Options:

We focus on options on stocks. We have two kind of options:

- CALL OPTIONS: options to buy $S(t)$
- PUT OPTIONS: " " sell $S(t)$

European Options: (At first analysis we'll focus only on these)

Options that can be exercised only at maturity T.

- European call option: is a contract giving the holder the right to buy an asset, called underlying, for a price K fixed in advance, known as the exercise price, or strike price, at a specified future time T, called the exercise or expiry time.

- European put option: gives the right to sell the underlying asset for the strike price K at the exercise time.

American options:

Options that can be exercise at any time between 0 and maturity T.

- American call option: gives the right to buy the underlying asset for the strike price K at any time between 0 and T.

- American put option: gives the right to sell the underlying asset for the strike price K at any time between 0 and T.

Different positions: (We will make the examples with European options)

For each option we have two possible positions:

CALL: → Long
→ Short

PUT: → Long
→ Short

CALL - Long: I buy the right of buying an underlying asset at a pre-specified strike price. (He has the asset)

CALL - Short: I sell the right of buying an underlying asset at a pre-specified strike price. (I have the asset)

PUT - Long: I buy the right of selling an underlying asset at a pre-specified strike price. (I have the asset)

PUT - Short: I sell the right of selling an underlying asset at a pre-specified strike price. (He has the asset)

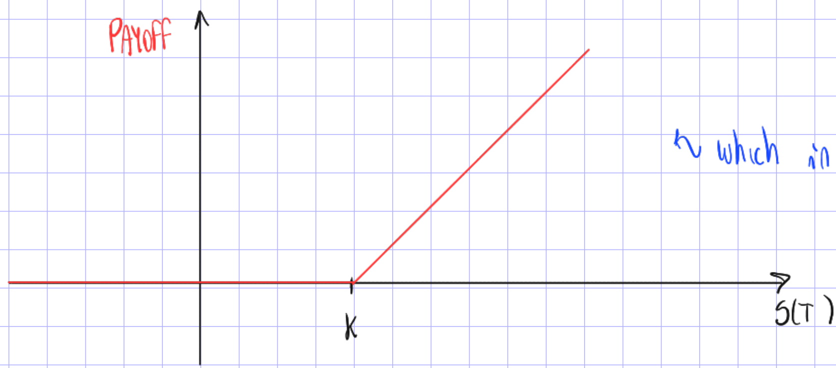
Now we will focus on call options:

CALL OPTIONS:

- For today we assume $r=0$ (no time value of the money)
- Notation:
  - $K :=$ strike price;
  - $S(t) =$ stock with maturity T.
  - $(x)^+ := \max\{x, 0\}$

Long Call:

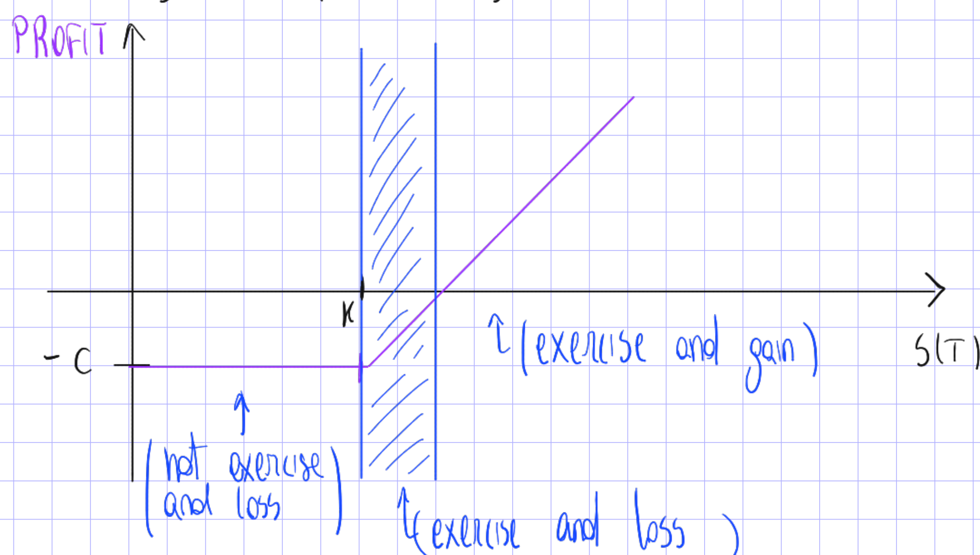
Payoff: $(S(T) - K)^+$ $\quad$ $\max\{0, S(T) - K\}$



↖ which is not linear everywhere.

Profit: Payoff - Call price $= (S(T) - K)^+ - C$

Having an option is not free, since it gives the right and not the obligation of exercising, thus buying an options is costly. We denote this cost as: C, call price.



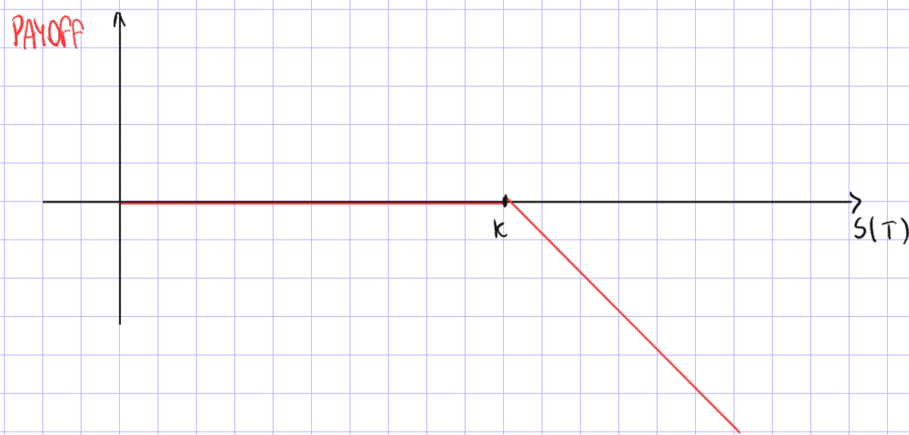
PROFIT Scheme (Profit = Payoff - Call price)

- If $S_T < K$ not exercise, then loss C
- If $S_T > K$ exercise:
  - If $S(T) > K + C$ PROFIT
  - If $S(T) < K + C$ LOSS

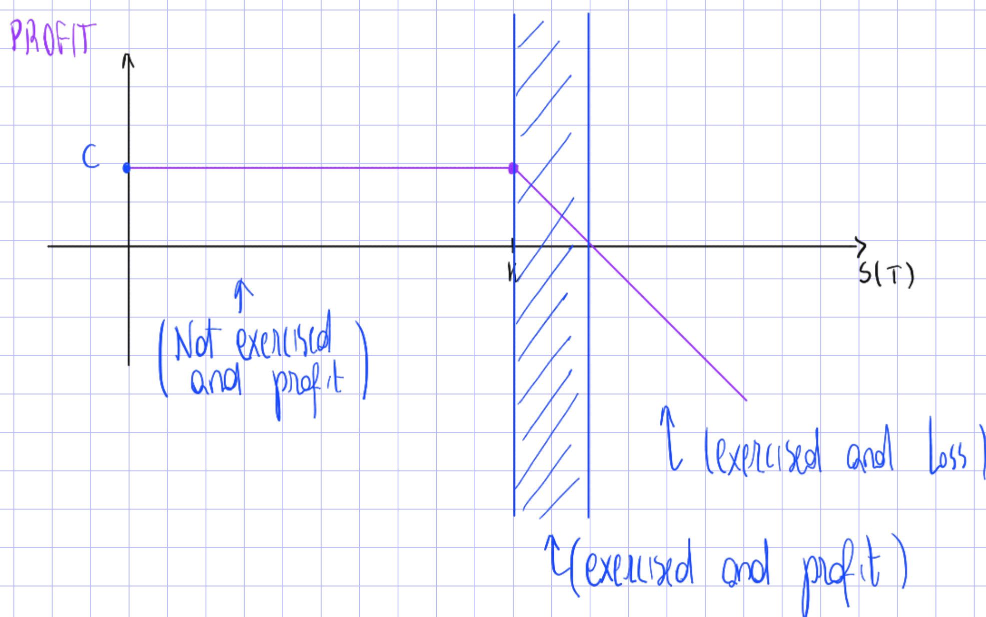
$[K \le S(T) \le K + C]$

Short Call :

Payoff : $-(S(T)-K)^{+} \quad -\max\{0, S(T)-K\} = \min\{0, K-S(T)\}$



Profit : $\text{Call price} + \text{Payoff} = C - (S(T)-K)^{+}$



PROFIT Scheme (Profit = Payoff + Call price)

- If $S_T < K$ not exercise and profit
- If $S_T > K$ exercise :
  - If $S(T) > K+C$ loss
  - If $S(T) < K+C$ Profit

$[K \leq S(T) \leq K+C]$

SUMMING UP (CALL)

POSITION	PAYOFF	PROFIT
LONG	$(S(T)-K)^+$	$(S(T)-K)^+ - C$
SHORT	$-(S(T)-K)^+$	$-(S(T)-K)^+ + C$

(Short = - Long)

Put Options:

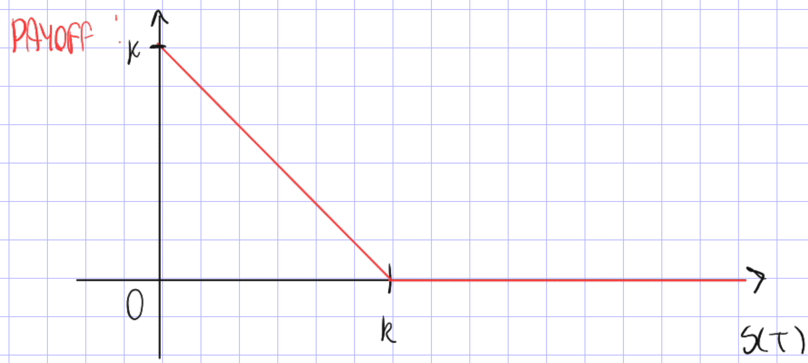
- For today we assume $r=0$ (no time value of the money)

- Notation :
  - $K :=$ strike price ;
  - $S(t) =$ stock with maturity T.
  - $(x)^+ := \max\{x, 0\}$

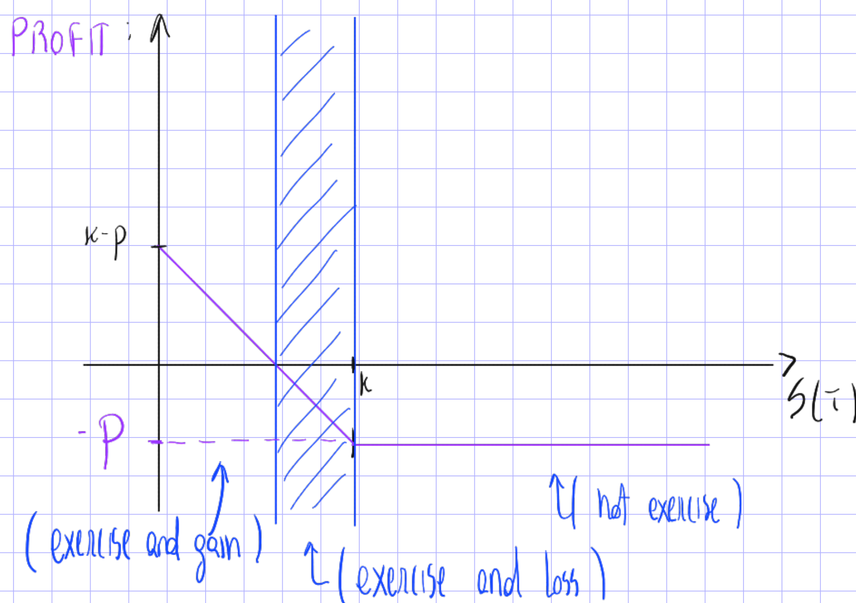
Long Put:

(I want to sell higher than its actual value)

Payoff: $(K-S(T))^+$ $\max\{K-S(T),\ 0\}$



Profit: payoff - put price $= (K-S(T))^+ - p$



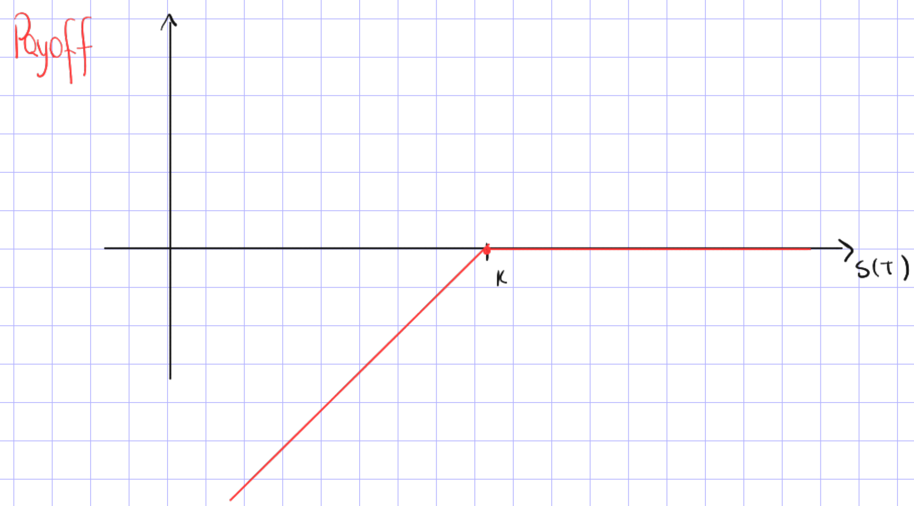
Profit scheme:

If $K-S(T)-p>0$ GAIN $S(T)<K-p$

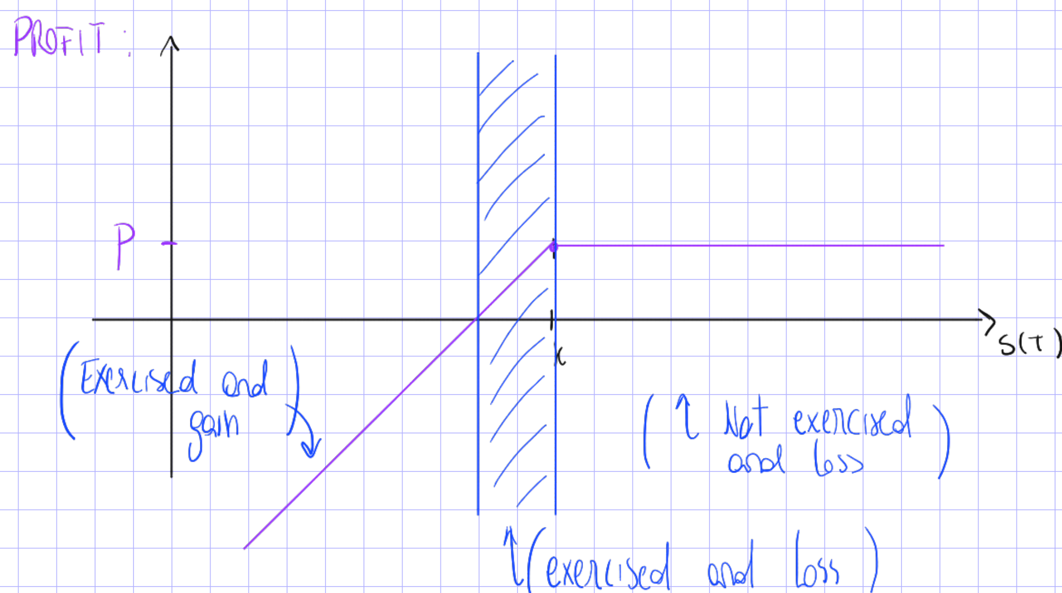
If $S(T)>K-p$ Loss also if $S(T)<K$ and the option is exercised.

Short Put:

Payoff: $-(K - S(T))^+$



Profit: Put price + Payoff $= p - (K - S(T))^+$



SUMMING UP (PUT)

POSITION	PAYOFF	PROFIT
LONG	$(K - S(T))^+$	$(K - S(T))^+ - p$
(Short = - Long) SHORT	$-(K - S(T))^+$	$p - (K - S(T))^+$

Terminology:

Definition: (Intrinsic Value of an option)

The intrinsic value of an option is the value of an option if it were exercised.

Definition: (Time value)

Time Value = Market value - Intrinsic value.

Other terms:

- At the money: If exercising the option corresponds to a null cashflow. ($S(T) = K$)
- In the money: If exercising the option corresponds to a positive cashflow.
  - ↳ examples: CALL: $S_T > K$
    PUT: $S_T < K$
- Out of money: If exercising the option corresponds to a negative cashflow.
  - ↳ examples: CALL: $S_T < K$
    PUT: $S_T > K$

PUT - CALL PARITY : (European Options)

Purpose : In this section we are going to find a link between the price of a call option and the price of a put option having same asset K and maturity T.

Scenario : Consider a portfolio costructed by : writing and selling one put (short-put) and buying one call option (long-call), both with the same strike price K and exercise date T ; portfolio $p = (1, -1)$

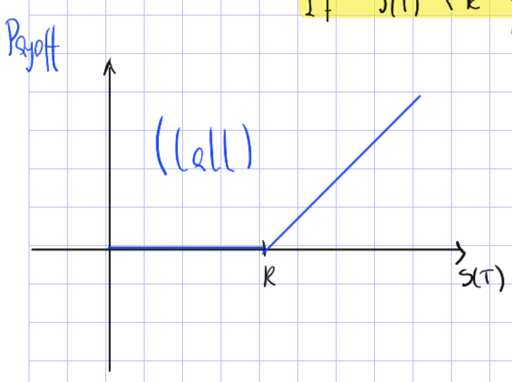
In order to find the payoff of this portfolio we add the payoff of the long position in call and the short position in put :

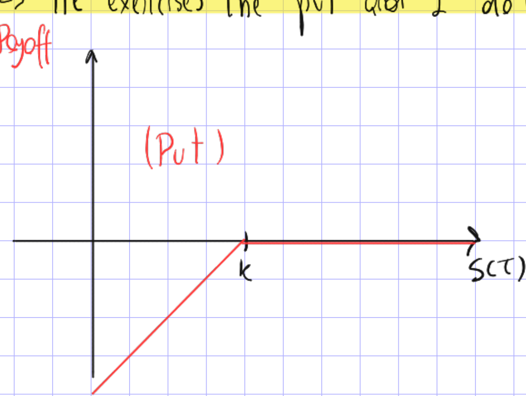
$$\begin{cases} \text{Short put} \xrightarrow[\text{(Payoff)}]{} -(K - S(T))^+ \\ \text{long call} \xrightarrow{} (S(T) - K)^+ \end{cases}$$

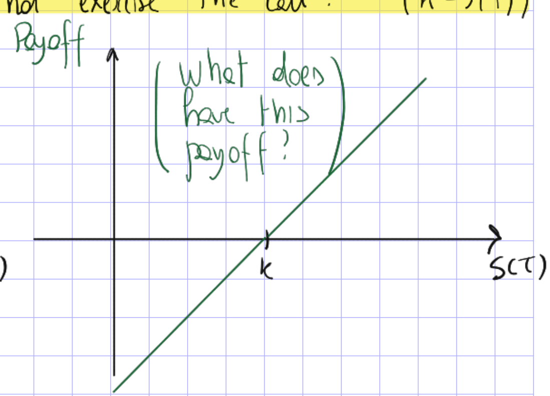
Payoff of P :

If $S(T) \geq K \Rightarrow$ I exercise the call and he does not exercise the put : $(S(T) - K)^+$

If $S(T) < K \Rightarrow$ He exercises the put and I do not exercise the call : $-(K - S(T))^+$







- Answer -

In either case the payoff of the portfolio will be $S(T) - K$, the same as a long forward position with forward price K and expire time T.

"The "forward price" in the portfolio $(1,-1)$

OBSERVATION:

- Typically K is decided by the two parties and it is not necessarly the risk-neutral forward price $F(0,T)$ given by the usual formula: $F(0,T) = S_0 e^{rT}$

In detail the value of a forward contract written at time 0 evaluated at time t:

$$\bullet)\quad V_F(t) = \Big[F(t,T) - F(0,T) \Big] \cdot e^{-r(T-t)}$$

If we choose instead of $F(0,T)$ the agreed forward price K then:

$$\bullet)\quad V_{F_K}(t) = \Big[F(t,T) - K \Big] \cdot e^{-r(T-t)}$$

In particular: $t=0$

$F(0,T) = S_0 e^{rT}$

$$\bullet)\quad V_{F_K}(0) = \Big[F(0,T) - K \Big] e^{-rT} = S_0 - K e^{-rT}$$

OBSERVATION: In order to the no-arbitrage to hold it must hold $V_P(0) = V_F(0)$, This leads to the following proposition.

$V_P(0) = C - P$

long call, short put

Proposition: For a stock that pays no dividends the following relation holds between the prices of European Call and Put options, both with exercise price K and exercise time T:

$$\boxed{C - P = S_0 - K e^{-rT}}$$

PUT - CALL PARITY

Proof: We prove it with no-arbitrage strategy.

1) If : $C - P \geq S_0 - K e^{-rT}$

At time 0 :

(They must be together) [•) If $S(0) + P - C < 0$ then borrow $S(0) + P - C$: $(+S(0) + P - C)$
•) If $S(0) + P - C > 0$ then invest $S(0) + P - C$: $(-(S(0) + P - C))$]
•) Write and sell a call option for C : $(+C)$
•) Buy a put option for P : $(-P)$
•) Buy a share for $S(0)$: $(-S(0))$

$V(0) = 0$

(Let's assume $S(0) + P - C > 0$, the other follows analogously)

At time T : (1^{st} scenario)
•) I take back the invested amount : $+((C - P - S(0))e^{rT}$
•) I sell the asset bought for $S(0)$: $+S(T)$
(1^{st} scenario) •) If the price of the asset at time T, $S(T)$, is $S(T) < K$ (the strike price) then I can exercise the put option I have bought for P, resulting in the following payoff : $(+K - S(T))$
•) The buyer of the call option does not exercise the option : (-0)

$V(T) = (C - P - S(0))e^{rT} + \cancel{S(T)} + K - \cancel{S(T)}$

$\Rightarrow V(T) > 0$ because $V(T)e^{-rT} > 0$ since $C - P - S(0) + K e^{-rT} > 0$

\# Arbitrage

At time T : (2^{nd} scenario)

(2nd scenario)

- I take back the invested amount : $+\left((c-p-S(0)\right)e^{rT}$
- I sell the asset bought for $S(0)$; $+(S(T))$
- If the price of the asset at time T $S(t)$ is s.t. $S(T) > K$, then I do not exercise the put option bought for p ; $(+0)$
- The buyer of the call option exercises which leads to : $-(S(T)-K)$

$$V(T) = (c-p-S(0))e^{rT} + S(T) - K - S(T)$$

$\Rightarrow V(T) > 0$ because $V(T)\cdot e^{-rT} > 0$ since

$$c - p - S(0) + Ke^{-rT} > 0$$

Arbitrage

2) If : $c - p < S_0 - e^{-rT}$

At time $t=0$:

(They must be together)

- If the amount $S(0)-c+p$ is >0 I invest it. $(-(S(0)-c+p))$
- If the amount $S(0)-c+p$ is <0 I borrow it. $(+S(0)-c+p)$
- I sell short one share $(+S(0))$
- Write and sell a put option $(+p)$
- Buy one call option $(-c)$

$$V(0) = 0$$

(Let's assume that $S(0)-c+p > 0$, the other case is analogous)

At time $t=T$; (1^{st} scenario)

(1^{st} scenario)

- Take back the investment $\left(+(S(0) - C + p)e^{rT} \right)$
- If the price of the asset at time T, $S(T)$ is such that $S(T) < K$ the owner of the put option will not exercise. (-0)
- In the 1^{st} scenario I will exercise the call option buying the asset at $K < S(T)$ $\left(+ S(T) - K \right)$
- I close the short selling position $\left(- S(T) \right)$

$$V(T) = (S(0) - C + p)e^{rT} + \cancel{S(T)} - K - \cancel{S(T)}$$

$V(T) > 0$ since $V(T)e^{-rT} > 0$ indeed

$$V(T)e^{-rT} = S(0) - C + p - Ke^{-rT} \overset{?}{>} 0$$

$$\downarrow$$

$$S(0) - Ke^{-rT} > C - p \quad \underline{\text{Ok}}$$

$\#$ Arbitrage.

At time $t=T$; (2^{nd} scenario)

(2^{nd} scenario)

- Take back the investment $\left(+(S(0) - C + p)e^{rT} \right)$
- If the price of the asset at time T, $S(T)$ is such that $S(T) > K$ the owner of the put option will exercise: $\left(-(K - S(T)) \right)$
- In the 2^{nd} scenario I will not exercise the call option $(+0)$
- I close the short selling position $\left(- S(T) \right)$

$$V(T) = (S(0) - C + p)e^{rT} - K + \cancel{S(T)} - \cancel{S(T)}$$

$V(T) > 0$ since $V(T)e^{-rT} > 0$ indeed

$S(0) - C + p - Ke^{-rT} > 0$ as already checked above

$\#$ Arbitrage

Put / Call parity with Dividends:

If an asset $S(t)$ pays dividends between 0 and T, being $div(0)$ the current value of dividends it holds:

$$C - P = S(0) - div(0) - Ke^{-rT}$$

(Proof. left for home)

Put / Call parity bounds (American Options)

Theorem:

The put, p^A, and call, c^A, prices of an american option with the same underlying asset, strike price K and expiry time T that does not pay dividends satisfy:

$$S(0) - Ke^{-rT} \geq c^A - p^A \geq S(0) - K$$

If the asset pays dividends:

$$S(0) - Ke^{-rT} \geq c^A - p^A \geq S(0) - div(0) - K$$

Proof. (Sugli extra)

Bounds on option prices: $c = c^E$, $p = p^E$

•) Obviously $c \le c^A$ $\wedge$ $p \le p^A$ since the european options are a sub-case of the americans.

Proof.

1) $c > c^A$ (or $p > p^A$ is literally the same)

At time $t=0$

•) I sell short an european option $(+c)$ $(+p)$

•) I buy an american option $(-c^A)$ $(-p^A)$

•) I invest $c - c^A > 0$ $(-(c-c^A))$ $(-(p-p^A))$

$V(0) = 0$

At time $t = T$

•) I take back my investment $(+(c-c^A)e^{rT})$ $(p-p^A)e^{rT}$

•) I copy whatever the other does $(+x-x)$ $(+x-x)$

$V(T) = (c-c^A)e^{rT} > 0$ $\quad$ $\left(V(T) = (p-p^A)e^{rT} > 0 \right)$

$\square$

Bounds on European Call options:

We are going to prove these two inequalities holding in the case of European call options: (If $S(t)$ pays dividends replace $S(0)$ with $S(0) + div(0)$)

1) $C \leq S(0)$
2) $C \geq S(0) - K e^{-rT}$

Proof.

1) $C \leq S(0)$

We prove this by showing that $C > S(0)$ leads to an arbitrage strategy.

At time $t=0$:

·) I write and sell an european call option $(+ C^E)$

·) I buy the asset $S(0)$ $(-S(0))$

·) I invest the amount $C^E - S(0)$ $(-(C^E - S(0)))$

$V(0) = 0$

At time $t=T$

·) I take back my investment, cashing $(+(C^E - S(0))e^{rT})$

·) I sell the asset $(+S(T))$

·) If the owner of the call option exercises I lose $S(T)-K$ $(-(S(T)-K))$

$V(T) = (C^E - S(0))e^{rT} + \cancel{S(T)} - \cancel{S(T)} + K$

$V(T) > 0$ since $V(T)e^{-rT} > 0$ indeed $C^E - S(0) > 0$ by hp.

and $K > 0$. # Arbitrage.

2) $C \geq S(0) - Ke^{-rT}$

The proof of this is immediate since $p^E \geq 0$ trivially, since it guarantees ≥ 0 payoff.

And it holds in general $c^E - p^E = S(0) - Ke^{-rT}$ by the put/call parity. Thus $c^E \geq S(0) - Ke^{-rT}$.

$\boxtimes$

Bounds on european put options:

We are going to prove these two inequalities holding in the case of European put options: (If $S(t)$ pays dividends replace $S(0)$ with $S(0) + div(0)$)

1) $p \leq Ke^{-rT}$
2) $-S(0) + Ke^{-rT} \leq p$

Proof.

1) $p \leq Ke^{-rT}$

- $C - p = S(0) - Ke^{-rT}$
- $C \leq S(0)$
- $-p \geq -Ke^{-rT} \Rightarrow p \leq Ke^{-rT}$

2) $-S(0) + Ke^{-rT} \leq p$

- $C \geq 0$
- $p - C = Ke^{rT} - S(0)$
- $p \geq Ke^{rT} - S(0)$

$\boxtimes$

No dividend american option theorem:

Theorem:

The price of an american call option on an asset that does not pay dividends is equal to the price of an european call option on the same underlying asset with strike price K and maturity T, i.e.: $C^A = C^E$

Proof.

- $C^A \geq C^E$ trivial and already proved.
- $C^A \leq C^E$

$\hookrightarrow$ Let's assume: $C^A > C^E$

At time $t=0$

- I sell the american call option (short) $(+C^A)$
- I buy the european call option (long) $(-C^E)$
- I invest the positive difference $(-(C^A - C^E))$

$V(0) = 0$

At time $t = t^*$ $\quad t^* \in (0, T)$

- The american call option is exercised, I suffer a loss $(-(S(t) - K))$
- I short sell the asset at $S(t)$ $(+S(t))$
- I invest the difference $(-(S(t) - S(t) - K) = K)$

$V(t^*) = 0$

At time $t = T$

·) I buy the share at K by using the european call option and I give back the short-sold asset : $(-K)$

·) I take back my investments : $+(C^A - C^E)e^{rT} + Ke^{r(T-t)}$

$$V(T) = \underbrace{(C^A - C^E)}_{\text{(by assumption)}} e^{rT} + \underbrace{K \underbrace{e^{r(T-t)}}_{>1} - K}_{>0}$$

$\Rightarrow V(T) > 0$ # Arbitrage

$\boxtimes$

OBS. This implies that is never convenient to exercise at $t < T$.

7 Recap Lezione 7 – Options, Payoff, Parity and Bounds

Idea generale della lezione La lezione introduce le **opzioni su azioni**, analizzandone payoff e profitto per le diverse posizioni, e sviluppa i principali risultati di **non arbitraggio**, in particolare la **put–call parity**, i **price bounds** per opzioni europee e americane e il teorema di equivalenza tra call americana ed europea in assenza di dividendi.

Tipologie di opzioni Si considerano opzioni su azioni, distinguendo

- **Call option**: diritto di acquistare l'asset sottostante
- **Put option**: diritto di vendere l'asset sottostante

European e American options

- **European options**: esercitabili solo a maturity T
- **American options**: esercitabili in qualunque istante $t \in [0, T]$

Assunzioni e notazione Si assume inizialmente $r = 0$. La notazione utilizzata è

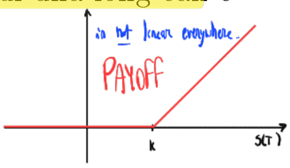
- K strike price
- $S(T)$ prezzo dell'asset a maturity
- $(x)^+ = \max\{x, 0\}$

Posizioni long e short Per ciascuna opzione si distinguono due posizioni

- **Long**: acquisto (buy) del diritto
- **Short**: vendita (sell) del diritto

Vale sempre la relazione short $= -$long.

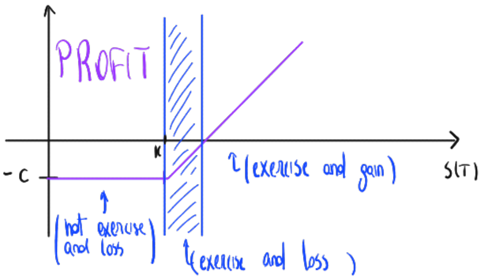
Long call Il payoff di una long call è



$$(S(T) - K)^+$$

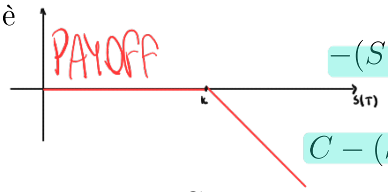
Il profit è

$$(S(T) - K)^+ - C$$



dove C è il call price. Si ha perdita limitata pari a C e profitto potenzialmente illimitato. Il break-even è $S(T) = K + C$.

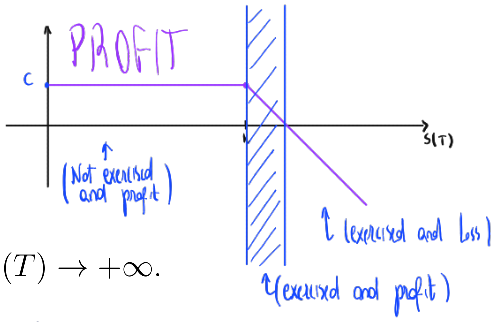
Short call Il payoff è



$$-(S(T) - K)^+$$

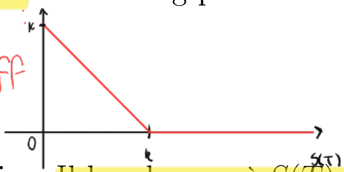
Il profit è

$$C - (S(T) - K)^+$$



Il guadagno massimo è limitato a C, mentre la perdita è illimitata per $S(T) \to +\infty$.

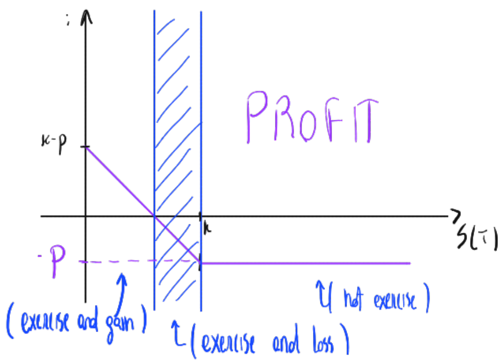
Long put Il payoff di una long put è



$$(K - S(T))^+$$

Il profit è

$$(K - S(T))^+ - P$$



dove P è il put price. Il break-even è $S(T) = K - P$.

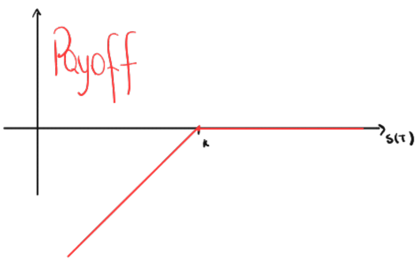
Short put Il payoff è

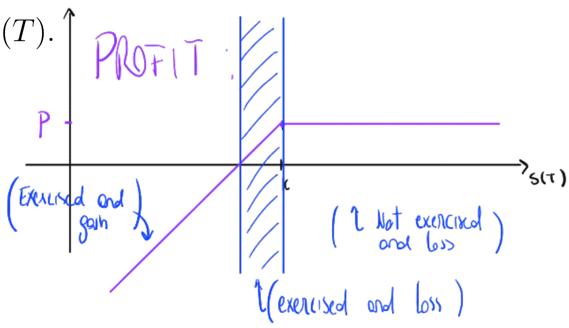
$$-(K - S(T))^+$$

Il profit è

$$P - (K - S(T))^+$$

Il guadagno massimo è P, mentre la perdita cresce al diminuire di $S(T)$.





Terminologia Si introducono le seguenti definizioni

- **Intrinsic value**: valore dell'opzione se fosse esercitata immediatamente
- **Time value**: differenza tra market value e intrinsic value

Inoltre

- **At the money**: cashflow nullo
- **In the money**: cashflow positivo
- **Out of the money**: cashflow negativo

Put–call parity per European options Per opzioni europee su asset che non paga dividendi vale

$$C - P = S(0) - Ke^{-rT}$$

La relazione è ottenuta confrontando il payoff di un portafoglio long call e short put con quello di una long forward position.

Put–call parity con dividendi Se l'asset paga dividendi con valore attuale div(0), la parità diventa

$$C - P = S(0) - \mathrm{div}(0) - Ke^{-rT}$$

Bounds per European call Il prezzo di una European call soddisfa

$$S(0) - Ke^{-rT} \leq C \leq S(0)$$

I bounds derivano da argomenti di non arbitraggio.

Bounds per European put Il prezzo di una European put soddisfa

$$-S(0) + Ke^{-rT} \leq P \leq Ke^{-rT}$$

Bounds e parità per American options Per opzioni americane su asset senza dividendi vale

$$S(0) - Ke^{-rT} \geq C^A - P^A \geq S(0) - K$$

Con dividendi

$$S(0) - Ke^{-rT} \geq C^A - P^A \geq S(0) - \mathrm{div}(0) - K$$

Relazione tra European e American prices Le opzioni europee sono un caso particolare di quelle americane, quindi

$$C \leq C^A, \qquad P \leq P^A$$

No-dividend American call theorem Se l'asset non paga dividendi, il prezzo della call americana coincide con quello della call europea

$$C^A = C^E$$

Ne segue che non è mai ottimale esercitare anticipatamente una American call in assenza di dividendi.