The binomial model:

Incipit: In this chapter we will study, in some detail, the simplest possible and non-trivial model of a financial market - The binomial model -, which is a discrete time model.
The model itself is very easy to understand, almost all important concepts which we will study later on already appear in the binomial case.
The mathematics required to analyse it is high school level, and, last but not least, the binomial model is often used in practice.

The One Period Binomial Model:

Model Description:

(Time) •) Time is denoted with 't' and we consider two points in time: $t=0$ (today), $t=1$ (tomorrow), might also be written $t+\Delta t$.

(Assets) •) We model a market with two assets:

a) Risk-less asset BOND with interest rate r. Bond: $B(t)$ (deterministic)

b) Risky STOCK Traded asset. Stock: $S(t)$ (random)

(Dynamics) •) Furthermore we assume that $S(t)$ has a Binomial dynamics

$B(0)=1 \longrightarrow B(1)=1+r$ (Linear compounding) It would have been $e^{r\Delta t}$

$S(0)=S \longrightarrow S(1)=$
- $S\cdot u$ (up), p_u ← probability of going up
- $S\cdot d$ (down), $p_d = 1-p_u$ ← probability of going down

↳ $p_d + p_u = 1$

(Assumption: the today's price of the stock is known)

(Interpretation of B(t))

In this model B(t) can be easily interpreted as a bank account which grows linear interest for small amount of time or a Z.C.B..

(Graphically)

$B(t)$: $\underset{t=0}{1} \longrightarrow \underset{t=1}{1+r}$

$S(t)$:
$$\begin{tikzcd} & S_u \\ S_0 \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & S_d \end{tikzcd}$$

(Z r.v.) OBS. It is often convenient to write the risky asset as:

$$\begin{cases} S(0) = s \\ S(1) = s \cdot Z \end{cases}$$

where Z is a r.v. s.t. $Z = \begin{cases} u & \text{, with probability } p_u \\ d & \text{, with probability } p_d \end{cases}$

(u & d) We assume $u > d$ they are called "up" and "down" factors:

$u > 1$, $d < 1$ and $p_u + p_d = 1$

(Returns) Let's talk about returns:

·) Linear return : $R = \dfrac{S(1) - S(0)}{S(0)} = \begin{cases} \dfrac{u\,S(0) - S(0)}{S(0)} = u - 1 \\ \dfrac{d\,S(0) - S(0)}{S(0)} = d - 1 \end{cases}$

·) Log return : $\ln\left(\dfrac{S(1)}{S(0)}\right) = \begin{cases} \ln\left(\dfrac{S(0)\cdot u}{S(0)}\right) = \ln(u) \\ \ln\left(\dfrac{S(0)\cdot d}{S(0)}\right) = \ln(d) \end{cases}$

Portfolios and Arbitrage:

Scenario:

We will study the behaviour of various portfolios on the (B,S) market, and to this end we define the portfolio as a vector $h = (x,y)$. The interpretation is that x is the number of bonds we hold in our portfolio, whereas y is the number of units of the stock held by us.
Note that is quite accettable for x and y to be positive as well as negative. If for example $x = 3$, this means that we have bought three bonds at time $t=0$. If on the other hand $y=-2$ this means that we have sold two shares of the stock at time $t=0$. (We would have a long position in the bond and a short position in the stock.)

In short:

We study portfolio in the market (B,S)

$h = (x,y)$ portfolio $\quad$ $x \to$ num. of bonds
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\;$ $y \to$ num. of shares

$x,y \in \mathbb{R}$: negative values correspond to short position

Assumptions:

1) Short position and fractions are allowed.
2) No Bid-Ask spread. (Selling price = buying price).
3) No transaction cost.
4) Market is completely liquid.

Portfolio:

$h = (x,y)$ it has:

- Deterministic value at time 0 $\quad(t)$
- Stochastic value at time 1 $\quad(T)$ or $(t+\Delta t)$

Consider now a fixed portfolio $h = (x, y)$. This portfolio has a deterministic market value at time $t=0$ and a stochastic value at $t=1$.

Definition: (Value process of the portfolio h)

The value process of the portfolio h is defined by:

$$V^h(t) = x\,B(t) + y\,\underbrace{S(t)}_{\text{(only source of randomness)}}, \quad t = 0,1$$

Or, in more detail:

$$V^h(0) = x + yS$$

$$V^h(1) = x(1+R) + ySZ$$

Scheme:

$t=0$	$t=1$	$\mathbb{P}$
$V^h(0) = x\cdot 1 + y\cdot S$	$V^h(1) = x(1+r) + yS\cdot u$	p_u
	$V^h(1) = x(1+r) + yS\cdot d$	p_d
		$\boxed{p_u + p_d = 1}$

Definition: (Arbitrage)

An arbitrage portfolio is a portfolio h with the following properties:

$$V^h(0) = 0$$

$$V^h(1) > 0 \quad \text{with probability } 1$$

Prop. (Condition of arbitrage free market)

The market (B,S) is free of arbitrage if and only if : $\boxed{d \leq 1+r \leq u}$ (*)

(it means the future price of the lower is not dominate or does not dominate the stochastic component.)

Proof: ($\Leftrightarrow$, thus we prove 1) No arbitrage $\Rightarrow$ (*) then 2) (*) $\Rightarrow$ No arbitrage.)

1) No arbitrage $\Rightarrow$ (*) (We prove the opposite $\overline{(*)} \Rightarrow$ arbitrage)

We assume that (*) does not hold and we build an arbitrage strategy.

a) Firstly : we assume $u < (1+r)$

$\Downarrow$

Then $S(1+r) > Su > Sd$ (since $u > d$)

(Idea) $\Rightarrow$ It is always better to invest in the bond, the strategy chosen is the following:

Arbitrage strategy:

- Sell the stock
- Invest in the bond

$\Rightarrow$ Resulting portfolio : $h = (S, -1)$

At time $t=0$

.) Sell short the stock $(+S)$

.) Invest S in bonds $(-S)$

$$V^h(0) = \underbrace{S \cdot 1}_{B(0)=1} - \underbrace{S}_{S(0)} = 0$$

At time $t=1$: (1st scenario: The stock goes "up")

(2nd scenario: The stock goes "down")

.) I take back the investment in bonds $(+S(1+r))$

.) I pay $S \cdot u$ to buy back the stock I need to cover the short selling action at $t=0$ $(-S \cdot u)$ P_u

.) I pay $S \cdot d$ to buy back the stock I need to cover the short selling action at $t=0$ $(-S \cdot d)$ P_d

$$V^h(1) = S(1+r) - Su \quad \boxed{>0}$$

$$V^h(1) = S(1+r) - S \cdot d \quad \boxed{>0}$$

By assumption

Arbitrage

$V^h(0) = 0$

$V^h(1) > 0$ with prob. $= 1$

$\mathbb{P}$

$P_u + P_d = 1$

(First step)

$\downarrow$

1.a) $u < 1+r$ leads to arbitrage.

b) Secondly : $1+r < d$

$\Downarrow$

Then $S(1+r) < S \cdot d$

(Idea) $\Rightarrow$ It is always better to invest in the stock the strategy chosen is the following :

Arbitrage strategy:

- Borrow S $(+S)$
- Buy a stock $(-S)$

$\Rightarrow$ Resulting portfolio : $h = (-S, 1)$

At time $t=0$	At time $t=1$ (1st scenario: The stock goes "up") (2nd scenario: The stock goes "down")	$\mathbb{P}$
·) I borrow S to buy the stock : $(+S)$	·) I take back $S \cdot u$: $(+S \cdot u)$	p_u
·) I buy the stock $(-S)$	·) I take back $S \cdot d$: $(+S \cdot d)$	p_d
	·) I pay back the loan : $(-S \cdot (1+r))$	
$V^h(0) = \underset{\substack{x=-1 \\ (\# \text{ of bonds})}}{-1} \cdot S + \underset{\substack{y=1 \\ (\# \text{ of stocks})}}{1} \cdot S = 0$	$V^h(1) = S \cdot u - S(1+r) \boxed{> 0}$ $V^h(1) = S \cdot d - S(1+r) \boxed{> 0}$	$p_u + p_d = 1$

By assumption

$\Rightarrow$ Arbitrage $\Rightarrow$ $V^h(0) = 0$, $V^h(1) > 0$ with prob. $= 1$

(Second step)

$\downarrow$

1.b) $1+r < d$ leads to arbitrage.

Conclusion : " $\Rightarrow$ " 1.a) + 1.b) translate into : $u < 1+r$, $1+r < d$ which are $\overline{d \leq 1+r \leq u}$ and this leads to arbitrage 1.a) + 1.b) $\Rightarrow$ arbitrage. Then it must hold no arbitrage $\Rightarrow$ $\overline{1.a) + 1.b)} = d \leq 1+r \leq u$

2) (*) $\Rightarrow$ No arbitrage

To show that this implies absence of arbitrage let us consider an arbitrary portfolio such that $V^h(0) = 0$. I.e.,

assume : $d \leq 1+r \leq u$ and construct a portfolio V^h : $V^h(0) = 0$

If $V^h(0) = 0$ then : $V^h(0) = x \cdot 1 + ys = 0 \Rightarrow h = (x, y)$: $x = -ys$

namely :

$$V^h(0) = -ys \cdot 1 + ys = 0$$

In the binomial dynamics this translates in :

$$V^h(1) = \begin{cases} V^h_u(1) = -yS(1+r) + ySu, & p_u \\ V^h_d(1) = -yS(1+r) + ySd, & p_d \end{cases}$$

Case 1: $y > 0$

$V^h(1) > 0$ iff (with prob. 1)
- $ySu > yS(1+r)$ iff $u > (1+r)$
- $ySd > yS(1+r)$ iff $\boxed{d > (1+r)}$

but this is not allowed by assumption, thus in the case of $y > 0$ no arbitrage is possible.

(we divide both sides by a negative value $\frac{1}{y} < 0$.)

Case 2: $y < 0$

$V^h(1) > 0$ iff (with prob. 1)
- $\cancel{y}Su \overset{<}{>} \cancel{y}S(1+r)$ iff $\boxed{u < (1+r)}$
- $\cancel{y}Sd \overset{<}{>} \cancel{y}S(1+r)$ iff $d < (1+r)$

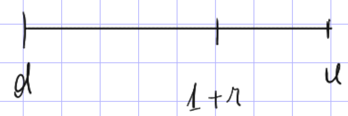
but this is not allowed by assumption, thus in the case of $y < 0$ no arbitrage is possible.

$\boxtimes$

Crucial Consequence:

At first glance this result is perhaps only moderately exciting, but we may write it in a more suggestive form:

The market is arbitrage free iff: $d \leq 1+r \leq u$



any point here is a convex combination of u and d

there exists positive: q_u, q_d s.t. $\begin{cases} 1+r = q_u \cdot u + q_d \cdot d \\ q_u + q_d = 1 \end{cases}$

In particular we see that the weights q_u and q_d can be interpreted as probabilities for a new probability measure Q with the property $Q(Z=u) = q_u$, $Q(Z=d) = q_d$.

$\Rightarrow$ They can be seen as a new distribution for $S(t)$ considering $S(t) = S \cdot Z$ $\quad (S = S(0))$

Let Q be a probability measure:

$$Q(S(1) = Su) = q_u$$
$$Q(S(1) = Sd) = q_d$$

If S has distribution Q

$$\mathbb{E}^Q[S(1)] = q_u Su + q_d Sd = S(q_u \cdot u + q_d \cdot d) = S(1+r)$$

$$\mathbb{E}^Q\left[\frac{S(1)}{1+r}\right] = S$$

Dynamics:

$t=0$	$t=1$	P	Q (Risk Neutral)
$S(0) = S$	$S(1) = Su$	p_u	q_u
	$S(1) = Sd$	p_d	q_d

Considerations on the just found formula: (Risk neutral valuation formula)

$$S = \mathbb{E}^{Q}\left[\frac{S(1)}{1+r}\right]$$ it is called risk-neutral valuation formula

in the sense that it gives today's stock price as the discounted expected value of tomorrow's stock price in this new probability measure.

Observation:

Of course we do not assume that the agents in our market are actually risk-neutral, what we have shown is only that if we use the Q-probabilities instead of the objective ones then we have in fact a risk-neutral valuation of the stock (given the absence of arbitrage)

Definition: (Risk Neutral Measure or Martingale Measure)

A probability measure Q is called a martingale measure if the following condition holds:

$$S = \mathbb{E}^{Q}\left[\frac{S(1)}{1+r}\right]$$

I.e. The today's stock price is given by the expected value of tomorrow's discounted price.

Prop.

The market model (B,S) is arbitrage free if and only if it exists a martingale measure Q.

Proof.

The proof is immediate since $\exists$ martingale measure if and only if $d \leq 1+r \leq u$, since we need to perform the convex combination but $d \leq 1+r \leq u \iff$ the market model is arbitrage free.

$\boxtimes$

Prop.

For the one-period Binomial model $\mathbb{Q}$ is unique and given by:

$$\begin{cases} q_u = \dfrac{(1+r)-d}{u-d} \\ q_d = \dfrac{u-(1+r)}{u-d} \end{cases}$$

Proof.

It is the solution of $\begin{cases} 1+r = q_u \cdot u + q_d \cdot d \\ 1 = q_u + q_d \end{cases}$

$$\begin{cases} 1+r = q_u \cdot u + q_d \cdot d \\ 1 = q_u + q_d \end{cases} \Rightarrow \begin{cases} 1+r = q_u \cdot u + (1-q_u) \cdot d \\ q_d = 1 - q_u \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 1+r-d = q_u (u-d) \\ \longrightarrow \end{cases} \Rightarrow \begin{cases} \boxed{q_u = \dfrac{(1+r)-d}{u-d}} \\ \longrightarrow \end{cases} \Rightarrow \begin{cases} \longrightarrow \\ q_d = 1 - \dfrac{(1+r)-d}{u-d} \end{cases}$$

$$\Rightarrow \begin{cases} \longrightarrow \\ q_d = \dfrac{u-\cancel{d}-(1+r)+\cancel{d}}{u-d} \end{cases} \Rightarrow \begin{cases} \longrightarrow \\ \boxed{q_d = \dfrac{u-(1+r)}{u-d}} \end{cases}$$

$\square$

Example:

$S = 100, \quad u = 1.2 \quad d = 0.8 \quad p_u = 0.6 \quad p_d = 0.4 \quad r = 0.01$

$t=0$	$t=1$	$\mathbb{P}$
	$S(1) = 120$	0.6
$S(0) = 100$		
	$S(1) = 80$	0.4

$$\mathbb{E}^P\left[\frac{S(1)}{1+r}\right] = \frac{1}{1.01}\,\mathbb{E}^P\left[S(1)\right] = 0.91 \cdot \left[120 \cdot 0.6 + 8 \cdot 0.4\right] = 102.97$$ (Market is risk adverse)

$$\begin{cases} q_u \cdot 1.2 + q_d \cdot 0.8 = 1 + 0.01 \\ q_u + q_d = 1 \end{cases}$$

$$q_u = \frac{(1+r) - d}{u-d} = \frac{1.01 - 0.8}{0.4} = 0.525$$

$$q_d = \frac{u - (1+r)}{u-d} = \frac{1.2 - 1.01}{0.4} = 0.475$$

$$\mathbb{E}^Q\left[\frac{S(1)}{1+r}\right] = \frac{1}{1.01}\left[0.525 \cdot 120 + 0.475 \cdot 80\right] = 100$$ (market is risk-neutral)

Contingent Claims:

Let us now assume that the market is arbitrage free. We go on study pricing problems for contingent claims.

Definition: (Contingent Claims)

A contingent claim (financial derivative) is any stochastic variable X of the form $X = \phi(Z)$ (written also: $X = \Phi(S(T))$ $^{(T=1 \text{ for us})}$ or $X = \Phi(s \cdot Z)$), where Z is the stochastic variable driving the stock price process aforementioned.

Scenario:

We interpret a given claim X as a contract which pays X to the holder of the contract at time $t=1$.

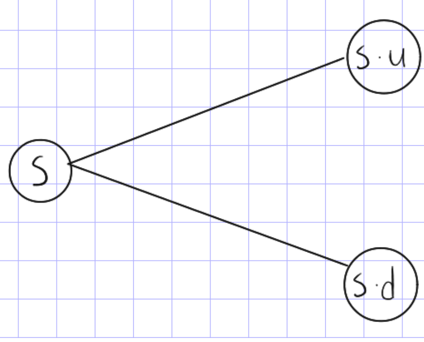
(Def.) The function ϕ is called the contract function. A typical example would be an european call option on the stock with strike price K. For this option to be interesting we assume that $s \cdot d < K < s \cdot u$. If $S_1 > K$ then we use the option, paying K to get the stock just to sell it on the market for $s \cdot u$, thus having a payoff of $s \cdot u - K$. If $S_1 < K$ then the option is obviously worthless. In this example we thus have:

$$X = \begin{cases} s \cdot u - K & \text{if } Z = u \\ 0 & \text{if } Z = d \end{cases}$$

and the contract function is given by:

$$X = \Phi(S(1)) = \Phi(s \cdot Z) = \phi(Z) = \begin{cases} \phi(u) = s \cdot u - K \\ \phi(d) = 0 \end{cases}$$

(Graphically)



, here happened $\phi(u)$

, here happened $\phi(d)$

Pricing the contingent claim:

Task: Our main problem is now to determine the "fair" price, if such an object exists at all for a given contingent claim X.

Notation: If we denote the price of X at time t by $\Pi(t; X)$, then it can be seen that at time $t=1$ the problem is trivial, since for no arbitrage reasons we must have: $\Pi(1; X) = X$.
The hard part of the problem is to determine $\Pi(0; X)$.
To tackle this problem we make a slight excursus.

Idea: Since we have assumed absence of arbitrage we know that we cannot make money out of nothing, this is intuitively analogous of saying that two entities that worth the same must cost the same. Let's see where this statement leads us.

Definition: (Reachability of a Contingent Claim + Replicating Portfolio + Completeness)
A given contingent claim X is said to be reachable if there exists a portfolio h such that $V^h(1) = X$ with probability 1.
In that case we say that the portfolio h is a replicating portfolio (or hedging portfolio). If all claims can be replicated, we say that the market is complete.

Setting: If a certain claim X is reachable with replicating portfolio h, then, from a financial point of view, there is no difference between holding the claim and holding the portfolio. No matter what happens on the stock market, the value of the claims at time $t=1$ will be exactly equal to the value of the portfolio at $t=1$. Thus the price of the claim should equal the market value of the portfolio, and we have the following basic pricing principle.

Pricing Principle 1:

If a claim X is reachable with replicating portfolio h, then the only reasonable price process for X is given by:

$$\Pi(t;X) = V^h(t) \quad , \quad t = 0,1$$

Where the word "reasonable" above can be given a more precise meaning as in the following proposition.

Prop. Suppose that a claim X is reachable with replicating portfolio h. Then any price at $t=0$ of the claim X, other than $V^h(0)$, will lead to an arbitrage possibility.

Purpose: We see that in a complete market we can in fact price all contingent claims, so it is of great interest to investigate when a given market is complete.

Prop. (Completeness of the general binomial model)

Assume that the general binomial model is free of arbitrage.

Then it is also complete.

Proof. We fix an arbitrary claim X with contract function $\Phi(\cdot)$ $(X = \Phi(Z))$, and we want to show that there exists a portfolio $h = (x,y)$ such that:

$$V^h(1) = \Phi(u) \quad , \quad \text{if } Z = u$$
$$V^h(1) = \Phi(d) \quad , \quad \text{if } Z = d$$

If we write this out in detail we are looking for a solution (x,y) to the following system of equations:

$$(1+r)x + (s \cdot u)y = \Phi(u)$$
$$(1+r)x + (s \cdot d)y = \Phi(d)$$

Since by assumption $d < u$, this linear system has a unique solution and a simple calculation shows that it given by:

$$x = \frac{1}{1+r} \cdot \frac{u \cdot \Phi(d) - d \cdot \Phi(u)}{u-d} \quad ; \quad y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}$$

$\boxtimes$

Risk Neutral Validation of Derivatives in the Binomial model:

Process:

Since the binomial model is shown to be complete we can now price any contingent claim in this setting. Indeed, according to the pricing principle 1 the price at time $t=0$ of a contingent claim replicated by a portfolio h is given by:

$$\Pi(0; X) = V^h(0)$$

But now we can find the actual x, y defining the replicating portfolio of the contingent claim $X = \Phi(Z)$, which will consequently lead us to the price $\Pi(0;X)$.

Namely:

$$\left(\frac{-d + (1+r)}{u-d} , \frac{u-(1+r)}{u-d} \right)$$

(price of a bond at time 0) (price of the stock at time 0)

$$\Pi(0;X) = x \cdot 1 + y \cdot S =$$

(completeness of the binomial model.)

$$= \frac{1}{1+r} \left\{ \underbrace{\boxed{\frac{(1+r)-d}{u-d}}}_{q_u} \cdot \Phi(u) + \underbrace{\boxed{\frac{u-(1+r)}{u-d}}}_{q_d} \cdot \Phi(d) \right\}$$

Here we recognize the martingale probabilities q_u and q_d. If we assume that the model is free of arbitrage, these are true probabilities (they are both ≥ 0), so we can write the pricing formula above as:

$$\Pi(0;X) = \frac{1}{1+r} \left\{ \Phi(u) \cdot q_u + \Phi(d) \cdot q_d \right\}$$

The right-hand side can now be interpreted as an expected value under the martingale probability measure Q.

So we proved the following proposition:

Prop. (Pricing of a contingent claim in the Binomial model)

If the binomial model is free of arbitrage, then the arbitrage free price of a contingent claim X is given by:

$$\Pi(0; X) = \frac{1}{1+r} \mathbb{E}^{Q}[X]$$

Which is a risk-neutral formula as the price of the contingent claim at time $t=0$ is given by the discounted expected value of the contingent claim at time $t=1$.

Here the martingale measure Q is uniquely determined by q_u and q_d, which lead to:

$$S(0) = \frac{1}{1+r} \mathbb{E}^{Q}[S(1)]$$

And the replicating portfolio $h = (x, y)$ of the contingent claim can be found by:

$$x = \frac{1}{1+r} \cdot \frac{u \cdot \Phi(d) - d\,\Phi(u)}{u-d}$$

$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}$$

Conclusions:

-) The only role played by the objective probabilities is that they determine which events are possible and which are impossible.

-) When we compute the arbitrage free price of a financial derivative we carry out the computations as if we lived in a risk neutral world. This does not mean that we de facto live (neither believe to live) in a risk-neutral world.

-) The valuation formula holds for all investors, regardless of their attitude toward risk, as long as they prefer more deterministic money to less.

-) The formula above is therefore often referred to as a "preference free" valuation formula.

Contingent Claims based on options:

OPTIONS:

The contingent claim is the payoff: $X = \phi(S(T))$ where $T = 1$, i.e. the contract pays X to the holder at time T.

The market is still: (B, S)

And the notation might not be always the same, hence: $\phi(S \cdot u) = \phi(u)$, $\phi(S \cdot d) = \phi(d)$

The usual dynamic follows:

$$
\begin{array}{lcl}
t = 0 & & t = 1 \\
B(0) = 1 & & B(1) = (1 + r) \\
S(0) = S & \longrightarrow & \begin{cases} S(1) = u \cdot S, & \text{here happened } \phi(u) \\ S(1) = d \cdot S, & \text{here happened } \phi(d) \end{cases}
\end{array}
$$

CALL OPTION:

Consistentely to what we introduced, $X = \phi(S(T)) = (S(T) - K)^+$ is the contingent claim i.e. the payoff induced by the r.v. $S(1)$. Where the dynamic is:

Asset dynamics:

$$
\begin{array}{ccc|c}
t = 0 & & t = 1 & \overset{\text{Probabilities}}{\mathbb{P}} \\
 & \nearrow & Su & p_u \\
S(0) = S & & & \\
 & \searrow & Sd & p_d
\end{array}
$$

Call dynamics:

$$
\begin{array}{ccc|c}
t = 0 & & t = 1 & \mathbb{P} \\
 & \nearrow & (S \cdot u - K)^+ & p_u \\
\text{(call option)} \; C & & & \\
 & \searrow & (S \cdot d - K)^+ & p_d
\end{array}
$$

(We want to price the option.)

Example :

$S = 100 \quad u = 1.2 \quad d = 0.8 \quad r = 0.01 \quad p_u = 0.6 \quad p_d = 0.4$

- Price of the stock at time 0 : $S = 100$
- Multiplying constants : $u = 1.2$, $d = 0.8$
- Annual interest rate : $r = 0.01$ (1 %)
- Outcome's probabilities of the 1-period stochastic process : $p_u = 0.6$, $p_d = 0.4$

Stock : $t=0$	$t=1$	(Payoffs)	(Probabilities)
		$(S(t) - K)^+$	$\mathbb{P}$
$S = 100$	120	$(120 - K)^+$	0.6
	80	$(80 - K)^+$	0.4

Bank account $B(0) = 1$ $\quad B(1) = (1+r)$

Theoretical remarks :

1) A derivative is reachable if $\exists$ portfolio h such that : $V^h(T) = \phi(S(T))$ with prob. 1 (h is a replicating portfolio)

2) If all the derivatives can be replicated , then the market is complete.

3) (Pricing Principle) If a derivative X is reachable with replicating portfolio h, then the only reasonable price for X is : $\Pi(0, X) = V^h(0)$ (No arbitrage)

If $K = 100$

C	20 ,	0.6
	0 ,	0.4

(We are looking for the price of the call option C)

We look for $h \,/\, V^h(1) = \begin{cases} 20 & u \\ 0 & d \end{cases}$, $h = (x, y)$

Then : $V^h(1) = x(1+r) + y\,S(1)$

Which leads to the following linear system:

$$\begin{cases} x\,(1+r) + y\,Su = 20 \\ x\,(1+r) + y\,Sd = 0 \end{cases} \Rightarrow \begin{cases} x\,(1.01) + y\,120 = 20 \\ x\,(1.01) + y\,80 = 0 \end{cases} \overset{(i)}{\underset{(\dots)}{\Rightarrow}} \quad x = -39.604, \quad y = \frac{1}{2}$$

2 eq with 2 unknown

(Borrow money) (Investing in stocks)

The Replicating Portfolio is: $h = (-39.604, \frac{1}{2})$

And its value at time $t=0$ is: $V^h(0) = -39.604 \cdot 1 + \frac{1}{2} \cdot 100 \cong 10.396$

which indeed also the call option price, i.e.: $C = 10.396$

Theoretical remarks: (In general)

(i)

$t=0$	$t=1$	Payoff	$\mathbb{P}$
S	Su	$\phi(u)$	p_u
	Sd	$\phi(d)$	p_d

$h = (x,y)$ replicate ϕ if

$V^h(1) = x(1+r) + y\,S(1) = \phi(S(1))$

$$\begin{cases} x\,(1+r) + y\,Su = \phi(u) & p_u \\ x\,(1+r) + y\,Sd = \phi(d) & p_d \end{cases}$$

$$\begin{cases} x = \frac{1}{1+r} \cdot \frac{u\,\phi(d) - d\,\phi(u)}{u-d} \\ y = \frac{1}{S} \cdot \frac{\phi(u) - \phi(d)}{u-d} \end{cases}$$

then $\Pi(0,X) = V^h(0)$

In our example:

(i)

$$x = \frac{1}{1+1.01} \cdot \frac{1.2 \cdot 0 - 0.8 \cdot 20}{1.2 - 0.8} = -39.604$$

$$y = \frac{1}{100} \cdot \frac{20 - 0}{1.2 - 0.8} = 0.5$$

(ii)

$$q_u = \frac{(1+r) - d}{u-d} = \frac{(1+0.01) - 0.8}{1.2 - 0.8} = 0.525$$

$$q_d = \frac{u - (1+r)}{u-d} = \frac{1.2 - (1+0.01)}{1.2 - 0.8} = 0.475$$

$$\Pi(0;X) = \frac{1}{1+r}\left\{ q_u \cdot \phi(u) + q_d \cdot \phi(d) \right\}$$

$$= \frac{1}{1+0.01}\left\{ 0.525 \cdot 20 + 0.475 \cdot 0 \right\} = 10.396$$

(ii) (Risk Neutral Valuation)

$$\Pi(0,X) = 1 \cdot x + Sy =$$

$$= \frac{1}{1+r}\,\frac{u\,\phi(d) - d\,\phi(u)}{u-d} + \frac{\phi(u) - \phi(d)}{u-d}$$

$$= \frac{1}{1+r}\left\{ \underbrace{\boxed{\frac{(1+r)-d}{u-d}}}_{q_u} \phi(u) + \underbrace{\boxed{\frac{u-(1+r)}{u-d}}}_{q_d} \phi(d) \right\}$$

$$= \frac{1}{1+r}\left\{ q_u\,\phi(u) + q_d\,\phi(d) \right\}$$

$$= \mathbb{E}^Q\left[\frac{\phi(S(1))}{1+r}\right]$$

Risk Neutral Valuation formula

q_u and q_d are uniquely determined by

$$S(0) = \mathbb{E}^Q\left[\frac{S(1)}{1+r}\right]$$

Conclusions :

Summing up we have two ways to find the price of X.

1) Find the replicating portfolio h and compute $V^h(0)$

2) Find the risk neutral probability Q.

$$\left(Q : \mathbb{E}^Q\left[\frac{S(1)}{1+r}\right] = s \right)$$

and compute : $\mathbb{E}^Q\left[\frac{\phi(S(1))}{1+r}\right]$

8 Recap Lezione 8 – One-Period Binomial Model, Arbitrage and Risk-Neutral Valuation

Idea generale della lezione La lezione introduce il **modello binomiale a un periodo** come primo modello discreto per la valutazione di asset rischiosi e derivati finanziari. Vengono formalizzati i concetti di **arbitrage** e **assenza di arbitrage** e si mostra come, in tale contesto, emergano in modo naturale la **misura risk-neutral**, i **contingent claims**, i **replicating portfolios** e la **completezza del mercato**. Nel modello binomiale privo di arbitraggio, ogni derivato ammette un **prezzo unico**.

Asset e dinamica del modello Si considera un mercato con due asset

- Un **bond risk-free** B
- Uno **stock rischioso** S

La dinamica è a un solo periodo $t = 0, 1$

$$B(0) = 1, \qquad B(1) = 1 + r$$

$$S(0) = S, \qquad S(1) = \begin{cases} Su & \text{se } p_u \\ Sd & \text{se } p_d = 1 - p_u \end{cases}$$

È spesso conveniente riscrivere la dinamica dello stock introducendo una variabile aleatoria Z

$$S(0) = S, \qquad S(1) = S \cdot Z$$

dove Z è una **variabile aleatoria** tale che

$$Z = \begin{cases} u & \text{con probabilità } p_u, \\ d & \text{con probabilità } p_d, \end{cases} \qquad p_u + p_d = 1.$$

I parametri u e d sono detti rispettivamente **up factor** e **down factor** e si assume $u > 1,\ d < 1$. La variabile aleatoria Z rappresenta quindi il **fattore moltiplicativo** che governa l'evoluzione dello stock tra $t = 0$ e $t = 1$.

Portafoglio e value process Un portafoglio fisso è descritto da $h = (x, y)$, dove

- x rappresenta la quantità investita nel bond
- y rappresenta la quantità investita nello stock

Il **value process** del portafoglio è

$$V^h(t) = xB(t) + yS(t), \qquad t = 0, 1$$

e in particolare

$$V^h(0) = x + yS, \qquad V^h(1) = x(1 + r) + ySZ$$

Arbitrage Un portafoglio h è detto un **arbitrage** se

$$V^h(0) = 0$$

$$V^h(1) > 0 \quad \text{con probabilità } 1$$

Assenza di arbitrage Il modello binomiale è **arbitrage-free** se e solo se

$$d < 1 + r < u$$

In tal caso, il fattore risk-free $1 + r$ è una **combinazione convessa** di u e d

Probabilità risk-neutral Se vale $d < 1 + r < u$, esistono **uniche** probabilità $q_u, q_d > 0$ tali che

$$\begin{cases} 1 + r = q_u u + q_d d \\ q_u + q_d = 1 \end{cases}$$

con

$$q_u = \frac{(1+r) - d}{u - d}, \qquad q_d = \frac{u - (1+r)}{u - d}$$

Misura risk-neutral La misura di probabilità Q definita da

$$Q(Z = u) = q_u, \qquad Q(Z = d) = q_d$$

è detta **risk-neutral measure** (o **martingale measure**)
Sotto Q vale

$$S = \mathbb{E}^Q\left[\frac{S(1)}{1+r}\right]$$

cioè il **prezzo scontato dello stock è una martingala**

Risk-neutral valuation formula Nel modello binomiale privo di arbitraggio il prezzo dello stock soddisfa

$$S = \mathbb{E}^Q\left[\frac{S(1)}{1+r}\right]$$

La formula è **indipendente dalle probabilità oggettive** P. Il modello di mercato (B, S) è **arbitrage-free** se e solo se esiste una **martingale measure** Q.

Contingent claims Un **contingent claim** è una variabile aleatoria della forma

$$X = \phi(Z)$$

equivalentemente

$$X = \phi(S(1))$$

e rappresenta il payoff di un contratto al tempo $t = 1$

Replicating portfolio e raggiungibilità Un claim X è detto **raggiungibile** se esiste un portafoglio $h = (x, y)$ tale che

$$V^h(1) = X \quad \text{con probabilità } 1$$

In tal caso h è detto **replicating portfolio**

Completezza del mercato Un mercato è detto **completo** se ogni contingent claim è raggiungibile Nel modello binomiale a un periodo, **assenza di arbitraggio implica completezza**

Pricing principle Se un claim X è replicabile da un portafoglio h, allora il suo **unico prezzo privo di arbitraggio** è

$$\Pi(t, X) = V^h(t), \qquad t = 0, 1$$

in particolare

$$\Pi(0, X) = V^h(0)$$

Pricing tramite sistema lineare Dato $X = \phi(Z)$, il portafoglio replicante $h = (x, y)$ risolve il sistema

$$\begin{cases} x(1+r) + ySu = \phi(u) \\ x(1+r) + ySd = \phi(d) \end{cases}$$

che ammette soluzione unica

$$x = \frac{1}{1+r}\frac{u\phi(d) - d\phi(u)}{u - d}, \qquad y = \frac{1}{S}\frac{\phi(u) - \phi(d)}{u - d}$$

Risk-neutral pricing dei derivati Nel modello binomiale arbitrage-free, il prezzo di un contingent claim è

$$\Pi(0, X) = \frac{1}{1+r}\mathbb{E}^Q[X]$$

ossia il **valore atteso scontato del payoff sotto la misura risk-neutral**

Call option Una call europea con strike K ha payoff

$$X = (S(1) - K)^+$$

Il suo prezzo è

$$C = \frac{1}{1+r}\Big(q_u(Su - K)^+ + q_d(Sd - K)^+\Big)$$

Conclusioni

- Le probabilità oggettive determinano solo quali eventi sono possibili
- I prezzi si calcolano come se il mondo fosse **risk-neutral**
- La valutazione è indipendente dalle preferenze individuali
- Il pricing è detto **preference-free valuation**
- **Replicating portfolio** e **risk-neutral valuation** sono metodi equivalenti