Financial Engineering

Springer Semester 2025

Lecturer: Patrizia Semeraro, Assistant: Tommaso Vanzan

Problem set 2

Topics: Coupon bonds, spot rates, forward rates, duration, convexity

---

Exercise 1 An interest rate is quoted as 5% per annum with semiannual compounding. Find the equivalent rates with annual compounding, monthly compounding and continuous compounding.

Exercise 2 When compounded annually an interest rate is equal to 11%. Find the equivalent rates expressed with semiannual compounding, quarterly compounding, monthly compounding, weekly compounding and daily compounding.

Exercise 3 Suppose that the 6-month, 12-month, 18-month, 24-month and 30-month zero rates are, respectively, 4%, 4.2%, 4.4%, 4.6% and 4.8% per annum, with continuous compounding. Estimate the cash price of a bond with a face value of 100 that will mature in 30 months and pay a coupon of 4% per annum semiannualy.
Consider then a 3-year bond that provides a coupon of 8% semiannually and that has a cash price of 104. Find the bond's yield to maturity. Hint: you may use a numerical routine to compute the yield.

Exercise 4 Suppose that the 6-month, 12-month, 18-month and 24-month zero rates are 5%, 6%, 6.5% and 7%, respectively. What is the 2-year par yield?

Exercise 5 The Treasury zero rates and the cash flows on a Treasury bond are reported in Table 1.

Maturity (years)	Zero rate (% per annum)	Coupon payment	Principal
0.5	2	20	
1	2.3	20	
1.5	2.7	20	
2	3.2	20	1000

Table 1:

Assume that zero rates are continuously compounded.

a) What is the bond's theoretical price?

b) What is the bond's yield assuming it sells for its theoretical price?

Exercise 6 In this exercise, we compute the implied spot interest rates $r(0,t)$, for different times instant t, given observed bond prices. Assume that the cash prices of 6-month and 1-year Treasury bills are 94$ and 89$. Further, a 1.5-year Treasury bond that will pay coupons of 4$ every 6 months currently sells for 94.84$, and a 2-year Treasury bond that will pay coupons of 5$ every 6 months currently seels for 97.12$. Calculate the 6-month, 1-year, 1.5-year and 2-year Treasury zero rates.

Exercise 7 Table 2 reports the risk-free zero interest rates with continuous compounding. Cal-

Maturity (months)	Zero Rate (% per annum)
3	3.0
6	3.2
9	3.4
12	3.5
15	3.6
18	3.7

Table 2:

culate the forward interest rates for the second, third, fourth, fifth and sixth quarters.

Exercise 8 Table 3 reports the risk-free zero interest rates with continuous compounding. Cal-

↳ Uguale all'esercizio 7

Maturity (years)	Zero Rate (% per annum)
1	2.0
2	3.0
3	3.7
4	4.2
5	4.5

Table 3: Table 3

culate the forward interest rates for the second, third, fourth, fifth years.

Exercise 9 Consider two zero-coupon bonds, both with a 5% yield continuously compounded. The first bond has maturity in 5 years, the second in 20 years.

1 Compute the bonds' prices if the yield decreases of 1%. Which bond is more sensible to variations of the yield?

2 If you believe the interest rates are going to decrease, which bond would you buy? Describe an investing strategy.

Exercise 10 Consider two bonds, both with maturity in 5 years and a 4% yield continuously compounded. The first bond is zero-coupon, while the second pays an annual coupon of 2%. Which bond is more sensible to variations of the yield?

Exercise 11 A 5-year bond with a yield of 7% continuously compounded pays an 8% coupon at the end of each year.

1 What is the bond's price?

2 What is the bond's duration?

3 Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.

4 Ricalculate the bond's price on the basis of a 6.8% per annum yield and verify that the result is in agreement with the point 3).

Exercise 12 Portoflio A consists of a 1-year zero-coupon bond with a face value of 2000\$ and a 10-year zero-coupon bond with face value of 6000\$. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of 5000\$. The current yield on all bonds is 10% annum.

1 Show that both porfolios have the same duration.

2 Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.

3 Find the percentage changes in the values of the two portfolios for a 5% per annum increase in yields.

4 Use duration and convexity to calculate the approximated percentange changes in the values of the two portfolios in case 2) and 3) and comment the results.

Financial Engineering

Springer Semester 2025

Lecturer: Patrizia Semeraro, Assistant: Tommaso Vanzan

Problem set 2

Topics: Coupon bonds, spot rates, forward rates, duration, convexity

---

Solution 1 Given the rate with semi-annual compounding $r_2 = 5\%$, equivalent rates are:

- Considering annual compounding:

$$1 + r = \left(1 + \frac{r_2}{2}\right)^2$$
$$1 + r = \left(1 + \frac{0.05}{2}\right)^2$$
$$r = \left(1 + \frac{0.05}{2}\right)^2 - 1 = 5.0625\%$$

- Considering monthly compounding:

$$\left(1 + \frac{r_{12}}{12}\right)^{12} = \left(1 + \frac{r_2}{2}\right)^2$$
$$\left(1 + \frac{r_{12}}{12}\right)^{12} = \left(1 + \frac{0.05}{2}\right)^2$$
$$r_{12} = 12\left[\left(1 + \frac{0.05}{2}\right)^{\frac{1}{6}} - 1\right] = 4.949\%$$

- Considering continuous compounding:

$$e^r = \left(1 + \frac{r_2}{2}\right)^2$$
$$e^r = \left(1 + \frac{0.05}{2}\right)^2$$
$$r = 2\ln\left(1 + \frac{0.05}{2}\right) = 4.939\%$$

Solution 2 Given an annually compounded interest rate $r = 11\%$, equivalent rates are:

- Considering semi-annual compounding

$$\left(1 + \frac{r_2}{2}\right)^2 = (1 + r)$$
$$r_2 = 2\left[(1 + 0.11)^{\frac{1}{2}} - 1\right] = 10.71\%$$

- Considering quarterly compounding

$$\left(1+\frac{r_4}{4}\right)^4 = (1+r)$$
$$r_4 = 4\left[(1+0.11)^{\frac{1}{4}}-1\right] = 10.57\%$$

- Considering monthly compounding

$$\left(1+\frac{r_{12}}{12}\right)^{12} = (1+r)$$
$$r_{12} = 12\left[(1+0.11)^{\frac{1}{12}}-1\right] = 10.48\%$$

- Considering weekly compounding

$$\left(1+\frac{r_{52}}{52}\right)^{52} = (1+r)$$
$$r_{52} = 52\left[(1+0.11)^{\frac{1}{52}}-1\right] = 10.45\%$$

- Considering daily compounding

$$\left(1+\frac{r_{365}}{365}\right)^{365} = (1+r)$$
$$r_{365} = 365\left[(1+0.11)^{\frac{1}{365}}-1\right] = 10.44\%$$

Solution 3 In general, the price of a bond is given by the net present value of the cash flows generated by it, discounted using the appropriate rates. In this case, the rates are given and the cash flows consist of coupons of 4% per annum paid semi-annually, i.e. 2 $ every six months, and the face value at the maturity:

$$P = 2e^{-0.04\frac{6}{12}} + 2e^{-0.042\frac{12}{12}} + 2e^{-0.044\frac{18}{12}} + 2e^{-0.046\frac{24}{12}} + 102e^{-0.048\frac{30}{12}} = 98.04\,\$$$

The cash flows of the second bond consist of the face value at maturity and coupons of 8% per annum paid semi-annually, i.e. 4 every six months. The bond's yield to maturity is the rate y that satisfies the equivalence between the market price and the net present value of the cash flows:

$$104 = 4e^{-y\frac{6}{12}} + 4e^{-y\frac{12}{12}} + 4e^{-y\frac{18}{12}} + 4e^{-y\frac{24}{12}} + 4e^{-y\frac{30}{12}} + 104e^{-y\frac{36}{12}}$$

Solving the equation for y we obtain:

$$y = 6.407\%$$

Solution 4 In this case the 2 year par yield is the coupon rate that equals the price of a bond to the par value:

$$\frac{c}{2}e^{-0.05\frac{6}{12}} + \frac{c}{2}e^{-0.06\frac{12}{12}} + \frac{c}{2}e^{-0.065\frac{18}{12}} + \left(100 + \frac{c}{2}\right)e^{-0.07\frac{24}{12}} = 100$$

where the price of the bond is given by the net present value of the cash flows, discounted using the given zero rates, and the par value is the face value 100. Therefore:

$$\begin{aligned}
\frac{c}{2}\left(e^{-0.05\frac{6}{12}} + e^{-0.06\frac{12}{12}} + e^{-0.065\frac{18}{12}} + e^{-0.07\frac{24}{12}}\right) + 100e^{-0.07\frac{24}{12}} &= 100 \\
c\left(e^{-0.05\frac{6}{12}} + e^{-0.06\frac{12}{12}} + e^{-0.065\frac{18}{12}} + e^{-0.07\frac{24}{12}}\right) &= 200 - 200e^{-0.07\frac{24}{12}} \\
c = \frac{200 - 200e^{-0.07\frac{24}{12}}}{\left(e^{-0.05\frac{6}{12}} + e^{-0.06\frac{12}{12}} + e^{-0.065\frac{18}{12}} + e^{-0.07\frac{24}{12}}\right)} &= 7.0740
\end{aligned}$$

Finally, since the face value is 100 the coupon rate is:

$$\frac{c}{100} = \frac{7.0740}{100} = 7.074\% \text{ per annum}$$

The same result can be obtained observing that:

$$\begin{aligned}
d &= \text{present value of } 1\,\$ \text{ received at the maturity of the bond} \\
A &= \text{present value of the annuity that pays } 1\,\$ \text{ on each coupon date} \\
m &= \text{number of coupon payments per year}
\end{aligned}$$

thus:

$$\begin{aligned}
d = e^{-0.07\frac{24}{12}} = 0.8694 \\
A = e^{-0.05\frac{6}{12}} + e^{-0.06\frac{12}{12}} + e^{-0.065\frac{18}{12}} + e^{-0.07\frac{24}{12}} = 3.6935 \\
m = 2
\end{aligned}$$

and the par yield is:

$$c = \frac{(100 - 100d)m}{A} = \frac{(100 - 100 \cdot 0.8694)2}{3.6935} = 7.0740$$

Solution 5 Assuming that zero rates are continuously compounded, the bond's theoretical price is given by the present value of all the cash flows that will be received by the owner of the bond, hence:

$$P = 20e^{-0.02 \cdot 0.5} + 20e^{-0.023 \cdot 1} + 20e^{-0.027 \cdot 1.5} + 1020e^{-0.032 \cdot 2} = 1015.32\,\$$$

The bond's yield is the single discount rate that, when applied to all cash flows, gives a bond price equal to its market price. Assuming the bond sells for its theoretical price, we are looking for the single discount rate that gives a bond price equal to its theoretical price, when applied to all cash flows:

$$1015.32 = 20e^{-y \cdot 0.5} + 20e^{-y \cdot 1} + 20e^{-y \cdot 1.5} + 1020e^{-y \cdot 2}$$

Solving the equation for y we obtain:

$$y = 0.0318 = 3.18\%$$

Solution 6 The 6 month zero rate is:

$$
\begin{aligned}
94 &= 100e^{-r\frac{6}{12}} \\
e^{\frac{1}{2}r} &= \frac{100}{94} \\
r &= 2\ln\frac{100}{24} = 0.1238 = 12.38\%
\end{aligned}
$$

The 1 year zero rate is:

$$
\begin{aligned}
89 &= 100e^{-r} \\
e^{r} &= \frac{100}{89} \\
r &= \ln\frac{100}{89} = 0.1165 = 11.65\%
\end{aligned}
$$

Using these results, we are able to compute the 1.5 year zero rate:

$$
\begin{aligned}
94.84 &= 4e^{-0.1238\frac{6}{12}} + 4e^{-0.1165\frac{12}{12}} + 104e^{-r\frac{18}{12}} \\
94.84 - 4e^{-0.1238\frac{6}{12}} - 4e^{-0.1165\frac{12}{12}} &= 104e^{-r\frac{18}{12}} \\
e^{-r\frac{18}{12}} &= 0.8415 \\
r &= -\frac{12}{18}\ln 0.8415 = 0.1150 = 11.50\%
\end{aligned}
$$

Using the same procedure, the 2 year zero rate is:

$$
\begin{aligned}
97.12 &= 5e^{-0.1238\frac{6}{12}} + 5e^{-0.1165\frac{12}{12}} + 5e^{-0.1150\frac{18}{12}} + 105e^{-r\frac{24}{12}} \\
97.12 - 5e^{-0.1238\frac{6}{12}} - 5e^{-0.1165\frac{12}{12}} - 5e^{-0.1150\frac{18}{12}} &= 105e^{-r\frac{24}{12}} \\
e^{-2r} &= 0.7977 \\
r &= -\frac{1}{2}\ln 0.7977 = 0.1130 = 11.30\%
\end{aligned}
$$

Solution 7 The 2nd quarter forward rate is:

$$
\begin{aligned}
e^{0.03\frac{3}{12}}e^{r\frac{3}{12}} &= e^{0.032\frac{6}{12}} \\
e^{0.03\frac{3}{12}+r\frac{3}{12}} &= e^{0.032\frac{6}{12}} \\
0.03\frac{3}{12} + r\frac{3}{12} &= 0.032\frac{6}{12} \\
r &= 0.034 = 3.4\%
\end{aligned}
$$

The 3rd quarter forward rate is:

$$
\begin{aligned}
e^{0.032\frac{6}{12}}e^{r\frac{3}{12}} &= e^{0.034\frac{9}{12}} \\
e^{0.032\frac{6}{12}+r\frac{3}{12}} &= e^{0.034\frac{9}{12}} \\
0.032\frac{6}{12} + r\frac{3}{12} &= 0.034\frac{9}{12} \\
r &= 0.038 = 3.8\%
\end{aligned}
$$

The 4th quarter forward rate is:

$$e^{0.034\frac{9}{12}}e^{r\frac{3}{12}} = e^{0.035\frac{12}{12}}$$
$$e^{0.034\frac{9}{12}+r\frac{3}{12}} = e^{0.035}$$
$$0.034\frac{9}{12} + r\frac{3}{12} = 0.035$$
$$r = 0.038 = 3.8\%$$

The 5th quarter forward rate is:

$$e^{0.035\frac{12}{12}}e^{r\frac{3}{12}} = e^{0.036\frac{15}{12}}$$
$$e^{0.035\frac{12}{12}+r\frac{3}{12}} = e^{0.036\frac{15}{12}}$$
$$0.035 + r\frac{3}{12} = 0.036\frac{15}{12}$$
$$r = 0.04 = 4\%$$

The 6th quarter forward rate is:

$$e^{0.036\frac{15}{12}}e^{r\frac{3}{12}} = e^{0.037\frac{18}{12}}$$
$$e^{0.036\frac{15}{12}+r\frac{3}{12}} = e^{0.037\frac{18}{12}}$$
$$0.036\frac{15}{12} + r\frac{3}{12} = 0.037\frac{18}{12}$$
$$r = 0.042 = 4.2\%$$

Solution 8 The forward rate for the 2nd year is:

$$e^{0.02}e^{r} = e^{0.03\cdot 2}$$
$$e^{0.02+r} = e^{0.03\cdot 2}$$
$$0.02 + r = 0.06$$
$$r = 0.04 = 4\%$$

The forward rate for the 3rd year is:

$$e^{0.03\cdot 2}e^{r} = e^{0.037\cdot 3}$$
$$e^{0.03\cdot 2+r} = e^{0.037\cdot 3}$$
$$0.06 + r = 0.111$$
$$r = 0.051 = 5.1\%$$

The forward rate for the 4th year is:

$$e^{0.037\cdot 3}e^{r} = e^{0.042\cdot 4}$$
$$e^{0.037\cdot 3+r} = e^{0.042\cdot 4}$$
$$0.111 + r = 0.168$$
$$r = 0.057 = 5.7\%$$

The forward rate for the 5th year is:

$$\begin{aligned} e^{0.042\cdot 4} e^{r} &= e^{0.045\cdot 5} \\ e^{0.042\cdot 4 + r} &= e^{0.045\cdot 5} \\ 0.168 + r &= 0.225 \\ r &= 0.057 = 5.7\% \end{aligned}$$

Solution 9

1 Considering 5% yield continuously compounded and assuming unit face value, the bond prices are:

$$\begin{aligned} P_1 &= e^{-0.05\cdot 5} = 0.7788\,\$ \\ P_2 &= e^{-0.05\cdot 20} = 0.3678\,\$ \end{aligned}$$

Considering a 1% decrease in the yield, the new bond prices are:

$$\begin{aligned} P_1' &= e^{-0.04\cdot 5} = 0.8187\,\$ \\ P_2' &= e^{-0.04\cdot 20} = 0.4493\,\$ \end{aligned}$$

In relative terms, the variations in prices are given by:

$$\begin{aligned} \frac{\Delta P_1}{P_1} &= \frac{P_1' - P_1}{P_1} = \frac{0.8187 - 0.7788}{0.7788} = 5.12\% \\ \frac{\Delta P_2}{P_2} &= \frac{P_2' - P_2}{P_2} = \frac{0.4493 - 0.3678}{0.3678} = 22.15\% \end{aligned}$$

As expected, using the first order approximation involving the duration:

$$\frac{\Delta P}{P} \approx -D\Delta r$$

we obtain similar results:

$$\begin{aligned} \frac{\Delta P_1}{P_1} &\approx -D_1 \Delta r = 5 \cdot 1\% = 5\% \\ \frac{\Delta P_2}{P_2} &\approx -D_2 \Delta r = 20 \cdot 1\% = 20\% \end{aligned}$$

The most sensible to variation of the yields is the bond with higher duration that is, in case of ZCB, the bond with longer time to maturity, i.e. the second bond.

2 Having an investment horizon of one year, if the interest rates are going to be lower in one year a possible investment strategy is to buy the second bond in $t = 0$ and sell it in $t = 1$. Using the data of the previous point:

$t = 0$	long position second bond	$-e^{-0.05\cdot 20} = -0.3678$
$t = 1$	short position second bond	$e^{-0.04\cdot 19} = 0.4676$
		$0.4676 - 0.3678 = 0.0998$

Using this strategy it is possible to earn 0.09$ for each bond bought in $t = 0$ and sold in $t = 1$.

Solution 10 The sensibility to variations of the yield is given by the duration. For the first bond, the duration is exactly the time to maturity since it is a ZCB:

$$D_1 = 5 \text{ years}$$

In general, the duration of a bond is given by the weighted average of the instants t_i of the generated cash flows, considering as weights the discounted amount of the cash flows c_i:

$$D = \frac{\sum_{i=1}^{n} t_i\, c_i\, e^{-t_i\, r}}{\sum_{i=1}^{n} c_i\, e^{-t_i\, r}} = \frac{\sum_{i=1}^{n} t_i\, c_i\, e^{-t_i\, r}}{P}$$

Considering the first bond we obtain:

$$D_1 = \frac{5 \cdot 100 \cdot e^{-0.04 \cdot 5}}{100 \cdot e^{-0.04 \cdot 5}} = 5 \text{ years}$$

which is exactly its time to maturity. Considering the second bond we obtain:

$$D_2 = \frac{\sum_{i=1}^{n} t_i\, c_i\, e^{-t_i\, r}}{\sum_{i=1}^{n} c_i\, e^{-t_i\, r}} = \frac{\sum_{i=1}^{4} i \cdot 2 \cdot e^{-0.04 \cdot i} + 5 \cdot 102 \cdot e^{-0.04 \cdot 5}}{\sum_{i=1}^{4} 2 \cdot e^{-0.04 \cdot i} + 102 \cdot e^{-0.04 \cdot 5}} = 4.79641 \text{ years}$$

The most sensible to variations of the yield is the first bond since it has higher duration.

Solution 11

1 The bond's price is given by:

$$P = \sum_{t=1}^{4} 8 \cdot e^{-0.07 \cdot t} + 108 \cdot e^{-0.07 \cdot 5} = 103.05\,\$$$

2 The bond's duration is given by:

$$D = \frac{\sum_{t=1}^{4} t \cdot 8 \cdot e^{-0.07 \cdot t} + 5 \cdot 108 \cdot e^{-0.07 \cdot 5}}{103.05} = 4.3235 \text{ years}$$

3 The effect on the bond's price of a 0.2% decrease in its yield can be computed using the first order approximation involving the duration:

$$\Delta P \approx -D \cdot \Delta r \cdot P = -4.3235 \cdot (-0.002) \cdot 103.05 = 0.8910$$

Hence the price increases of 0.8910 \$:

$$P' = P + 0.8910 = 103.05 + 0.8910 = 103.941\,\$$$

4 Recalculating the bond's price on the basis of a 6.8% per annum yield, we obtain:

$$P' = \sum_{t=1}^{4} 8 \cdot e^{-0.068 \cdot t} + 108 \cdot e^{-0.068 \cdot 5} = 103.947\,\$$$

The result is consistent with the one above and the first order approximation is a good approximation of the new price.

Solution 12

1 The duration of the portfolio A is:

$$D_A = \frac{\sum_{i=1}^{n} t_i\, c_i\, e^{-t_i\, r}}{\sum_{i=1}^{n} c_i\, e^{-t_i\, r}} = \frac{1 \cdot 2000 \cdot e^{-0.1 \cdot 1} + 10 \cdot 6000 \cdot e^{-0.1 \cdot 10}}{2000 \cdot e^{-0.1 \cdot 1} + 6000 \cdot e^{-0.1 \cdot 10}} = 5.95 \text{ years}$$

The duration of B is:

$$D_B = \frac{\sum_{i=1}^{n} t_i\, c_i\, e^{-t_i\, r}}{\sum_{i=1}^{n} c_i\, e^{-t_i\, r}} = \frac{5.95 \cdot 5000 \cdot e^{-0.1 \cdot 5.95}}{5000 \cdot e^{-0.1 \cdot 5.95}} = 5.95 \text{ years}$$

As expected, the duration of the ZCB is its time to maturity. Moreover, the two portfolios A and B have the same duration.

2 The value of the portfolio A is:

$$V_A = 2000 \cdot e^{-0.1 \cdot 1} + 6000 \cdot e^{-0.1 \cdot 10} = 4016.95\,\$$$

and if the yield increases by 0.1 % the value of A becomes:

$$V_A' = 2000 \cdot e^{-0.101 \cdot 1} + 6000 \cdot e^{-0.101 \cdot 10} = 3993.18\,\$$$

hence the percentage change in the value of A is:

$$\frac{\Delta V_A}{V_A} = \frac{3993.18 - 4016.95}{4016.95} = -0.0059 = -0.59\,\%$$

The value of the portfolio B is:

$$V_B = 5000 \cdot e^{-0.1 \cdot 5.95} = 2757.81\,\$$$

and if the yield increases by 0.1 % the value of B becomes:

$$V_B' = 5000 \cdot e^{-0.101 \cdot 5.95} = 2741.45\,\$$$

hence the percentage change in the value of B is:

$$\frac{\Delta V_B}{V_B} = \frac{2741.45 - 2757.81}{2757.81} = -0.0059 = -0.59\,\%$$

therefore the percentage changes in the values of the portfolios are the same.

3 If the yield increases by 5 % the value of A becomes:

$$V_A'' = 2000 \cdot e^{-0.15 \cdot 1} + 6000 \cdot e^{-0.15 \cdot 10} = 3060.20\,\$$$

hence the percentage change in the value of A is:

$$\frac{\Delta V_A}{V_A} = \frac{3060.20 - 4016.95}{4016.95} = -0.2382 = -23.82\,\%$$

While value of the portfolio B becomes:

$$V_B'' = 5000 \cdot e^{-0.15 \cdot 5.95} = 2048.15\,\$$$

hence the percentage change in the value of B is:

$$\frac{\Delta V_B}{V_B} = \frac{2048.15 - 2757.81}{2757.81} = -0.2573 = -25.73\,\%$$

4 The duration of the two portfolios, as seen above, is the same. The convexity of portfolio A, then, is:

$$C_A = \frac{\sum_{i=1}^{n} t_i^2\, c_i\, e^{-t_i\, r}}{\sum_{i=1}^{n} c_i\, e^{-t_i\, r}} = \frac{1^2 \cdot 2000 \cdot e^{-0.1 \cdot 1} + 10^2 \cdot 6000 \cdot e^{-0.1 \cdot 10}}{2000 \cdot e^{-0.1 \cdot 1} + 6000 \cdot e^{-0.1 \cdot 10}} = 55.40$$

and the convexity of portfolio B is:

$$C_B = \frac{\sum_{i=1}^{n} t_i^2\, c_i\, e^{-t_i\, r}}{\sum_{i=1}^{n} c_i\, e^{-t_i\, r}} = \frac{5.95^2 \cdot 5000 \cdot e^{-0.1 \cdot 5.95}}{5000 \cdot e^{-0.1 \cdot 5.95}} = 35.40$$

Considering a 0.1 % per annum increase in yields, the approximated effect on the value of portfolio A using only the duration is:

$$\frac{\Delta V_A}{V_A} \approx -D_A \cdot \Delta r = -5.95 \cdot 0.001 = -0.595\,\%$$

while, considering also the convexity, it is:

$$\frac{\Delta V_A}{V_A} \approx -D_A \cdot \Delta r + \frac{1}{2} \cdot C_A \cdot (\Delta r)^2 = -5.95 \cdot 0.001 + \frac{1}{2} \cdot 55.40 \cdot (0.001)^2 = -0.592\,\%$$

that is closer to the true change. In a similar way, the approximated effect on the values of portfolio B using only the duration is:

$$\frac{\Delta V_B}{V_B} \approx -D_B \cdot \Delta r = -5.95 \cdot 0.001 = -0.595\,\%$$

while, considering also the convexity, it is:

$$\frac{\Delta V_B}{V_B} \approx -D_B \cdot \Delta r + \frac{1}{2} \cdot C_B \cdot (\Delta r)^2 = -5.95 \cdot 0.001 + \frac{1}{2} \cdot 35.40 \cdot (0.001)^2 = -0.593\,\%$$

that is closer to the true change. Considering a 5 % per annum increase in yields, the approximated effect on the value of portfolio A using only the duration is:

$$\frac{\Delta V_A}{V_A} \approx -D_A \cdot \Delta r = -5.95 \cdot 0.05 = -29.75\,\%$$

while, considering also the convexity, it is:

$$\frac{\Delta V_A}{V_A} \approx -D_A \cdot \Delta r + \frac{1}{2} \cdot C_A \cdot (\Delta r)^2 = -5.95 \cdot 0.05 + \frac{1}{2} \cdot 55.40 \cdot (0.05)^2 = -22.825\,\%$$

that is closer to the true change. In a similar way, the approximated effect on the values of portfolio B using only the duration is:

$$\frac{\Delta V_B}{V_B} \approx -D_B \cdot \Delta r = -5.95 \cdot 0.05 = -29.75\,\%$$

while, considering also the convexity, it is:

$$\frac{\Delta V_B}{V_B} \approx -D_B \cdot \Delta r + \frac{1}{2} \cdot C_B \cdot (\Delta r)^2 = -5.95 \cdot 0.05 + \frac{1}{2} \cdot 35.40 \cdot (0.05)^2 = -25.325\,\%$$

that is closer to the true change. In conclusion, the approximation obtained using also the convexity are better than those obtained using only the duration since they are based on a 2nd order Taylor approximation, that is always better than a 1st order one.

Financial Engineering

Springer Semester 2025

Lecturer: Patrizia Semeraro, Assistant: Tommaso Vanzan

Problem set 2

Topics: Coupon bonds, spot rates, forward rates, duration, convexity

Exercise 1 An interest rate is quoted as 5% per annum with semiannual compounding. Find the equivalent rates with annual compounding, monthly compounding and continuous compounding.

Ask: convert an interest rate quoted with one compounding convention into equivalent interest rates expressed with other compounding conventions

- Two interest rate are equivalent if they produce the same accumulation factor over one year, i.e. the value of 1 invested today after 1 year must be the same

- Accumulation factors by compounding convention → i. Semi-Annual: $\left(1+\frac{r_2}{2}\right)^2$ iii. Monthly: $\left(1+\frac{r_{12}}{12}\right)^{12}$ ii. Annual: $1+r$ iv. Continuous: e^r

Hence, i. Semi-Annual vs Annual: $\left(1+\frac{r_1}{2}\right)^2 = 1+r_2$ $r_1 = 0.05$ $r_2 = 5.0625\%$

ii. Semi-Annual vs Monthly: $\left(1+\frac{r_1}{2}\right)^2 = \left(1+\frac{r_2}{12}\right)^{12}$ $r_1 = 0.05$ $r_2 = \left(\left(1+\frac{r_1}{2}\right)^{2\cdot\frac{1}{12}} - 1\right)\cdot 12 = 4.949\%$

iii. Semi-Annual vs Continuous: $\left(1+\frac{r_1}{2}\right)^2 = e^{r_2}$ $r_1 = 0.05$ $r_2 = \ln\left(\left(1+\frac{r_1}{2}\right)^2\right) = 4.939\%$

Exercise 2 When compounded annually an interest rate is equal to 11%. Find the equivalent rates expressed with semiannual compounding, quarterly compounding, monthly compounding, weekly compounding and daily compounding.

As before, i. Annual vs Semi-Annual $(1+r_1) = \left(1+\frac{r_2}{2}\right)^2$ $r_1 = 0.11$ $r_2 = \left((1+r_1)^{\frac{1}{2}} - 1\right)\cdot 2 = 10.71\%$

ii. Annual vs Quaranterly (4) $(1+r_1) = \left(1+\frac{r_2}{4}\right)^4$ $r_1 = 0.11$ $r_2 = \left((1+r_1)^{\frac{1}{4}} - 1\right)\cdot 4 = 10.57\%$

iii. Annual vs Monthly (12) $(1+r_1) = \left(1+\frac{r_2}{12}\right)^{12}$ $r_1 = 0.11$ $r_2 = \left((1+r_1)^{\frac{1}{12}} - 1\right)\cdot 12 = 10.48\%$

iv. Annual vs Weekly (52) $(1+r_1) = \left(1+\frac{r_2}{52}\right)^{52}$ $r_1 = 0.11$ $r_2 = \left((1+r_1)^{\frac{1}{52}} - 1\right)\cdot 52 = 10.45\%$

v. Annual vs Daily (365) $(1+r_1) = \left(1+\frac{r_2}{365}\right)^{365}$ $r_1 = 0.11$ $r_2 = \left((1+r_1)^{\frac{1}{365}} - 1\right)\cdot 365 = 10.44\%$

Exercise 3 Suppose that the 6-month, 12-month, 18-month, 24-month and 30-month zero rates are, respectively, 4%, 4.2%, 4.4%, 4.6% and 4.8% per annum, with continuous compounding. Estimate the cash price of a bond with a face value of 100 that will mature in 30 months and pay a coupon of 4% per annum semiannualy.
Consider then a 3-year bond that provides a coupon of 8% semiannually and that has a cash price of 104. Find the bond's yield to maturity. Hint: you may use a numerical routine to compute the yield.

- The price of a bond is the net present value of its cash flows, discounted using the appropriate zero rates. When zero rate are given, each cash flow must be discounted using the zero rate corresponding to its maturity, not a single yield

↓ With continuous compounding, the discount factor for a cash flow at time t is: $e^{-r(0,t)t}$ (interest rate, fixed today (0))

- The yield to maturity is defined as the constant interest rate y that makes the present value of all cash flows equal to the market price of the bond

↓ $P = \sum_i c_i e^{-y t_i}$

Hence, the price of a bond is given by the net present value of a cash flows generated by it, discounting using the appropriate rates

$$V(0,T) = c_1 e^{-r_1 t_1} + c_2 e^{-r_2 t_2} + c_3 e^{-r_3 t_3} + c_4 e^{-r_4 t_4} + (c_5 + F) e^{-r_5 T}$$

i.e. 2\$ every six mounth

$$V(0,T) = 2e^{-0.04 \cdot \frac{6}{12}} + 2e^{-0.042 \cdot 1} + 2e^{-0.044 \cdot \frac{18}{12}} + 2e^{-0.046 \cdot 2} + (2+100)e^{-0.048 \cdot \frac{30}{12}} = 98.04\ \$$$

Bond yield y: $V(0,T) = c\, e^{-y t_1} + \dots + (c+F)e^{-yT}$

8% semiamnaly => 4% ; $\frac{8}{2} = 4$, $8 = c$, $2 = m$; i.e. 4\$

$$104 = 4e^{-y\frac{6}{12}} + 4e^{-y} + 4e^{-y\frac{18}{12}} + 4e^{-y2} + 4e^{-y \cdot \frac{30}{12}} + (4+100)e^{-y\frac{36}{12}}$$

3years = 36 mounth

↓ $y = 6.407\%$ ⟶ risolto sicuramente numericamente

Exercise 4 Suppose that the 6-month, 12-month, 18-month and 24-month zero rates are 5%, 6%, 6.5% and 7%, respectively. What is the 2-year par yield?

$$V(0,T) = \frac{c}{m}e^{-r_1 t_1} + \frac{c}{m}e^{-r_2 t_2} + \frac{c}{m}e^{-r_3 t_3} + \left(\frac{c}{m} + F\right)e^{-r_4 t_4}$$

$V(0,T)$: 100 ; c = incognita m := frequenta di pagamento annua per cedola

Question: find c, indeed the par yield is the coupon rate c that makes the bond price equal to its face value

Per convenzione il face value = 100

6-12-18-24 mounth vuol dire 2 per anno => $m = 2$

$$100 = \frac{c}{2}\left(e^{-0.05 \cdot \frac{6}{12}} + e^{-0.06} + e^{-0.065 \cdot \frac{18}{12}} + e^{-0.07 \cdot 2}\right) + 100e^{-0.07 \cdot 2} \Rightarrow c = 7.074$$

Exercise 5 The Treasury zero rates and the cash flows on a Treasury bond are reported in Table 1.

Maturity (years)	Zero rate (% per annum)	Coupon payment	Principal
0.5	2	20	
1	2.3	20	
1.5	2.7	20	
2	3.2	20	1000

Table 1:

↓ Tasso per periodo NON annuo

Assume that zero rates are continuously compounded.

a) What is the bond's theoretical price?

b) What is the bond's yield assuming it sells for its theoretical price?

a. Assuming that zero rates are continously compounded, the bond's theoretical price is given by the present value of all the cash flows that will be received by the owner of the bond

$$V(0,T) = c_1 \cdot e^{-r_1 t_1} + c_2 e^{-r_2 t_2} + c_3 e^{-r_3 t_3} + (c_4 + F) e^{-r_4 t_4} = 20 e^{-0.02 \cdot \frac{6}{12}} + 20 e^{-0.023} + 20 e^{-0.027 \cdot \frac{18}{12}} + (20 + 1000) e^{-0.032 \cdot 2} = 1015.32 \ \$$$

b. Il theoretical price è $V(0,T)$ ⟶ bond yield y: $V(0,T) = 20 e^{-y 0.5} + 20 e^{-y} + 20 e^{-1.5y} + 120 e^{2y}$

$V(0,T) = 1016.32$

↓ numerically $y = 3.18\%$

Exercise 6 In this exercise, we compute the implied spot interest rates $r(0,t)$, for different times instant t, given observed bond prices. Assume that the cash prices of 6-month and 1-year Treasury bills are 94$ and 89$. Further, a 1.5-year Treasury bond that will pay coupons of 4$ every 6 months currently sells for 94.84$, and a 2-year Treasury bond that will pay coupons of 5$ every 6 months currently seels for 97.12$. Calculate the 6-month, 1-year, 1.5-year and 2-year Treasury zero rates.

6-mounth: $V(0,1) = c e^{-r_1 t_1}$ $94 = c e^{-r \frac{6}{12}}$ assuming $c = 100$ ⟶ $-\ln\left(\frac{94}{100}\right)\frac{12}{6} = r_1 = 12.38\%$

1-year: $V(0,T) = c e^{-rt}$ $89 = 100 e^{-r}$ ⟶ $-\ln\left(\frac{89}{100}\right) = r_2 = 11.65\%$

1.5-year: $V(0,T) = 4 e^{-r_1 \frac{6}{12}} + 4 e^{-r_2} + (4+100) e^{-r \frac{18}{12}} = 94.84$ $r = -\ln(0.84)\frac{12}{18} = 11.50\%$

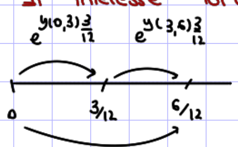
2-year: $V(0,T) = 97.12 = 5 e^{-0.1238 \cdot \frac{6}{12}} + 5 e^{-0.1165} + 5 e^{-0.1150 \cdot \frac{18}{12}} + 105 e^{-r \cdot 2}$ $r = -\ln\left(\frac{97.12 - (\ )}{105}\right) \cdot \frac{1}{2} = 11.30\%$

Exercise 7 Table 2 reports the risk-free zero interest rates with continuous compounding. Calculate the forward interest rates for the second, third, fourth, fifth and sixth quarters.

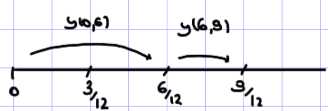
Maturity (months)	Zero Rate (% per annum)
3	3.0
6	3.2
9	3.4
12	3.5
15	3.6
18	3.7

Table 2:

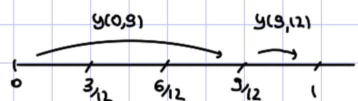
Tassi di interesse forward: tassi concordati oggi per un prestito/ investimento futuro

2°


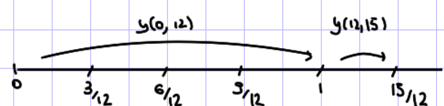
$e^{y(0,3)\frac{3}{12}} \cdot e^{y(3,6)\frac{3}{12}} = e^{y(0,6)\frac{6}{12}}$ $y(0,3) = 0.03$, $y(0,6) = 0.032$ $y = (3,6) = y = 3.4\%$

3°


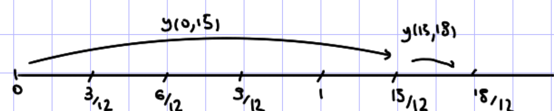
$e^{0.032\cdot 0.5} \cdot e^{y(6,9)\frac{3}{12}} = e^{0.034\cdot\frac{9}{12}}$ $y(6,9) = 3.8\%$

4°


$e^{0.034\frac{9}{12}} \cdot e^{y(9,12)\cdot\frac{3}{12}} = e^{0.035}$ $y(9,12) = 3.8\%$

5°


$e^{0.035} \cdot e^{y(12,15)\frac{3}{12}} = e^{0.036\frac{15}{12}}$ $y(12,15) = 4\%$

6°


$e^{0.036\frac{15}{12}} \cdot e^{y(15,18)\frac{3}{12}} = e^{0.037\frac{18}{12}}$ $y(15,18) = 4.2\%$

Exercise 9 Consider two zero-coupon bonds, both with a 5% yield continuously compounded. The first bond has maturity in 5 years, the second in 20 years.

1 Compute the bonds' prices if the yield decreases of 1%. Which bond is more sensible to variations of the yield?

2 If you believe the interest rates are going to decrease, which bond would you buy? Describe an investing strategy.

1. Consider 5% yield continuously compounded assuming $F = 1$, the bond prices are

$P_1 = e^{-0.05\cdot 5} = 0.7788\,\$$ $P_2 = e^{-0.05\cdot 20} = 0.3678\,\$$

Considering 4% yield : $P_1' = e^{-0.04\cdot 5} = 0.8187\,\$$ $P_2' = e^{-0.04\cdot 20} = 0.4493\,\$$

In relative terms, the variation in price are given by

$\frac{\Delta P_1}{P_1} = \frac{P_1' - P_1}{P_1} = 5.23\%$ $\frac{\Delta P_2}{P_2} = \frac{P_2' - P_2}{P_2} = 22.16\%$ ⟶ The 2° bond is the most sensitive to variation bacause has higher duration $\frac{\Delta P}{P} \simeq -D\Delta r$

2. Having an investment horizon of one year, if the investment rate are going to be lower in one year a possible investment strategy is to buy the second bond in $t=0$ and sell in $t=1$

$t=0$	long position	2°	$-e^{-0.05\cdot 2.0} = -0.3678$		$t=0$	long position	1°	$-e^{-0.05\cdot 5} = -0.7788$
$t=1$	short position	2°	$e^{-0.04\cdot 1.9} = 0.4676$		$t=1$	short position	1°	$e^{-0.04\cdot 4} = 0.8521$
			$+0.0999$ ✓					$+0.0733$

Exercise 10 Consider two bonds, both with maturity in 5 years and a 4% yield continuously compounded. The first bond is zero-coupon, while the second pays an annual coupon of 2%. Which bond is more sensible to variations of the yield?

The sensitivity to variations of the yield is given by the duration $D(y) = \frac{\sum_{i=1}^{n} t_i c_i e^{-yt_i}}{\sum_{i=1}^{n} c_i e^{-yt_i}}$

ZCB: $D(y) = \frac{5\cdot 100\cdot e^{-0.04\cdot 5}}{100\cdot e^{-0.04\cdot 5}} = 5$ years

CB: c_i = rate annuo × face value = $0.02\cdot 100 = 2$

$$D(y) = \frac{1\cdot 2\cdot e^{-0.04\cdot 1} + 2\cdot 2 e^{-0.04\cdot 2} + 3\cdot 2 e^{-0.04\cdot 3} + 4\cdot 2 e^{-0.04\cdot 4} + 5\cdot 102 e^{-0.04\cdot 5}}{2e^{-0.04} + 2e^{-0.04\cdot 2} + 2e^{-0.04\cdot 3} + 2e^{-0.04\cdot 4} + 102 e^{-0.04\cdot 5}} = 4.796411 \text{ years}$$

The most sensitive to variation of the yield is the first bond since it has higher duration

Exercise 11 A 5-year bond with a yield of 7% continuously compounded pays an 8% coupon at the end of each year.

$t_i = 1,\dots,5$ $\quad r = 0.07$ $\quad c_i = 0.08\cdot 100 = 8$

1 What is the bond's price?

2 What is the bond's duration?

3 Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.

4 Ricalculate the bond's price on the basis of a 6.8% per annum yield and verify that the result is in agreement with the point 3).

Solution 11

1 The bond's price is given by:

$$P = \sum_{t=1}^{4} 8\cdot e^{-0.07\cdot t} + 108\cdot e^{-0.07\cdot 5} = 103.05\,\$ \checkmark$$

$$P = \sum_{i=1}^{4} c_i e^{-r t_i} + (c_i + F) e^{-r\cdot 5}$$

2 The bond's duration is given by:

$$D = \frac{\sum_{t=1}^{4} t\cdot 8\cdot e^{-0.07\cdot t} + 5\cdot 108\cdot e^{-0.07\cdot 5}}{103.05} = 4.3235 \text{ years} \checkmark$$

3 The effect on the bond's price of a 0.2% decrease in its yield can be computed using the first order approximation involving the duration:

$$\Delta P \approx -D\cdot \Delta r\cdot P = -4.3235\cdot(-0.002)\cdot 103.05 = 0.8910 \checkmark$$

$\frac{\Delta P}{P} \approx -D\cdot\Delta r$ $\quad \Delta r = 0.2\%$

Hence the price increases of 0.8910 $:

$$P' = P + 0.8910 = 103.05 + 0.8910 = 103.941\,\$$$

Prezzo e rendimento si muovono in direzione opposta e la duration misura quanto e' forte questa relazione

4 Recalculating the bond's price on the basis of a 6.8% per annum yield, we obtain:

$$P' = \sum_{t=1}^{4} 8\cdot e^{-0.068\cdot t} + 108\cdot e^{-0.068\cdot 5} = 103.947\,\$ \checkmark$$

The result is consistent with the one above and the first order approximation is a good approximation of the new price.

$\Delta r \downarrow \Rightarrow$ prezzo $\uparrow$ & $\Delta r \uparrow \Rightarrow$ prezzo $\downarrow$

Exercise 12 Portoflio A consists of a 1-year zero-coupon bond with a face value of 2000$ and a 10-year zero-coupon bond with face value of 6000$. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of 5000$. The current yield on all bonds is 10% annum.

↳ r = 0.1 capitalizzazione continua

1 Show that both porfolios have the same duration.

2 Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.

3 Find the percentage changes in the values of the two portfolios for a 5% per annum increase in yields.

4 Use duration and convexity to calculate the approximated percentange changes in the values of the two portfolios in case 2) and 3) and comment the results.

1 The duration of the portfolio A is:

$$D_A = \frac{\sum_{i=1}^n t_i\, c_i\, e^{-t_i r}}{\sum_{i=1}^n c_i\, e^{-t_i r}} = \frac{1 \cdot 2000 \cdot e^{-0.1 \cdot 1} + 10 \cdot 6000 \cdot e^{-0.1 \cdot 10}}{2000 \cdot e^{-0.1 \cdot 1} + 6000 \cdot e^{-0.1 \cdot 10}} = 5.95 \text{ years} \checkmark$$

The duration of B is:

$$D_B = \frac{\sum_{i=1}^n t_i\, c_i\, e^{-t_i r}}{\sum_{i=1}^n c_i\, e^{-t_i r}} = \frac{5.95 \cdot 5000 \cdot e^{-0.1 \cdot 5.95}}{5000 \cdot e^{-0.1 \cdot 5.95}} = 5.95 \text{ years} \checkmark$$

As expected, the duration of the ZCB is its time to maturity. Moreover, the two portfolios A and B have the same duration.

2 The value of the portfolio A is:

$$V_A = 2000 \cdot e^{-0.1 \cdot 1} + 6000 \cdot e^{-0.1 \cdot 10} = 4016.95\,\$ \checkmark$$

and if the yield increases by 0.1 % the value of A becomes:

$$V'_A = 2000 \cdot e^{-0.101 \cdot 1} + 6000 \cdot e^{-0.101 \cdot 10} = 3993.18\,\$ \checkmark$$

hence the percentage change in the value of A is:

$$\frac{\Delta V_A}{V_A} = \frac{3993.18 - 4016.95}{4016.95} = -0.0059 = -0.59\,\% \checkmark$$

The value of the portfolio B is:

$$V_B = 5000 \cdot e^{-0.1 \cdot 5.95} = 2757.81\,\$ \checkmark$$

and if the yield increases by 0.1 % the value of B becomes:

$$V'_B = 5000 \cdot e^{-0.101 \cdot 5.95} = 2741.45\,\$ \checkmark$$

hence the percentage change in the value of B is:

$$\frac{\Delta V_B}{V_B} = \frac{2741.45 - 2757.81}{2757.81} = -0.0059 = -0.59\,\% \checkmark$$

therefore the percentage changes in the values of the portfolios are the same.

3 If the yield increases by 5 % the value of A becomes:

$$V''_A = 2000 \cdot e^{-0.15 \cdot 1} + 6000 \cdot e^{-0.15 \cdot 10} = 3060.20\,\$ \checkmark$$

hence the percentage change in the value of A is:

$$\frac{\Delta V_A}{V_A} = \frac{3060.20 - 4016.95}{4016.95} = -0.2382 = -23.82\,\% \checkmark$$

While value of the portfolio B becomes:

$$V''_B = 5000 \cdot e^{-0.15 \cdot 5.95} = 2048.15\,\$ \checkmark$$

hence the percentage change in the value of B is:

$$\frac{\Delta V_B}{V_B} = \frac{2048.15 - 2757.81}{2757.81} = -0.2573 = -25.73\,\% \checkmark$$

4 The duration of the two portfolios, as seen above, is the same. The convexity of portfolio A, then, is:

$$C_A = \frac{\sum_{i=1}^n t_i^2\, c_i\, e^{-t_i r}}{\sum_{i=1}^n c_i\, e^{-t_i r}} = \frac{1^2 \cdot 2000 \cdot e^{-0.1 \cdot 1} + 10^2 \cdot 6000 \cdot e^{-0.1 \cdot 10}}{2000 \cdot e^{-0.1 \cdot 1} + 6000 \cdot e^{-0.1 \cdot 10}} = 55.40 \checkmark$$

and the convexity of portfolio B is:

$$C_B = \frac{\sum_{i=1}^n t_i^2\, c_i\, e^{-t_i r}}{\sum_{i=1}^n c_i\, e^{-t_i r}} = \frac{5.95^2 \cdot 5000 \cdot e^{-0.1 \cdot 5.95}}{5000 \cdot e^{-0.1 \cdot 5.95}} = 35.40 \checkmark$$

Considering a 0.1 % per annum increase in yields, the approximated effect on the value of portfolio A using only the duration is:

$$\frac{\Delta V_A}{V_A} \approx -D_A \cdot \Delta r = -5.95 \cdot 0.001 = -0.595\,\%$$

while, considering also the convexity, it is:

$$\frac{\Delta V_A}{V_A} \approx -D_A \cdot \Delta r + \frac{1}{2} \cdot C_A \cdot (\Delta r)^2 = -5.95 \cdot 0.001 + \frac{1}{2} \cdot 55.40 \cdot (0.001)^2 = -0.592\,\%$$

that is closer to the true change. In a similar way, the approximated effect on the values of portfolio B using only the duration is:

$$\frac{\Delta V_B}{V_B} \approx -D_B \cdot \Delta r = -5.95 \cdot 0.001 = -0.595\,\% \checkmark$$

while, considering also the convexity, it is:

$$\frac{\Delta V_B}{V_B} \approx -D_B \cdot \Delta r + \frac{1}{2} \cdot C_B \cdot (\Delta r)^2 = -5.95 \cdot 0.001 + \frac{1}{2} \cdot 35.40 \cdot (0.001)^2 = -0.593\,\% \checkmark$$

that is closer to the true change. Considering a 5 % per annum increase in yields, the approximated effect on the value of portfolio A using only the duration is:

$$\frac{\Delta V_A}{V_A} \approx -D_A \cdot \Delta r = -5.95 \cdot 0.05 = -29.75\,\% \checkmark$$

while, considering also the convexity, it is:

$$\frac{\Delta V_A}{V_A} \approx -D_A \cdot \Delta r + \frac{1}{2} \cdot C_A \cdot (\Delta r)^2 = -5.95 \cdot 0.05 + \frac{1}{2} \cdot 55.40 \cdot (0.05)^2 = -22.825\,\% \checkmark$$

that is closer to the true change. In a similar way, the approximated effect on the values of portfolio B using only the duration is:

$$\frac{\Delta V_B}{V_B} \approx -D_B \cdot \Delta r = -5.95 \cdot 0.05 = -29.75\,\% \checkmark$$

while, considering also the convexity, it is:

$$\frac{\Delta V_B}{V_B} \approx -D_B \cdot \Delta r + \frac{1}{2} \cdot C_B \cdot (\Delta r)^2 = -5.95 \cdot 0.05 + \frac{1}{2} \cdot 35.40 \cdot (0.05)^2 = -25.325\,\% \checkmark$$

that is closer to the true change. In conclusion, the approximation obtained using also the convexity are better than those obtained using only the duration since they are based on a 2nd order Taylor approximation, that is always better than a 1st order one.