Politecnico di Torino
Financial Engineering-Exam 07-10-2025
P. Semeraro

SURNAME and NAME

Student number

All the answers must be clearly motivated, the numerical results are not sufficient. All the answers that will be considered in the correction MUST be written here. If the student number and the name are not filled in the text will not be corrected.

Answers written with a pencil are null.

Exercise 1 (10 points) A stock is currently priced at S(0)=4\$. In one year, it is believed that the stock's price may go up to 8\$ or down to 2\$.

1. Find a range for the risk-free interest rate r, with simple compounding, such that the market composed by the stock S and a riskless bond B is arbitrage free.
2. Set $r = 25\%$. Compute the price of a call option with strike price $K = 5\$$ and maturity in one year using the risk-neutral evaluation method.
3. A bank holds a long position in the call option of point 2), and wishes to hedge its position by trading the stock S and the bond B in the market. Describe a hedging strategy that allows the bank to earn the risk-free interest rate over the capital invested in the call option.

1) $\mathcal{M} = (S, B)$ is arbitrage free iff (see theory and you should argue)

$$d \leq 1+r \leq u.$$

For this problem, $\begin{cases} d = \frac{1}{2} \\ u = 2 \end{cases} \Rightarrow \boxed{-0.5 \leq r \leq 1}$

2) Using the risk-neutral evaluation method, we first compute the martingale measure of $S(t)$ s.t. $S(0) = \mathbb{E}^{Q}\left[\frac{S(T)}{1+r}\right]$. Then, the option price is $P = \frac{1}{1+r}\mathbb{E}^{Q}[\phi(S_T)]$

Computing Q reduces to find (q_u, q_d) s.t. $\begin{cases} 1+r = q_u u + q_d d \\ q_u + q_d = 1. \end{cases}$

$$\Rightarrow \begin{cases} q_u = \dfrac{1+r-d}{u-d} = 0.5. \\ q_d = \dfrac{u-(1+r)}{u-d} = 0.5. \end{cases}$$

S_T $\qquad \phi(S_T) = (S_T - K)_+ \qquad K = 5\$.$

Next,

$$S(0) = 4 \begin{array}{l} \xrightarrow{u} 8 \qquad 3 \\ \xrightarrow{d} 2 \qquad 0 \end{array}$$

Hence, $P = \frac{1}{1+r}\left[0.5 \cdot 3 + 0.5 \cdot 0\right] = 1.2.$

3) To hedge the position, the bank should sell short $\Delta = \frac{\phi(\text{"u"}) - \phi(\text{"d"})}{S(u-d)} = 0.5$ unit invest the $\Delta \cdot 4 = 2\$$ in the bond market.

Then at time $T=1$, the CASH flows are:

IF event "u" happens

$+3$ (PAYOFF long position)

-4 (close short position

$2\cdot(1+r)$ (return bond)

$1.5 = 1.2(1+r)$

IF event "d" happens

0 (zero payoff from call)

-1 (close short position)

$2(1+r)$ (return of the bond)

$1.5 = 1.2(1+r)$

1.2 being the capital invested in the call option.

Exercise 2 (10 points)

Table 1 reports the prices of a set of bonds available for trading in the market. All coupons are paid with a semiannual frequency.

Maturity (years)	Coupon rate	Face value	Price
0.5	0	100	98.46
1	6%	100	102.31
1.5	3%	100	98.51
2	5%	100	101.46

Table 1

1. Compute the implied continuously compounded spot interest rates $r(0,t)$, $t \in \{0.5, 1, 1.5, 2\}$.
2. Consider a 5-year zero-coupon bond and 2-year bond that pays a coupon of 2% on an annual basis. The current, constant, continuously compounded interest rate is $y = 5\%$. Both bonds have face value of 100\$. Compute their prices.
3. Which one of the two bonds is more sensitive to small decrease of y?

1) To deduce the structure $r(0,T)$ of the spot rates we use the bootstrap method.

Setting $r_1 = r(0,\frac{1}{2})$, $r_2 = r(0,1)$, $r_3 = r(0,\frac{3}{2})$, $r_4 = r(0,2)$, we obtain the linear system

$$\begin{cases} e^{-\frac{1}{2}r_1} 100 = 98.46 \\ 3e^{-\frac{1}{2}r_1} + 103 e^{-r_2} = 102.37 \\ 1.5 e^{-\frac{1}{2}r_1} + 1.5 e^{-r_2} + 101.5 e^{-\frac{3}{2}r_3} = 98.51 \\ 2.5 e^{-\frac{1}{2}r_1} + 2.5 e^{-r_2} + 2.5 e^{-\frac{3}{2}r_3} + 102.5 e^{-2r_4} = 101.46 \end{cases}$$

From which,

$$\begin{cases} r_1 = 3.1\% \\ r_2 = 3.6\% \\ r_3 = 4.0\% \\ r_4 = 4.2\% \end{cases}$$

(2) Zero coupon with $T=5$. $Z = e^{-0.05 \cdot 5} \cdot 100 = 77.88$

C, coupon rate 2%, $T=2$. $C = 2e^{-0.05 \cdot 1} + 102 \cdot e^{-0.05 \cdot 2} = 94.19$

(3) We compute the duration of the two bonds in order to understand which one is more sensitive to a small decrease of y.

$$D_Z = -\frac{\partial_y Z}{Z} = -\frac{\partial_y e^{-y \cdot T}}{e^{-y \cdot T}} = T = 5,$$

$$D_C = -\frac{1}{C}\left[-1 \cdot 2 e^{-0.05 \cdot 1} - 2 \cdot 102 \cdot e^{-0.05 \cdot 2}\right] = 2.03,$$

hence the price of the zero coupon bond is more sensitive to small variations of y.

Exercise 3 (10 points) Let $dS = 0.06Sdt + 0.12SdW$ and risk-free interest rate is 2% per annum (all rates are continuously compounded).
(a) Price a European call option with strike price 60, maturing in 2 years, written on the stock $S(t)$ (no dividends) whose current price $S(0)$ is 57.
(b) If an investor holds 100 such calls and wishes a portfolio that is not sensitive to variations of the underlying asset (on the short term), how many stock shares should she hold?
(c) Find the probability that the option is exercised.

(a) $S_0 = 57$

$K = 60$

$T = 2$

$r = 0.02$

$\sigma = 0.12$

$$e^{-rT} = e^{-0,02 \times 2} = 0,96$$

$$d_1 = \frac{\ln(57/60) + \left(0.02 + \frac{0.12^2}{2}\right) \cdot 2}{0.12\sqrt{2}} = 0.018$$

$$d_2 = d_1 - 0.12\sqrt{2} = -0.151$$

$N(d_1) = 0.507$; $N(d_2) = 0.4398$

the call price is

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2) =$$
$$= 57 \times 0,507 - 60\ 0.96 \times 0.4398 = 3.56$$

$$\boxed{C = 3.56}$$

(b) $P = 100C + nS$

A portfolio is not sensitive to variations of S if $\Delta_P = 0$

$\Delta_P = 100\Delta_C + n$ because $\Delta_S = 1$

Since $\Delta_C = N(d_1) = 0.507$

$\Delta_P = 50.7 + n$ and $\Delta_P = 0$

if $n = -50.7$

we have to short 50.7 shares

(c) $P(S(T) > 60) =$

$P(\ln S(T) > \ln 60)$

$\ln S(T) \sim N\left(\ln S(0) + (\mu - \sigma^2/2)T,\ \sigma^2 T\right) = N(4.1487, 0.03)$

$$P\left(Z > \frac{\ln 60 - \ln \overset{57}{S(0)} - (\overset{0.06}{\mu} - \overset{0.12}{\sigma^2}/2)T}{\sigma\sqrt{T}}\right) =$$

$= P(Z > -0.32) = 1 - P(Z < 0.32) = 0.6255$