Financial Engineering

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Problem set 7

Topics: Ito's lemma, Black-Scholes-Merton model and option pricing

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Exercise 1

Suppose that a stock price S follows a geometric Brownian motion with expected return μ and volatility σ,

$$dS = \mu S dt + \sigma S dz$$

where dz is a Wiener process. What is the process followed by the variable S^n? Show that S^n also follows a geometric Brownian motion.

Exercise 2

Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off 1$ at time T. Assume that x follows the process:

$$dx = a\left(x_0 - x\right) dt + sxdz$$

where a, x_0 and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

Exercise 3

Suppose that x is the yield on a perpetual government bond that pays interest at the rate of 1$ per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process:

$$dx = a\left(x_0 - x\right) dt + sxdz$$

where a, x_0 and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

Hint: The price of a perpetual governement bond with annual coupon I and yield x is $P = \frac{I}{x}$.

Exercise 4

Calculate the price of a European call option on a non-dividend-paying stock when the stock price is 52$, the strike price is 50$, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is 3 months.

Exercise 5

Calculate the price of a European put option on a non-dividend-paying stock when the stock price is 69$, the strike price is 70$, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is 6 months.

Exercise 6

Calculate the price of a 3-month European put option on a non-dividend-paying stock with a strike price of 50$ when the current stock price is 50$, the risk-free interest rate is 10% per annum, and

the volatility is 30% per annum.
$P(0) = Ke^{-rT}N(-d2) - S_0N(-d1)$.

Exercise 7
Show that the Black-Scholes-Merton formulas for call and put options satisfy put-call parity.

Exercise 8
Show that the probability that a European call option will be exercised in a risk-neutral world is $N(d_2)$. What is an expression for the value of a derivative that pays off 100$ if the price of a stock at time T is greater than K?

Exercise 9
A stock price follows a geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is 38$.
(i) What is the probability that a European call option on the stock with an exercise price of 40$ and a maturity date in 6 months will be exercised?
(ii) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

Exercise 10
Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security that pays off a dollar amount equal to $\ln S_T$ at time T, where S_T denotes the value of the stock price at time T.
(i) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t.
(ii) Verify that the price satisfies the Black-Scholes-Merton differential equation.

Exercise session 7 - Solutions

Exercise 1
Suppose that a stock price S follows a geometric Brownian motion with expected return μ and volatility σ,

$$dS = \mu S dt + \sigma S dz$$

where dz is a Wiener process. What is the process followed by the variable S^n? Show that S^n also follows a geometric Brownian motion.

EXERCISE 1

We have:

$$dS = \mu S dt + \sigma S dz$$

$$G(S,t) = S^n$$

and by Itô's lemma:

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

and since:

$$\frac{\partial G}{\partial S} = n S^{n-1} \qquad \frac{\partial G}{\partial t} = 0 \qquad \frac{\partial^2 G}{\partial S^2} = n(n-1) S^{n-2}$$

we have:

$$dG = \left[n S^{n-1} \mu S + 0 + \frac{1}{2} n(n-1) S^{n-2} \sigma^2 S^2 \right] dt + n S^{n-1} \sigma S dz =$$

$$= \left[\mu n S^n + \frac{1}{2} n(n-1) \sigma^2 S^n \right] dt + \sigma n S^n dz$$

That is:

$$dG = \left[\mu n G + \frac{1}{2} n(n-1) \sigma^2 G \right] dt + \sigma n G dz =$$

$$= \left[\mu n + \frac{1}{2} n(n-1) \sigma^2 \right] G dt + \sigma n G dz$$

Therefore $G = S^n$ also follows a geometric Brownian motion with:

expected return $= \mu n + \frac{1}{2} n(n-1)\sigma^2$

volatility $= n\sigma$

Exercise 2
Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off 1$ at time T. Assume that x follows the process:

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0 and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

EXERCISE 2

We have:

x = yield to maturity with continuous compounding on a zero-coupon bond that pays off 1 $ at time T

$$dx = a(x_0 - x) dt + sx dz$$

a, x_0, s = positive constants

dz = Wiener process

In this case the bond price is:

dz = Wiener process

In this case the bond price is:

$$B = 1 \cdot e^{-x(T-t)}$$

and the process followed by B, from Itô's lemma, is:

$$dB = \left[\frac{\partial B}{\partial x} a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2} \frac{\partial^2 B}{\partial x^2} s^2 x^2 \right] dt + \frac{\partial B}{\partial x} sx\, dz$$

and since:

$$\frac{\partial B}{\partial x} = -(T-t) e^{-x(T-t)} = -(T-t) B$$

$$\frac{\partial B}{\partial t} = x e^{-x(T-t)} = xB$$

$$\frac{\partial^2 B}{\partial x^2} = -(T-t) e^{-x(T-t)} \left(-(T-t)\right) = (T-t)^2 e^{-x(T-t)} = (T-t)^2 B$$

we obtain:

$$dB = \left[-a(x_0 - x)(T-t) B + xB + \frac{1}{2} (T-t)^2 s^2 x^2 B \right] dt - sx(T-t) B\, dz =$$

$$= \left[-a(x_0 - x)(T-t) + x + \frac{1}{2} (T-t)^2 s^2 x^2 \right] B\, dt - sx(T-t) B\, dz$$

Exercise 3
Suppose that x is the yield on a perpetual government bond that pays interest at the rate of 1\$ per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process:

$$dx = a(x_0 - x)\, dt + sx dz$$

where a, x_0 and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?
Hint: The price of a perpetual governement bond with annual coupon I and yield x is $P = \frac{I}{x}$.

EXERCISE 3

We have:

x = yield with continuous compounding on a perpetual bond that pays 1 \$ per annum

$$dx = a(x_0 - x)\, dt + sx\, dz$$

a, x_0, s = positive constants
dz = Wiener process

In this case the bond price is:

$$B = \frac{1}{x}$$

and the process followed by B, from Itô's lemma, is:

$$dB = \left[\frac{\partial B}{\partial x} a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2} \frac{\partial^2 B}{\partial x^2} s^2 x^2 \right] dt + \frac{\partial B}{\partial x} sx\, dz$$

and since:

$$\frac{\partial B}{\partial x} = -\frac{1}{x^2} \qquad \frac{\partial B}{\partial t} = 0 \qquad \frac{\partial^2 B}{\partial x^2} = \frac{2}{x^3}$$

we obtain:

$$dB = \left[-\frac{1}{x^2} a (x_0 - x) + 0 + \frac{1}{2} \cdot \frac{2}{x^3} s^2 x^2 \right] dt - \frac{1}{x^2} s x \, dz =$$

$$= \left[-a (x_0 - x) \frac{1}{x^2} + \frac{s^2}{x} \right] dt - \frac{s}{x} dz$$

$$= \left[-a (x_0 - x_0) \frac{1}{x} + s^2 \right] B \, dt - sB \, dz$$

Exercise 4

Calculate the price of a European call option on a non-dividend-paying stock when the stock price is 52\$, the strike price is 50\$, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is 3 months.

EXERCISE 4

We have:

$S_0 = 52 \qquad r = 0,12$ per annum $\qquad T = \frac{3}{12}$

$K = 50 \qquad \sigma = 0,30$ per annum

and according to the Black-Scholes-Merton formula for the price of a European call option:

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

with:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

hence:

$$d_1 = \frac{\ln\left(\frac{52}{50}\right) + \left(0,12 + \frac{0,3^2}{2}\right) \cdot \frac{3}{12}}{0,30 \cdot \sqrt{\frac{3}{12}}} = 0,5365$$

$$d_2 = 0,5365 - 0,30 \cdot \sqrt{\frac{3}{12}} = 0,3865$$

and then:

$$c = 52 \cdot N(0,5365) - 50 \cdot e^{-0,12 \cdot \frac{3}{12}} \cdot N(0,3865) =$$

$$= 52 \cdot 0,7042 - 50 \cdot e^{-0,12 \cdot \frac{3}{12}} \cdot 0,6504 =$$

$$= 5,06$$

i.e. the price of the call option is 5,06 \$.

Exercise 5

Calculate the price of a European put option on a non-dividend-paying stock when the stock price is 69$, the strike price is 70$, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is 6 months.

EXERCISE 5

We have:

$S_0 = 69$ $\quad r = 0,05$ per annum $\quad T = \frac{6}{12}$

$K = 70$ $\quad \sigma = 0,35$ per annum

and according to the Black - Scholes - Merton formula for the price of a European put option:

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

with:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

hence:

$$d_1 = \frac{\ln\left(\frac{69}{70}\right) + \left(0,05 + \frac{0,35^2}{2}\right)\cdot\frac{6}{12}}{0,35\cdot\sqrt{\frac{6}{12}}} = 0,1666$$

$$d_2 = 0,1666 - 0,35\cdot\sqrt{\frac{6}{12}} = -0,0809$$

and then:

$$p = 70\cdot e^{-0,05\cdot\frac{6}{12}}\cdot N(0,0809) - 69\cdot N(-0,1666) =$$

$$= 70\cdot e^{-0,05\cdot\frac{6}{12}}\cdot 0,5323 - 69\cdot 0,4338 =$$

$$= 6,40$$

i.e. the price of the put option is 6,40 $.

Exercise 6

Calculate the price of a 3-month European put option on a non-dividend-paying stock with a strike price of 50$ when the current stock price is 50$, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

$P(0) = Ke^{-rT}N(-d2) - S_0N(-d1)$.

EXERCISE 6

We have:

$S_0 = 50$ $\quad r = 0,10$ per annum $\quad T = \frac{3}{12}$

$K = 50$ $\quad \sigma = 0,30$ per annum

and according to the Black - Scholes - Merton formula for the price of a European put option:

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

with:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

hence:

$$d_1 = \frac{\ln\left(\frac{50}{50}\right) + \left(0,1 + \frac{0,3^2}{2}\right)\cdot\frac{3}{12}}{0,3\sqrt{\frac{3}{12}}} = 0,2417$$

$$d_2 = 0,2417 - 0,3\sqrt{\frac{3}{12}} = 0,0917$$

and then:

$$p = 50 \cdot e^{-0,1\cdot\frac{3}{12}} \cdot N(-0,0917) - 50 \cdot N(-0,2417) =$$

$$= 50 \cdot e^{-0,1\cdot\frac{3}{12}} \cdot 0,4634 - 50 \cdot 0,4045 =$$

$$= 2,37$$

i.e. the price of the put option is 2,37 \$

EXERCISE 7

Exercise 7

Show that the Black-Scholes-Merton formulas for call and put options satisfy put-call parity.

We have:

$$c + Ke^{-rT} = p + S_0 \qquad \text{put-call parity}$$

The Black-Scholes-Merton formula for a European call option is:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

so that:

$$c + Ke^{-rT} = S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT}$$

and also:

$$c + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT}[1 - N(d_2)]$$

that is:

that is:

$$c + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT} N(-d_2)$$

The Black-Scholes-Merton formula for a European put option, on the other hand, is:

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

so that:

$$p + S_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1) + S_0$$

and also:

$$p + S_0 = Ke^{-rT} N(-d_2) + S_0 [1 - N(-d_1)]$$

that is:

$$p + S_0 = Ke^{-rT} N(-d_2) + S_0 N(d_1)$$

We have therefore:

$$c + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT} N(-d_2)$$

$$p + S_0 = Ke^{-rT} N(-d_2) + S_0 N(d_1)$$

hence:

$$c + Ke^{-rT} = p + S_0$$

i.e. the put-call parity holds.

EXERCISE 8

Exercise 8

Show that the probability that a European call option will be exercised in a risk-neutral world is $N(d_2)$. What is an expression for the value of a derivative that pays off 100\$ if the price of a stock at time T is greater than K?

We have:

- European call option
- Derivative that pays off 100 \$ if $S_T > K$

and the probability that the call option will be exercised is the probability that $S_T > K$. Since:

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T \,;\, \sigma^2 T\right]$$

in a risk-neutral world (where $\mu = r$) we have:

$$\ln S_T \sim \phi\left[\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T \,;\, \sigma^2 T\right]$$

and then:

$$P(S_T > K) = P(\ln S_T > \ln K)$$

that is:

$$P\left(\frac{\ln S_T - \mu}{\sigma} > \frac{\ln K - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) =$$

$$= P\left(Z > \frac{\ln K - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) =$$

$$= 1 - P\left(Z < \frac{\ln K - \ln S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) =$$

$$= 1 - P\left(Z < -\frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) =$$

$$= 1 - P(Z < -d_2) =$$

$$= 1 - N(-d_2)$$

$$= N(d_2)$$

The expected value at time T in a risk-neutral world of a derivative that pays off 100 \$ when $S_T > K$ is therefore:

$$100 \cdot N(d_2)$$

and the value of this derivative at time 0 is equal to its discounted expected payoff, that is:

$$e^{-rT} \cdot 100 \cdot N(d_2)$$

Exercise 9

A stock price follows a geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is 38\$.

(*i*) What is the probability that a European call option on the stock with an exercise price of 40\$ and a maturity date in 6 months will be exercised?

(*ii*) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?

EXERCISE 9

We have:

$$dS = \mu S\,dt + \sigma S\,dz \qquad S_0 = 38$$

$$\mu = 0{,}16 \qquad K = 40$$

$$\sigma = 0{,}35 \qquad T = \frac{6}{12}$$

The probability that a European call option on the stock with $K = 40$ and $T = \frac{6}{12}$ will be exercised is the probability that $S_T > K$, i.e.:

$P\left(S_{\frac{6}{12}} > 40\right)$

Since S follows a geometric Brownian motion we have that:

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T \,;\, \sigma^2 T\right] =$$

$$= \phi\left[\ln 38 + \left(0{,}16 - \frac{0{,}35^2}{2}\right)\cdot \frac{6}{12} \,;\, 0{,}35^2 \cdot \frac{6}{12}\right] =$$

$$= \phi\left(3{,}6870 \,;\, 0{,}06125\right)$$

and then:

$$P(S_T > 40) = P(\ln S_T > \ln 40) = P(\ln S_T > 3{,}689)$$

that is:

$$P\left(\frac{\ln S_T - \mu}{\sigma} > \frac{3{,}689 - 3{,}6870}{\sqrt{0{,}06125}}\right) = P(Z > 0{,}008) =$$

$$= 1 - P(Z < 0{,}008) =$$

$$= 1 - N(0{,}008) =$$

$$= 1 - 0{,}5032 =$$

$$= 0{,}4968$$

and the probability that the call option will be exercised is 49,68%.

The probability that a European put option on the stock with $K = 40$ and $T = \frac{6}{12}$ will be exercised is the probability that $S_T < K$, i.e.:

$$P\left(S_{\frac{6}{12}} < 40\right) = 1 - P\left(S_{\frac{6}{12}} > 40\right) =$$

$$= 1 - 0{,}4968 =$$

$$= 0{,}5032$$

hence the probability that the put option will be exercised is 50,32%.

Exercise 10

Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a security that pays off a dollar amount equal to $\ln S_T$ at time T, where S_T denotes the value of the stock price at time T.

(i) Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t.

(ii) Verify that the price satisfies the Black-Scholes-Merton differential equation.

EXERCISE 10

We have:

μ = expected return

σ = volatility

S_T = stock price at time T

$\ln S_T$ = payoff of the derivative at time T

and since $\ln S_T$ follows a normal distribution:

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T \;;\; \sigma^2 T\right]$$

at time t the expected value of $\ln S_T$ is:

$$E[\ln S_T] = \ln S + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)$$

and in a risk-neutral world (where $\mu = r$) the expected value of $\ln S_T$ is:

$$\hat{E}[\ln S_T] = \ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)$$

and using risk-neutral valuation the value of the derivative at time t is equal to its discounted expected payoff, that is:

$$f = e^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]$$

From this expression we have:

$$\frac{\partial f}{\partial t} = -e^{-r(T-t)}\left(r - \frac{\sigma^2}{2}\right) + re^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$\frac{\partial f}{\partial S} = e^{-r(T-t)} \cdot \frac{1}{S}$$

$$\frac{\partial^2 f}{\partial S^2} = -e^{-r(T-t)} \cdot \frac{1}{S^2}$$

and the left-hand side of the Black-Scholes-Merton differential equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$$

becomes:

$$-e^{-r(T-t)}\left(r - \frac{\sigma^2}{2}\right) + re^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] +$$

$$+ r\cancel{S}e^{-r(T-t)} \cdot \frac{1}{\cancel{S}} + \frac{1}{2}\sigma^2 \cancel{S^2}\left(-e^{-r(T-t)}\right) \cdot \frac{1}{\cancel{S^2}} =$$

$$= e^{-r(T-t)}\left\{-\cancel{r} + \cancel{\frac{\sigma^2}{2}} + r\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] + \cancel{r} - \cancel{\frac{\sigma^2}{2}}\right\} =$$

$$= re^{-r(T-t)}\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] =$$

$= 2f$

that is equal to the right-hand side of the same equation, hence the differential equation is satisfied.