Politecnico di Torino
Financial Engineering-Exam 02-07-2023
P. Semeraro

SURNAME AND NAME

Student number

All the answers must be clearly motivated, the numerical results are not sufficient.

Answers written with a pencil are null.
Exercise 1 (10 points)
Suppose that 3-month, 6-month and 12-month zero rates are, respectively, $4\%, 4.2\%$ and 4.3% per annum with continuous compounding. Estimate the cash price of a bond with a face value of 100 that will mature in 12 months and pay a coupon of \$1 in three months and in six months.
A portfolio consists of a 5-year zero-coupon bond with face value of 5000 \$ and a 6-year zero-coupon bond with face value of 8000 \$. The current yield on all bonds is 6% per annum.
(a) Compute the duration and the convexity of the portfolio (using continuous compounding).
(b) Compute the percentage change in the value of the portfolio in the case of a 0.5% per annum increase. Discuss with the results in (a).

Exercise 3 (10 points) Let $dS = 0.07Sdt + 0.12SdW$ and risk-free interest rate is 2% per annum (all rates are continuously compounded).
(a) Price a European call option with strike price 40, maturing in 2 years, written on the stock $S(t)$ (no dividends) whose current price $S(0)$ is 37.
(b) If an investor holds 100 such calls and wishes a portfolio that is not sensitive to variations of the underlying asset (on the short term), how many stock shares should she hold?
(c) Find the Γ of that portfolio.

Exercise 3 (12 points) A call option with strike 8 euros is written on a stock with current price 8 euros and without dividends. At each ensuing two years the stock price can move up by 40% with probability $p = 0.8$ or down by 60% with probability $1 - p$. Denote by $S_t, t = 0, 1, 2$ (t is measured in years) the stochastic process representing the stock. The risk free rate $R = 4\%$ per year.

1. Compute the conditional expectation of S_2 at the present date. Is S_t a martingale with respect to the probability p?

2. Let S_t^* the discounted price process, is it a martingale with respect to p? If not, find a martingale measure Q for S_t^*.

3. Find the price of the European call, of the European put options and of an American call option with the same parameters of the European call option.

4. Assume that only the stock S and a risk free bound are traded. Is the market arbitrage free? why? Comment on the result. Consider now a market where only the stock S^* and a risk free bound are traded. in the next two years S^* can rise by 4% with $p = 0.8$ or fall by 4% with $p = 0.2$ and $R = 0.2$. Is the market arbitrage free? why? Comment on the result.

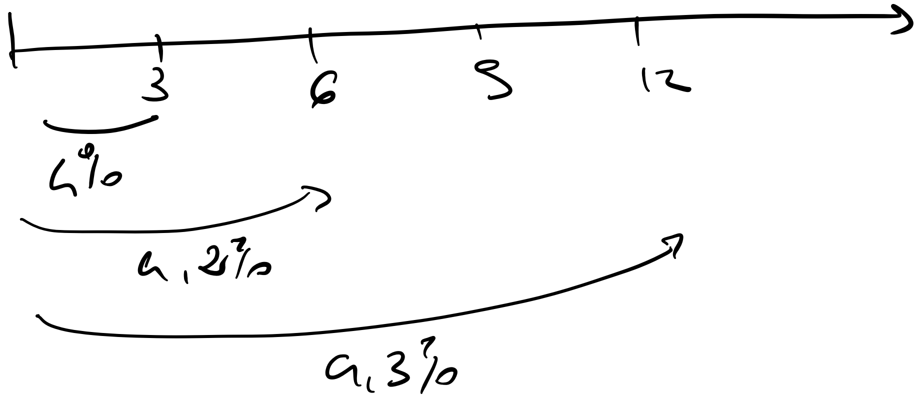
EXERCISE 1

The bond pays:

1€ 3 months

1€ 6 months

100€ 12 months



cash price C

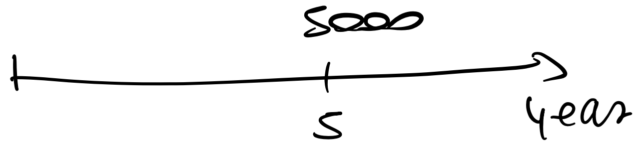
$$C = 1\,e^{-0.04\times 0.25} + 1\,e^{-0{,}042\times 0{,}5} + 100\,e^{-0{,}043\times 1} =$$

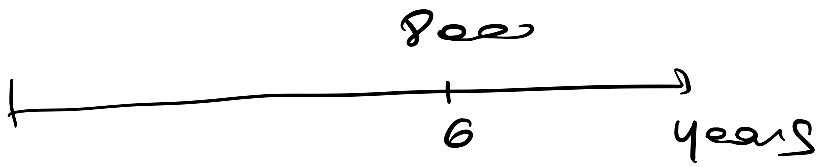
$$= 0{,}99 + 0{,}98 + 95{,}79 =$$

$$= 97{,}75$$

$$\boxed{C = 97{,}76}$$

(a)





current yeld = 6% per annum

Duration:

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-y t_i}}{\sum_{i=1}^{n} c_i e^{-y t_i}} =$$

$$= \frac{5 \cdot 5000\, e^{-0.06 \times 5} + 6 \times 8000\, e^{-0.06 \times 6}}{\underbrace{5000\, e^{-0.06 \times 5}}_{3704{,}081} + \underbrace{8000\, e^{-0.06 \times 6}}_{5581{,}41}}$$

$$= \frac{52008{,}92}{9285{,}502} \cong 5{,}60 \text{ years}$$

Convexity:

$$C = \frac{\sum_{i=1}^{n} t_i^2 c_i e^{-y_i t_i}}{\sum_{i=1}^{n} c_i e^{-y_i t_i}} =$$

$$= \frac{293533,1}{9285,502} = 31,61$$

(b) Value of the portfolio

$$V = 5000 e^{-0.06 \times 5} + 8000 e^{-0.06 \times 6}$$

$$= 9285,502$$

If the yield increases 0.5%

it becomes 0.065 and

$$V' = 5000 e^{-0.065 \times 5} + 8000 e^{-0.065 \times 6} =$$

$$= 9029,092$$

$$\frac{V' - V}{V} = \frac{-256,4}{9285,502} = -0,0271$$

Using first order approx with duration in (a)

$$\frac{\Delta V}{V} = -D\Delta y = -5{,}60 \cdot 0{,}005$$

$$= -0{,}028$$

Using second order approx with convexity in (a)

$$\frac{\Delta V}{V} = -D\Delta y + \frac{1}{2}C(\Delta y)^2 =$$

$$= -0{,}028 + 0{,}000395 =$$

$$= -0{,}0276$$

with convexity the approx improves since it is based on second order Taylor expansion

EXERCISE 2

(a) $S_0 = 37$

$K = 40$

$T = 2$

$r = 0.02$

$\sigma = 0.12$

$$e^{-rT} = e^{-0{,}02 \times 2} = 0{,}96$$

$$d_1 = \frac{\ln(37/40) + \left(0.02 + \frac{0.12^2}{2}\right) \cdot 2}{0.12\sqrt{2}} = -0{,}138$$

$$d_2 = d_1 - 0.12\sqrt{2} = -0{,}31$$

$N(d_1) = 0{,}44$; $N(d_2) = 0{,}38$

the call price is

$$C = S_0 N(d_1) - K e^{-rT} N(d_2) =$$

$$= 37 \cdot 0{,}44 - 40 \cdot 0.96 \cdot 0.38 = 1{,}90$$

$$\boxed{C = 1{,}90}$$

(b) $P = 100\,C + nS$

A portfolio is not sensitive to variation of S if $\Delta_P = 0$

We have $\Delta_P = 100\,\Delta_C + n \cdot 1$,

since the Δ of the stock is 1, i.e.

$$\Delta_S = 1$$

we have $\Delta_C = N(d_1) = 0{,}44$

thus

$$\Delta_P = 100 \cdot 0{,}44 + n$$

and

$$n = -100 \cdot \Delta_C = -100 \cdot 0{,}44 \overset{\checkmark}{\simeq} -44{,}48$$

$$-44$$

$$\boxed{\text{To hedge 100 long call we have to short 44 stock shares.}}$$

(c) the Γ of the portfolio

$$P = 100\,C - 44\,S$$

$$\Gamma_P = 100\,\Gamma_C$$

EXERCISE 3

$S_0 = 8$ no dividends

$u = 1{,}4$ $p_u = 0.8$ every year

$d = 0{,}4$ $p_d = 0.2$ → value if $d = 0.6$

$r_2 = 0.04$

$t=0$	$t=1$	$t=2$	$\mathbb{P}$
		$S_{uu} = 15.68$	0.64
	$S_u = 11.2$		
$S_0 = 8$		$S_{ud} = 4.48$ (6.72)	0,32
	$S_d = 3.2$ (4.8)		
		$S_{dd} = 1.18$ (2.88)	0.04

Branch probabilities: 0.8 (up), 0.2 (down).

1]

$p_{uu} = p^2 = 0.64$

$p_{ud} = 2p(1-p) = 0{,}32$

$p_{dd} = (1-p)^2 = 0.04$

$$E_p[S(2)] = 15.68 \times 0.64 + \overset{6.72}{4.48} \times 0.32 + \overset{2.88}{1.18} \times 0.04 = \overset{12.23}{11{,}52}$$

$E_p[S(2)] \neq S_0$ we do not have a martingale!

$\Rightarrow E_p[S(2)] = E_p[S(2) \mid S_0 = 8]$

2) Let $\tilde{S}(t) = \frac{S(t)}{1+R}$ a martingale?

$$E_P[\tilde{S}(t)] = \frac{E_P[S(t)]}{1+r} = \frac{11.52}{(1.04)^2} \neq 8 \quad \text{no!}$$

12.23 $= 11.3991 \neq 8$

Martingale measure

$$q_u = \frac{(1+r)-d}{u-d} = \frac{1.04-0.4}{1.4-0.4} = 0.62$$

0.6 0.6 0.55

$$q_d = 1 - q_u = 0.38$$

0.45

BINOMIAL MODEL COMPLETE

STRIKE 8

3) Call PRICE

t=0	t=1	t=2	P	CALL PAYOFF
	0.8: $S_u = 11.2$	0.8: $S_{uu} = 15.68$	0.64	7,68
$S_0 = 8$		0.2 / 0.8: $S_{ud} = 4.48$ (6.72)	0.32	0
	0.2: $S_d = 3.2$ (4.8)	0.2: $S_{dd} = 1.28$ (2.88)	0.04	0

$$C = \frac{1}{(1+r)^2}\left[q_u^2 \cdot 7{,}68 + 2p_u(1-p_u)\cdot 0 + (1-p_u)^2 \cdot 0 \right]$$

2.1479

$= 2{,}48$ Euros

Put price

1.564

$$P = C - S_0 + \frac{K}{(1+r)^2} = 2.53$$

PUT-CALL PARITY

<u>American</u> Option CALL

early exercise never optimal if S_t does not pay dividends

$$\Rightarrow \quad C^a = C = 2{,}48$$

4) The market is arbitrage free because $d < 1+R < u$

$0.4 < 1.04 < 1.4$

0.6

S^* has $up = 1.04$ $\quad d = 0.96$

and $R = 0.2$

$1+R = 1.2 > 1.04$

the market is not arbitrage free

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Exercise 1 (10 points)

The current price of a stock is $S(0) = 8$ euros. In one year, such price may move up to 12 euros with probability 2/3 or down to 7 euros with probability 1/3.

1. Find a range for the risk-free interest rate r that makes the market composed by the stock S and a riskless bond B arbitrage free and motivate the answer. Choose $r = 4\%$ and show that the market is arbitrage free and complete.

2. Find the price of a call option with payoff strike $K = 11$ and maturity $T = 2$ and the probability that the option is exercised.

3. Assume now that in one year, S may move up to 16 euros or to 12 euros, or move down to 8 euros. Each event may happen with probability 1/3. Suppose now that on the market it is possible to sell (at the price of one euro) a European Call option written on the stock above, with maturity of 1 year and with strike of 11 euros.

   Consider the following strategy:

   - buy one share of the stock;
   - sell the Call;
   - borrow 7 euros at the risk-free rate.

   Verify that such a strategy represents an arbitrage opportunity.

SOLUTION

$S_1 = 12 \quad S_0 = 8 \quad S_1^u / S_0 = \frac{12}{8} = 1.5 \Rightarrow u = 1.5$

$S_1^d / S_0 = 7/8 = 0,875 \Rightarrow d = 0.875$

1) Arbitrage free means that there are not arbitrage opportunities in the market

the market is arbitrage free iff $1+r \in [u, d]$

therefore if

$$0.875 < 1+r < 1.5$$

Otherwise the future asset price is always higher than the risk free asset or viceversa and these two scenarios allows us to construct arbitrage strategies → you should show one example

We can prove this condition since it is equivalent to require that there is a risk neutral martingale measure, i.e

$Q: \quad \mathbb{E}^Q\left[\frac{S(1)}{1+r}\right] = S(0)$ that means

that the system

$$\begin{cases} u q_u + d q_d = 1+r \\ q_u + q_d = 1 \end{cases}$$

must have a positive solution

If $r = 0.04$ we have

$$1+r = 1.04 \in (0.875; 1.5) \Rightarrow \text{the}$$

market is arbitrage free and it is also complete because the binomial model is complete means that every cont. claim is reachable and we have a unique price.
In fact if we look for a risk neutral Q we look for $Q = (q_u, q_d)$ so that

$$\begin{cases} u q_u + d q_d = 1+r \\ q_u + q_d = 1 \end{cases}$$

this system has a (unique) solution iff

$$\begin{vmatrix} u & d \\ 1 & 1 \end{vmatrix} = u - d \neq 0$$

, and this is the case

then

$$\begin{cases} q_u = \dfrac{(1+r) - d}{u-d} \\ q_d = \dfrac{u - (1+r)}{u-d} \end{cases}$$

and this is positive iff

$$d < 1+r < u, \text{ thus}$$
$$0.875 < 1+r < 1.5$$

In this case, since the solution is unique the market is also complete.

If $r = 0.04$, we have

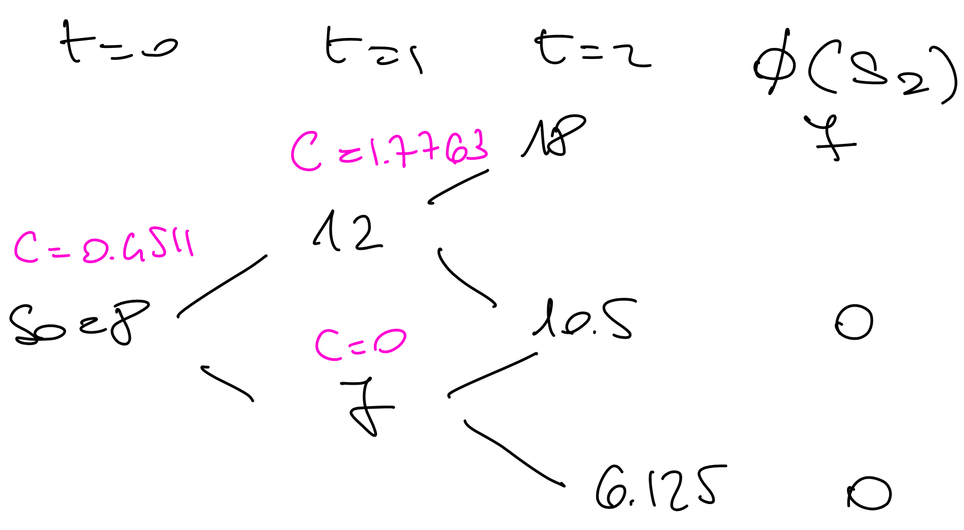
$$0.875 < 1.04 < 1.5$$

then
$$\begin{cases} q_u = \dfrac{(1+r) - d}{u-d} = 0.2640 \\[2ex] q_d = \dfrac{u - (1+r)}{u-d} = 0.7360 \end{cases}$$

is positive.

Therefore the market is arbitrage free (there is a positive solution) and complete (the solution is unique).

2] The binomial tree:



$$C_0 = \frac{1}{(1+r)^2} \sum_{k=0}^{2} \binom{2}{k} q_u^k q_d^{T-k} \phi(S u^k d^{3-k}) =$$

$$= \frac{1}{(1.04)^2} (0.2640)^2 \cdot 7$$

$$= 0.4511$$

the option is exercised if $S_2 > 11$

$$P(S_2 \geq 11) = P(S_2 = 18) = p^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

3] Consider the value of the strategy at time $t=0$ and $t=1$ year

$t=0$		$t=1$	
buy one stock	$-S_0 = -8$	sell the stock	S_T
sell the call	$+C_0 = 1$	call payoff	$-(S_T - 11)^+$
borrow 7 euro	$+7$	return the loan	$-7(1+0.04)$
(to buy the stock you need 7 euro)	//		$S_T - (S_T - k)^+ - 7.28$

$$S_T - (S_T - \overset{\overset{11}{\uparrow}}{k})^+ - 7.28 = \begin{cases} 16 - 5 - 7.28 = 3.72 & 1/3 \\ 12 - 1 - 7.28 = 3.72 & 1/3 \\ 8 - 0 - 7.28 = 0.72 & 1/3 \end{cases} \quad \mathbb{P}$$

$\to$ ARBITRAGE

Exercise 2 (10 points)

A 4-year bond with a yield of 5% (continuously compounded) pays a 6% coupon at the end of each year.

1) Determine the current price of the bond.

2) Find the duration and the convexity of the bond.

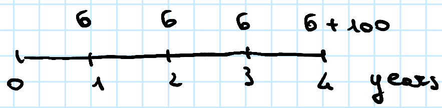
3) Use first only the duration, then both the duration and the convexity, to calculate the effect on the bond's price of a 0.3% decrease in its yield.

4) Ricalculate the bond's price on the basis of the new yield and compare the result with that obtained at point 3), commenting on it.

Solutions

EXERCISE 2

1) In this case the cash-flows are:



and the current price of the bond is:

$$P = 6 \cdot e^{-0{,}05 \cdot 1} + 6 \cdot e^{-0{,}05 \cdot 2} + 6 \cdot e^{-0{,}05 \cdot 3} + 106 \cdot e^{-0{,}05 \cdot 4} =$$

$$= 103{,}09$$

2) The duration of the bond is:

$$D = \frac{1 \cdot 6 \cdot e^{-0{,}05 \cdot 1} + 2 \cdot 6 \cdot e^{-0{,}05 \cdot 2} + 3 \cdot 6 \cdot e^{-0{,}05 \cdot 3} + 4 \cdot 106 \cdot e^{-0{,}05 \cdot 4}}{103{,}09} =$$

$$= 3{,}68 \text{ years}$$

and the convexity is:

$$C = \frac{1^2 \cdot 6 \cdot e^{-0{,}05 \cdot 1} + 2^2 \cdot 6 \cdot e^{-0{,}05 \cdot 2} + 3^2 \cdot 6 \cdot e^{-0{,}05 \cdot 3} + 4^2 \cdot 106 \cdot e^{-0{,}05 \cdot 4}}{103{,}09} =$$

$$= 14{,}19$$

3) If the yield decreases by 0,3% the effect on the price of the bond, using only the duration, is:

$$(\Delta P)_I = -D \cdot \Delta y \cdot P \rightarrow \Delta P = -3{,}68 \cdot (-0{,}003) \cdot 103{,}09 = 1{,}13$$

hence the bond price becomes:

$$P' = P + (\Delta P)_I = 103{,}09 + 1{,}13 = 104{,}22$$

The effect on the price of the bond using also the convexity is:

$$(\Delta P)_{II} = -D \cdot \Delta y \cdot P + \frac{1}{2} \cdot C \cdot (\Delta y)^2 \cdot P =$$

$$= -3{,}68 \cdot (-0{,}003) \cdot 103{,}09 + \frac{1}{2} \cdot 14{,}19 \cdot (-0{,}003)^2 \cdot 103{,}09 =$$

$$= 1{,}14$$

hence the bond price becomes:

$$P' = P + (\Delta P)_{II} = 103,09 + 1,14 = 104,23$$

4) If the yield decreases by 0,3%, i.e. from 5% to 4,7%, the price of the bond becomes:

$$P' = 6 \cdot e^{-0,047 \cdot 1} + 6 \cdot e^{-0,047 \cdot 2} + 6 \cdot e^{-0,047 \cdot 3} + 106 \cdot e^{-0,047 \cdot 4} =$$

$$= 104,23$$

That is consistent with the results found above (in particular, the 2nd order approximation is more precise than the 1st order one).

Exercise 3 (10 points)

An investor considers the possibility of creating both a bull spread and a bear spread using put options on a stock. The strike prices of the options are 40 euros and 45 euros and the costs of these options are 5 euros and 8 euros, respectively.

1) Describe how the investor can create a bull spread, then find the payoff and the profit for this spread.

2) Describe how the investor can create a bear spread, then find the payoff and the profit for this spread.

3) Determine for which prices of the underlying stock at maturity is possible to obtain a positive profit in the case of the bull spread and in the case of the bear spread.

EXERCISE 3

1) The investor can create a bull spread by:

- buying 1 put option with strike price $K_1 = 40$ that costs $p_1 = 5$
- selling 1 put option with strike price $K_2 = 45$ that costs $p_2 = 8$

The initial inflow is:

$$p_2 - p_1 = 8 - 5 = 3$$

and the payoff and the profit are:

Stock price range	Payoff from long put with strike K_1	Payoff from short put with strike K_2	Total payoff	Profit
$S_T \leq 40$	$40 - S_T$	$-(45 - S_T)$	-5	-2
$40 < S_T < 45$	0	$-(45 - S_T)$	$S_T - 45$	$S_T - 42$
$S_T \geq 45$	0	0	0	3

2) The investor can create a bear spread by:

- selling 1 put option with strike price $K_1 = 40$ that costs $p_1 = 5$
- buying 1 put option with strike price $K_2 = 45$ that costs $p_2 = 8$

The initial investment is:

$$p_2 - p_1 = 8 - 5 = 3$$

and the payoff and the profit are:

Stock price range	Payoff from long put with strike K_2	Payoff from short put with strike K_1	Total payoff	Profit
$S_T \leq 40$	$45 - S_T$	$-(40 - S_T)$	5	2
$40 < S_T < 45$	$45 - S_T$	0	$45 - S_T$	$42 - S_T$
$S_T \geq 45$	0	0	0	-3

3) In the case of the bull spread the profit is positive when:

$$S_T - 42 \geq 0 \quad \rightarrow \quad S_T > 42$$

while in the case of the bear spread the profit is positive when:

$$42 - S_T > 0 \quad \rightarrow \quad S_T < 42$$

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Answers written with a pencil are null.

Exercise 1 (10 points)

A trader considers the following two contracts:

(i) a 1-year long forward contract on a non-dividend-paying stock;

(ii) a 1-year long futures contract on a commodity that is an investment asset with storage costs.

The stock price relative to the first contract is 60 euros, while the spot price of the commodity relative to the second contract is 30 euros. The risk-free rate of interest is 5% per annum with continuous compounding for all maturities for both contracts.

1) Explain what is a forward contract and what are the forward price and the initial value of a forward contract.

2) Find the forward price and the initial value of the forward contract in point (i).

3) Find the forward price and the value of the forward contract after 6 months, assuming that the price of the stock is now 65 euros and the risk-free interest rate is still 5%.

4) Find the futures price of the commodity for delivery in 1 year assuming the presence of storage costs of 0.80 euros per year payable quarterly in advance. Comment on the difference between the forward price in case of commoditities and in case of stocks.

EXERCISE 1

1) See the theory

2) The forward price in the case of a non-dividend-paying stock is:

$$F_0 = S_0 e^{rT}$$

hence in this case:

$$F_0 = 60 \cdot e^{0,05 \cdot 1} = 63,08$$

while the initial value of the forward contract is always 0.

3) After 6 months the forward price is:

$$F_0 = S_0 e^{rT} = 65 \cdot e^{0,05 \cdot \frac{6}{12}} = 66,65$$

while the value of the forward contract is:

$$f = S_0 - K e^{-rT}$$

that is:

$$f = 65 - 63,08 \cdot e^{-0,05 \cdot \frac{6}{12}} = 3,48$$

4) The futures price in the case of a commodity that is an investment asset with storage costs is:

$$F_0 = (S_0 + U) e^{rT}$$

and in this case the present value of the storage costs for 1 year is:

$$U = 0,20 + 0,20 \cdot e^{-0,05 \cdot \frac{3}{12}} + 0,20 \cdot e^{-0,05 \cdot \frac{6}{12}} + 0,20 \cdot e^{-0,05 \cdot \frac{9}{12}} = 0,79$$

hence the futures price of the commodity is:

$$F_0 = (30 + 0,79) \cdot e^{0,05 \cdot 1} = 32,37$$

Storage costs: see theory.

It is important to specify that storage costs are added because the owner holding the asset (short) means we storage costs. In fact the long part does not pay storage costs until it own the asset, so the short part requires the long part to compensate storage costs and the forward price is higher.

Exercise 2(10 points)

A stock (indexed by A) is available on the market at the current price $S_A(0) = 8$ euros. In one year, the price may increase by 25% or decrease by 25%. Another stock (indexed by B) is also available. Its current price is $S_B(0) = 12$ euros that, in one year, may increase by 25% (when also the price of stock A is increased) or decrease by 25% (when also the price of stock A is decreased). The risk-free interest rate on the market is 4% per year (simple compounding).

1. Consider a European Call option with maturity of one year, with strike of 8 euros and written on stock A. Verify if it is possible to replicate such an option only by means of stock A and of cash invested or borrowed at risk free rate. If yes explain the replicating strategy.

2. Verify if it is possible to replicate the European Call option above by taking long positions in stock A, short positions in stock B and cash. If yes, find a replicating portfolio and compute the cost of the replicating strategy.

3. Verify if it is possible to replicate the European Call option above only by means of stock A, of cash and of one (and only one) short position in stock B. If yes, find the replicating portfolio and compute the cost of the replicating strategy.

SOLUTION EXERCISE 2

$T=1$

$r=0.04$

$\phi(S_A(T)) = (S_A(T)-8)^+$

$$\begin{array}{lll} & S_A(1)=10 & 2 \\ S_A(0)=8 & S_B(1)=15 & \\ S_B(0)=12 & S_A(1)=6 & 0 \\ & S_B(1)=9 & \end{array}$$

$$\begin{array}{ll} & S_B(1)=15 \\ S_B(0)=12 & \\ & S_B(1)=9 \end{array}$$

$C(0)=1 \longrightarrow C(1)=1.04$

cash

(a)

$P=(h_S, h_B)$

$$V_p(1) = \begin{cases} 2 & \text{if } S_A(1)=10 \\ 0 & \text{if } S_A(1)=6 \end{cases} \quad ; \quad V_p(1) = h_S S(1) + h_C (C(1))$$

$$V_p(1) = \begin{cases} h_S 10 + 1.04 \cdot h_C & \text{if } S_A(1)=10 \\ h_S 6 + 1.04 \cdot h_C & \text{if } S_A(1)=6 \end{cases}$$

$$\begin{cases} h_S 10 + h_C \cdot 1.04 = 2 \\ h_S \cdot 6 + h_C \cdot 1.04 = 0 \end{cases}$$

$4h_S = 2 \qquad h_S = 1/2 \quad \text{and} \quad h_C = -2.88$

$\boxed{\text{buy } \tfrac{1}{2} \text{ share and borrow 2.88 euros}}$

ⓑ long A, short B and cash

$P(h_A, h_B, h_C)$

$V_P(1) = h_A S_A(1) + h_B S_B(1) + h_C S_C(1)$ $\quad \phi(S_A(1))$

$$V_P(1) = \begin{cases} h_A \, 10 + h_B 15 + h_C \, 1.04 = 2 \\ h_A \cdot 6 + h_B \cdot 9 + h_C \, 1.04 = 0 \end{cases}$$

N.B. we want a solution with $h_A > 0$ and $h_B < 0$

not unique

the system has 1 ∞ solutions $\quad 3-2$

since $\begin{vmatrix} 15 & 1.04 \\ 9 & 1.04 \end{vmatrix} \neq 0 \Rightarrow \operatorname{rg} A = \operatorname{rg} A^c = 2$

Parametrize with h_B that I can choose $h_B \leq 0$

$$h_A \, 10 + h_B 15 + h_C \, 1.04 = 2$$

$$h_A \cdot 6 + h_B \cdot 9 + h_C \, 1.04 = 0 \qquad \text{subtraction}$$

$$4h_A + 6h_B + \quad = 2$$

$\Rightarrow h_A = 1/2 - \frac{3}{2} h_B$

$1.04 \, h_C = -6h_A - 9h_B = -3 + 9h_B - 9h_B = -3$

and $h_C = -3/1.04 = -2.88$

For example

$h_B = -1/3 \Rightarrow h_A = 1$

$1.04 h_C = -6h_A - 9h_B = -6 + 9/3 = -3$

$h_C = -3/1.04 = -2.88$

Replication cost

$V_P(0) = h_A S_A(0) + h_B S_B(0) + h_C =$

$= (+1/2 - 3/2\, h_B)\, 8 + h_B \cdot 12 - 2.88 =$

$= 1.12$ euros

c) the answer of this point is yes, you choose

$h_B = -1$

$h_A = 1/2 + 3/2 = 2$, $(h_C = -2{,}88)$

and $V_P(0)$ is unchanged since it is a a constant function of h_B

$V_P(0) = (+1/2 - 3/2\, h_B)\, 8 + h_B \cdot 12 - 2.88 = 1.12$ euros

Exercise 3 (10 points) Consider a market model where it is possible to trade on a stock (with current price $S_0 = 20$ euros) and on different European Call written on such a stock. The price of the stock follows a geometric Brownian motion, namely $dS(t) = 0.8S(t)dt + 0.3S(t)dW(t)$ with $S(0) = 20$. Our goal is an investment with one year as horizon of time. The risk free rate is 4% per year continuously compounded. Consider two European Call options $C1$ and $C2$ (both with maturity T = 1 year and written on the stock above), with strikes of $K1 = 20$ euro and $K2 = 25$ euro, respectively.

1. Find the Black and Scholes price of call option $C1$.
2. Find the price of a put option on the same stock with strike $K1$.
3. Assume that the two options have prices: $C1_0 = 2.75$ euro $C2_0 = 1$ euro, respectively.

   Find the the probability that a bear spread formed by the previous options is non-negative.

1) The Black and Scholes price is:

$$C_0^{E} = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} =$$

$$= \frac{\ln(20/20) + \left(0.04 + \frac{0.3^2}{2}\right)1}{0.3\sqrt{1}} = 0.2833$$

and

$$d_2 = d_1 - 0.3\sqrt{1} = -0.0167$$

Thus

$$N(d_1) = 0.6115$$

$$N(d_2) = 0.4934$$

and

$$C1_0 = 20 \cdot 0.6115 - 20\, e^{-0.04 \cdot 1} \cdot 0.4934 =$$

$$= 2.75$$

2) By using put-call parity we have:

$$P1_0 = C1_0 - S_0 + K e^{-rT} \cong 1.96$$

3)

A bear spread can be built - by means of call options - as follows:

- sell the call option with the lower strike
- buy the call option with the higher strike

the payoff is:

	A	B	TOTAL	TOTAL PROFIT
$S_T < K_1 = 20$	0	0	0	$C_0^1 - C_0^2$
$K_1 = 20 < S_1 < K_2 = 25$	$-(S_T - K_1)$	0	$K_1 - S_T$	$K_1 - S_T + C_0^1 - C_0^2$
$S_1 > 25$	$-(S_T - K_1)$	$S_T - K_2$	$K_1 - K_2$	$K_1 - K_2 + C_0^1 - C_0^2$

thus

	BEAR SPREAD
$S_T < 20$	$2.75 - 1 = 1.75$
$20 < S_T < 25$	$20 - S_T + 1.75 = 21.75 - S_T$
$S_T > 25$	$20 - 25 + 1.75 = -5 + 1.75 = -3.25$

$$P(\text{profit} \geq 0) =$$

$$P\left(\{S_T < 20\} \cup \{20 \leq S_T \leq 25 \cap \underbrace{21.75 - S_T > 0}^{S_T \leq 21.5}\}\right)$$

$$= P(S_T < 20) + P(20 \leq S_T \leq 21.75) =$$

$= P(S_1 \leq 21.75) =$
$\uparrow$
$T=1$

$= P\left(20 e^{(\mu - \frac{1}{2}\sigma^2)\cdot 1 + 0.3 W_1} \leq 21.75\right) =$

$= P\left(0.8 - \frac{1}{2} 0.3^2 + 0.3 W_1 \leq \ln \frac{21.75}{20}\right) =$ $\ln 1.0875$

$= P\left(0.7550 + 0.3 W_1 \leq 0.0839\right) =$

$= P\left(W_1 \leq \frac{0.0839 - 0.7550}{0.3}\right) = P\left(W_1 \leq -2.2370\right)$

$= N(-2.2370) \cong 0.012$

Similarly:

$S_1 = S_0 e^{(\mu - \frac{1}{2}\sigma^2) + \sigma W}$

$\ln S_1 = \ln S_0 + (\mu - \frac{1}{2}\sigma^2) + \sigma W$

etc...

Politecnico di Torino
Financial Engineering-Exam ~~07-21-2023~~ 09 - 22 - 2023
P. Semeraro

SURNAME and NAME

Student number

All the answers must be clearly motivated, the numerical results are not sufficient. All the answers that will be considered in the correction MUST be written here. If the student number and the name are not filled in the text will not be corrected.

Answers written with a pencil are null.

Exercise 1

We take a long position in a European Call option on a stock S with strike $K = 80$ euros and maturity $T = 1$ year. The price of the underlying follows a geometric Brownian motion with $S_0 = 80$ euros, $\mu = 0.2$ and $\sigma = 0.4$ per year. The risk-free interest rate available on the market is $r = 0.04\%$ per year continuously compunded.

1. Find the price of the Call option.

1.Compute the Delta Δ of the Call option and, accordingly, establish how many shares of stock S we need to buy/sell in order to make our long position Delta-neutral.

3. Compute the Delta Δ of a put option with the same underlying, maturity and strike and, accordingly, establish how many shares of stock S we need to buy/sell in order to make our short position Delta-neutral.

SOLUTION

$K_1 = 80 \quad T = 1\text{ year} \quad S_0 = 80 \quad r = 0.04 \quad \sigma = 0.4$

1] The Black and Scholes price is:

$$C_0^{E} = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} =$$

$$= \frac{\ln(80/80) + \left(0.04 + \frac{0.4^2}{2}\right)\cdot 1}{0.4\sqrt{1}} = 0.3$$

and

$$d_2 = d_1 - 0.4\sqrt{T} = -0.1$$

Thus

$N(d_1) = 0.6179$

$N(d_2) = 0.4602$

$$C_0 = 80 \cdot 0.618 - 80\, e^{-0.04}\, 0.4602 = 14.062$$

2] $\Delta_1 = N(d_1) = 0.6179$

Short the call : $h(x_S, x_C) = P$ with $x_C = 1$

$\underset{S(t)}{\uparrow}$ $\underset{\text{Call}}{\uparrow}$

$P = x_S S + 1 \cdot C$

$\Delta_P = + x_S + \Delta_1$

To have P Δ neutral $\Delta_1 = -x_S$

we have to short 0.6179 shares to make a long position in the call Δ neutral.

3] $\Delta_1^P = \Delta_C - 1 = 0.618 - 1 = -0.382$

$h = (x_S, x_P) = (x_S, -1)$

$P = x_S S - P$

$\Delta P = x_S + 0.382 \rightarrow x = -0.382$

we have to short 0.382 shares

Exercise 2

Your current wealth is 100,000 euro, which you invest in a financial portfolio S whose value process follows a geometric Brownian motion with drift coefficient 7% and volatility coefficient 25%. The risk-free rate (with continuous compounding) is 3%.

1. Your target is to double your wealth in 10 years. Find the shortfall probability, i.e., the probability you will not achieve your target.

2. Define the loss of a financial portfolio (use logreturns) its value at risk.

3. Find the value at risk of the portfolio loss after 10 years at level 0.95.

SOLUTION

$\mu = 0.07 \quad \sigma = 0.25$

1] $P(T) = P(0)\, e^{(\mu - \frac{\sigma^2}{2})T + \sigma W(T)}$ with $W(T)$ standard brownian motion

The target is $P(T) = 2P(0)$, thus

$$P(P(T) \leq 2P(0)) = P\left(P(0)\, e^{(\mu - \sigma^2/2)T + \sigma W(T)} \leq 2P(0)\right) =$$

$$= P\left(e^{(\mu - \sigma^2/2)T + \sigma W(T)} \leq 2\right) =$$

$$= P\left((\mu - \sigma^2/2)T + \sigma W(T) \leq \ln 2\right) =$$

$$= P\left(W(T) \leq \frac{\ln 2 - (\mu - \sigma^2/2)T}{\sigma}\right) =$$

$$= P\left(W(1) \leq \frac{\ln 2 - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\right) =$$

$$= \phi\left(\frac{\ln 2 - (0.07 - \frac{0.25^2}{2})10}{0.25 \cdot \sqrt{10}}\right) = \phi(0.38661) = 0.65048$$

0.7905

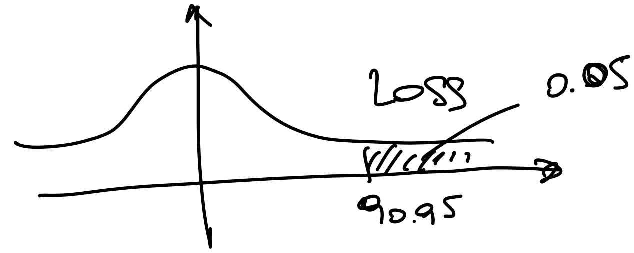
the shortfall probability is 0.65048

2] R portfolio log return $R(t+h) = \ln \frac{V(t+h)}{V(t)}$

Loss $\quad L(t+h) = -R(t+h)$

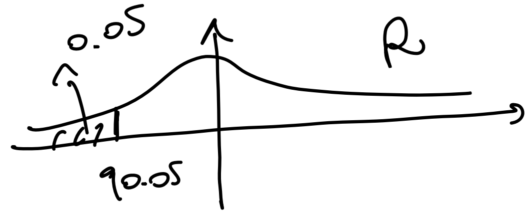
VaR : see theory

3] $t=0 \quad h=10$ years $\quad L(10) = -R(10) = -\ln \frac{V(10)}{V(0)}$



$P(L(10) \le y) = 0.95$

$R(10) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right) = (0.3875,\ 0.25^2 \cdot 10)$



$\to L(10) \le y$

$L(10) \sim N(-0.3875,\ 0.25^2 \cdot 10)$

$VaR_{0.95}(L(10)) = -0.3875 + 0.25(1.64485)\sqrt{10} = 0.91694$

$= \mu_L + \sigma_L Q(\alpha) = \mu_L + \sigma_L Q(0.95)$

μ_L is the mean of the loss $L(10)$

σ_L " " st. dev. " " "

You then consider $VaR(\tilde{L}(10))$ with

$\tilde{L}(10) = V(0) \cdot L(10) = V(0) \cdot (-R(10)) \cong 91694$

Exercise 3

Let $W(t)$ be a standard Wiener process.

1. Show that
$$cov(W(t), W(s)) = min\{s, t\} \tag{0.1}$$
and that
$$\rho(W(t), W(s)) = \sqrt{\frac{min\{s, t\}}{max\{s, t\}}}$$

2. Write down the SDE for the process
$$X(t) = W(t)^4$$

3. Show that
$$Var(Y(t)) = \frac{t^3}{3},$$
where
$$Y(t) = \int_0^t (t - s)\ dW(s)$$

Solution -

1. See the theory: main passages below

$$cov(W(t), W(s)) = E[W(t)W(s)] - \underbrace{E[W(t)]}_{=0}\underbrace{E[W(s)]}_{=0} = E[W(t)W(s)]$$

assume w.l.g. $s \leq t$

$$E[W(t)W(s)] = E[W(t)W(s) - W(s)^2 + W^2(s)] =$$

$$= E[W(s)(W(t) - W(s))] + E[W^2(s)] =$$

$$= \underbrace{E[W(s)]}_{=0} E[W(t) - W(s)] + \underbrace{s}_{\text{because } W(s) \sim N(0,s)} = s$$

$$\Rightarrow cov(W(t), W(s)) = s$$

Similarly $t \le s$ $\quad Cov(W(t), W(s)) = t$

$$\rho = \frac{Cov(W(t), W(s))}{\sqrt{var(W(t))}\sqrt{var W(s)}} = \frac{\min\{s,t\}}{\sqrt{t}\sqrt{s}} = \text{se } t < s$$

$$= \frac{t}{\sqrt{t}\sqrt{s}} = \sqrt{t}/\sqrt{s} = \frac{\sqrt{\min\{s,t\}}}{\sqrt{\max\{s,t\}}}$$

se $s < t$ analogo

2) let $f(t,x) = x^4$

$$\partial_t f = 0$$
$$\partial_x f = 4x^3$$
$$\partial_{xx} f = 12x^2$$

$$df(t, W(t)) = f_t(t, W(t))\,dt + f_x(t, W(t))\,dW(t) +$$
$$+ \frac{1}{2} f_{xx}(t, W(t))\,dt$$
$$= 0 \cdot dt + 4W^3 dW(t) + \frac{1}{2} 12 W^2(t)\,dt =$$
$$= 6W^2 dt + 4W^3 dW$$

3) by Ito Isometry

$$E[Y_t^2] = E\left(\int_0^t (t-s)\,dW_s\right)^2 = \int_0^t (t-s)^2\,ds$$

$$= -\frac{(t-s)^3}{3}\Big|_0^t = -\frac{0}{3} + \frac{t^3}{3} = \frac{t^3}{3}$$

since $E(Y_t) = E\left[\int_0^t (t-s)\,dW_s\right] = 0$

We have

$$V(Y_t) = EY_t^2 = \frac{t^3}{3}$$