is not positive and therefore there not exist a martingale measure and the market is not arbitrage free.

Exercise 3

An investor currently has a untary wealth, which is fully invested in a financial portfolio P whose value process $V_P(t)$ follows a geometric Brownian motion (GBM) with a drift coefficient of 5% per annum and a volatility coefficient of 10% per annum. The risk-free rate (continuously compounded) is 2% per annum.

1. The investor's goal is to double their wealth in 15 years.
   Compute the probability that this goal will be achieved.

2. Define the loss of a financial portfolio P' with initial value $V(0)$ in terms of log-returns, its value at risk (VaR) and its expected shortfall (ES).
   Comment on the difference between these two risk measures.

3. Calculate the expected shortfall (ES) of the portfolio's loss after 10 years at the 95% confidence level.

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Solution

1. **Probability of doubling the wealth**

   The portfolio value S_t follows:

   $$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \text{where} \quad \mu = 0.05,\ \sigma = 0.10.$$

   The solution is:

   $$S_t = S_0 \exp\big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\big).$$

   Let:

   $$X = \ln\left(\frac{S_T}{S_0}\right) \sim \mathcal{N}\big((\mu - \tfrac{1}{2}\sigma^2)T,\, \sigma^2 T\big).$$

   To double the wealth:

   $$P(S_T \geq 2S_0) = P(X \geq \ln 2).$$

   So:

   $$P = P\Big(Z \geq \frac{\ln 2 - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\Big), \quad Z \sim \mathcal{N}(0,1).$$

   Substituting:

   $$\mu - \tfrac{1}{2}\sigma^2 = 0.05 - 0.005 = 0.045, \quad \sigma = 0.10, \quad T = 15.$$

   Then:

   $$d = \frac{\ln 2 - 0.045 \times 15}{0.10\sqrt{15}} = \frac{0.6931 - 0.675}{0.3873} \approx 0.0468.$$

   Hence,

   $$P(S_{15} \geq 2S_0) = P(Z \geq 0.0468) = 1 - \Phi(0.0468) \approx 0.4813.$$

   Therefore, the probability of doubling the wealth in 15 years is **approximately 48.13%**.

2. **Definition of Loss, VaR and ES**

   The *loss* over a given period is defined as:

   $$L(t+h) = -V(t)R(t+h) = -V(t)\ln\left(\frac{S_T}{S_0}\right).$$

   The *Value at Risk* (VaR) at confidence level α is:

   $$\mathrm{VaR}_\alpha(L) = \inf\{l \in \mathbb{R} \mid P(L > l) \leq 1 - \alpha\}.$$

   The *Expected Shortfall* (ES) at level α is:

   $$\mathrm{ES}_\alpha(L) = E[L \mid L > \mathrm{VaR}_\alpha(L)].$$

   Difference: See the theory

3. **Expected Shortfall after 10 years at 95%**

Over $T = 10$ years: $X = R(10)$ and $L = L(10)$

$$X \sim \mathcal{N}(\mu_X, \sigma_X^2), \quad \mu_X = 0.045 \times 10 = 0.45, \quad \sigma_X^2 = 0.10^2 \times 10 = 0.10.$$

Then:

$$L = -V(0)X, \qquad X \sim \mathcal{N}(-0.45, 0.10),$$

Since $V(0) = 1$,

$$L = \sim \mathcal{N}(-0.45, 0.10)..$$

The ES for a normal loss is:

$$\mathrm{ES}_\alpha(X) = -0.45 + \sqrt{0.10}\frac{\varphi(z_{0.95})}{1 - 0.95}, \quad \text{where} \quad \varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}.$$

Compute:

$$\varphi(1.645) \approx \frac{1}{\sqrt{2\pi}}e^{-1.645^2/2} \approx 0.103.$$

So:

$$\mathrm{ES}_{0.95}(X) = -0.45+0.3162\times\frac{0.103}{0.05} = -0.45+0.3162\times 2.06 \approx -0.45+0.65 = 0.20.$$

Therefore, the expected shortfall at the 95% confidence level after 10 years is **approximately 20%** of the initial wealth.