

Politecnico di Torino
Financial Engineering-Exam 06-19-2025
P. Semeraro

SURNAME and NAME

Student number

All the answers must be clearly motivated, the numerical results are not sufficient. All the answers that will be considered in the correction MUST be written here. If the student number and the name are not filled in the text will not be corrected.

Answers written with a pencil are null.

Exercise 1 (10 points)

A non-dividend paying stock price $S(0)$ is \$100, and the risk-free rate of interest is 5% per annum with continuous compounding for all maturities.

- (a) A trader observes that the forward price of a six month forward contract written on S is $F_0 = 105$. Can he make an arbitrage? Why? Construct an arbitrage strategy.
(b) The stock is expected to pay a dividend of \$0.6 per share in 2 months and in 5 months. What are the no arbitrage forward price and the initial value of a six month forward contract on the stock?

Solution

$$S_0 = 100 \quad r = 0.05$$

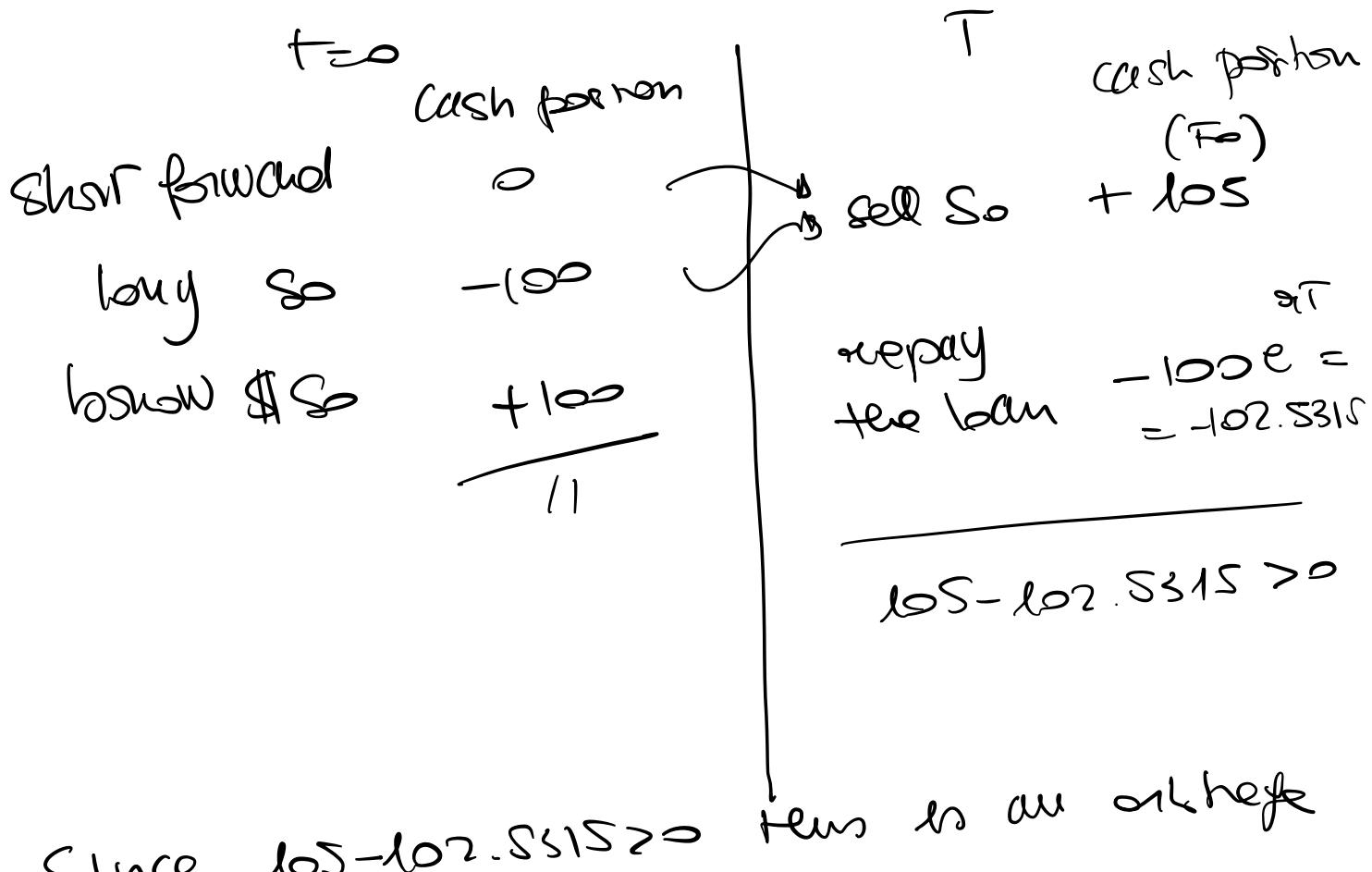
(a) $F_0 = 105$

No arbitrage price:

$$\tilde{F}_0 = S_0 e^{rT} = 100 e^{0.05 \frac{1}{2}} = 102.5315$$

Since $\tilde{F}_0 < F_0$ there is an arbitrage opportunity

ST strategy:



Strategy

$$(b) D_1 = 0.6 \text{ in 2 months } t=2/12$$

$$D_2 = 0.6 \text{ " " } t=5/12$$

$$-2/12 \cdot 0.05$$

$$D_1 = 0.6 e^{-2/12 \cdot 0.05} = 0.5950$$

$$D_2 = 0.6 e^{-5/12 \cdot 0.05} = 0.5876$$

$$D \Rightarrow D_1 + D_2 = 1.1826$$

the no outside price is the cost is

$$\tilde{F}_o = (S - D) e^{-\alpha T} = (100 - 1.1826) e^{0.05 \frac{1}{2}} = \\ = 101.3190$$

\tilde{F}_o is the no outside price and
the actual value of the correct is 80.

Exercise 2(10 points)

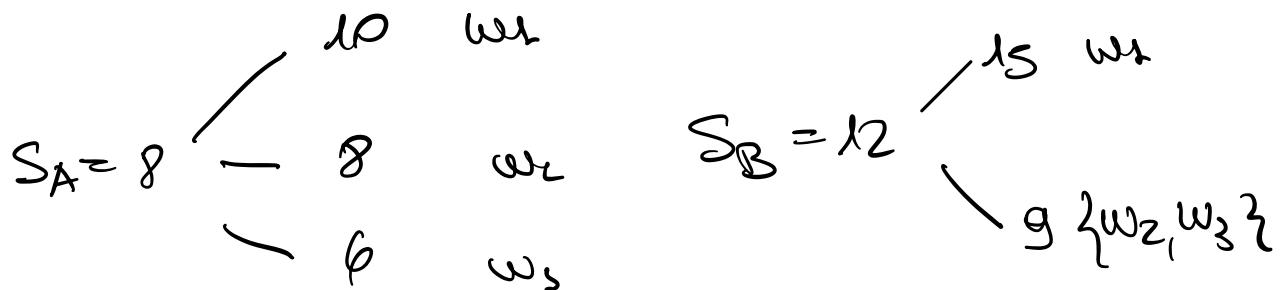
A stock (indexed by A) is available on the market at the current price $S_A(0) = 8$ euros. In one year, the price may increase by 25% or decrease by 25% or stay unchanged. Another stock (indexed by B) is also available. Its current price is $S_B(0) = 12$ euros that, in one year, may increase by 25% (when also the price of stock A is increased) or decrease by 25% (when also the price of stock A is decreased or unchanged). The risk-free interest rate on the market is 4% per year (simple compounding).

1. Consider a European put option with maturity of one year, with strike of 8 euros and written on stock A. Verify if it is possible to replicate such an option only by means of stock A and of cash invested or borrowed at risk free rate. If yes explain the replicating strategy, if no explain why.
2. Verify if it is possible to replicate the European put option above investing in stock A, stock B and cash. If yes, find a replicating portfolio and compute the cost of the replicating strategy. Comment on the result.
3. Discuss whether the market (A, B, C) , where C is the risk-free asset is complete. It is possible to find a risk neutral measure.

Solution

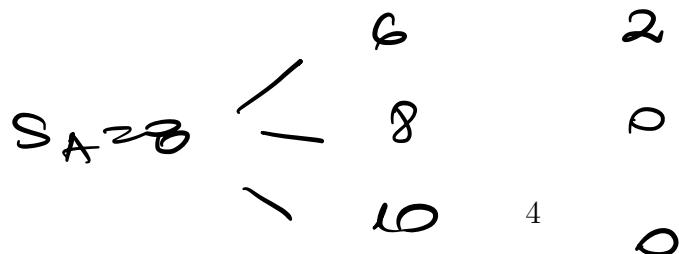
$t=1$

$t=1$



$$r = 0.04$$

$$C(0) = 1 \rightarrow C(0) = 1.04 \\ \phi(S_A) = (8 - 8\tau)^+$$



(1) $P(\ell_A, \ell_C)$ where A asset A and C is cash
 $V^{\ell_h}(\delta) = \ell_A S_A(\cdot) + \ell_C C(\cdot)$

Given ℓ_A, ℓ_C t.c.

$$V^{\ell_h}(1) = \begin{cases} 2 & w_1 \\ 0 & 2w_2 w_3 \end{cases}$$

$$\left. \begin{array}{l} \ell_A 6 + \ell_C 1.0h = 2 \\ \ell_A 8 + \ell_C 1.0h = 0 \\ \ell_A 10 + \ell_C 1.0h = 0 \end{array} \right\}$$

This system does not have solution

Since $\det \begin{vmatrix} 6 & 1.0h & 2 \\ 8 & 1.0h & 0 \\ 10 & 1.0h & 0 \end{vmatrix} =$
 $= 2 \begin{vmatrix} 8 & 1.0h \\ 10 & 1.0h \end{vmatrix} = 2(8 - 10)1.0h \neq 0$

$$\det A^C = 3$$

$$\text{and } \det A = 2$$

We cannot replicate the option on S_A
 only with cash and S_A because
 the market $\{S_A, C\}$ is not complete!

you find a rule needed needs should
verify

$$\begin{cases} q_u 12 + q_s 8 + q_d 6 = 8(1.04) \\ q_u + q_s + q_d = 1 \end{cases}$$

that has a solution (the market
is abs. free since

$$0.75 \leq 1.04 \leq 1.25$$

therefore the system admits a solution)

$$(2) \quad h = (h_A, h_B, h_C)$$

$$V_h(1) = h_A S_A(1) + h_B S_B(1) + h_C C(1)$$

$$\begin{cases} h_A 6 + h_B 9 + h_C 1.04 = 2 \\ h_A 8 + h_B 9 + h_C 1.04 = 0 \\ h_A 10 + h_B 15 + h_C 1.04 = 0 \end{cases}$$

$$\text{Since } \det A = 12.8$$

the system admit a unique solution that is

$$h = (-1, 0.33, 4.8)$$

The cost of the strategy is

$$V_h(0) = -S_A(0) + 0.33 S_B(0) - h \cdot 8 = 0.8$$

(3) the system

$$\left\{ \begin{array}{l} q_u 10 + q_s 8 + q_d 6 = 8(1.04) \\ q_u 15 + (q_s + q_d) 9 = 12(1.04) \\ q_u + q_s + q_d = 1 \end{array} \right.$$

has a solution ($\det A \neq 0$) that is

$$q_u = 0.58 \quad q_s = 0 \quad q_d = 0.42$$

the measure Q is not equivalent
to P

Exercise 3 (10 points)

Consider a market model where it is possible to trade on a stock with current price $S(0) = 20$ euro. Assume that the stock price follows a geometric Brownian motion with $\mu = 0.03$, $\sigma = 0.1$ and the risk free-interest rate is $r = 0.05$.

1. Consider a European call option with strike $K = 20$ euro and maturity in one year. Find its price.
2. Establish how many shares of the stock we need to buy/sell in order to make our short position Delta-neutral.
3. Consider now two European call options written on a stock that follows a geometric Brownian motion with undefined parameters μ and σ . The strikes are $K_1 = 20$ euro and $K_2 = 40$ euro, and the call prices are 6 and 2 euro, respectively. Both options have maturity in one year. Establish under which condition on μ and $\sigma > 0$ the profit of a bear spread from these two call options is positive with a probability of at least 50%.

1) From Black-Scholes, the price of a European call option is

$$C = S N(d_1) - K e^{-rT} N(d_2), \text{ where}$$

$$d_1 = \left[\ln \frac{S}{K} + \left(r + \frac{\sigma^2}{2} \right) T \right] / \sigma \sqrt{T},$$

$$d_2 = d_1 - \sigma \sqrt{T}.$$

Plugging in the numbers, we get $d_1 = 0.55$, $d_2 = 0.45$ and

$$C = 1.3610.$$

2) Consider the portfolio $P = (x_s, x_c)$
 ↓
 # of shares ↑ # of calls option.

If we own one share, then $x_c = -1$ and the value of portfolio $P = x_s \cdot s - C$.
 To make it neutral w.r.t. variation of s we ask that $\frac{\partial P}{\partial s} = 0 \Rightarrow x_s - \Delta^c = 0$

$$\Rightarrow x_s = \Delta^c.$$

We then compute $\Delta^c = N(d_1) = 0.7088$, i.e. to make our short position neutral we need to buy 0.7088 shares.

3) Recall that a bear spread is obtained by:

- selling the call option with lower strike. (A)
- buying the " " with higher strike. (B)

We have the following scheme:

	A	B	TOTAL PAYOFF	TOTAL PROFIT
$S_T \leq k_1$	0	0	0	$C_1 - C_2 \neq 0$
$k_1 < S_T < k_2$	$-(S_T - k_1)$	0	$K_1 - S_T$	$K_1 - S_T + C_1 - C_2 = 24 - S_T$
$S_T \geq k_2$	$-(S_T - k_1)$	$S_T - K_2$	$K_1 - K_2$	$K_1 - K_2 + C_1 - C_2 = -16$

We further now that $S_T \sim S_0 \exp\left((\mu - \frac{\sigma^2}{2})T + \sigma W_T\right)$.

Hence, $0.5 \leq \mathbb{P}(\text{PROFIT} \geq 0)$

$$= \mathbb{P}([0 \leq S_T \leq 20]) \cup \{20 \leq S_T \leq 24; 24 - S_T \geq 0\}$$

$$= \mathbb{P}(0 \leq S_T \leq 24)$$

$$\geq \mathbb{P}(S_T \leq 24)$$

$$= \mathbb{P}\left(20 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} \leq 24\right)$$

$$\stackrel{T=1}{=} \mathbb{P}\left((\mu - \frac{1}{2}\sigma^2)\bar{T} + \sigma W_1 \leq \ln \frac{24}{20}\right)$$

$$\stackrel{N(0,1)}{=} N\left(\frac{\ln(1.2) - (\mu - \frac{1}{2}\sigma^2)}{\sigma}\right).$$

We now look for a condition on (μ, σ) : $N\left(\frac{\ln(1.2) - (\mu - \frac{1}{2}\sigma^2)}{\sigma}\right) \geq 0.5$, ≈ 0.620

which implies that $\frac{\ln(1.2) - (\mu - \frac{1}{2}\sigma^2)}{\sigma} \geq 0 \Rightarrow$

$$\boxed{\mu \leq \frac{1}{2}\sigma^2 + \ln(1.2)}$$

≈ 0.182