Financial Engineering

Springer Semester 2025

Lecturer: Patrizia Semeraro, Assistant: Tommaso Vanzan

Problem set 6

Topics: Binomial model and option pricing

Exercise 1 A stock price is currently 50$. It is known that at the end of 2 months it will be either 53$ or 48$. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 2-month European call option with a strike price of 49$? Verify that an hedging argument and risk-neutral valuation argument give the same answer.

Exercise 2 A stock price is currently 80$. It is known that at the end of 4 months it will be either 75$ or 85$. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 4-month European put option with a strike price of 80$? Verify that an hedging argument and risk-neutral valuation arguments give the same answer.

Exercise 3 A stock price is currently 50$. Over each of the next two 3-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% annum with continuous compounding. What is the value of a 6-month European call option with a strike price of 51$? What is the value of a 6-month European put option with a strike price of 51$? Verify that the European call and European put prices satisfy put-call parity. If the put option were American, would it ever be optimal to exercise early at any of the nodes on the tree?

Exercise 4 A stock price is currently 25$. It is known that at the end of 2 months it will be either 23$ or 27$. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of 2 months. What is the value of a derivative that pays off S_T^2 at this time?

Exercise 5 A stock price is currently 50$. It is known that at the end of 6 months it will be either 60$ or 42$. The risk-free interest rate is 12% per annum with continuous compounding. Calculate the value of a 6-month European call option on the stock with an exercise price of 48$. Verify that hedging arguments and risk-neutral valuation arguments give the same answer.

Exercise 6 A stock price is currently 40$. Over each of the next two 3-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% annum with continuous compounding. What is the value of a 6-month European call option with a strike price of 42$? What is the value of a 6-month European put option with a strike price of 42$? What is the value of a 6-month American put option with a strike price of 42$?

Exercise 7 Consider a European call option on a non-dividend-paying stock where the stock price is 40$, the strike price is 40$, the risk-free rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is 6 months. Calculate u, d and p for a two-step tree, and value the option using a two-step tree.

Exercise 8 A stock price is currently 20\$. It is known that at the end of 3 months it will be either 22\$ or 18\$. The risk-free interest rate is 12% per annum with continuous compounding. Calculate the value of a 3-month European call option on the stock with an exercise price of 21\$. Find the answer using both the risk-neutral valuation method, and the replicating portfolio method.

Exercise 9 A stock price is currently 50\$. It is known that at the end of 6 months it will be either 55\$ or 45\$. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 6-month European put option with a strike price of 50\$? Find the answer using both the risk-neutral valuation method and the replicating portfolio method.

Exercise 10 Consider a 6-month European call option with a strike price of 21\$ on a stock whose current price is 20\$. Suppose that there are 2 time steps of 3 months and in each time step the stock price either moves up by a factor of 1.1 or moves down by a factor of 0.9. Suppose that the risk-free interest rate is 12% per annum with continuous compounding. Find the price of the option using both the risk-neutral valuation method and the replicating portfolio method.

Exercise 11 Consider a 2-year European put option with a strike price of 52\$ on a stock whose current price is 50\$. Suppose that there are 2 time steps of 1 year each, and in each time step the stock price either moves up by a factor of 1.2 or moves down by a factor of 0.8. Suppose that the risk-free interest rate is 5% per annum with continuous compounding. Find the price of the option using both the risk-neutral valuation method and the replicating portfolio method.

Exercise 12 Consider a 3-year European call option with a strike price of 80\$ on a stock whose current price is 80\$. Suppose that there are 3 time steps of 1 year each, and in each time step the stock price either moves up by a factor of 1.5 or moves down by a factor of 0.5. Suppose that the risk-free interest rate is 0% per annum with continuous compounding. Find the price of the option using both the risk-neutral valuation method and the replicating portfolio method.

Exercise 1 A stock price is currently 50\$. It is known that at the end of 2 months it will be either 53\$ or 48\$. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 2-month European call option with a strike price of 49\$? Verify that an hedging argument and risk-neutral valuation argument give the same answer.

Financial Engineering

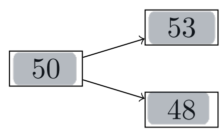
Springer Semester 2025

Lecturer: Patrizia Semeraro, Assistant: Tommaso Vanzan

Problem set 6

Topics: Binomial model and option pricing

Solution 1 In this case the price dynamics of the stock is:



Considering a portfolio consisting of a long position in Δ shares of the stock and a short position in one call option, if the stock price rises from 50\$ to 53\$, the value of the portfolio will be:

$$\begin{array}{lr} \text{value of the shares} & 53\cdot\Delta \\ \text{value of the call option} & -\max(53-49,0) = -4 \\ \hline & 53\cdot\Delta - 4 \end{array}$$

While if the stock price falls from 50\$ to 48\$, the value of the portfolio will be:

$$\begin{array}{lr} \text{value of the shares} & 48\cdot\Delta \\ \text{value of the call option} & -\max(48-49,0) = 0 \\ \hline & 48\cdot\Delta \end{array}$$

The portfolio is risk less if the value of the portfolio is the same in both cases, i.e.:

$$53\cdot\Delta - 4 = 48\cdot\Delta \implies \Delta = \frac{4}{5} = 0.8$$

and its value in two months is:

$$53\cdot 0.8 - 4 = 38.4$$

for both strike prices. Its value today must be the present value of 38.4\$ at the risk free interest rate of 10% per annum, i.e.:

$$38.4e^{-0.1\cdot\frac{2}{12}} = 37.77$$

The stock price today is so and if the option price is denoted by f the value of the portfolio today is:

$$50\cdot\Delta - f = 50\cdot 0.8 - f = 40 - f$$

hence it follows that:

$$40 - f = 37.77 \implies f = 2.23$$

i.e. the call price today is 2.23\$.

In alternative, the price dynamics of the call option is:

$$\begin{tikzcd}[row sep=small] & \boxed{c_u} \\ \boxed{c_0} \arrow[ur] \arrow[dr] & \\ & \boxed{c_d} \end{tikzcd}$$

where

$$c_u = \max(53 - 49, 0) = 4$$

$$c_d = \max(48 - 49, 0) = 0$$

then, using the risk-neutral valuation, we recover the risk-neutral probabilities from the stock price dynamics:

$$S_0 = e^{-r\Delta t} \left[S_u p + S_d(1-p) \right]$$

i.e.:

$$\begin{gathered}
50 = e^{-0.1 \cdot \frac{2}{12}} \left[53p + 48(1-p) \right] \\
50 = e^{-0.1 \cdot \frac{2}{12}} \left[53p + 48 - 48p \right] \\
50 = e^{-0.1 \cdot \frac{2}{12}} \left[(53-48)p + 48 \right] \\
50 = e^{-0.1 \cdot \frac{2}{12}} \left[5p + 48 \right] \\
p = \frac{e^{0.1 \cdot \frac{2}{12}} \cdot 50 - 48}{5} = 0.5681
\end{gathered}$$

that can also be obtained observing that:

$$u = \frac{S_u}{S_0} = \frac{53}{50} = 1.06$$

$$d = \frac{S_d}{S_0} = \frac{48}{50} = 0.96$$

and the risk-neutral probability is given by:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.1 \cdot \frac{2}{12}} - 0.96}{1.06 - 0.96} = 0.5681$$

Finally, the value of the option today is given by:

$$c_0 = e^{-r\Delta t} \left[c_u p + c_d(1-p) \right]$$

i.e.:

$$\begin{aligned}
c_0 &= e^{-0.1 \cdot \frac{2}{12}} \left[4 \cdot 0.5681 + 0(1 - 0.5681) \right] \\
&= e^{-0.1 \cdot \frac{2}{12}} \left[4 \cdot 0.5681 \right] \\
&= e^{-0.1 \cdot \frac{2}{12}} \cdot 2.2724 \\
&= 2.23
\end{aligned}$$

as found above.

Exercise 2 A stock price is currently 80\$. It is known that at the end of 4 months it will be either 75\$ or 85\$. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 4-month European put option with a strike price of 80\$? Verify that an hedging argument and risk-neutral valuation arguments give the same answer.

Solution 2 In this case the price dynamics of the stock is:

$$\begin{tikzcd} & \boxed{85} \\ \boxed{80} \arrow[ur] \arrow[dr] & \\ & \boxed{75} \end{tikzcd}$$

Considering a portfolio consisting of a short position in Δ shares of the stock and a long position in one put option, if the stock price rises from 80\$ to 85\$, the value of the portfolio will be:

value of the shares	$-85 \cdot \Delta$
value of the put option	$\max(80-85, 0) = 0$
	$-85 \cdot \Delta$

While if the stock price falls from 80\$ to 75\$, the value of the portfolio will be:

value of the shares	$-75 \cdot \Delta$
value of the put option	$\max(80-75, 0) = 5$
	$-75 \cdot \Delta + 5$

The portfolio is risk less if the value of the portfolio is the same in both cases, i.e.:

$$-85 \cdot \Delta = -75 \cdot \Delta + 5 \implies \Delta = -\frac{5}{10} = -0.5$$

and its value in four months is:

$$-75 \cdot (-0.5) + 5 = -85 \cdot (-0.5) = 42.5$$

for both strike prices. Its value today must be the present value of 42.5\$ at the risk free interest rate of 5% per annum, i.e.:

$$42.5e^{-0.05 \cdot \frac{4}{12}} = 41.80$$

The stock price today is 80\$ and if the option price is denoted by f the value of the portfolio today is:

$$-80 \cdot \Delta + f = -80 \cdot (-0.5) + f = 40 + f$$

hence it follows that:

$$40 + f = 41.80 \implies f = 1.80$$

i.e. the put price today is 1.80\$. In alternative, the price dynamics of the put option is:

$$\begin{tikzcd} & \boxed{p_u} \\ \boxed{p_0} \arrow[ur] \arrow[dr] & \\ & \boxed{p_d} \end{tikzcd}$$

where

$$p_u = \max(80-85, 0) = 0$$

$$p_d = \max(80-75, 0) = 5$$

then, using the risk-neutral valuation, we recover the risk-neutral probabilities from the stock price dynamics:

$$S_0 = e^{-r\Delta t}\left[S_u p + S_d(1-p)\right]$$

i.e.:

$$\begin{aligned}
80 &= e^{-0.05\cdot\frac{4}{12}}\left[85p + 75(1-p)\right] \\
80 &= e^{-0.05\cdot\frac{4}{12}}\left[85p + 75 - 75p\right] \\
80 &= e^{-0.05\cdot\frac{4}{12}}\left[(85-75)p + 75\right] \\
80 &= e^{-0.05\cdot\frac{4}{12}}\left[10p + 75\right] \\
p &= \frac{e^{0.05\cdot\frac{4}{12}}\cdot 80 - 75}{10} = 0.6345
\end{aligned}$$

that can also be obtained observing that:

$$u = \frac{S_u}{S_0} = \frac{85}{80} = 1.0625$$

$$d = \frac{S_d}{S_0} = \frac{75}{80} = 0.9375$$

and the risk-neutral probability is given by:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05\cdot\frac{4}{12}} - 0.9375}{1.0625 - 0.9375} = 0.6345$$

Finally, the value of the option today is given by:

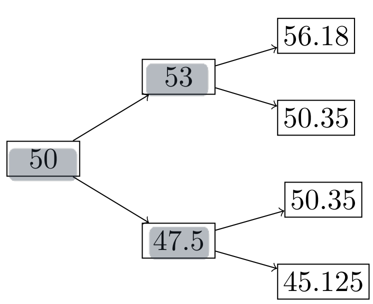
$$p_0 = e^{-r\Delta t}\left[p_u p + p_d(1-p)\right]$$

i.e.:

$$\begin{aligned}
p_0 &= e^{-0.05\cdot\frac{4}{12}}\left[0\cdot 0.6345 + 5(1-0.6345)\right] \\
&= e^{-0.05\cdot\frac{4}{12}}\left[5(1-0.6345)\right] \\
&= e^{-0.05\cdot\frac{4}{12}}\cdot 1.825 \\
&= 1.80
\end{aligned}$$

as found above.

Solution 3 In this case the price dynamics of the stock is:



while the price dynamics of the call option is:

Exercise 3 A stock price is currently 50$. Over each of the next two 3-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% annum with continuous compounding. What is the value of a 6-month European call option with a strike price of 51$? What is the value of a 6-month European put option with a strike price of 51$? Verify that the European call and European put prices satisfy put-call parity. If the put option were American, would it ever be optimal to exercise early at any of the nodes on the tree?

$$
\begin{tikzcd}
 & & \boxed{c_{uu}} \\
 & \boxed{c_u} \arrow[ur] \arrow[dr] & \\
 & & \boxed{c_{ud}} \\
\boxed{c_0} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{c_{ud}} \\
 & \boxed{c_d} \arrow[ur] \arrow[dr] & \\
 & & \boxed{c_{dd}}
\end{tikzcd}
$$

where

$$
\begin{aligned}
c_{uu} &= \max(56.18 - 51, 0) = 5.18 \\
c_{ud} &= \max(50.35 - 51, 0) = 0 \\
c_{dd} &= \max(45.125 - 51, 0) = 0
\end{aligned}
$$

Using the risk neutral evaluation, the probability of an up movement in a risk neutral world is given by:

$$
p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05 \cdot \frac{3}{12}} - 0.95}{1.06 - 0.95} = 0.5689
$$

then at $t = \frac{3}{12}$ we have:

$$
\begin{aligned}
c_u &= e^{-r\Delta t}\left[c_{uu}p + c_{ud}(1-p)\right] \\
&= e^{-0.05 \cdot \frac{3}{12}}\left[5.18 \cdot 0.5689 + 0(1 - 0.5689)\right] \\
&= 2.91
\end{aligned}
$$

and also:

$$
\begin{aligned}
c_d &= e^{-r\Delta t}\left[c_{ud}p + c_{dd}(1-p)\right] \\
&= e^{-0.05 \cdot \frac{3}{12}}\left[0 \cdot 0.5689 + 0(1 - 0.5689)\right] \\
&= 0
\end{aligned}
$$

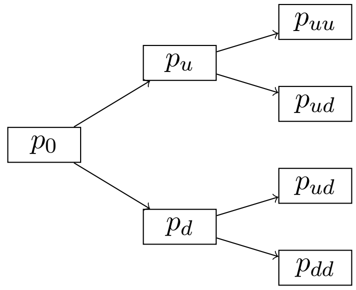
and at $t = 0$ we have:

$$
\begin{aligned}
c_0 &= e^{-r\Delta t}\left[c_u p + c_d(1-p)\right] \\
&= e^{-0.05 \cdot \frac{3}{12}}\left[2.91 \cdot 0.5689 + 0(1 - 0.5689)\right] \\
&= 1.635
\end{aligned}
$$

This value can also be calculated directly using the equation:

$$
\begin{aligned}
c_0 &= e^{-2r\Delta t}[c_{uu}p^2 + 2c_{ud}(1-p)p + c_{dd}(1-p)^2] \\
&= e^{-2 \cdot 0.05 \cdot \frac{3}{12}}[5.18 \cdot 0.5689^2 + 2 \cdot 0 \cdot (1 - 0.5689) \cdot 0.5689 + 0 \cdot (1 - 0.5689)^2] \\
&= e^{-0.05 \cdot \frac{3}{12}}[5.18 \cdot 0.3237] \\
&= 1.635
\end{aligned}
$$

The price dynamics of the put option then is:



where

$$p_{uu} = \max(51 - 56.18, 0) = 0$$
$$p_{ud} = \max(51 - 50.35, 0) = 0.65$$
$$p_{dd} = \max(51 - 45.125, 0) = 5.875$$

The value p is the one found above, then at $t = \frac{3}{12}$ we have:

$$\begin{aligned} p_u &= e^{-r\Delta t}\left[p_{uu}p + p_{ud}(1-p)\right] \\ &= e^{-0.05\cdot\frac{3}{12}}\left[0 \cdot 0.5689 + 0.65(1 - 0.5689)\right] \\ &= 0.277 \end{aligned}$$

and also:

$$\begin{aligned} p_d &= e^{-r\Delta t}\left[p_{ud}p + p_{dd}(1-p)\right] \\ &= e^{-0.05\cdot\frac{3}{12}}\left[0.65 \cdot 0.5689 + 5.875(1 - 0.5689)\right] \\ &= 2.866 \end{aligned}$$

and at $t = 0$ we have:

$$\begin{aligned} p_0 &= e^{-r\Delta t}\left[p_u p + p_d(1-p)\right] \\ &= e^{-0.05\cdot\frac{3}{12}}\left[0.277 \cdot 0.5689 + 2.866(1 - 0.5689)\right] \\ &= 1.376 \end{aligned}$$

that is the price of the put option today. This value can also be calculated directly using the equation:

$$\begin{aligned} p_0 &= e^{-2r\Delta t}[p_{uu}p^2 + 2p_{ud}(1-p)p + p_{dd}(1-p)^2] \\ &= e^{-2\cdot 0.05\cdot\frac{3}{12}}[0 \cdot 0.5689^2 + 2 \cdot 0.65(1 - 0.5689) \cdot 0.5689 + 5.875 \cdot (1 - 0.5689)^2] \\ &= e^{-0.05\cdot\frac{3}{12}}[2 \cdot 0.65(1 - 0.5689) \cdot 0.5689 + 5.875 \cdot (1 - 0.5689)^2] \\ &= 1.376 \end{aligned}$$

The put-call parity is given by:

$$c_0 + Ke^{-rT} = p + S_0$$

and since:

$$c_0 + Ke^{-rT} = 1.635 + 51e^{-0.05\cdot\frac{6}{12}} = 51.37$$
$$p + S_0 = 1.376 + 50 = 51.37$$

this relation is satisfied. If the put option were American, to establish if it is optimal to exercise it early at any of the nodes it is necessary to compare the value calculated for the option at each node with the payoff from the immediate exercise. At node p_d we have:

- Value of the option: 2.866
- Payoff from immediate exercise: $\max(51 - 47.5, 0) = 3.5$

and since $3.5 > 2.866$ it is optimal to exercise the option at this node and $p_d = 3.5$. At node p_u we have:

- Value of the option: 0.277
- Payoff from immediate exercise: $\max(51 - 53, 0) = 0$

and since $0.277 > 0$ it is not optimal to exercise the option at this node and $p_u = 0.277$. At node p_0 we have:

- Value of the option: $e^{-0.05 \cdot \frac{3}{12}}[0.277 \cdot 0.5689 + 3.5(1 - 0.5689)] = 1.646$
- Payoff from immediate exercise: $\max(51 - 50, 0) = 1$

and since $1.646 > 1$ it is not optimal to exercise the option at this node and $p_0 = 1.646$.

Solution 4 In this case the price dynamics of the stock is:

$$\begin{tikzcd} & \boxed{27} \\ \boxed{25} \arrow[ur] \arrow[dr] & \\ & \boxed{23} \end{tikzcd}$$

Considering a portfolio consisting of a long position in Δ shares of the stock and a short position in one derivative, if the stock price rises from 25\$ to 27\$, the value of the portfolio will be:

$$\begin{array}{ll} \text{value of the shares} & 27 \cdot \Delta \\ \text{value of the derivative} & -(27^2) = -729 \\ \hline & 27 \cdot \Delta - 729 \end{array}$$

While if the stock price falls from 25\$ to 23\$, the value of the portfolio will be:

$$\begin{array}{ll} \text{value of the shares} & 23 \cdot \Delta \\ \text{value of the derivative} & -(23^2) = -529 \\ \hline & 23 \cdot \Delta - 529 \end{array}$$

The portfolio is risk less if the value of the portfolio is the same in both cases, i.e.:

$$27 \cdot \Delta - 729 = 23 \cdot \Delta - 529 \implies \Delta = \frac{200}{4} = 50$$

and its value in two months is:

$$27 \cdot 50 - 729 = 1350 - 729 = 621$$

for both strike prices. Its value today must be the present value of 621\$ at the risk free interest rate of 10% per annum, i.e.:

$$621e^{-0.1 \cdot \frac{2}{12}} = 610.74$$

The stock price today is 25\$ and if the derivative price is denoted by f the value of the portfolio today is:

$$25 \cdot \Delta - f = 25 \cdot 50 - f = 1250 - f$$

hence it follows that:

$$1250 - f = 610.74 \implies f = 639.26$$

i.e. the derivative price today is 639.26\$. In alternative, the price dynamics of the derivative is:

Exercise 4 A stock price is currently 25\$. It is known that at the end of 2 months it will be either 23\$ or 27\$. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of 2 months. What is the value of a derivative that pays off S_T^2 at this time?

$$
\begin{tikzcd}[column sep=small]
 & \boxed{f_u} \\
\boxed{f_0} \arrow[ur] \arrow[dr] & \\
 & \boxed{f_d}
\end{tikzcd}
$$

where

$$f_u = (27^2) = 729$$

$$f_d = (23^2) = 529$$

then, using the risk-neutral valuation, we recover the risk-neutral probabilities from the stock price dynamics:

$$S_0 = e^{-r\Delta t}\left[S_u p + S_d(1-p)\right]$$

i.e.:

$$
\begin{aligned}
25 &= e^{-0.1\cdot\frac{2}{12}}\left[27p + 23(1-p)\right] \\
25 &= e^{-0.1\cdot\frac{2}{12}}\left[27p + 23 - 23p\right] \\
25 &= e^{-0.1\cdot\frac{2}{12}}\left[(27-23)p + 23\right] \\
25 &= e^{-0.1\cdot\frac{2}{12}}\left[4p + 23\right] \\
p &= \frac{e^{0.1\cdot\frac{2}{12}}\cdot 25 - 23}{4} = 0.6050
\end{aligned}
$$

that can also be obtained observing that:

$$u = \frac{S_u}{S_0} = \frac{27}{25} = 1.08$$

$$d = \frac{S_d}{S_0} = \frac{23}{25} = 0.92$$

and the risk-neutral probability is given by:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.1\cdot\frac{2}{12}} - 0.92}{1.08 - 0.92} = 0.6050$$

Finally, the value of the option today is given by:

$$f_0 = e^{-r\Delta t}\left[f_u p + f_d(1-p)\right]$$

i.e.:

$$f_0 = e^{-0.1\cdot\frac{2}{12}}\left[729\cdot 0.6050 + 529(1-0.6050)\right] = 639.26$$

as found above.

Solution 5 In this case the price dynamics of the stock is:

$$
\begin{tikzcd}[column sep=small]
 & \boxed{60} \\
\boxed{50} \arrow[ur] \arrow[dr] & \\
 & \boxed{42}
\end{tikzcd}
$$

Exercise 5 A stock price is currently 50$. It is known that at the end of 6 months it will be either 60$ or 42$. The risk-free interest rate is 12% per annum with continuous compounding. Calculate the value of a 6-month European call option on the stock with an exercise price of 48$. Verify that hedging arguments and risk-neutral valuation arguments give the same answer.

Considering a portfolio consisting of a long position in Δ shares of the stock and a short position in one call option, if the stock price rises from 50$ to 60$, the value of the portfolio will be:

value of the shares	$60 \cdot \Delta$
value of the call option	$-\max(60-48, 0) = -12$
	$60 \cdot \Delta - 12$

While if the stock price falls from 50$ to 42$, the value of the portfolio will be:

value of the shares	$42 \cdot \Delta$
value of the call option	$-\max(42-48, 0) = 0$
	$42 \cdot \Delta$

The portfolio is risk less if the value of the portfolio is the same in both cases, i.e.:

$$60 \cdot \Delta - 12 = 42 \cdot \Delta \implies \Delta = \frac{12}{18} = 0.6667$$

and its value in six months is:

$$60 \cdot 0.6667 - 12 = 42 \cdot 0.6667 = 28$$

for both strike prices. Its value today must be the present value of 28$ at the risk free interest rate of 12% per annum, i.e.:

$$28e^{-0.12 \cdot \frac{6}{12}} = 26.37$$

The stock price today is 50$ and if the option price is denoted by f the value of the portfolio today is:

$$50 \cdot \Delta - f = 50 \cdot 0.6667 - f = 33.33 - f$$

hence it follows that:

$$33.33 - f = 26.37 \implies f = 6.96$$

i.e. the call price today is 6.96$. In alternative, the price dynamics of the call option is:

$$\begin{tikzcd} & \boxed{c_u} \\ \boxed{c_0} \arrow[ur] \arrow[dr] & \\ & \boxed{c_d} \end{tikzcd}$$

where

$$c_u = \max(60-48, 0) = 12$$
$$c_d = \max(42-48, 0) = 0$$

then, using the risk-neutral valuation, we recover the risk-neutral probabilities from the stock price dynamics:

$$S_0 = e^{-r\Delta t}\left[S_u p + S_d(1-p)\right]$$

i.e.:

$$50 = e^{-0.12 \cdot \frac{6}{12}}\left[60p + 42(1-p)\right]$$
$$50 = e^{-0.12 \cdot \frac{6}{12}}\left[60p + 42 - 42p\right]$$
$$50 = e^{-0.12 \cdot \frac{6}{12}}\left[(60-42)p + 42\right]$$
$$50 = e^{-0.12 \cdot \frac{6}{12}}\left[18p + 42\right]$$
$$p = \frac{e^{0.12 \cdot \frac{6}{12}} \cdot 50 - 42}{18} = 0.6161$$

that can also be obtained observing that:

$$u = \frac{S_u}{S_0} = \frac{60}{50} = 1.2$$

$$d = \frac{S_d}{S_0} = \frac{42}{50} = 0.84$$

and the risk-neutral probability is given by:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.12 \cdot \frac{6}{12}} - 0.84}{1.2 - 0.84} = 0.6161$$

Finally, the value of the option today is given by:

$$c_0 = e^{-r\Delta t}\left[c_u p + c_d(1-p)\right]$$

i.e.:

$$\begin{aligned} c_0 &= e^{-0.12 \cdot \frac{6}{12}}\left[12 \cdot 0.6161 + 0(1 - 0.6161)\right] \\ &= e^{-0.12 \cdot \frac{6}{12}}\left[12 \cdot 0.6161\right] \\ &= 6.96 \end{aligned}$$

as found above.

Solution 6 In this case the price dynamics of the stock is:

Exercise 6 A stock price is currently 40$. Over each of the next two 3-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% annum with continuous compounding. What is the value of a 6-month European call option with a strike price of 42$? What is the value of a 6-month European put option with a strike price of 42$? What is the value of a 6-month American put option with a strike price of 42$?

$$\begin{tikzcd}
 & & \boxed{48.4} \\
 & \boxed{44} \arrow[ur] \arrow[dr] & \\
 & & \boxed{39.6} \\
\boxed{40} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{39.6} \\
 & \boxed{36} \arrow[ur] \arrow[dr] & \\
 & & \boxed{32.4}
\end{tikzcd}$$

while the price dynamics of the put option is:

$$\begin{tikzcd}
 & & \boxed{p_{uu}} \\
 & \boxed{p_u} \arrow[ur] \arrow[dr] & \\
 & & \boxed{p_{ud}} \\
\boxed{p_0} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{p_{ud}} \\
 & \boxed{p_d} \arrow[ur] \arrow[dr] & \\
 & & \boxed{p_{dd}}
\end{tikzcd}$$

where

$$p_{uu} = \max(42 - 48.4, 0) = 0$$

$$p_{ud} = \max(42 - 39.6, 0) = 2.4$$

$$p_{dd} = \max(42 - 32.4, 0) = 9.6$$

Using the risk neutral evaluation, the probability of an up movement in a risk neutral world is given by:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.12 \cdot \frac{3}{12}} - 0.9}{1.1 - 0.9} = 0.6523$$

then at $t = \frac{3}{12}$ we have:

$$\begin{aligned} p_u &= e^{-r\Delta t}\left[p_{uu}p + p_{ud}(1-p)\right] \\ &= e^{-0.12 \cdot \frac{3}{12}}\left[0 \cdot 0.6523 + 2.4(1 - 0.6523)\right] \\ &= 0.810 \end{aligned}$$

and also:

$$\begin{aligned} p_d &= e^{-r\Delta t}\left[p_{ud}p + p_{dd}(1-p)\right] \\ &= e^{-0.12 \cdot \frac{3}{12}}\left[2.4 \cdot 0.6523 + 9.6(1 - 0.6523)\right] \\ &= 4.759 \end{aligned}$$

and at $t = 0$ we have:

$$\begin{aligned} p_0 &= e^{-r\Delta t}\left[p_u p + p_d(1-p)\right] \\ &= e^{-0.12 \cdot \frac{3}{12}}\left[0.810 \cdot 0.6523 + 4.759(1 - 0.6523)\right] \\ &= 2.118 \end{aligned}$$

that is the price of the put option today. This value can also be calculated directly using the equation:

$$\begin{aligned} p_0 &= e^{-2r\Delta t}[p_{uu}p^2 + 2p_{ud}(1-p)p + p_{dd}(1-p)^2] \\ &= e^{-2 \cdot 0.12 \cdot \frac{3}{12}}[0 \cdot 0.6523^2 + 2 \cdot 2.4(1 - 0.6523) \cdot 0.6523 + 9.6 \cdot (1 - 0.6523)^2] \\ &= 2.118 \end{aligned}$$

Considering the American put option, the values at the final nodes are always:

$$\begin{aligned} p_{uu} &= \max(42 - 48.4, 0) = 0 \\ p_{ud} &= \max(42 - 39.6, 0) = 2.4 \\ p_{dd} &= \max(42 - 32.4, 0) = 9.6 \end{aligned}$$

At node p_d we have:

- Value of the option: 4.759
- Payoff from immediate exercise: $\max(42 - 36, 0) = 6$

and since $6 > 4.759$ it is optimal to exercise the option at this node and $p_d = 6$. At node p_u we have:

- Value of the option: 0.810
- Payoff from immediate exercise: $\max(42 - 44, 0) = 0$

and since $0.810 > 0$ it is not optimal to exercise the option at this node and $p_u = 0.810$. At node p_0 we have:

- Value of the option: $e^{-0.12\cdot\frac{3}{12}}\left[0.810\cdot 0.6523+6(1-0.6523)\right]=2.537$
- Payoff from immediate exercise: $\max(42-40,0)=2$

and since $2.537>2$ it is not optimal to exercise the option at this node and $p_0=2.537$. Finally, the value of the European call option can be obtained using the put-call parity:

$$c_0+Ke^{-rT}=p+S_0$$

hence:

$$c_0+42e^{-0.12\frac{6}{12}}=2.118+40 \implies c_0=2.118-42e^{-0.12\frac{6}{12}}+40=2.5639$$

Exercise 7 Consider a European call option on a non-dividend-paying stock where the stock price is 40\$, the strike price is 40\$, the risk-free rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is 6 months. Calculate u, d and p for a two-step tree, and value the option using a two-step tree.

Solution 7 In this case we have:

$$S_0=40$$
$$K=40$$
$$r=0.04$$
$$\sigma=0.3$$
$$T=\frac{6}{12}$$
$$\Delta t=\frac{3}{12}$$

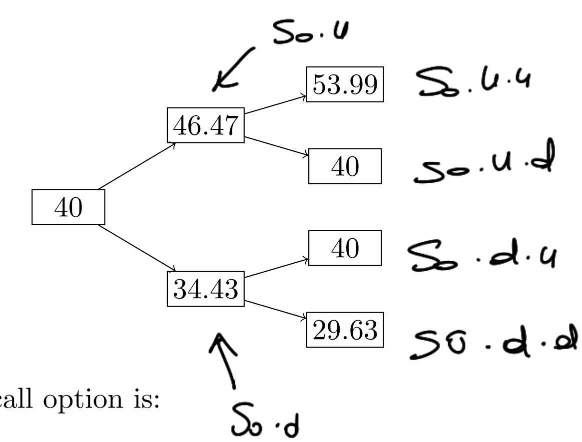
The up and down factors are given by:

$$u=e^{\sigma\sqrt{\Delta t}}=e^{0.3\sqrt{\frac{3}{12}}}=1.1618$$
$$d=e^{-\sigma\sqrt{\Delta t}}=e^{-0.3\sqrt{\frac{3}{12}}}=0.8607$$

and the risk-neutral probability is given by:

$$p=\frac{e^{r\Delta t}-d}{u-d}=\frac{e^{0.04\cdot\frac{3}{12}}-0.8607}{1.1618-0.8607}=0.4960$$

The stock price dynamics is:



while the price dynamics of the call option is:

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
 & & \boxed{c_{uu}} \\
 & \boxed{c_u} \arrow[ur] \arrow[dr] & \\
 & & \boxed{c_{ud}} \\
\boxed{c_0} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{c_{ud}} \\
 & \boxed{c_d} \arrow[ur] \arrow[dr] & \\
 & & \boxed{c_{dd}}
\end{tikzcd}
$$

where

$$
\begin{aligned}
c_{uu} &= \max(53.99 - 40, 0) = 13.99 \\
c_{ud} &= \max(40 - 40, 0) = 0 \\
c_{dd} &= \max(29.63 - 40, 0) = 0
\end{aligned}
$$

Then at $t = \frac{3}{12}$ we have:

$$
\begin{aligned}
c_u &= e^{-r\Delta t}\left[c_{uu}p + c_{ud}(1-p)\right] \\
&= e^{-0.04\cdot\frac{3}{12}}\left[13.99 \cdot 0.4960 + 0(1 - 0.4960)\right] \\
&= 6.87
\end{aligned}
$$

and also:

$$
\begin{aligned}
c_d &= e^{-r\Delta t}\left[c_{ud}p + c_{dd}(1-p)\right] \\
&= e^{-0.04\cdot\frac{3}{12}}\left[0 \cdot 0.4960 + 0(1 - 0.4960)\right] \\
&= 0
\end{aligned}
$$

and at $t = 0$ we have:

$$
\begin{aligned}
c_0 &= e^{-r\Delta t}\left[c_{u}p + c_{d}(1-p)\right] \\
&= e^{-0.04\cdot\frac{3}{12}}\left[6.87 \cdot 0.4960 + 0(1 - 0.4960)\right] \\
&= 3.37
\end{aligned}
$$

that is the price of the call option today.

Solution 8 In this case the price dynamics of the stock is:

$$
\begin{tikzcd}[row sep=tiny]
 & \boxed{22} \\
\boxed{20} \arrow[ur] \arrow[dr] & \\
 & \boxed{18}
\end{tikzcd}
$$

while the price dynamics of the call option is:

$$
\begin{tikzcd}[row sep=tiny]
 & \boxed{c_u} \\
\boxed{c_0} \arrow[ur] \arrow[dr] & \\
 & \boxed{c_d}
\end{tikzcd}
$$

Exercise 8 A stock price is currently 20\$. It is known that at the end of 3 months it will be either 22\$ or 18\$. The risk-free interest rate is 12% per annum with continuous compounding. Calculate the value of a 3-month European call option on the stock with an exercise price of 21\$. Find the answer using both the risk-neutral valuation method, and the replicating portfolio method.

where

$$c_u = \max(22 - 21, 0) = 1$$

$$c_d = \max(18 - 21, 0) = 0$$

then, using the risk-neutral valuation, we recover the risk-neutral probabilities from the stock price dynamics:

$$S_0 = e^{-r\Delta t}\left[S_u p + S_d(1-p)\right]$$

i.e.:

$$20 = e^{-0.12\cdot\frac{3}{12}}\left[22p + 18(1-p)\right]$$

$$20 = e^{-0.12\cdot\frac{3}{12}}\left[4p + 18\right]$$

$$p = \frac{e^{0.12\cdot\frac{3}{12}}\cdot 20 - 18}{4} = 0.6523$$

that can also be obtained observing that:

$$u = \frac{S_u}{S_0} = \frac{22}{20} = 1.1$$

$$d = \frac{S_d}{S_0} = \frac{18}{20} = 0.9$$

and the risk-neutral probability is given by:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.12\cdot\frac{3}{12}} - 0.9}{1.1 - 0.9} = 0.6523$$

Finally, the value of the option today is given by:

$$c_0 = e^{-r\Delta t}\left[c_u p + c_d(1-p)\right]$$

i.e.:

$$\begin{aligned} c_0 &= e^{-0.12\cdot\frac{3}{12}}\left[1\cdot 0.6523 + 0(1-0.6523)\right] \\ &= e^{-0.12\cdot\frac{3}{12}}\left[0.6523\right] \\ &= 0.633 \end{aligned}$$

that is the price of the call option today. Using the replicating portfolio we have at $t = \frac{3}{12}$:

$$\begin{cases} V_u = xe^{rt} + yS_u = c_u \\ V_d = xe^{rt} + yS_d = c_d \end{cases}$$

i.e.:

$$\begin{cases} V_u = xe^{0.12\cdot\frac{3}{12}} + y22 = 1 \\ V_d = xe^{0.12\cdot\frac{3}{12}} + y18 = 0 \end{cases}$$

and subtracting the second equation from the first one:

$$y(22 - 18) = 1 \implies y = \frac{1}{4} = 0.25$$

and then:

$$xe^{0.12\cdot\frac{3}{12}} + 0.25 \cdot 18 = 0 \implies x = -0.25 \cdot 18 e^{-0.12\cdot\frac{3}{12}} = -4.367$$

hence:

$$\begin{aligned} x &= -4.367 \text{ borrowing} \\ y &= 0.25 \text{ shares of the stock} \end{aligned}$$

and then at $t = 0$ we have:

$$\begin{aligned} V_0 &= x + yS_0 \\ &= -4.367 + 0.25 \cdot 20 \\ &= -4.367 + 5 = 0.633 = c_0 \end{aligned}$$

that is the price of the call option and is the same as found with the risk-neutral valuation method. The replicating portfolio can also be found with the formula:

$$x = e^{-rt}\frac{uc_d - dc_u}{u-d} = e^{-0.12\cdot\frac{3}{12}}\frac{1.1\cdot 0 - 0.9 \cdot 1}{1.1 - 0.9} = -4.367$$

$$y = \frac{1}{S_0}\frac{c_u - c_d}{u - d} = \frac{1}{20}\frac{1-0}{1.1-0.9} = 0.25$$

Solution 9 In this case the price dynamics of the stock is:

$$\begin{tikzcd} & \boxed{55} \\ \boxed{50} \arrow[ur] \arrow[dr] & \\ & \boxed{45} \end{tikzcd}$$

while the price dynamics of the put option is:

$$\begin{tikzcd} & \boxed{p_u} \\ \boxed{p_0} \arrow[ur] \arrow[dr] & \\ & \boxed{p_d} \end{tikzcd}$$

where

$$\begin{aligned} p_u &= \max(50 - 55, 0) = 0 \\ p_d &= \max(50 - 45, 0) = 5 \end{aligned}$$

then, using the risk-neutral valuation, we recover the risk-neutral probabilities from the stock price dynamics:

$$S_0 = e^{-r\Delta t}\left[S_u p + S_d(1-p)\right]$$

i.e.:

$$\begin{aligned} 50 &= e^{-0.1\cdot\frac{6}{12}}\left[55p + 45(1-p)\right] \\ 50 &= e^{-0.1\cdot\frac{6}{12}}\left[10p + 45\right] \\ p &= \frac{e^{0.1\cdot\frac{6}{12}} \cdot 50 - 45}{10} = 0.7564 \end{aligned}$$

Exercise 9 A stock price is currently 50$. It is known that at the end of 6 months it will be either 55$ or 45$. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 6-month European put option with a strike price of 50$? Find the answer using both the risk-neutral valuation method and the replicating portfolio method.

that can also be obtained observing that:

$$u = \frac{S_u}{S_0} = \frac{55}{50} = 1.1$$

$$d = \frac{S_d}{S_0} = \frac{45}{50} = 0.9$$

and the risk-neutral probability is given by:

$$p = \frac{e^r - d}{u - d} = \frac{e^{0.1 \cdot \frac{6}{12}} - 0.9}{1.1 - 0.9} = 0.7564$$

Finally, the value of the option today is given by:

$$p_0 = e^{-r\Delta t}\left[p_u p + p_d(1-p)\right]$$

i.e.:

$$\begin{aligned} p_0 &= e^{-0.1 \cdot \frac{6}{12}}\left[0 \cdot 0.7564 + 5(1 - 0.7564)\right] \\ &= e^{-0.1 \cdot \frac{6}{12}}\left[5(1 - 0.7564)\right] \\ &= 1.16 \end{aligned}$$

that is the price of the put option today. Using the replicating portfolio we have at $t = \frac{6}{12}$:

$$\begin{cases} V_u = xe^{rt} + yS_u = p_u \\ V_d = xe^{rt} + yS_d = p_d \end{cases}$$

i.e.:

$$\begin{cases} V_u = xe^{0.1 \cdot \frac{6}{12}} + y55 = 0 \\ V_d = xe^{0.1 \cdot \frac{6}{12}} + y45 = 5 \end{cases}$$

and subtracting the second equation from the first one:

$$y(55 - 45) = -5 \implies y = \frac{-5}{10} = -0.5$$

and then:

$$xe^{0.1 \cdot \frac{6}{12}} - 0.5 \cdot 55 = 0 \implies x = 0.5 \cdot 55 e^{-0.1 \cdot \frac{6}{12}} = 26.16$$

hence:

$$\begin{aligned} x &= 26.16 \text{ amount of the bond} \\ y &= -0.5 \text{ shares of the stock shorted} \end{aligned}$$

and then at $t = 0$ we have:

$$\begin{aligned} V_0 &= x + yS_0 \\ &= 26.16 - 0.5 \cdot 50 \\ &= 26.16 - 25 = 1.16 = p_0 \end{aligned}$$

that is the price of the put option and is the same as found with the risk-neutral valuation method. The replicating portfolio can also be found with the formula:

$$x = e^{-rt}\frac{up_d - dp_u}{u - d} = e^{-0.1 \cdot \frac{6}{12}}\frac{1.1 \cdot 5 - 0.9 \cdot 0}{1.1 - 0.9} = 26.16$$

$$y = \frac{1}{S_0}\frac{p_u - p_d}{u - d} = \frac{1}{50}\frac{0 - 5}{1.1 - 0.9} = -0.5$$

Exercise 10 Consider a 6-month European call option with a strike price of 21\$ on a stock whose current price is 20\$. Suppose that there are 2 time steps of 3 months and in each time step the stock price either moves up by a factor of 1.1 or moves down by a factor of 0.9. Suppose that the risk-free interest rate is 12% per annum with continuous compounding. Find the price of the option using both the risk-neutral valuation method and the replicating portfolio method.

Solution 10 In this case the price dynamics of the stock is:

$$\begin{tikzcd}
 & & \boxed{24.2} \\
 & \boxed{22} \arrow[ur] \arrow[dr] & \\
 & & \boxed{19.8} \\
\boxed{20} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{19.8} \\
 & \boxed{18} \arrow[ur] \arrow[dr] & \\
 & & \boxed{16.2}
\end{tikzcd}$$

while the price dynamics of the call option is:

$$\begin{tikzcd}
 & & \boxed{c_{uu}} \\
 & \boxed{c_u} \arrow[ur] \arrow[dr] & \\
 & & \boxed{c_{ud}} \\
\boxed{c_0} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{c_{ud}} \\
 & \boxed{c_d} \arrow[ur] \arrow[dr] & \\
 & & \boxed{c_{dd}}
\end{tikzcd}$$

where

$$c_{uu} = \max(24.2 - 21, 0) = 3.2$$
$$c_{ud} = \max(19.8 - 21, 0) = 0$$
$$c_{dd} = \max(16.2 - 21, 0) = 0$$

Using the risk neutral valuation, first of all, we recover the risk neutral probabilities from the stock price dynamics; we can start from any of the nodes, for example:

$$S_0 = e^{-rt}[S_u p + S_d(1-p)]$$

i.e.:

$$20 = e^{-0.12 \cdot \frac{6}{12}} [22p + 18(1-p)]$$
$$20 = e^{-0.12 \cdot \frac{6}{12}} [4p + 18]$$
$$p = \frac{e^{0.12 \cdot \frac{6}{12}} \cdot 20 - 18}{4} = 0.6523$$

or:

$$S_u = e^{-rt}[S_{uu} p + S_{ud}(1-p)]$$

i.e.:

$$22 = e^{-0.12 \cdot \frac{3}{12}} [24.2p + 19.8(1-p)]$$
$$22 = e^{-0.12 \cdot \frac{3}{12}} [4.4p + 19.8]$$
$$p = \frac{e^{0.12 \cdot \frac{3}{12}} \cdot 22 - 19.8}{4.4} = 0.6523$$

or:

$$S_d = e^{-rt}[S_{ud} p + S_{dd}(1-p)]$$

i.e.:

$$
\begin{aligned}
18 &= e^{-0.12\cdot\frac{3}{12}}\left[19.8p + 16.2(1-p)\right] \\
18 &= e^{-0.12\cdot\frac{3}{12}}\left[3.6p + 16.2\right] \\
p &= \frac{e^{0.12\cdot\frac{3}{12}}\cdot 18 - 16.2}{3.6} = 0.6523
\end{aligned}
$$

The same risk neutral probability can be obtained as:

$$
p = \frac{e^{rt}-d}{u-d} = \frac{e^{0.12\cdot\frac{3}{12}}-0.9}{1.1-0.9} = 0.6523
$$

At this point the price of the call option can be determined starting from the last period going backward. At $t = \frac{3}{12}$ we have:

$$
\begin{aligned}
c_u &= e^{-rt}\left[c_{uu}p + c_{ud}(1-p)\right] \\
&= e^{-0.12\cdot\frac{3}{12}}\left[3.2\cdot 0.6523 + 0(1-0.6523)\right] \\
&= 2.0257
\end{aligned}
$$

and also:

$$
\begin{aligned}
c_d &= e^{-rt}\left[c_{ud}p + c_{dd}(1-p)\right] \\
&= e^{-0.12\cdot\frac{3}{12}}\left[0\cdot 0.6523 + 0(1-0.6523)\right] \\
&= 0
\end{aligned}
$$

and at $t = 0$ we have:

$$
\begin{aligned}
c_0 &= e^{-rt}\left[c_u p + c_d(1-p)\right] \\
&= e^{-0.12\cdot\frac{3}{12}}\left[2.0257\cdot 0.6523 + 0(1-0.6523)\right] \\
&= 1.2823
\end{aligned}
$$

that is the price of the call option today. Using the replicating portfolio we start from the last period and again we go backward. At $t = \frac{6}{12}$:

$$
\begin{cases}
V_{uu} = xe^{rt} + yS_{uu} = c_{uu} \\
V_{ud} = xe^{rt} + yS_{ud} = c_{ud}
\end{cases}
$$

i.e.:

$$
\begin{cases}
V_{uu} = xe^{0.12\cdot\frac{3}{12}} + y24.2 = 3.2 \\
V_{ud} = xe^{0.12\cdot\frac{3}{12}} + y19.8 = 0
\end{cases}
$$

and subtracting the second equation from the first one:

$$
y(24.2 - 19.8) = 3.2 \implies y = \frac{3.2}{4.4} = 0.7273
$$

and then:

$$
xe^{0.12\cdot\frac{3}{12}} + 0.7273\cdot 19.8 = 0 \implies x = -0.7273\cdot 19.8e^{-0.12\cdot\frac{3}{12}} = -13.9744
$$

hence:

$$x = -13.9744 \text{ borrowing}$$
$$y = 0.7273 \text{ shares of the stock}$$

and then at $t = \frac{3}{12}$ we have:

$$\begin{aligned} V_u &= x + yS_u \\ &= -13.9744 + 0.7273 \cdot 22 \\ &= 2.0257 = c_u \end{aligned}$$

that is the price of the call in the "up" state and is the same as found with the risk-neutral valuation method. The replicating portfolio can also be found with the formulas:

$$x = e^{-rt}\frac{uc_{ud} - dc_{uu}}{u-d} = e^{-0.12\cdot\frac{3}{12}}\frac{1.1\cdot 0 - 0.9\cdot 3.2}{1.1-0.9} = -13.9744$$

$$y = \frac{1}{S_0}\frac{c_{uu} - c_{ud}}{u-d} = \frac{1}{20}\frac{3.2-0}{1.1-0.9} = 0.7273$$

At $t = \frac{6}{12}$ we have:

$$\begin{cases} V_{ud} = xe^{rt} + yS_{ud} = c_{ud} \\ V_{dd} = xe^{rt} + yS_{dd} = c_{dd} \end{cases}$$

i.e.:

$$\begin{cases} V_{ud} = xe^{0.12\cdot\frac{3}{12}} + y19.8 = 0 \\ V_{dd} = xe^{0.12\cdot\frac{3}{12}} + y16.2 = 0 \end{cases}$$

and subtracting the second equation from the first one:

$$y(19.8 - 16.2) = 0 \implies y = 0$$

and then:

$$xe^{0.12\cdot\frac{3}{12}} + 0\cdot 19.8 = 0 \implies x = 0$$

hence:

$$x = 0 \text{ borrowing}$$
$$y = 0 \text{ shares of the stock}$$

and then at $t = \frac{3}{12}$ we have:

$$\begin{aligned} V_d &= x + yS_d \\ &= 0 + 0\cdot 18 \\ &= 0 = c_d \end{aligned}$$

that is the price of the call in the "down" state and is the same as found with the risk-neutral valuation method. The replicating portfolio can also be found with the formulas:

$$x = e^{-rt}\frac{uc_{dd} - dc_{ud}}{u-d} = e^{-0.12\cdot\frac{3}{12}}\frac{1.1\cdot 0 - 0.9\cdot 0}{1.1-0.9} = 0$$

$$y = \frac{1}{S_0}\frac{c_{ud} - c_{dd}}{u - d} = \frac{1}{20}\frac{0 - 0}{1.1 - 0.9} = 0$$

At $t = \frac{3}{12}$ we have:

$$\begin{cases} V_u = xe^{rt} + yS_u = c_u \\ V_d = xe^{rt} + yS_d = c_d \end{cases}$$

i.e.:

$$\begin{cases} V_u = xe^{0.12 \cdot \frac{3}{12}} + y22 = 2.0257 \\ V_d = xe^{0.12 \cdot \frac{3}{12}} + y18 = 0 \end{cases}$$

and subtracting the second equation from the first one:

$$y(22 - 18) = 2.0257 \implies y = \frac{2.0257}{4} = 0.5064$$

and then:

$$xe^{0.12 \cdot \frac{3}{12}} + 0.5064 \cdot 18 = 0 \implies x = -0.5064 \cdot 18e^{-0.12 \cdot \frac{3}{12}} = -8.8462$$

hence:

$$x = -8.8462 \text{ borrowing}$$
$$y = 0.5064 \text{ shares of the stock}$$

and then at $t = 0$ we have:

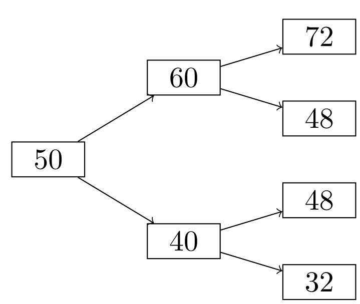
$$\begin{aligned} V_0 &= x + yS_0 \\ &= -8.8462 + 0.5064 \cdot 20 \\ &= -8.8462 + 10.128 = 1.2823 = c_0 \end{aligned}$$

that is the price of the call option and is the same as found with the risk-neutral valuation method. The replicating portfolio can also be found with the formulas:

$$x = e^{-rt}\frac{uc_d - dc_u}{u - d} = e^{-0.12 \cdot \frac{3}{12}}\frac{1.1 \cdot 0 - 0.9 \cdot 3.2}{1.1 - 0.9} = -8.8462$$

$$y = \frac{1}{S_0}\frac{c_u - c_d}{u - d} = \frac{1}{20}\frac{2.0257 - 0}{1.1 - 0.9} = 0.5064$$

Solution 11 In this case the price dynamics of the stock is:



while the price dynamics of the put option is:

Exercise 11 Consider a 2-year European put option with a strike price of 52$ on a stock whose current price is 50$. Suppose that there are 2 time steps of 1 year each, and in each time step the stock price either moves up by a factor of 1.2 or moves down by a factor of 0.8. Suppose that the risk-free interest rate is 5% per annum with continuous compounding. Find the price of the option using both the risk-neutral valuation method and the replicating portfolio method.

$$
\begin{tikzcd}
 & & \boxed{p_{uu}} \\
 & \boxed{p_u} \arrow[ur] \arrow[dr] & \\
 & & \boxed{p_{ud}} \\
\boxed{p_0} \arrow[uur] \arrow[ddr] & & \\
 & & \boxed{p_{ud}} \\
 & \boxed{p_d} \arrow[ur] \arrow[dr] & \\
 & & \boxed{p_{dd}}
\end{tikzcd}
$$

where

$$p_{uu} = \max(52 - 72, 0) = 0$$

$$p_{ud} = \max(52 - 48, 0) = 4$$

$$p_{dd} = \max(52 - 32, 0) = 20$$

Using the risk neutral valuation, first of all, we recover the risk neutral probabilities:

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{0.05 \cdot 1} - 0.8}{1.2 - 0.8} = 0.6282$$

At this point the price of the put option can be determined starting from the last period going backward. At $t = 1$ we have:

$$\begin{aligned}
p_u &= e^{-rt}\left[p_{uu}p + p_{ud}(1-p)\right] \\
&= e^{-0.05 \cdot 1}\left[0 \cdot 0.6282 + 4(1 - 0.6282)\right] \\
&= 1.4147
\end{aligned}$$

and also:

$$\begin{aligned}
p_d &= e^{-rt}\left[p_{ud}p + p_{dd}(1-p)\right] \\
&= e^{-0.05 \cdot 1}\left[4 \cdot 0.6282 + 20(1 - 0.6282)\right] \\
&= 9.4636
\end{aligned}$$

and at $t = 0$ we have:

$$\begin{aligned}
p_0 &= e^{-rt}\left[p_u p + p_d(1-p)\right] \\
&= e^{-0.05 \cdot 1}\left[1.4147 \cdot 0.6282 + 9.4636(1 - 0.6282)\right] \\
&= 4.1923
\end{aligned}$$

that is the price of the put option today. Using the replicating portfolio we start from the last period and again we go backward. At $t = 2$:

$$\begin{cases}
V_{uu} = xe^{rt} + yS_{uu} = p_{uu} \\
V_{ud} = xe^{rt} + yS_{ud} = p_{ud}
\end{cases}$$

i.e.:

$$\begin{cases}
V_{uu} = xe^{0.05 \cdot 1} + y72 = 0 \\
V_{ud} = xe^{0.05 \cdot 1} + y48 = 4
\end{cases}$$

and subtracting the second equation from the first one:

$$y(72-48) = -4 \implies y = \frac{-4}{24} = -0.1667$$

and then:

$$xe^{0.05\cdot 1} - 0.1667 \cdot 72 = 0 \implies x = 11.4148$$

hence:

$$x = 11.4148 \text{ amount of the bond}$$
$$y = -0.1667 \text{ shares of the stock shorted}$$

and then at $t = 1$ we have:

$$\begin{aligned} V_u &= x + yS_u \\ &= 11.4148 - 0.1667 \cdot 60 \\ &= 1.4147 = p_u \end{aligned}$$

that is the price of the put in the "up" state and is the same as found with the risk-neutral valuation method. The replicating portfolio can also be found with the formulas:

$$x = e^{-rt}\frac{up_{ud} - dp_{uu}}{u-d} = e^{-0.05\cdot 1}\frac{1.2\cdot 4 - 0.8 \cdot 0}{1.2 - 0.8} = 11.4148$$

$$y = \frac{1}{S_0}\frac{p_{uu} - p_{ud}}{u-d} = \frac{1}{50}\frac{0-4}{1.2-0.8} = -0.1667$$

At $t = 2$ we also have:

$$\begin{cases} V_{ud} = xe^{rt} + yS_{ud} = p_{ud} \\ V_{dd} = xe^{rt} + yS_{dd} = p_{dd} \end{cases}$$

i.e.:

$$\begin{cases} V_{ud} = xe^{0.05\cdot 1} + y48 = 4 \\ V_{dd} = xe^{0.05\cdot 1} + y32 = 20 \end{cases}$$

and subtracting the second equation from the first one:

$$y(48-32) = -16 \implies y = \frac{-16}{16} = -1$$

and then:

$$xe^{0.05\cdot 1} - 1 \cdot 32 = 20 \implies x = 49.4636$$

hence:

$$x = 49.4636 \text{ amount of the bond}$$
$$y = -1 \text{ shares of the stock shorted}$$

and then at $t = 1$ we have:

$$\begin{aligned} V_d &= x + yS_d \\ &= 49.4636 - 1 \cdot 40 \\ &= 9.4636 = p_d \end{aligned}$$

that is the price of the put in the "down" state and is the same as found with the risk-neutral valuation method. The replicating portfolio can also be found with the formulas:

$$x = e^{-rt}\frac{up_{dd} - dp_{ud}}{u-d} = e^{-0.05\cdot 1}\frac{1.2\cdot 20 - 0.8\cdot 4}{1.2-0.8} = 49.4636$$

$$y = \frac{1}{S_0}\frac{p_{ud} - p_{dd}}{u-d} = \frac{1}{50}\frac{4-20}{1.2-0.8} = -1$$

At $t = 1$ finally we have:

$$\begin{cases} V_u = xe^{rt} + yS_u = p_u \\ V_d = xe^{rt} + yS_d = p_d \end{cases}$$

i.e.:

$$\begin{cases} V_u = xe^{0.05\cdot 1} + y60 = 1.4147 \\ V_d = xe^{0.05\cdot 1} + y40 = 9.4636 \end{cases}$$

and subtracting the second equation from the first one:

$$y(60-40) = -8.0489 \implies y = \frac{-8.0489}{20} = -0.4024$$

and then:

$$xe^{0.05\cdot 1} - 0.4024\cdot 40 = 9.4636 \implies x = 24.3148$$

hence:

$$\begin{aligned} x &= 24.3148 \text{ amount of the bond} \\ y &= -0.4024 \text{ shares of the stock shorted} \end{aligned}$$

and then at $t = 0$ we have:

$$\begin{aligned} V_0 &= x + yS_0 \\ &= 24.3148 - 0.4024\cdot 50 \\ &= 24.3148 - 20.12 = 4.1923 = p_0 \end{aligned}$$

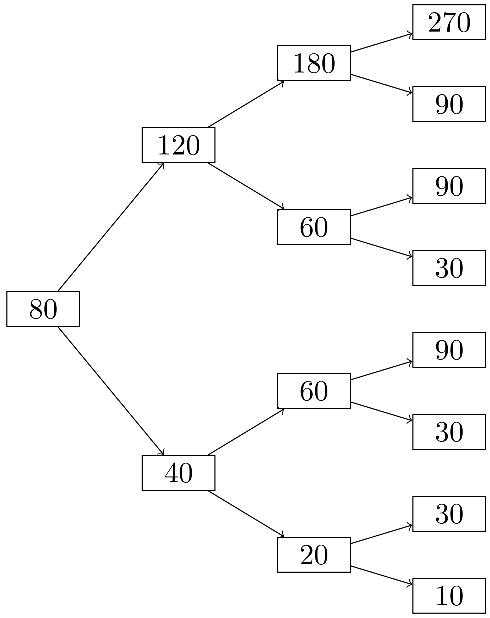
that is the price of the put option and is the same as found with the risk-neutral valuation method. The replicating portfolio can also be found with the formulas:

$$x = e^{-rt}\frac{up_d - dp_u}{u-d} = e^{-0.05\cdot 1}\frac{1.2\cdot 4 - 0.8\cdot 0}{1.2-0.8} = 24.3148$$

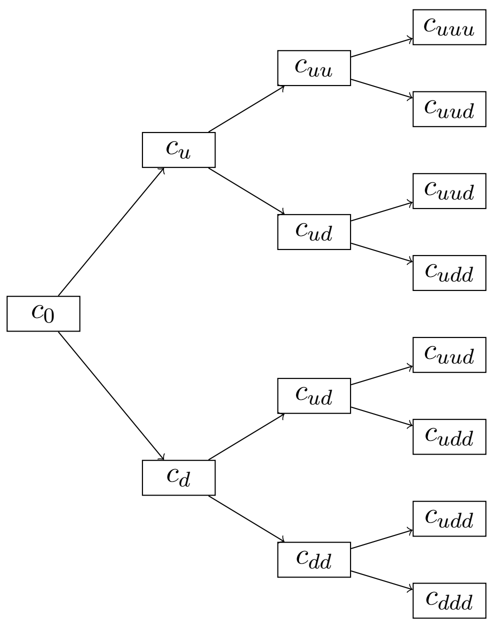
$$y = \frac{1}{S_0}\frac{p_u - p_d}{u-d} = \frac{1}{50}\frac{1.4147 - 9.4636}{1.2-0.8} = -0.4024$$

Solution 12 In this case the price dynamics of the stock is:

Exercise 12 Consider a 3-year European call option with a strike price of 80$ on a stock whose current price is 80$. Suppose that there are 3 time steps of 1 year each, and in each time step the stock price either moves up by a factor of 1.5 or moves down by a factor of 0.5. Suppose that the risk-free interest rate is 0% per annum with continuous compounding. Find the price of the option using both the risk-neutral valuation method and the replicating portfolio method.



while the price dynamics of the call option is:



where

$$c_{uuu} = \max(270 - 80, 0) = 190$$

$$c_{uud} = \max(90 - 80, 0) = 10$$

$$c_{udd} = \max(30 - 80, 0) = 0$$

$$c_{ddd} = \max(10 - 80, 0) = 0$$

Using the risk neutral valuation, first of all, we recover the risk neutral probabilities:

$$p = \frac{(1 + rt) - d}{u - d} = \frac{(1 + 0 \cdot 1) - 0.5}{1.5 - 0.5} = 0.5$$

At this point the price of the call option can be determined starting from the last period going backward. At $t = 2$ we have:

$$\begin{aligned} c_{uu} &= \frac{1}{1 + r \cdot t}[c_{uuu}p + c_{uud}(1 - p)] \\ &= \frac{1}{1 + 0 \cdot 1}\left[190 \cdot 0.5 + 10(1 - 0.5)\right] \\ &= 100 \end{aligned}$$

and also:

$$\begin{aligned} c_{ud} &= \frac{1}{1 + r \cdot t}[c_{uud}p + c_{udd}(1 - p)] \\ &= \frac{1}{1 + 0 \cdot 1}\left[10 \cdot 0.5 + 0(1 - 0.5)\right] \\ &= 5 \end{aligned}$$

and also:

$$\begin{aligned} c_{dd} &= \frac{1}{1 + r \cdot t}[c_{udd}p + c_{ddd}(1 - p)] \\ &= \frac{1}{1 + 0 \cdot 1}\left[0 \cdot 0.5 + 0(1 - 0.5)\right] \\ &= 0 \end{aligned}$$

At $t = 1$ we have:

$$\begin{aligned} c_u &= \frac{1}{1 + r \cdot t}[c_{uu}p + c_{ud}(1 - p)] \\ &= \frac{1}{1 + 0 \cdot 1}\left[100 \cdot 0.5 + 5(1 - 0.5)\right] \\ &= 52.5 \end{aligned}$$

and also:

$$\begin{aligned} c_d &= \frac{1}{1 + r \cdot t}[c_{ud}p + c_{dd}(1 - p)] \\ &= \frac{1}{1 + 0 \cdot 1}\left[5 \cdot 0.5 + 0(1 - 0.5)\right] \\ &= 2.5 \end{aligned}$$

and at $t = 0$ we have:

$$\begin{aligned} c_0 &= \frac{1}{1 + r \cdot t}[c_u p + c_d(1 - p)] \\ &= \frac{1}{1 + 0 \cdot 1}\left[52.5 \cdot 0.5 + 2.5(1 - 0.5)\right] \\ &= 27.5 \end{aligned}$$

that is the price of the call option today. Using the replicating portfolio we start from the last period and again we go backward. At $t = 3$:

$$\begin{cases} V_{uuu} = x(1 + r \cdot t) + yS_{uuu} = c_{uuu} \\ V_{uud} = x(1 + r \cdot t) + yS_{uud} = c_{uud} \end{cases}$$

i.e.:

$$\begin{cases} V_{uuu} = x(1 + 0 \cdot 1) + y270 = 190 \\ V_{uud} = x(1 + 0 \cdot 1) + y90 = 10 \end{cases}$$

and subtracting the second equation from the first one:

$$y(270 - 90) = 180 \implies y = \frac{180}{180} = 1$$

and then:

$$x(1 + 0 \cdot 1) + 1 \cdot 90 = 10 \implies x = -80$$

hence:

$$x = -80 \text{ borrowing}$$
$$y = 1 \text{ shares of the stock}$$

and then at $t = 2$ we have:

$$\begin{aligned} V_{uu} &= x + yS_{uu} \\ &= -80 + 1 \cdot 180 \\ &= 100 = c_{uu} \end{aligned}$$

as found with the risk-neutral valuation method. At $t = 3$ we also have:

$$\begin{cases} V_{uud} = x(1 + r \cdot t) + yS_{uud} = c_{uud} \\ V_{udd} = x(1 + r \cdot t) + yS_{udd} = c_{udd} \end{cases}$$

i.e.:

$$\begin{cases} V_{uud} = x(1 + 0 \cdot 1) + y90 = 10 \\ V_{udd} = x(1 + 0 \cdot 1) + y30 = 0 \end{cases}$$

and subtracting the second equation from the first one:

$$y(90 - 30) = 10 \implies y = \frac{10}{60} = 0.1667$$

and then:

$$x(1 + 0 \cdot 1) + 0.1667 \cdot 30 = 0 \implies x = -5$$

hence:

$$x = -5 \text{ borrowing}$$
$$y = 0.1667 \text{ shares of the stock}$$

and then at $t = 2$ we have:

$$\begin{aligned} V_{ud} &= x + yS_{ud} \\ &= -5 + 0.1667 \cdot 60 \\ &= 5 = c_{ud} \end{aligned}$$

In $t = 3$ we also have:

$$\begin{cases} V_{udd} = x(1 + r \cdot t) + yS_{udd} = c_{udd} \\ V_{ddd} = x(1 + r \cdot t) + yS_{ddd} = c_{ddd} \end{cases}$$

i.e.:

$$\begin{cases} V_{udd} = x(1 + 0 \cdot 1) + y30 = 0 \\ V_{ddd} = x(1 + 0 \cdot 1) + y10 = 0 \end{cases}$$

and subtracting the second equation from the first one:

$$y(30 - 10) = 0 \implies y = 0$$

and then:

$$x(1 + 0 \cdot 1) + 0 \cdot 10 = 0 \implies x = 0$$

hence:

$$x = 0 \text{ borrowing}$$
$$y = 0 \text{ shares of the stock}$$

and then at $t = 2$ we have:

$$\begin{aligned} V_{dd} &= x + yS_{dd} \\ &= 0 + 0 \cdot 40 \\ &= 0 = c_{dd} \end{aligned}$$

At $t = 2$ we then have:

$$\begin{cases} V_{uu} = x(1 + r \cdot t) + yS_{uu} = c_{uu} \\ V_{ud} = x(1 + r \cdot t) + yS_{ud} = c_{ud} \end{cases}$$

i.e.:

$$\begin{cases} V_{uu} = x(1 + 0 \cdot 1) + y180 = 100 \\ V_{ud} = x(1 + 0 \cdot 1) + y60 = 5 \end{cases}$$

and subtracting the second equation from the first one:

$$y(180 - 60) = 95 \implies y = \frac{95}{120} = 0.7917$$

and then:

$$x(1 + 0 \cdot 1) + 0.7917 \cdot 60 = 5 \implies x = -42.5$$

hence:

$$x = -42.5 \text{ borrowing}$$
$$y = 0.7917 \text{ shares of the stock}$$

and then at $t = 1$ we have:

$$\begin{aligned} V_u &= x + yS_u \\ &= -42.5 + 0.7917 \cdot 120 \\ &= 52.5 = c_u \end{aligned}$$

At $t = 2$ we also have:

$$\begin{cases} V_{ud} = x(1 + r \cdot t) + yS_{ud} = c_{ud} \\ V_{dd} = x(1 + r \cdot t) + yS_{dd} = c_{dd} \end{cases}$$

i.e.:

$$\begin{cases} V_{ud} = x(1 + 0 \cdot 1) + y60 = 5 \\ V_{dd} = x(1 + 0 \cdot 1) + y20 = 0 \end{cases}$$

and subtracting the second equation from the first one:

$$y(60 - 20) = 5 \implies y = \frac{5}{40} = 0.125$$

and then:

$$x(1 + 0 \cdot 1) + 0.125 \cdot 20 = 0 \implies x = -2.5$$

hence:

$$x = -2.5 \text{ borrowing}$$
$$y = 0.125 \text{ shares of the stock}$$

and then at $t = 1$ we have:

$$\begin{aligned} V_d &= x + yS_d \\ &= -2.5 + 0.125 \cdot 40 \\ &= 2.5 = c_d \end{aligned}$$

At $t = 1$ finally we have:

$$\begin{cases} V_u = x(1 + r \cdot t) + yS_u = c_u \\ V_d = x(1 + r \cdot t) + yS_d = c_d \end{cases}$$

i.e.:

$$\begin{cases} V_u = x(1 + 0 \cdot 1) + y120 = 52.5 \\ V_d = x(1 + 0 \cdot 1) + y40 = 2.5 \end{cases}$$

and subtracting the second equation from the first one:

$$y(120 - 40) = 50 \implies y = \frac{50}{80} = 0.625$$

and then:

$$x(1 + 0 \cdot 1) + 0.625 \cdot 40 = 2.5 \implies x = -22.5$$

hence:

$$x = -22.5 \text{ borrowing}$$
$$y = 0.625 \text{ shares of the stock}$$

and then at $t = 0$ we have:

$$\begin{aligned} V_0 &= x + yS_0 \\ &= -22.5 + 0.625 \cdot 80 \\ &= -22.5 + 50 = 27.5 = c_0 \end{aligned}$$

that is the price of the call option and is the same as found with the risk-neutral valuation method.