Financial Engineering

Springer Semester 2025

Lecturer: Patrizia Semeraro, Assistant: Tommaso Vanzan

Problem set 1

Topics: compounding and discounting, zero-coupon bonds, coupon bonds

---

Exercise 1 A sum of 9000\$ paid into a bank account for two months (61 days) to attract simple interest will produce 9020\$ at the end of the term. Find the simple interest rate and the linear return of the investment.

Exercise 2 How much would you pay today to receive 1000\$ at a certain future date if you require a linear return of 2%

Exercise 3 How long will it take for a sum of 800\$ attracting simple interest to become 830\$ if the rate is 9%? Compute the return on this investment.

Exercise 4 Find the principal to be deposited initially in an account attracting simple interest rate at a rate of 8% if 1000\$ is needed after three months (91 days).

Exercise 5 How long does it take to double a capital attracting interest at 6% compounded daily?

Exercise 6 What is the interest rate if a deposit subject to annual compounding is double after ten years?

Exercise 7 Find and compare the future value after two years of a deposit of 100\$ subjected to an interest at 10% compounded either annually or semi-annually.

Exercise 8 Consider a deposit of 1000\$ attracting an interest of 15% compounded daily or of 15.5% compounded semi-annually. Which options is more profitable?

Exercise 9 What is the initial investment subjected to an annual compounding interest of 12% needed to produce 1000\$ after two years?

Exercise 10 Find the present value of 10^5\$ to be received after 100 years if the interest rate is assumed to be constant at 5% for the whole period and daily or annual compounding applies.

Exercise 11 In 1759, Arthur Guinness signed a deal to rent the land that hosts nowadays the factory of the Guinness beer in Dublin for the next nine thousand years. The deal consisted in an annual payment of 45£. Assuming an interest rate of 2% compounded annually, what would have been the initial investment to receive 45£ today?. Viceversa, what would be the value today of 45£ earned in 1759?

Exercise 12 Find the return over one year under monthly compounding with an interest of 10%.

Exercise 13 How long will it take to earn 1$ in interest if $10^6$$ is deposited at 10% compounded continously?

Exercise 14 In 1626 the governor of the colony of New Netherlands bought the island of Manhattan for 24$. Find the value of this sum today (2025) at 5% compounded continously and annually.

Exercise 15 What will be the difference between the value after one year of 100$ deposited at 10% compounded monthly and compounded continously? How frequent should the periodic compounding be for the difference to be less than 0.01$?

Exercise 16 Find the present valye a milion dollars to be receveived after 20 years assuming continuous compounding at 6%.

Exercise 17 Given that the future value of 950$ subjected to continuous compounding will be 1000$ after half a year, find the interest rate.

Exercise 18 Find the rate for continuous compounding equivalent to monthly compounding at 12%.

Exercise 19 Find the frequency of periodic compounding at 20% to be equivalent to annual compounding at 21%.

Exercise 20 An investor paid 95$ for a bond with face value 100$ maturing in six months. When will the bond value reach 99% if the interest rate remains constant?

Exercise 21 Find the interest rates for annual, semi-annual and continuous compounding implied by a unit bond with $B(0.5, 1) = 0.9455$.

Exercise 22 Find the price of a bond with face value 100$ and annual coupons 5$ that matures in four years, given that the continuous compounding rate is 8% or 5%/

Exercise 23 A bond with face value 100$ and annual coupons of 8$ maturing after three years is trading at par. Find the implied continuous compounding rate.

Exercise 24 Find the return on a 75-days investment in zero coupon if $B(0, 1) = 0.89$.

Exercise 25 The return on a bond over six months is 7%. Find the implied continuous compounding rate.

Exercise 26 After how many days will a bond purchased for $B(0, 1) = 0.92$ produce a return of 5%?

Issuer	Face Value	Price	Coupon	Frequency
Italy	100	101.86	3.8	Semi-Annual
France	100	101.63	3.5	Annual
Germany	100	99.05	0.1	Annual

Table 1: List of bonds

Exercise 27 On the first of December 2024, the bonds reported in Table 1 are quoted with maturity on April 26, 2026.

- Compute the linear return of the investment considering the undiscounted sum of cash flows.
- **Numerical** Compute the yield to maturity of each bond using the routine `fsolve` in Matlab (or any equivalent routine in another language). For simplicity assume that the maturity is 31 May, 2026. Which one would you buy, assuming all Issuers have the same financial solidity?

Financial Engineering

Springer Semester 2025

Lecturer: Patrizia Semeraro, Assistant: Tommaso Vanzan

Problem set 1

Topics: compounding and discounting, zero-coupon bonds, coupon bonds

---

Solution 1 In simple compounding we have:

$$P(1 + r\,t) = V$$

Substituting the given values $t = 61$ days , $P = 9000\,\$$ and $V = 9020\,\$$:

$$9000\left(1 + r\,\frac{61}{365}\right) = 9020$$

where $\frac{61}{365}$ is the conversion into years of 61 days, which is needed since the given interest rate r is annual. Otherwise, we could have used 61 days and r converted in a daily interest rate. Finally, solving for the simple interest rate r:

$$1 + r\,\frac{61}{365} = \frac{9020}{9000}$$

$$r = \left(\frac{9020}{9000} - 1\right)\frac{365}{61} = 0.0133 = 1.33\%$$

The linear return of the investment is given by:

$$R(t_1, t_2) = \frac{V(t_2) - V(t_1)}{V(t_1)}$$

Considering the instants $t_1 = 0$ and $t_2 = \frac{61}{365}$, we obtain:

$$V(0) = P = 9000$$

$$V\left(\frac{61}{365}\right) = V = 9020$$

Hence:

$$R\left(0, \frac{61}{365}\right) = \frac{9020 - 9000}{9000} = 0.0022 = 0.22\%$$

Solution 2 The linear return of the investment is given by:

$$R(t_1, t_2) = \frac{V(t_2) - V(t_1)}{V(t_1)}$$

Denoting the amount to be paid today by P, we have $V(t_1) = P$, $V(t_2) = 1000\,\$$ and $R(t_1, t_2) = 2\%$. Therefore:

$$0.02 = \frac{1000 - P}{P}$$

$$0.02P = 1000 - P$$

$$1.02P = 1000$$

$$P = \frac{1000}{1.02} = 980.39\,\$$$

Solution 3 In simple compounding we have:

$$P(1 + r\,t) = V$$

Substituting the given values $P = 800\,\$$, $V = 830\,\$$ and $r = 9\%$:

$$800\,(1 + 0.09\,t) = 830$$

Finally, solving for the lifetime of the investment t:

$$(1 + 0.09\,t) = \frac{830}{800}$$
$$t = \left(\frac{830}{800} - 1\right)\frac{1}{0.09} = 0.4167 \text{ years}$$

As long as the interest rate is annual, t is expressed in years. For example, converting t in days we obtain:

$$t = 0.4167 \text{ years } = 0.4167 \cdot 365 \text{ days } = 152 \text{ days}$$

The linear return of the investment is given by:

$$R(t_1, t_2) = \frac{V(t_2) - V(t_1)}{V(t_1)}$$

Considering the instants $t_1 = 0$ and $t_2 = 0.4167$, we obtain:

$$V(t_1) = 800$$
$$V(t_2) = 830$$

Hence:

$$R\,(0, 0.4167) = \frac{830 - 800}{800} = 0.0375 = 3.75\%$$

Solution 4 In simple compounding we have:

$$P(1 + r\,t) = V$$

Substituting the given values $r = 8\%$, $V = 1000\,\$$, $t = 91$ days :

$$P\left(1 + 0.08\,\frac{91}{365}\right) = 1000$$

Finally, solving for the principal P:

$$P = \frac{1000}{\left(1 + 0.08\,\frac{91}{365}\right)} = 980.44\,\$$$

Solution 5 In periodic, or compounded, compounding we have:

$$P\left(1+\frac{r}{m}\right)^{mt} = V$$

In this exercise we have a daily compounding, $m = 365$, an interest rate $r = 6\%$ and it is required to double the initial principal P, i.e. $V = 2P$:

$$\begin{aligned}
P\left(1+\frac{0.06}{365}\right)^{365\,t} &= 2P \\
\left(1+\frac{0.06}{365}\right)^{365\,t} &= 2 \\
365\,t\,\ln\left(1+\frac{0.06}{365}\right) &= \ln 2 \\
t = \frac{\ln 2}{365\ln\left(1+\frac{0.06}{365}\right)} &= 11.5534 \text{ years}
\end{aligned}$$

As long as the interest rate is annual, t is expressed in years. For example, converting t in days we obtain:

$$t = 11.5534 \text{ years } = 11.5534 \cdot 365 \text{ days } = 202 \text{ days}$$

Solution 6 In periodic, or compounded, compounding we have:

$$P\left(1+\frac{r}{m}\right)^{mt} = V$$

In this exercise we have an annual compounding, $m = 1$ and it is required to double the initial principal P, i.e. $V = 2P$ after the time $t = 10$ years :

$$\begin{aligned}
P\,(1+r)^{10} &= 2P \\
(1+r)^{10} &= 2 \\
10\,\ln(1+r) &= \ln 2 \\
1+r &= e^{\frac{\ln 2}{10}} \\
r = e^{\frac{\ln 2}{10}} - 1 &= 0.0718 = 7.18\%
\end{aligned}$$

Solution 7 In periodic, or compounded, compounding we have:

$$P\left(1+\frac{r}{m}\right)^{mt} = V$$

Substituting the given values $t = 2$ years , $P = 100\,\$$ and $r = 10\%$ we have two situations:

- Annual compounding, i.e. $m = 1$:

$$100(1+0.1)^2 = 121\,\$$$

- Semi-annual compounding, i.e. $m = 2$:

$$100\left(1+\frac{0.1}{2}\right)^{2\cdot 2} = 121.55\,\$$$

Solution 8 In periodic, or compounded, compounding we have:

$$P\left(1+\frac{r}{m}\right)^{mt}=V$$

Let us compare the two options after $t=1$ years :

a) The first option has $P=1000\,\$$, $r=15\%$ and $m=365$:

$$1000\left(1+\frac{0.15}{365}\right)^{365\cdot 1}=1161.80\,\$$$

b) The second option has $P=1000\,\$$, $r=15.5\%$ and $m=2$:

$$1000\left(1+\frac{0.155}{2}\right)^{2\cdot 1}=1161.01\,\$$$

Solution 9 In periodic, or compounded, compounding we have:

$$P\left(1+\frac{r}{m}\right)^{mt}=V$$

Substituting the given values $r=12\%$, $V=1000\,\$$, $t=2$ years and $m=1$:

$$P\,(1+0.12)^{2}=1000$$

Solving for the principal P we obtain:

$$P=\frac{1000}{(1+0.12)^{2}}=797.19\,\$$$

Solution 10 In periodic, or compounded, compounding we have:

$$P\left(1+\frac{r}{m}\right)^{mt}=V$$

Substituting the given values $V=10^{5}\,\$$, $t=100$ years and $r=5\%$ we have:

- Daily compounding option:

$$P\left(1+\frac{0.05}{365}\right)^{365\cdot 100}=10^{5}$$
$$P=10^{5}\left(1+\frac{0.05}{365}\right)^{-365\cdot 100}=674.03\$$$

- Annual compounding option:

$$P\,(1+0.05)^{100}=10^{5}$$
$$P=10^{5}\,(1+0.05)^{-100}=760.45\$$$

Solution 11 The linear return of the investment is given by:

$$R(t_1, t_2) = \frac{V(t_2) - V(t_1)}{V(t_1)}$$

Considering one year under monthly compounding with an interest of 10% we obtain:

$$\begin{aligned} R(0,1) &= \frac{V(1) - V(0)}{V(0)} = \\ &= \frac{P\left(1 + \frac{0.1}{12}\right)^{12 \cdot 1} - P}{P} = \\ &= \left(1 + \frac{0.1}{12}\right)^{12 \cdot 1} - 1 = 1.1047 - 1 = 10.47\% \end{aligned}$$

Solution 12 In continuous compounding we have:

$$Pe^{r\,t} = V$$

In general, the earning in interest I is given by:

$$I = V - P$$

Substituting the given values $I = 1\,\$$, $P = 10^6\$$ and $r = 10\%$:

$$1 = 10^6 e^{0.1t} - 10^6$$

Solving by the maturity t we obtain:

$$\begin{aligned} \frac{1 + 10^6}{10^6} &= e^{0.1t} \\ \ln \frac{1 + 10^6}{10^6} &= 0.1t \\ t = \frac{1}{0.1} \ln \frac{1 + 10^6}{10^6} &= 0.00001 \text{ years} \end{aligned}$$

Solution 13 In continuous compounding we have:

$$Pe^{r\,t} = V$$

Substituting the given values $P = 24\,\$$, $t = 2025 - 1626 = 399$ years and $r = 5\%$:

$$V = 24e^{0.05 \cdot 399} = 1.10 \cdot 10^{10}\,\$$$

In periodic, or compounded, compounding we have:

$$P\left(1 + \frac{r}{m}\right)^{mt} = V$$

Considering annual compounding, i.e. $m = 1$, and substituting the given values $P = 24\,\$$, $t = 2025 - 1626 = 399$ years and $r = 5\%$:

$$V = 24\,(1 + 0.05)^{399} = 6.83 \cdot 10^9\,\$$$

Solution 14 The value after one year of 100$ deposited at 10% compounded monthly is given by:

$$V_1 = 100\left(1 + \frac{0.1}{12}\right)^{12} = 110.47\,\$$$

The value after one year of 100$ deposited at 10% continuously compounded is given by:

$$V_2 = 100e^{0.1} = 110.52\,\$$$

The difference is given by $V_2 - V_1 = 110.52 - 110.47 = 0.05\,\$$. In order for difference to be less than $0.01\,\$$ the periodic compounding should be:

$$\begin{aligned}
100e^{0.1} - 100\left(1 + \frac{0.1}{m}\right)^m &< 0.01 \\
110.52 - 100\left(1 + \frac{0.1}{m}\right)^m &< 0.01 \\
100\left(1 + \frac{0.1}{m}\right)^m &> 110.52 - 0.01 \\
\left(1 + \frac{0.1}{m}\right)^m &> \frac{110.52 - 0.01}{100} \\
\left(1 + \frac{0.1}{m}\right)^m &> 1.1051 \\
m &\approx 78
\end{aligned}$$

Solution 15 In continuous compounding we have:

$$Pe^{r\,t} = V$$

Substituting the given values $V = 1\,000\,000\,\$$, $t = 20$ years and $r = 6\%$:

$$Pe^{0.06\cdot 20} = 1\,000\,000$$

Solving for the principal P:

$$P = 1\,000\,000e^{-0.06\cdot 20} = 301\,194.21\,\$$$

Solution 16 In continuous compounding we have:

$$Pe^{r\,t} = V$$

Substituting the given values $P = 950$, $V = 1000\$$ and $t = 0.5$ years:

$$950e^{r\cdot 0.5} = 1000$$

Solving for the interest rate r:

$$\begin{aligned}
e^{r\cdot 0.5} &= \frac{1000}{950} \\
0.5\,r &= \ln\frac{1000}{950} \\
r &= 2\ln\frac{1000}{950} = 0.1026 = 10.26\%
\end{aligned}$$

Solution 17 In continuous compounding with rate r we have:

$$Pe^{r\,t} = V$$

while in periodic, or compounded, compounding with rate r' we have:

$$P\left(1+\frac{r'}{m}\right)^{mt} = V$$

the rate r for continuous compounding equivalent to monthly compounding, i.e. $m = 12$, at $r' = 12\%$ satisfies the equivalence:

$$Pe^{r\,t} = P\left(1+\frac{0.12}{12}\right)^{12t}$$

Solving for r:

$$e^{r\,t} = \left(1+\frac{0.12}{12}\right)^{12t}$$
$$r\,t = 12\,t\,\ln\left(1+\frac{0.12}{12}\right)$$
$$r = 12\,\ln\left(1+\frac{0.12}{12}\right) = 0.1194 = 11.94\%$$

Solution 18 The value m in order for periodic compounding at 20% to be equivalent to annual compounding at 21% should be such that:

$$P\left(1+\frac{0.2}{m}\right)^{mt} = P\,(1+0.21)^{t}$$

Therefore, solving for m we obtain:

$$\left(1+\frac{0.2}{m}\right)^{mt} = (1+0.21)^{t}$$
$$\left(1+\frac{0.2}{m}\right)^{m} = (1+0.21) = 1.21$$
$$m = 2$$

Solution 19 Assuming annual compounding, the value of a bond with face value F and maturity T at time t is given by:

$$B(t,T) = F(1+r)^{-(T-t)}$$

Substituting the given values, the bond's value reaches 99$ when:

$$B(t,T) = 99 \iff 100(1+r)^{-\left(\frac{1}{2}-t\right)} = 99$$

The interest rate r is not explicitly given, but since it remains constant, it can be inferred from the initial price paid by the investor. By definition, we have:

$$B(0,T) = 95$$

Thus:

$$\begin{aligned}
100(1+r)^{-\frac{1}{2}} &= 95 \\
\frac{100}{95} &= (1+r)^{\frac{1}{2}} \\
\left(\frac{100}{95}\right)^2 &= 1+r \\
r = \left(\frac{100}{95}\right)^2 - 1 &= 0.1080 = 10.80\%
\end{aligned}$$

Now, we can determine the time t at which $B(t,T) = 99$:

$$\begin{aligned}
100(1+0.1080)^{-\left(\frac{1}{2}-t\right)} &= 99 \\
(1.1080)^{-\left(\frac{1}{2}-t\right)} &= \frac{99}{100} \\
-\left(\frac{1}{2}-t\right)\ln 1.1080 &= \ln\frac{99}{100} \\
t = \frac{\ln\frac{99}{100}}{\ln 1.1080} + \frac{1}{2} &= 0.402 \text{ years } = 147 \text{ days}
\end{aligned}$$

The same result could be obtained by considering the bond value $B(t,T)$ as the price compounded at t instead of the face value discounted at t:

$$B(t,T) = P(1+r)^t$$

Solution 20 The interest rate implied by a unit bond, i.e. $F = 1$, with $B(0.5,1) = 0.9455$ is given by:

- In case of annual compounding we have

$$B(t,T) = (1+r)^{-(T-t)}$$

thus:

$$\begin{aligned}
B(0.5,1) &= (1+r)^{-(1-0.5)} \\
0.9455 &= (1+r)^{-0.5} \\
r = \left(\frac{1}{0.9455}\right)^{0.5} - 1 &= 0.1186 = 11.86\%
\end{aligned}$$

- In case of semi-annual compounding we have

$$B(t,T) = (1+\frac{r}{2})^{-2(T-t)}$$

thus:

$$\begin{aligned}
B(0.5,1) &= \left(1+\frac{r}{2}\right)^{-2(1-0.5)} \\
0.9455 &= \left(1+\frac{r}{2}\right)^{-1} \\
r = 2\left(\frac{1}{0.9455} - 1\right) &= 0.1153 = 11.53\%
\end{aligned}$$

- In case of continuous compounding we have

$$B(t,T) = e^{-r(T-t)}$$

thus:

$$B(0.5,1) = e^{-r(1-0.5)}$$
$$0.9455 = e^{-\frac{r}{2}}$$
$$r = -2\ln 0.9455 = 0.1121 = 11.21\%$$

Solution 21 In the case of continuous compounding with rate r, the price of a bond with face value 100$ and annual coupons 5$ that matures in four years is given by:

$$V = 5e^{-r} + 5e^{-2r} + 5e^{-3r} + 105e^{-4r}$$

Thus we obtain:

$$r = 8\% \implies V = 5e^{-0.08} + 5e^{-2 \cdot 0.08} + 5e^{-3 \cdot 0.08} + 105e^{-4 \cdot 0.08} = 89.06\,\$$$
$$r = 5\% \implies V = 5e^{-0.05} + 5e^{-2 \cdot 0.05} + 5e^{-3 \cdot 0.05} + 105e^{-4 \cdot 0.05} = 99.55\,\$$$

Solution 22 A bond is traded at par when its price is equal to its face value. Therefore a bond with face value 100$ and annual coupons of 8$ maturing after three years is traded at par when:

$$100 = 8e^{-r} + 8e^{-2r} + 108e^{-3r}$$

Solving the equation by the interest rate r we find the implied continuous compounding rate:

$$r = 7.70\%$$

Solution 23 Firstly, we have to find the effective rate implied by the bond assuming annual compounding:

$$B(0,1)(1+r) = 1$$
$$0.89(1+r) = 1$$
$$r = \frac{1}{0.89} - 1 = 0.1236 = 12.36\%$$

The price of the bond after 75 days is given by:

$$B\left(\frac{75}{365}, 1\right) = B(0,1)(1+r)^{\frac{75}{365}} = 0.89(1+0.1236)^{\frac{75}{365}} = 0.9115$$

Finally the return is:

$$R\left(0, \frac{75}{365}\right) = \frac{B\left(\frac{75}{365}, 1\right) - B(0,1)}{B(0,1)} = \frac{0.9115 - 0.89}{0.89} = 0.0242 = 2.42\%$$

Solution 24 The initial price of a 6 months unit bond in continuous compounding is given by:

$$B\left(0, \frac{1}{2}\right) = e^{-\frac{1}{2}r}$$

where r is the implied continuous compounding rate. The return of the bond over six months is:

$$R\left(0, \frac{1}{2}\right) = \frac{B\left(\frac{1}{2}, \frac{1}{2}\right) - B\left(0, \frac{1}{2}\right)}{B\left(0, \frac{1}{2}\right)} = \frac{1 - e^{-\frac{1}{2}r}}{e^{-\frac{1}{2}r}}$$

Finally, if the return is 7%, we can find the implied continuous compounding rate solving the equation:

$$\frac{1 - e^{-\frac{1}{2}r}}{e^{-\frac{1}{2}r}} = 0.07$$
$$1 - e^{-\frac{1}{2}r} = 0.07e^{-\frac{1}{2}r}$$
$$1.07e^{-\frac{1}{2}r} = 1$$
$$e^{\frac{1}{2}r} = 1.07$$
$$r = 2\ln 1.07 = 0.1353 = 13.53\%$$

Solution 25 Let us consider a unit zero coupon bond ans, assuming continuous compounding, we have:

$$B(0,1)e^{r} = 1$$

Substituting the given values, we obtain the implied continuous compounding rate solving the equation:

$$0.92e^{r} = 1$$
$$r = -\ln 0.92 = 0.0834 = 8.34\%$$

The value of the bond at time t is:

$$B(t,1) = B(0,1)e^{rt} = 0.92e^{0.0834t}$$

and it will produce a 5% return at a time t given by:

$$\frac{B(t,1) - B(0,1)}{B(0,1)} = 0.05$$
$$\frac{0.92e^{0.0834t} - 0.92}{0.92} = 0.05$$
$$e^{0.0834t} = 1.05$$
$$0.084t = \ln 1.05$$
$$t = \frac{\ln 1.05}{0.0834} = 0.5850 \text{ years} = 214 \text{ days}$$

Solution 26 Assuming an interest rate of 2% compounded annually, the initial investment in 1759 to receive 45£ today, in 2025, would have been:

$$P(1 + 0.02)^{2025-1759} = 45$$
$$P = 45(1 + 0.02)^{-266} = 0.23£$$

Viceversa, the value today, in 2025, of 45£ earned in 1759 would be:

$$V = 45(1 + 0.02)^{2025-1759} = 45(1 + 0.02)^{266} = 8726.86£$$

Solution 27

- To compute the linear return of the investments, we consider the cash flows and the standard formula, obtaining,

$$
\begin{aligned}
LR_{\mathrm{IT}} &= \frac{1.9 + 1.9 + 101.9 - 101.86}{101.86} = 0.0377 = 3.8\%, \\
LR_{\mathrm{FR}} &= \frac{1.75 + 103.5 - 101.63}{101.63} = 0.0356 = 3.6\%, \\
LR_{\mathrm{GR}} &= \frac{0.05 + 100.1 - 99.05}{99.05} = 0.0111 = 1.1\%.
\end{aligned} \tag{1}
$$

- See Matlab file `Compute_Yield_to_Maturity.m`.

Financial Engineering

Springer Semester 2025

Lecturer: Patrizia Semeraro, Assistant: Tommaso Vanzan

Problem set 1

Topics: compounding and discounting, zero-coupon bonds, coupon bonds

Exercise 1 A sum of 9000\$ paid into a bank account for two months (61 days) to attract simple interest will produce 9020\$ at the end of the term. Find the simple interest rate and the linear return of the investment.

- **Simple interest rate**: is the rate r used in the simple compounding model, where interest grows linearly in time

principal or initial value

$$\boxed{V(t) = V(0)\cdot(1+rt)} \leftarrow t := \text{time expressed in year} \longrightarrow \boxed{V(t) = V(0)\left(1+\frac{r}{12}\cdot 12t\right)}$$

annual simple rate — monthly rate — time expressed in mounth

- **Linear return**: it measured the percentage gain of the investment over the considered time period, it depends only on the initial & final value, not on how the interest is modelled

$$\boxed{LR(s,t) = \frac{V(t)-V(s)}{V(s)}}$$

Hence, in simple compounding, we have: $V(t) = V(0) + V(0)rt$, $\frac{V(t)-V(0)}{V(0)\cdot t} = r \longrightarrow r = \frac{9020-9000}{9000\cdot(\underbrace{61/365}_{\text{conversion in year of 61 days}})} = 1.33\%$ annual

The linear return of the investment is $LR\left(0,\frac{61}{365}\right) = \frac{9020-9000}{9000} = 0.22\%$

Exercise 2 How much would you pay today to receive 1000\$ at a certain future date if you require a linear return of 2%

As before, the linear return is $LR(s,t) = \frac{V(t)-V(s)}{V(s)} \longrightarrow V(s) = \frac{V(t)}{1+LR(s,t)} = \frac{1000}{1+0.02} = 980{,}39$ \$

Exercise 3 How long will it take for a sum of 800\$ attracting simple interest to become 830\$ if the rate is 9%? Compute the return on this investment.

In simple compounding, we have $V(t) = V(0)(1+rt) \longrightarrow t = \frac{V(t)-V(0)}{V(0)\cdot r} = \frac{830-800}{800\cdot 0.09} = \frac{5}{12} = 0.4167$ years $\cdot 365 = 152$ days

Moreover, the linear return of the investment is $LR(s,t) = \frac{V(t)-V(s)}{V(s)} = \frac{830-800}{800} = 3.75\%$

Exercise 4 Find the principal to be deposited initially in an account attracting simple interest rate at a rate of 8% if 1000\$ is needed after three months (91 days).

In simple compounding, $V(t) = V(0)\cdot(1+rt) \longrightarrow V(0) = \frac{V(t)}{1+rt} = \frac{1000}{1+0.08\cdot 91/365} = 980{,}44$ \$

Exercise 5 How long does it take to double a capital attracting interest at 6% compounded daily?

• Periodic Compound: interest is credited at discrete, equally spaced times and each period's interest is added to the principal so that future interest is earned also on past interest

principal, annual nominal interest rate

$$\boxed{V(t) = V(0)\left(1+\frac{r}{m}\right)^{mt}}$$

time expressed in years

number of compounding periods per year $m=1$ annual, $m=2$ semi-annual, $m=12$ monthly, $m=365$ daily

Hence, we have $2V(0) = V(0)\left(1+\frac{0.06}{365}\right)^{365t} \longrightarrow \ln(2) = 365t \ln\left(1+\frac{0.06}{365}\right)$ $t = \frac{\ln(2)}{365} \cdot \frac{1}{\ln\left(1+\frac{0.06}{365}\right)} = 11{,}5534$ years

Exercise 6 What is the interest rate if a deposit subject to annual compounding is double after ten years?

In periodic, or compounded, compounding we have

$V(t) = V(0)\left(1+\frac{r}{m}\right)^{mt} \Rightarrow 2V(0) = V(0)(1+r)^{10}$ $2 = (1+r)^{10}$ $\sqrt[10]{2} - 1 = r$ $r = 7{,}18\%$

Exercise 7 Find and compare the future value after two years of a deposit of 100$ subjected to an interest at 10% compounded either annually or semi-annually.

I. Annually: $V(t) = V(0)\left(1+\frac{r}{m}\right)^{mt}$, $V(0)=100$ $r=0.1$ $t=2$ $m=1$ $V(t) = 100(1+0.1)^2 = 121$ \$

II. Semi-annually: $V(0)=100$, $r=0.1$, $m=2$, $t=2$ $V(t) = 100\left(1+\frac{0.1}{2}\right)^{2\cdot 2} = 121.55$ \$

Exercise 8 Consider a deposit of 1000$ attracting an interest of 15% compounded daily or of 15.5% compounded semi-annually. Which options is more profitable?

I. Compounded daily: $V(t) = V(0)\left(1+\frac{r}{m}\right)^{mt}$, $V(0)=1000$, $r=0.15$, $m=365$, $t=1$ year $\longrightarrow V(1) = 1000\left(1+\frac{0.15}{365}\right)^{365} = 1161{,}8$ \$

II. Compounded semi-annually: $V(0)=1000$, $r=0.155$, $m=2$, $t=1$ $\longrightarrow V(1) = 1000\left(1+\frac{0.155}{2}\right)^2 = 1161{,}01$ \$

Exercise 9 What is the initial investment subjected to an annual compounding interest of 12% needed to produce 1000$ after two years?

As before, we have $V(t) = V(0)\left(1+\frac{r}{m}\right)^{mt}$ $V(0)=?$ $m=1$ $r=0.12$ $V(t)=1000$ $t=2$

$\longrightarrow V(0) = \frac{V(t)}{\left(1+\frac{r}{m}\right)^{mt}} = \frac{1000}{\left(1+\frac{0.12}{1}\right)^2} = 797.19$ \$

Exercise 10 Find the present value of 10^5\$ to be received after 100 years if the interest rate is assumed to be constant at 5% for the whole period and daily or annual compounding applies.

In periodi compounding $\longrightarrow$ I. Daily: $V(0) = \frac{V(t)}{\left(1+\frac{r}{m}\right)^{mt}} = \frac{10^5}{\left(1+\frac{0.05}{365}\right)^{365\cdot 100}} = 674.025$ \$ II. Annual $V(0) = \frac{10^5}{(1+0.05)^{100}} = 760.449$ \$

Exercise 11 In 1759, Arthur Guinness signed a deal to rent the land that hosts nowadays the factory of the Guinness beer in Dublin for the next nine thousand years. The deal consisted in an annual payment of 45£. Assuming an interest rate of 2% compounded annually, what would have been the initial investment to receive 45£ today?. Viceversa, what would be the value today of 45£ earned in 1759?

(Sol 26) Considering periodic compound, we have $V(t) = V(0)\left(1+\frac{r}{m}\right)^{mt}$ $r = 0.02$, $m = 1$, $t = 2025 - 1759 = 266$ years

I. $V(t) = 45$ $\quad V(0) = \dfrac{V(t)}{(1+\frac{r}{m})^{mt}} = \dfrac{45}{(1+0.02)^{266}} = 0.23$ £

II. $V(0) = 45$ $\quad V(t) = 45\left(1+\frac{0.02}{1}\right)^{266} = 8726.86$ £

Exercise 12 Find the return over one year under monthly compounding with an interest of 10%.

(Sol 11) As before, the linear return is $LR(s,t) = \dfrac{V(t)-V(s)}{V(s)}$, $V(t) = V(s)\left(1+\frac{0.1}{12}\right)^{12\cdot 1}$

$\longrightarrow LR(0,1) = \dfrac{V(0)\left(1+\frac{0.1}{12}\right)^{12} - V(0)}{V(0)} = \left(1+\frac{0.1}{12}\right)^{12} - 1 = 10.47\%$

Exercise 13 How long will it take to earn 1$ in interest if $10^6$$ is deposited at 10% compounded continously?

• Continuos Compounding: interest is accrued continuously over time, rather than at discrete compounding dates, the growth of the investment is smooth and exponential

↓

$\boxed{V(t) = V(0)\, e^{rt}}$ — r: continuously compounded annual interest rate; t: time expressed in year; V(0): principal

$\longrightarrow \boxed{\lim_{m\to\infty} V(0)\left(1+\frac{r}{m}\right)^{mt} = V(0)\, e^{rt}}$

(Sol 12) $V(t) = V(0)\, e^{rt} \longrightarrow \ln(V(t)) = \boxed{\ln(V(0)\, e^{rt}) \overset{\text{logaritmic property}}{=} \ln(V_0) + \ln(e^{rt})} \longrightarrow \dfrac{\ln(V(t)) - \ln(V(0))}{r} = t$, $t = \dfrac{\ln(10^6+1) - \ln(10^6)}{0.1} = 0.1 \cdot 10^{-5}$

Exercise 14 In 1626 the governor of the colony of New Netherlands bought the island of Manhattan for 24$. Find the value of this sum today (2025) at 5% compounded continously and annually.

(Sol 13) I. Compounded continously: $V(t) = V(0)\, e^{rt} = 24 \cdot e^{0.05\cdot(2025-1626)} = 1.1 \cdot 10^{10}$

II. Compounded annually: $V(t) = V(0)\left(1+\frac{r}{m}\right)^{mt} = 24\left(1+0.05\right)^{(2025-1626)} = 6.8 \cdot 10^{9}$

Exercise 15 What will be the difference between the value after one year of 100$ deposited at 10% compounded monthly and compounded continously? How frequent should the periodic compounding be for the difference to be less than 0.01$?

(Sol 14) I. Monthly: $V(t) = V(0)\left(1+\frac{r}{m}\right)^{mt}$ $\quad t = 1$ year $\quad V(0) = 100$ $\quad r = 0.1$ $\quad m = 12$ $\quad V_m(t) = 100\left(1+\frac{0.1}{12}\right)^{12} = 110.471$ \$

II. Continously: $V(t) = V(0)\, e^{rt}$ $\quad V_c(t) = 100\, e^{0.1} = 110.517$ \$ $\quad V_c(t) - V_m(t) = 0.0457$

$V(0)e^{rt} - V(0)\left(1+\frac{r}{m}\right)^{mt} < 0.01$ $\quad e^{r} - \left(1+\frac{r}{m}\right)^{m} < 10^{-4}$ $\quad -\left(1+\frac{r}{m}\right)^{m} < 10^{-4} - e^{0.1}$ $\quad \left(1+\frac{r}{m}\right)^{m} > 1.1051 \xrightarrow{\text{per tentativi}} m \simeq 78$

Exercise 16 Find the present valye a milion dollars to be receveived after 20 years assuming continuous compounding at 6%.

(Sol 15) As before, $V(t) = V(0)e^{rt}$, $V(t) = 1\,000{,}000$ $t = 20$ years $r = 0.06$ $V(0) = \frac{V(t)}{e^{rt}} = 301194.21$ \$

Exercise 17 Given that the future value of 950\$ subjected to continuous compounding will be 1000\$ after half a year, find the interest rate.

(Sol 16) Continuous compound: $V(t) = V(0)e^{rt}$ $\ln(V(t)) = \ln(V(0)e^{rt}) = \ln(V(0)) + \ln(e^{rt}) = \ln(V(0)) + rt$

$\longrightarrow r = \frac{\ln(V(t)) - \ln(V(0))}{t} = \frac{\ln(1000) - \ln(950)}{0.5} = 10.26\,\%$

Exercise 18 Find the rate for continuous compounding equivalent to monthly compounding at 12%.

(Sol 17) $V(t) = V(0)e^{r}$ vs $V(t) = V(0)\left(1 + \frac{r}{12}\right)^{12}$ considering $t = 1$ year

$V(0)e^{r} = V(0)\left(1 + \frac{0.12}{12}\right)^{12}$ $\ln(e^{r}) = \ln\left(\left(1 + \frac{0.12}{12}\right)^{12}\right)$ $r = 11.94\,\%$

Exercise 19 Find the frequency of periodic compounding at 20% to be equivalent to annual compounding at 21%.

(Sol 18) $V(0)\left(1 + \frac{r}{m}\right)^{mt} = V(0)\left(1 + \frac{r}{m}\right)^{mt}$ $t = 1$ year $(1 + 0.21) = \left(1 + \frac{0.2}{m}\right)^{m} \longrightarrow 1.21 = \left(1 + \frac{0.2}{m}\right)^{m}$ $\ln(1.21) = m\ln\left(1 + \frac{0.2}{m}\right) \longrightarrow m = 2$

$m = 1,\ r = 0.21$ $m = ?\ r = 0.2$

Exercise 20 An investor paid 95\$ for a bond with face value 100\$ maturing in six months. When will the bond value reach 99% if the interest rate remains constant?

- Zero copon bonds: is a bond that pays no coupons during its life and pays a single amount the face value F at maturity T

The cash flows are: $\begin{cases} \text{at time } t: & -B(t,T) \\ \text{at maturity } T: & +F \end{cases}$

i. With continuous compounding the price of the bond at time $t \le T$ is $\boxed{B(t,T) = F e^{-r(T-t)}}$ face value; $T-t :=$ time to maturity; continuous interest rate

ii. With periodic compounding the price is $\boxed{B(t,T) = F(1+r)^{-(T-t)}}$ Face Value; $T :=$ maturity; interest rate

(Sol 19) $F = 100$, $T = \frac{1}{2}$ year, $t = ?$ $r =$ constant $B(t,T) = 99$

$B(0,\frac{1}{2}) = 95$ $B(\frac{1}{2},\frac{1}{2}) = 100$ $B(t,\frac{1}{2}) = 100\,(1+r)^{-(0.5-t)} = 99 \longrightarrow 100\,(1 + 0.108)^{t - \frac{1}{2}} = 99$ $t = 0.402$ years $= 147$ days

$B(0,\frac{1}{2}) = 95 = 100\,(1+r)^{-(0.5-0)} \longrightarrow \frac{400}{361} = (1+r)$ $r = 10.8\,\%$

Exercise 21 Find the interest rates for annual, semi-annual and continuous compounding implied by a unit bond with $B(0.5, 1) = 0.9455$.

(Sol 20) $\boxed{\text{Unit bond} \Rightarrow F = 1}$

i. Annual Compounding $\boxed{B(t,T) = F(1+r)^{-(T-t)}}$ $\longrightarrow$ $B(0.5,1) = 0.9455 = 1\cdot(1+r)^{-(1-0.5)}$ $\quad 1+r = 1.1186 \quad r = 11.86\,\%$

ii. Semi-Annual Compounding $\boxed{B(t,T) = F\left(1+\frac{r}{2}\right)^{-2(T-t)}}$ $\longrightarrow$ $B(0.5,1) = 0.9455 = \left(1+\frac{r}{2}\right)^{-2(1-0.5)}$ $\quad r = 11.53\,\%$

iii. Continuous Compounding $\boxed{B(t,T) = Fe^{-r(T-t)}}$ $\longrightarrow$ $B(0.5,1) = 0.9455 = e^{-r(1-0.5)}$ $\quad -\frac{r}{2} = \ln(0.9455) \quad r = 11.208\,\%$

Exercise 22 Find the price of a bond with face value 100$ and annual coupons 5$ that matures in four years, given that the continuous compounding rate is 8% or 5%/

• General coupon bond pricing — continuous compounding: the price of the bond at time 0 is $\boxed{V(0,T) = \sum_{i=1}^{n-1} C_i e^{-rt_i} + (F+C_n)e^{-rT}}$

coupon payment, rate, Face Value, coupon dates, maturity

(Sol 21) $F = 100 \quad T = 4$ years

i. $r = 0.08 \quad V(0,T) = 5e^{-0.08\cdot 1} + 5e^{-0.08\cdot 2} + 5e^{-0.08\cdot 3} + (100+5)e^{-0.08\cdot 4} = 89.06\ \$$

ii. $r = 0.05 \quad V(0,T) = 5e^{-0.05\cdot 1} + 5e^{-0.05\cdot 2} + 5e^{-0.05\cdot 3} + (100+5)e^{-0.05\cdot 4} = 99.55\ \$$

Exercise 23 A bond with face value 100$ and annual coupons of 8$ maturing after three years is trading at par. Find the implied continuous compounding rate.

• Trading at par ≡ the value of the face value is equal to the market price of the bond $\boxed{\text{Bond price} = \text{Face Value}}$

(Sol 22) As before, $F = 100$, $T = 3$ years, $C_i = 8\$$ annually $\quad V(0,T) = C_1e^{-r} + C_2e^{-r2} + (C_3+F)e^{-r\cdot 3} = 100 = 8e^{-r} + 8e^{-2r} + 108e^{-3r}$

$\longrightarrow e^{-r} = x \quad 108x^3 + 8x^2 + 8x - 100 = 0 \quad x_1 = 0.9259 \quad x_2, x_3 \in \mathbb{C}$, hence $\quad r = -\ln(x) \quad r = -\ln(0.9259) = 7.7\,\%$

Exercise 24 Find the return on a 75-days investment in zero coupon if $B(0, 1) = 0.89$.

(Sol 23) Considering annual compounding $B(t,T) = F(1+r)^{-(1-t)} \longrightarrow$ with $F=1$ $\quad B(0,1) = 0.89 = 1\cdot(1+r)^{-(1-0)} = \frac{1}{0.89} - 1 = r \quad r = 12.36\,\%$

Hence, $t = \frac{75}{365} = 0.21$ years $\quad B(0.21,1) = 1(1+0.1236)^{-(1-0.21)} = 0.9120$

The linear return is $\boxed{LR(0,0.21) = \frac{B(0.21,1) - B(0,1)}{B(0,1)}} = \frac{0.9120 - 0.89}{0.89} = 2.47\,\%$ $\quad$ (NB Approx too much $75/365$!!)

Exercise 25 The return on a bond over six months is 7%. Find the implied continuous compounding rate.

(Sol 24) As before, $LR(0,0.5) = \frac{B(0.5,0.5) - B(0,0.5)}{B(0,0.5)} = 0.07 = \frac{Fe^{-r(0.5-0.5)} - Fe^{-r(0.5-0)}}{Fe^{-r(0.5-0)}} = \frac{1 - e^{-r/2}}{e^{-r/2}} \quad 0.07e^{-r/2} + e^{-r/2} = 1$

$\longrightarrow e^{-r/2} = 0.9346 \quad -\frac{r}{2} = \ln(0.9346) \Rightarrow r = -2\ln(0.9346) = 13.53\,\%$

Exercise 26 After how many days will a bond purchased for $B(0,1) = 0.92$ produce a return of 5%?

(Sol 25) Let us consider a unit zero coupon bond ans, assuming continuous compounding: $B(0,1) = Fe^{-r(T-t)} = 0.92 = e^{-r}$

$\longrightarrow r = -\ln(0.92) = 8.34\%$

$LR(t,1) = \dfrac{B(t,1) - B(0,1)}{B(0,1)} = 0.05 \longrightarrow 0.05 = \dfrac{e^{-0.083(1-t)} - 0.92}{0.92}$ $\ln(0.966) = -0.0834 + 0.0834t$ $t = 0.58$ years

Issuer	Face Value	Price	Coupon	Frequency
Italy	100	101.86	3.8	Semi-Annual
France	100	101.63	3.5	Annual
Germany	100	99.05	0.1	Annual

Table 1: List of bonds

Exercise 27 On the first of December 2024, the bonds reported in Table 1 are quoted with maturity on April 26, 2026. $\longrightarrow$ c'è un anno e meno

- Compute the linear return of the investment considering the undiscounted sum of cash flows.
- *~~Numerical~~* Compute the yield to maturity of each bond using the routine `fsolve` in Matlab (or any equivalent routine in another language). For simplicity assume that the maturity is 31 May, 2026. Which one would you buy, assuming all Issuers have the same financial solidity?

As before,

I. IT: $B(t,T) = F\left(1+\frac{r}{2}\right)^{-2(T-t)} \longrightarrow LR_{IT} = \dfrac{\underbrace{\frac{3.8}{2}}_{6\text{ month}} + \underbrace{\frac{3.8}{2}}_{6\text{ month}} + \underbrace{\frac{3.8}{2}}_{6\text{ month}} + \underbrace{100}_{\text{Face Value}} - 101.86}{101.86} = 3.76\%$

II. FR: $B(t,T) = F(1+r)^{-(T-t)} \longrightarrow LR_{FR} = \dfrac{3.5 + \frac{3.5}{2} + 100 - 101.63}{101.63} = 3.56\%$

III. GM $\longrightarrow LR_{GM} = \dfrac{0.1 + \frac{0.1}{2} + 100 - 99.05}{99.05} = 1.11\%$

$LR = \dfrac{\text{Total cash flow} - \text{Initial price}}{\text{Initial price}}$