Politecnico di Torino
Financial Engineering-Exam 02-07-2022
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SURNAME and NAME

Student number

Exercise 1. (12 points)
Consider a stock price S_t that follows a Binomial tree. Let the stock current price be $S_0 = 100\$$. Suppose that there are 2 time steps of 1 year and in each time step the stock price either moves up by a factor of 1.5 or moves down by a factor of 0.5. Suppose that the risk-free interest rate is 1% per annum with simple compounding.
(a) Consider a 2-year option with payoff

$$\phi(S_T) = \Big(\frac{S_T - S_0}{2} - K\Big)^+, \tag{0.1}$$

where $K = \$40$. Find the delta of the option in $t = 1$ and the option price.
(b) Find the replicating strategy (horizon one year) of a portfolio formed by two short positions on a call option on S with maturity one year and strike K=60 and one long position in the underlying S.
(c) Suppose now that the stock price does not follow a binomial tree anymore, but that in the next two years it may either increase by a growth factor $u = 2.25$, decrease by $d = 0.25$ or remain unchanged. Show whether it is possible to find a unique martingale measure to price the option with payoff (0.1).

Exercise 2 (12 points)
A market is characterized by a flat term structure with annual interest rate of 4%. A bond with maturity 3 years and nominal value 100\$ pays annual coupons equal to 5\$.
(a) Find the current price of the bond.
(b) Find the duration and the convexity of the bond.
(c) Assume that the term structure shifs upward of 1 percentage point, find the new price at $t = 0$ of the bond and compare it with the price that can be obtained using a first order approximation and a second order approximation and comment on the result.
(d) Calculate the value of the bond at $t = D$ (where D is the duration) before and after the shift in the term structure, and comment on the result.

Exercise 3 (8 points)
A stock is quoted at the same time in the New York Stock Exchange and in the Tokyo Stock Exchange. The current price of this stock is 70\$ in New York and 8000 yen in Tokyo. The exchange rate is 114.10 yen for each \$.

(a) Describe the arbitrage opportunity that arises in this situation (for simplicity assume that there are no transaction costs) and the profit that can be obtained trading 5000 stocks.

(b) Describe what happens as long as traders exploit this opportunity.

(c) Keeping constant the quotation in Tokyo, find the quotation in New York that would avoid arbitrage opportunities.

All the answers must be clearly motivated.

EXERCISE 1

We have:

$$S_0 = 100 \qquad u = 1{,}5 \qquad r = 0{,}01 \text{ per annum with simple compounding}$$
$$K = 40 \qquad d = 0{,}5 \qquad T = 2 \text{ years with } \Delta t = 1$$
$$\phi(S_T) = \left(\frac{S_T - S_0}{2} - K \right)^+ \text{ option payoff}$$

and the price dynamics of the stock is:

$$\begin{tikzcd}[column sep=small]
 & & S_{uu} = 225 \\
 & S_u = 150 \arrow[ur] \arrow[dr] & \\
S_0 = 100 \arrow[ur] \arrow[dr] & & S_{ud} = 75 \\
 & S_d = 50 \arrow[ur] \arrow[dr] & \\
 & & S_{dd} = 25 \\
t=0 & t=1 & t=2
\end{tikzcd}$$

while the price dynamics of the option is:

$$\begin{tikzcd}[column sep=small]
 & & f_{uu} = \left(\frac{225-100}{2} - 40\right)^+ = 22{,}5 \\
 & f_u \arrow[ur] \arrow[dr] & \\
f_0 \arrow[ur] \arrow[dr] & & f_{ud} = \left(\frac{75-100}{2} - 40\right)^+ = 0 \\
 & f_d \arrow[ur] \arrow[dr] & \\
 & & f_{dd} = \left(\frac{25-100}{2} - 40\right)^+ = 0 \\
t=0 & t=1 & t=2
\end{tikzcd}$$

Using the risk-neutral valuation, the probability of an up movement is:

$$p = \frac{(1 + r\,\Delta t) - d}{u - d} = \frac{(1 + 0{,}01 \cdot 1) - 0{,}5}{1{,}5 - 0{,}5} = 0{,}51$$

Then at $t = 1$ we have:

$$f_u = \frac{1}{1 + r \cdot \Delta t} \left[f_{uu} \cdot p + f_{ud} \cdot (1-p) \right] =$$
$$= \frac{1}{1 + 0{,}01 \cdot 1} \left[22{,}5 \cdot 0{,}51 + 0 \cdot (1 - 0{,}51) \right] =$$
$$= 11{,}36$$

and also:

$$f_d = \frac{1}{1+r\cdot\Delta t}\left[f_{ud}\cdot p + f_{dd}\cdot(1-p)\right] =$$

$$= \frac{1}{1+0{,}01\cdot 1}\left[0\cdot 0{,}51 + 0\cdot(1-0{,}51)\right] =$$

$$= 0$$

and at $t=0$ we have:

$$f_0 = \frac{1}{1+r\cdot\Delta t}\left[f_u\cdot p + f_d\cdot(1-p)\right] =$$

$$= \frac{1}{1+0{,}01\cdot 1}\left[11{,}36\cdot 0{,}51 + 0\cdot(1-0{,}51)\right] =$$

$$= 5{,}74$$

The delta of the option in $t=1$ is:

$$\Delta_1 = \frac{f_u - f_d}{S_u - S_d} = \frac{11{,}36 - 0}{150 - 50} = 0{,}11$$

Considering a 1-year horizon the price dynamics of the stock is:

$$S_0 = 100 \quad \begin{cases} S_u = 150 \\ S_d = 50 \end{cases}$$

$t=0$ $\qquad$ $t=1$

while the price dynamics of the call option with $K=60$ is:

$$C_0 \quad \begin{cases} C_u = \max(150-60, 0) = 90 \\ C_d = \max(50-60, 0) = 0 \end{cases}$$

$t=0$ $\qquad$ $t=1$

and considering a portfolio formed by 2 short positions on the call and 1 long position in the stock, the payoff of this portfolio is:

$$\phi_\pi = -2\,\phi_{call} + S_T$$

that is (at $t=1$ in the "up" and "down" states of nature):

$$\phi_\pi^u = -2 \cdot 90 + 150 = -30$$
$$\phi_\pi^d = -2 \cdot 0 + 50 = 50$$

In order to replicate this portfolio we have to consider the following strategy:

$$\begin{cases} x(1+rt) + y\,S_u = \phi_\pi^u \\ x(1+rt) + y\,S_d = \phi_\pi^d \end{cases} \quad \text{with} \quad \begin{array}{l} x = \text{amount invested in the bond} \\ y = \text{units of the stock} \end{array}$$

i.e.:

$$\begin{cases} x \cdot (1+0{,}01 \cdot 1) + y \cdot 150 = -30 \\ x \cdot (1+0{,}1 \cdot 1) + y \cdot 50 = 50 \end{cases}$$

and subtracting the second equation from the first one:

$$100\,y = -80 \quad \rightarrow \quad y = -0{,}8$$

and then:

$$x \cdot 1{,}01 - 0{,}8 \cdot 150 = -30 \rightarrow 1{,}01 \cdot x = 90 \rightarrow$$
$$\rightarrow x = 89{,}11$$

hence:

$$\begin{array}{ll} x = 89{,}11 & \text{amount of the bond} \\ y = -0{,}8 & \text{units of the stock shorted} \end{array}$$

If the stock price follows the dynamics:

$$S_0 = 100 \quad \begin{array}{l} S_{uu} = 225 \\ S_{ud} = 100 \\ S_{dd} = 25 \end{array}$$

$$t=0 \qquad\qquad t=2$$

the price dynamics of the first option is:

$$f_0 \quad \begin{array}{l} f_{uu} : \left(\dfrac{225-100}{2} - 40 \right)^+ = 22{,}5 \\ f_{ud} = \left(\dfrac{100-100}{2} - 40 \right)^+ = 0 \end{array}$$

$$f_{ud} = \left(\frac{\quad}{2} - \ldots \right)$$

$$f_{dd} = \left(\frac{25 - 100}{2} - 40 \right)^+ = 0$$

In this case since:

$$d < 1+r < u \quad \longrightarrow \quad 0,25 < 1+0,1 < 2,25$$

the market is arbitrage free and we look for a martingale measure such that:

$$E\left[S_1 \mid S_0 = 100 \right] = 100$$

that is:

$$\begin{cases} 225 \cdot p_u + 100 \cdot p_c + 25 \cdot p_d = 100 \cdot (1 + 0,01 \cdot 2) \\ p_u + p_c + p_d = 1 \end{cases}$$

If A is the matrix of coefficients of the system, i.e.:

$$A = \begin{bmatrix} 225 & 100 & 25 \\ 1 & 1 & 1 \end{bmatrix}$$

we have:

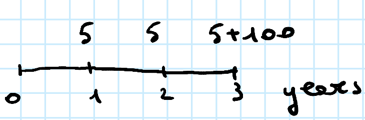
$$\begin{vmatrix} 225 & 100 \\ 1 & 1 \end{vmatrix} \neq 0$$

hence the matrix A has rank 2, moreover:

$$r(A) = r(A^c) < 3$$

Therefore the system has solution, but it is not unique.
In conclusion, the market is not complete because the system considered above doesn't have a unique solution, therefore the martingale measure is not unique.

EXERCISE 2
We have:



$y = 0,04$

and the current price of the bond is:

$$B = \sum^{m} c\, e^{-y t_i} = 5 \cdot e^{-0,04 \cdot 1} + 5 \cdot e^{-0,04 \cdot 2} + 105 \cdot e^{-0,04 \cdot 3} =$$

... price of the bond is:

$$B = \sum_{i=1}^{m} c_i \, e^{-y t_i} = 5 \cdot e^{-0{,}04 \cdot 1} + 5 \cdot e^{-0{,}04 \cdot 2} + 105 \cdot e^{-0{,}04 \cdot 3} =$$

$$= 102{,}55$$

The duration of the bond is:

$$D = \frac{\sum_{i=1}^{m} t_i \, c_i \, e^{-y t_i}}{B} = \frac{1 \cdot 5 \cdot e^{-0{,}04 \cdot 1} + 2 \cdot 5 \cdot e^{-0{,}04 \cdot 2} + 3 \cdot 105 \cdot e^{-0{,}04 \cdot 3}}{102{,}55} =$$

$$= 2{,}86 \text{ years}$$

and the convexity is:

$$C = \frac{\sum_{i=1}^{m} t_i^2 \, c_i \, e^{-y t_i}}{B} = \frac{1^2 \cdot 5 \cdot e^{-0{,}04 \cdot 1} + 2^2 \cdot 5 \cdot e^{-0{,}04 \cdot 2} + 3^2 \cdot 105 \cdot e^{-0{,}04 \cdot 3}}{102{,}55} =$$

$$= 8{,}40$$

If the term structure shifts upward of 1 percentage point, moving from 4% to 5%, the new price of the bond at $t=0$ becomes:

$$B' = \sum_{i=1}^{m} c_i \, e^{-y t_i} = 5 \cdot e^{-0{,}05 \cdot 1} + 5 \cdot e^{-0{,}05 \cdot 2} + 105 \cdot e^{-0{,}05 \cdot 3} =$$

$$= 99{,}65$$

Using the 1st order approximation based on the duration the change in the value of the bond is:

$$(\Delta B)_{I} = -D \cdot \Delta y \cdot B = -2{,}86 \cdot 0{,}01 \cdot 102{,}55 = -2{,}93$$

hence the new price of the bond is:

$$(B')_{I} = B + (\Delta B)_{I} = 102{,}55 - 2{,}93 = 99{,}62$$

while using the 2nd order approximation based on the convexity the change in the value of the bond is:

$$(\Delta B)_{II} = -D \cdot \Delta y \cdot B + \frac{1}{2} \cdot C \cdot (\Delta y)^2 \cdot B =$$

$$= -2{,}86 \cdot 0{,}01 \cdot 102{,}55 + \frac{1}{2} \cdot 8{,}40 \cdot 0{,}01^2 \cdot 102{,}55 =$$

$$= -2{,}89$$

hence the new price of the bond is:

$$(B')_{\Gamma} = B + (\Delta B)_{\Gamma} = 102,55 - 2,89 = 99,66$$

and the 2^{nd} order approximation is more precise than the 1^{st} order approximation, as always happens using Taylor expansion.

The value of the bond at $t = D = 2,86$ before the change in the interest rate is:

$$V = 5 \cdot e^{0,04 \cdot (2,86-1)} + 5 \cdot e^{0,04 \cdot (2,86-2)} + 105 \cdot e^{0,04 \cdot (2,86-3)} =$$

$$= 114,97$$

and the value always at $t = D = 2,86$ after the change in the interest rate is:

$$V' = 5 \cdot e^{0,05 \cdot (2,86-1)} + 5 \cdot e^{0,05 \cdot (2,86-2)} + 105 \cdot e^{0,05 \cdot (2,86-3)} =$$

$$= 114,97$$

hence the change in the interest rate does not change the value of the bond in correspondence of the duration, that is the property of immunization.

EXERCISE 3

We have:

price of the stock in New York = 70 \$
price of the stock in Tokyo = 8'000 yen
exchange rate = 114,10 yen for 1 \$

and then:

$$70 \cdot 114,10 = 7'987 \text{ yen for 1 stock in New York}$$

or:

$$\frac{8'000}{114,10} = 70,11 \text{ \$ for 1 stock in Tokyo}$$

hence it is possible to buy the stocks in New York and to sell them in Tokyo, obtaining for each stock a profit of:

$$8'000 - 7'987 = 13 \text{ yen}$$

and trading 5'000 stocks:

$$13 \cdot 5'000 = 65'000 \text{ yen}$$

or:

$$\frac{65'000}{114,10} = 569,68 \ \$$$

As long as traders exploit the arbitrage opportunity, the price of the stock increases in New York and decreases in Tokyo, so that the arbitrage opportunity disappears.

In order to avoid arbitrage opportunities the quotation of 1 stock in New York, keeping constant the quotation in Tokyo, should be:

$$\frac{8'000}{114,10} = 70,11 \ \$$$

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All the answers must be clearly motivated, the numerical results are not sufficient.

Answers written with a pencil are null.

Exercise 1 (12 points)

Consider a European call option and a European put option on the same stock, with strike price 45$ and maturity 2 years. The current price of the stock is 50$. Assume that there are 2 annual intervals and that in each interval the price of the stock either increases according to the factor 1.2 or decreases according to the factor 0.8. The riskless interest rate is 5% per year, with continuous compounding.

(a) Find the price of the call option using the risk-neutral valuation method.

(b) Find the price of the put option using the replicating portfolio method.

(c) Verify that the prices found satisfy the put-call parity.

(d) Consider the straddle that can be built using the two options, draw a table that shows the payoff and the profit that can be obtained and find for which intervals of the stock price the straddle produces a gain.

Exercise 1.

We have:

$$S_0 = 50 \qquad K = 45$$
$$u = 1{,}2 \qquad r = 5\% \text{ per year}$$
$$d = 0{,}8 \qquad T = 2 \text{ with } \Delta t = 1$$

(a) The dynamics of the stock price is:

$$
\begin{array}{ccccc}
 & & & & S_{uu} = 72 \\
 & & S_u = 60 & & \\
S_0 = 50 & & & & S_{ud} = 48 \\
 & & S_d = 40 & & \\
 & & & & S_{dd} = 32 \\
t=0 & & t=1 & & t=2
\end{array}
$$

and the dynamics of the call option price is:

$$
\begin{array}{ccccc}
 & & & & C_{uu} = \max(72-45, 0) = 27 \\
 & & C_u = & & \\
C_0 = & & & & C_{ud} = \max(48-45, 0) = 3 \\
 & & C_d = & & \\
 & & & & C_{dd} = \max(32-45, 0) = 0 \\
t=0 & & t=1 & & t=2
\end{array}
$$

Using the risk-neutral valuation we have first of all:

$$q_u = \frac{e^{rt} - d}{u-d} = \frac{e^{0{,}05 \cdot 1} - 0{,}8}{1{,}2 - 0{,}8} = 0{,}6282$$

$$q_d = 1 - q_u = 1 - 0{,}6282 = 0{,}3718$$

then the value of the option in correspondence of the intermediate nodes at $t = 1$ is:

$$C_u = e^{-rt}\left[C_{uu} \cdot q_u + C_{ud} \cdot q_d\right] =$$

$$= e^{-0{,}05 \cdot 1}\left[27 \cdot 0{,}6282 + 3 \cdot 0{,}3718\right] =$$

$$= 17{,}1952$$

$$C_d = e^{-rt}\left[C_{ud} \cdot q_u + C_{dd} \cdot q_d \right] =$$

$$= e^{-0,05 \cdot 1} \left[3 \cdot 0,6282 + 0 \cdot 0,3718 \right] =$$

$$= 1,7927$$

and finally the value of the option at the initial node at $t=0$ is:

$$C_0 = e^{-rt}\left[C_u \cdot q_u + C_d \cdot q_d \right] =$$

$$= e^{-0,05 \cdot 1} \left[17,1952 \cdot 0,6282 + 1,7927 \cdot 0,3718 \right] =$$

$$= 10,9092$$

(b) The dynamics of the put option price is:

$$\begin{array}{ccccc} & & & & P_{uu} = \max(45-72, 0) = 0 \\ & & P_u = & & \\ P_0 = & & & & P_{ud} = \max(45-48, 0) = 0 \\ & & P_d = & & \\ & & & & P_{dd} = \max(45-32, 0) = 13 \end{array}$$

$t=0$ $\quad$ $t=1$ $\quad$ $t=2$

and using the replicating portfolio valuation at time $t=2$ we have:

$$\begin{cases} V_{uu} = x e^{rt} + y S_{uu} = P_{uu} \\ V_{ud} = x e^{rt} + y S_{ud} = P_{ud} \end{cases}$$

that is:

$$\begin{cases} x e^{0,05 \cdot 1} + y \cdot 72 = 0 \\ x e^{0,05 \cdot 1} + y \cdot 48 = 0 \end{cases}$$

and subtracting the second equation from the first one:

$$24 y = 0 \quad \Rightarrow \quad y = 0$$

and then:

$$x e^{0,05 \cdot 1} + 0 = 0 \quad \Rightarrow \quad x = 0$$

hence:

$$x = 0$$
$$y = 0$$

Therefore at $t=1$ in the "up" state of nature we have:

$$V_u = x + y \cdot S_u = 0 + 0 \cdot 60 = 0 = P_u$$

Always at time $t=2$ we have:

$$\begin{cases} V_{ud} = x e^{rt} + y S_{ud} = P_{ud} \\ V_{dd} = x e^{rt} + y S_{dd} = P_{dd} \end{cases}$$

that is:

$$\begin{cases} x e^{0,05 \cdot 1} + y \cdot 48 = 0 \\ x e^{0,05 \cdot 1} + y \cdot 32 = 13 \end{cases}$$

and subtracting the second equation from the first one:

$$16 y = -13 \Rightarrow y = -0,8125$$

and then:

$$x e^{0,05} - 0,8125 \cdot 48 = 0 \Rightarrow x = 0,8125 \cdot 48 \cdot e^{-0,05} = 37,0979$$

hence:

$x = 37,0979$ amount of the bond

$y = -0,8125$ units of the stock shorted

Therefore at $t=1$ in the "down" state of nature we have:

$$V_d = x + y \cdot S_d = 37,0979 - 0,8125 \cdot 40 = 4,5979 = P_d$$

At time $t=1$, finally, the replicating portfolio must satisfy the relations:

$$\begin{cases} V_u = x e^{rt} + y S_u = P_u \\ V_d = x e^{rt} + y S_d = P_d \end{cases}$$

that is:

$$\begin{cases} x e^{0,05 \cdot 1} + y \cdot 60 = 0 \\ x e^{0,05 \cdot 1} + y \cdot 40 = 4,5979 \end{cases}$$

and subtracting the second equation from the first one:

$$20y = -4,5979 \Rightarrow y = -0,2299$$

and then:

$$xe^{0,05} - 0,2299 \cdot 60 = 0 \Rightarrow x = 0,2299 \cdot 60 \cdot e^{-0,05} = 13,1213$$

hence:

$x = 13,1213$ amount of the bond

$y = -0,2299$ units of the stock shorted

therefore at $t=0$ we have:

$$V_0 = x + y \cdot S_0 = 13,1213 - 0,2299 \cdot 50 = 1,6263 = P_0$$

(c) The put-call parity relation is:

$$c + Ke^{-rT} = p + S_0$$

and since:

$$c + Ke^{-rT} = 10,9092 + 45 \cdot e^{-0,05 \cdot 1} = 51,6263$$

$$p + S_0 = 1,6263 + 50 = 51,6263$$

this relation is satisfied.

(d) The straddle can be created buying the two options, and since:

$K = 45$ $\quad c = 10,91$ $\quad p = 1,63$

the initial investment is:

$$c + p = 10,91 + 1,63 = 12,54$$

and the payoff and the profit are:

Stock price range	Payoff from call	Payoff from put	Total payoff	Total profit
$S_T \leq 45$	0	$45 - S_T$	$45 - S_T$	$32,46 - S_T$
$S_T > 45$	$S_T - 45$	0	$S_T - 45$	$S_T - 57,54$

The straddle leads to a gain if:

$$32{,}46 - S_T > 0 \quad \Rightarrow \quad S_T < 32{,}46$$

$$S_T - 57{,}54 > 0 \quad \Rightarrow \quad S_T > 57{,}54$$

i.e. if the stock price S_T is less than 32,46 or larger than 57,54.

Exercise 2 (10 points)
A stock price is \$50, and the risk-free rate of interest is 6% per annum with continuous compounding for all maturities.
(a) A trader observes that the forward price of a six month forward contract is $F_0^m = 40$. Can he make an arbitrage? Why? If yes, find an arbitrage strategy.
(b) Find the risk neutral price of a six months forward contract if the stock is expected to pay a dividend of \$0.5 per share in 2 months and in 5 months.
(c) Consider the risk neutral price F_0 for F, under the assumption that the stock does not pay dividends. Assume that the 6-months asset volatility is $\sigma = 0.2$. Find the price of a 3-months call option defined of F with strike $k = 50$.

EXERCISE 2

$S_0 = 50$

$q = 0.06$

a) $F_0^m = 40$ F_0^m = market observed price

No arbitrage price:

$$F_0 = S_0 e^{qT} =$$

$$= 50\, e^{0.06 \cdot 1/2} = 51{,}52$$

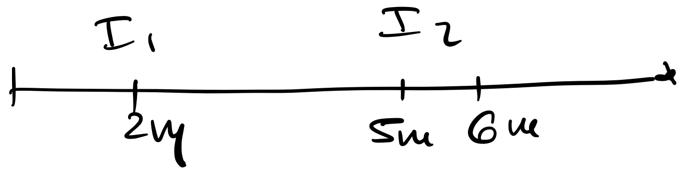
$F_0^m < F_0$, thus there is an arb. opportunity

Arbitrage strategy		
long forward	0	$\to\ -F_0^m = 40$
short	$+S_0$	close short on S_0
invest	$-S_0$	$\to\ S_0 e^{qT} = 51{,}52$
	0	$F_0 - F_0^m > 0$
		$51{,}52 - 40 = 11{,}52$

b]

$S_0 = 50$

$r = 0.06$



$d = 0.5$ per share in 2 months and 5 months

Present value of dividends:

$I_1 = e^{-0.06 \cdot 2/12} \times 0.5 = 0.495$

$I_2 = e^{-0.06 \cdot 5/12} \times 0.5 = 0{,}487$

$I = I_1 + I_2 = 0.982$

$F_0 = (S_0 - I)\, e^{rT} =$

$= (50 - 0.98)\, e^{0.06 \cdot 1/2} = 50.51$

Without dividends $\boxed{F_0 = 51{,}52}$

c) $\sigma = 0.2$ / 6 months $\quad T = 3/12 \qquad r = 0{,}06$

Annual vol. $\quad \sigma_A = 0.2\sqrt{2} = 0.28$

Black's Formula $\qquad K = 50$

$F_0 = 51{,}52$

$$c = e^{-rT}\left[F_0 N(d_1) - K N(d_2)\right]$$

$$d_1 = \frac{\ln(F_0/K) + \sigma_A^2/2\, T}{\sigma_A\sqrt{T}} \cong 0{,}28$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.14$$

$N(d_1) = 0.61$

$N(d_2) = 0.55$

$$c = e^{-0.06 \frac{3}{12}}\left[51{,}52\, N(d_1) - 50 N(d_2)\right] \cong 3{,}60$$

$c \cong 3{,}60$

Exercise 3 (10 points)
A stock price follows a geometric Brownian motion with an expected return (equal to the risk-free interest rate) of 11% per annum and a volatility of 30% per annum. The current stock price is 60 \$. Consider 5-months call and put options on this stock with strike $K = 50$ and maturity $T = 5$ months.
(a) At initial time an investor sells eight call options and he invests his income in a portfolio P with one long put option, shares and cash. How many shares of the stock would he be able to buy and how much money remain to be invested in a bank account?
(b) Find the delta of the Portfolio P in point (a).
(c) Consider another portfolio $h = (h_1, h_2)$, where h_1 is the number of call options and $h_2 = 10$ is the number of shares of the underlying stock. Find h_1 such that h is Δ-neutral. Is h Γ-neutral? Find the Γ of h. Comment on the results.

EXERCISE 3

a] $r = \mu = 0.11$ $\quad T = 5/12$

$\sigma = 0.3$

$S_0 = 60 \quad K = 50$

Call price :

$$C_0 = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} =$$

$$= \frac{\ln 60/50 + \left(0.11 + \frac{0.3^2}{2}\right)5/12}{0.3\sqrt{5/12}} = 1.27$$

$$d_2 = d_1 - 0.3\sqrt{5/12} = 1.08$$

$$N(d_1) = 0.89$$

$$N(d_2) = 0.86$$

$$C_0 = 60\, N(d_1) - 50 e^{-0.11 \cdot 5/12} N(d_2) = 12.85$$

the price of the call is 12.85

the corresponding put price:
PUT-CALL PARITY

$$p = c + K e^{-rT} - S_0$$

$$= 12.85 + 50 e^{-0.11 \cdot 5/12} - 60 = 0.61$$

				Cash position
he	sells	8	call options	+102.76
	buy	5	put options	-6.06
				96.7

cash 96.7 : he can buy 1 share

and he has to invest 96.7 - 60 = 36.7 \$

$$\text{Portfolio } P = (\overset{\text{PUT}}{1}, \overset{S}{1}, \overset{B}{36.7})$$

$$P = 1 \text{ Put} + 1 \text{ share} + 36.7 \$$$

(b) Δ Portfolio

$$\Delta_P = \Delta_{Put} + 1 = \Delta_C - 1 + 1 = \Delta_C = N(d_1)$$

$$= 0.89$$

(c)

$$P = h_2 C + 10S$$

$$\Delta_P = h_2 \Delta_C + 10$$

$$h_1 \Delta_C + 10 = 0 \qquad h_2 = -10/\Delta_C = -\frac{10}{0.88} = -11.14$$

$$\Gamma_P = h_2 \Gamma_C = 11\, \Gamma_C$$

$$\Gamma_C = \frac{\varphi(d_1)}{S\sigma\sqrt{T}} = 0.015$$

$$\Gamma_P = -0.169$$

P is not gamma neutral

→ Comments ....

Politecnico di Torino
Financial Engineering-Exam ~~12-05-2022~~ 13-06.2022
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Exercise 1 (12 points)
Consider a one-period (annual) model formed by a bond (paying a risk-free rate of 5% per year) and by two stocks with prices S^1 and S^2
evolving as follows:

$$S_0^1 = 10;$$

$$S_T^1(w) = \begin{cases} 12; \text{if } w = w_1 \\ 10; \text{if } w = w_2 \\ 6; \text{if } w = w_3 \end{cases} \tag{0.1}$$

and

$$S_0^2 = 10;$$

$$S_T^2(w) = \begin{cases} 15; \text{if } w = w_1 \\ 8; \text{if } w \in \{w_2, w_3\} \end{cases} \tag{0.2}$$

with $P(w_1), P(w_2), P(w_3) > 0$ and $P(w_1) + P(w_2) + P(w_3) = 1$.

1. Establish if the market is free of arbitrage and complete.
2. Consider two derivatives (A and B) with maturity T of one year and with

$$\Phi_A = (\frac{S_T^1 + S_T^2}{2} - 8)^+ \tag{0.3}$$

and

$$\Phi_B = (S_T^1 - S_T^2)^+, \tag{0.4}$$

respectively. Find their price.

3. Discuss whether the market formed only by the bond and by stock S^1 would remain free of arbitrage and complete.

EXERCISE 1]

1] S^1 mart. rpde

$$\begin{cases} \tilde{E}\left[S_1^1 / 1+r \right] = S_0^1 \\ \tilde{E}\left[S_1^2 / 1+r \right] = S_0^2 \end{cases}$$

$$\begin{cases} \frac{12}{1.05} q_1 + \frac{10}{1.05} q_2 + \frac{6}{1.05} q_3 = 10 \\ \frac{15}{1.05} q_1 + \frac{8}{1.05}(q_2 + q_3) = 10 \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 12 q_1 + 10 q_2 + 6 q_3 = 10.5 \\ 15 q_1 + 8 (q_2 + q_3) = 10.5 \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

$$\begin{vmatrix} 12 & 10 & 6 \\ 15 & 8 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 10 & 6 \\ 8 & 8 \end{vmatrix} - \begin{vmatrix} 12 & 6 \\ 15 & 8 \end{vmatrix} + \begin{vmatrix} 12 & 10 \\ 15 & 8 \end{vmatrix} =$$

$$= 80 - 48 - 96 + 90 + 96 - 150 = -28 \neq 0$$

the system has one sole solution $\Rightarrow$

Risolvendo il sistema:

$$\begin{cases} q_1 = 5/14 \\ q_2 = \dfrac{33}{56} \\ q_3 = 3/56 \end{cases}$$

Soluzione positiva $\rightarrow \exists!$ misura risk neutral quindi mercato Arb. free e completo

2]

$$\phi_A = \begin{cases} 5.5 & \text{if } \omega = \omega_1 \\ 1 & \text{if } \omega = \omega_2 \\ 0 & \text{if } \omega = \omega_3 \end{cases} \qquad \phi_B = \begin{cases} 0 & \text{if } \omega = \omega_1 \\ 2 & \text{if } \omega = \omega_2 \\ 0 & \text{if } \omega = \omega_3 \end{cases}$$

$$E[\phi_A/(1+r)] = \frac{1}{1.05}[5.5\, q_1 + q_2] = 2.432$$

$$E[\phi_B/(1+r)] = \frac{1}{1.05}[0 + 2q_2 + 0] = 1.1224$$

3] With only asset A

$$\begin{cases} \dfrac{12}{1.05} q_1 + \dfrac{10}{1.05} q_2 + \dfrac{6}{1.05} q_3 = 10 \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

the system has ∞ positive solutions

Set $q_1 = q$ we have $q_2 = \frac{9}{8} - \frac{3}{2} q$ $\quad q_3 = \frac{1}{2} q - 1/8$

If $q \in \left(\frac{1}{4} ; 3/4\right)$ the solutions are positive

$\rightarrow$ risk neutral and not complete

$\hookrightarrow$ with 1 asset $d < 1+r < u$

Exercise 2(10 points)

(a) Consider a stock share that does not pay any dividend, with current price 100. A call option with strike price 105, maturing in six months, is traded for 12. The corresponding put option trades for 18. The expected return on the risky stock share is 10% and the risk-free rate is 2% (both are annual, with continuous compounding). Is there an arbitrage opportunity? If yes, devise a trading strategy to take advantage of it.

(b)

Your current wealth is $100,000$, which you invest in a portfolio whose value process follows a geometric Brownian motion with drift coefficient 7% and volatility coefficient 25%. The risk-free rate (with continuous compounding) is 3%. Your target is to double your wealth in 10 years. Find the shortfall probability, i.e., the probability you will not achieve your target.

EXERCISE 2

(2) Using put call parity

$$p + S_0 = 118$$
$$c + Ke^{-rT} = 12 + 105e^{-0.02\times0.5} = 115.96$$

it is not verified : expensive side should be shorted

$T=0$	$T=1$, $S_T > 100$	$T=1$, $S_T < 100$
short put short stock	not exercised $-S_T$	exercised buy S_T from put $+S_T - 105$
buy Ke^{-rT}	105	+105
buy c	exercise -105 and give back stock	give back the asset $-S_T$

Solution

Using put-call parity:

$$P_0 + S_0 = €118$$
$$C_0 + Ke^{-rT} = 12 + 105e^{-0.02\times0.5} = €115.96$$

The protective put side (put + stock) is expensive and should be shorted. So, at time 0 we

1. Write put, short stock, buy call, buy bond with face value €105 maturing in six months. The net cash flow is €2.04, the difference of the two values above.
2. At maturity, if the call is in the money, we exercise the call using the €105 that we receive from the bond and close the short position on the stock. The put is not exercised.
3. If the put is in the money, we have to buy the stock from the holder of the put. We use again the €105 from the bond and use the stock to close the short position. The call is not exercised.
4. In any case, we break even at maturity and have a positive cash flow at time 0.

(b)

The quick way to solve the problem is to remember that the probability that a call option is in the money is given by $\Phi(d_2)$, where we should use the actual drift 7%, initial price 100,000, strike price 200,000. Alternatively, we may write the terminal wealth as a lognormal random variable:

$$W_T = W_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\varepsilon}$$

The shortfall probability is

$$P\{W_T < 2W_0\} = P\left\{W_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\varepsilon} < 2W_0\right\} = P\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\varepsilon < \log 2\right\}$$

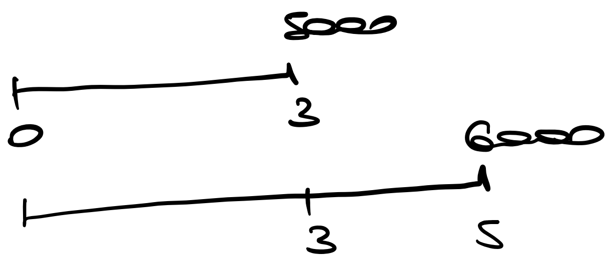
$$= \Phi\left(\frac{\log 2 - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = \Phi\left(\frac{\log 2 - \left(0.07 - \frac{0.25^2}{2}\right) \times 10}{0.25 \times \sqrt{10}}\right) = \Phi(0.3866) = 0.6505$$

Common mistakes

Exercise 3 (10 points)
A portfolio consists of a 3-year zero-coupon bond with face value of 5000 \$ and a 5-year zero-coupon bond with face value of 6000 \$. The current yield on all bonds is 5% per annum.
(a) Compute the duration of the portfolio (using continuous compounding).
(b) Compute the percentage change in the value of the portfolio in the case of a 0.2% per annum decrease in yields.
(c) Compare the result obtained in (b) with the 1st order and 2nd order approximations based on the use of duration and of convexity, and comment the results.

EXERCISE 3

(a) We have



current yield 0.05 per annum

Duration

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-y t_i}}{\sum_{i=1}^{n} c_i e^{-y t_i}} =$$

$$= \frac{3 \cdot 5000\, e^{-0.05 \cdot 3} + 5 \cdot 6000\, e^{-0.05 \cdot 5}}{5000\, e^{-0.05 \cdot 3} + 6000\, e^{-0.05 \cdot 5}} =$$

$$= \frac{36274{,}64}{8976{,}34} = 4{,}04$$

$$D = 4{,}04$$

(b) Value of portfolio

$$V = 5000\, e^{-0.05\times 3} + 6000\, e^{-0.05\times 5} = 8976$$

If the yield decreases of 0,2 % it becomes

$$V' = 5000\, e^{-0.048\times 3} + 6000\, e^{-0.048\times 5} = 9049$$

Percentage change

$$\frac{V'-V}{V} = 0.00811$$

(C) Using first order approx based on duration we have

$$\frac{\Delta V}{V} = -D\,\Delta y = -4{,}94(-0.002) = 0.0080$$

Convexity

$$C = \frac{\sum t_i^2 c_i e^{-y_i t_i}}{\sum c_i e^{-y_i t_i}} =$$

$$\frac{3^2 \cdot 5000 e^{-0.05\times 3} + 5^2\, 6000 e^{-0.05\times 5}}{8976{,}34} =$$

$= 17{,}32$

and the second order approximation is

$$\frac{\Delta V}{V} = -D\,\Delta y + \frac{1}{2}\,C(\Delta y^2) =$$

$$= 0.00808 + 0.00003 \approx 0.00811$$

the approx with convexity is better because it is a second order taylor expansion –

Politecnico di Torino
Financial Engineering-Exam 06-09-2022
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All the answers must be clearly motivated, the numerical results are not sufficient.

Answers written with a pencil are null.

Exercise 1 (10 points)

You hold a portfolio of (vanilla, European-style) options written on the same stock share, whose price follows a geometric Brownian motion with drift 9% and volatility 25%. At present, the stock price is 30 dollars, and the risk-free rate, with continuous compounding, is 3%. The portfolio consists of:

1. A short position in 1000 put options, strike 27, maturing in three months
2. A long position in 500 call options, strike 30, maturing in four months
3. A short position in 1500 call options, strike 28, maturing in two months

(a) How many stock shares do you need to make the portfolio delta-neutral?
(b) Can you also make the portfolio gamma-neutral by using stock shares? If so, explain how. Otherwise, how should you change the position in the last call to make the portfolio gamma neutral?

SOLUTION EXERCISE 1

We refer to the option at points 1, 2 and 3 as options a, b and c, respectively

the portfolio P is:

$$P = n_S \cdot S - 1000\, P^a + 500\, C^b - 1500\, C^c,$$

where n^S is the number of shares

a] To find Δ^P we need Δ^a, Δ^b and Δ^c:

we need the terms

$$d_1^a = \frac{\log(30/27) + (0.03 + 0.25^2/2)/4}{0.25 \cdot \sqrt{1/4}} = 0.9654$$

$$d_1^b = \frac{\log(30/30) + (0.03 + 0.25^2/2)/3}{0.25 \cdot \sqrt{1/3}} = 0.1415$$

$$d_1^c \quad \frac{\log(30/28) + (0.03 + 0.25^2/2)/6}{0.25 \cdot \sqrt{1/6}} = 0.7760$$

then we find the deltas:

$$\Delta^a = \phi(0.9654) - 1 = -0.1672$$

$$\Delta^b = \phi(0.1415) = 0.5562$$

$$\Delta^c = \phi(0.7760) = 0.7811$$

Notice the negative delta of the put and the large Δ of the last call, which is currently in-the-money the Δ of P is:

$$\Delta P = h_S - 1000\,\Delta^a + 500\,\Delta^b - 1500\,\Delta^c$$

To set $\Delta P = 0$ we need to hold

$$h_S = 1000\,\Delta^a - 500\,\Delta^b + 1500\,\Delta^c =$$
$$= 1000(-0.1672) - 500 \cdot 0.5562 + 1500 \cdot 0.711 =$$
$$= 726{,}395$$

Shares

b] We cannot achieve a Γ-neutral portfolio by using stock shares, because their Γ is zero.

We must use non-linear instruments and the call is a non-linear instruments. Nevertheless, if we change the position only in the last call the portfolio is no more Δ-neutral.
We therefore have to change also the position in the stock to preserve Δ-neutrality. We have to solve the system:

$$\begin{cases} h_S + h_C\,\Delta^c = 1000\,\Delta^a - 500\,\Delta^b \\ h_C\,\Gamma^c = 1000\,\Gamma^a - 500\,\Gamma^b \end{cases}$$

We need to find the gammas

$$\Gamma^a = \frac{\phi(0.9654)}{30 \times 0,25\sqrt{1/4}} = 0.0668$$

$$\Gamma^b = \frac{\phi(0.1415)}{30 \times 0,25\sqrt{1/3}} = 0.0912$$

$$\Gamma^c = \frac{\phi(0.7760)}{30 \times 0,25\sqrt{1/2}} = 0,0557$$

thus

$$h_c = \frac{1000\,\Gamma^a - 500\,\Gamma^b}{\Gamma^c} = \frac{1000 \cdot 0.0668 - 500 \cdot 0.0912}{0.0557}$$

$$\cong 379$$

and

$$h_S = 1000(-0.1672) - 500 \cdot 0.5562 - 219.37 \cdot 0.7811 =$$
$$\approx -741,3$$

the new portfolio is long 379 call, so we have to buy 1500+379 call c to offset the current position
the new portfolio is short 741 stocks, now we should have a short position in the stock -

Exercise 2 (10 points)
The risk-free rate of interest is 6% per annum with continuous compounding for all maturities.
(a) A stock price is \$50. A trader observes that the forward price of a six months month forward contract is $F_0^m = 55$. Can he make an arbitrage? Why? If yes, find an arbitrage strategy.
(b) Assume that the dividend yield on a stock index varies throughout the year. In february, May, August and November, dividends are paid at a rate of 5% per annum. In other months, dividend are paid at a rate of 2% per annum. Suppose that the value of the index on July 31 is 1300. What is the future price for a contract deliverable in December 31 of the same year?

EXERCISE 2

(a) $S_0 = 50$

$r = 0.06$

$F_0 = 55$

the risk neutral price for a forward contract is

$$\tilde{F}_0 = e^{0.06 \cdot 1/2} \cdot 50 = 50\, e^{0.03} = 51{,}52$$

thus, $F_0 > \tilde{F}_0$ and there is an arbitrage opportunity

Arbitrage strategy

	$t=0$	$t=1$
short forward	0	
borrow \$ S_0	$+S_0$	→ pay the loan $-S_0e^{rT}$
buy asset S_0	$-S_0$	sell the asset $+F_0$
	0	$F_0 - S_0e^{rt}$

$$F_0 - S_0e^{rT} = 55 - 50\,e^{0.03} =$$

$$= 55 - 51{,}52 > 0$$

(b) We have:

$S_0 = 1300$ on July 31

$r = 9\%$ per annum

$q = \begin{cases} 5\% \text{ in Feb.; May; August; November} \\ 2\% \text{ in other months} \end{cases}$

$T = \frac{5}{12}$

Since the contract lasts from August to December the dividend field is 5% for 2 months and 2% for the remaining 3 months, hence the average dividend yeld is $\frac{0.05 \times 2 + 0.02 \times 3}{5} = 0.032 = 3{,}2\%$ per annum

the Future price is

$$F_0 = S_0 e^{(r-q)T} = 1300 e^{(0.06 - 0.032) \cdot 5/12} =$$

$$= 1315{,}255$$

Exercise 3 (12 points)
Let $dS = \mu_t dt + \sigma_t dW$ and risk-free interest rate is 4% per annum (all rates are continuously compounded).

(a) When $\sigma_t = 0.01$ for $t \in [0, 1]$ and

$$\mu_t = \begin{cases} 0.02t;\ 0 \leq t \leq 1 \\ 0.01(10 - t);\ 1 < t \leq 2 \end{cases} \tag{0.1}$$

compute the probability of having at $t = 2$ a profit greater or equal than 0.2.

(b) When $\mu_t = 0.01$ for $t \in [0, 2]$ and

$$\sigma_t = \begin{cases} 0.2;\ 0 \leq t \leq 1 \\ 0.4;\ 1 < t \leq 2 \end{cases} \tag{0.2}$$

compute the probability of having at $t = 2$ a profit greater than or equal to 0.2.

(c) Assuming $\mu_t = 0.03S_t$ and $\sigma_t = 0.01S_t$ for $t \in [0, 1]$, in years, i.e.

$$dS = 0.03S_t dt + 0.01S_t dW$$

and that the risk free rate is 2% per annum, price an European call option with strike price 100, maturing in one years, written on the stock $S(t)$ (no dividends) whose current price $S(0)$ is 98.

EXERCISE 3

(a) by integrating $dS_t = \mu_t dt + \sigma_t dW_t$

we have

$$S_2 - S_0 = \int_0^2 \mu_s ds + \int_0^2 \sigma_s dW_s =$$

$$= \int_0^1 0.02\, s\, ds + \int_1^2 0.01(10-s)\, ds + \int_0^2 0.01\, dW_s$$

$$= 0.02 \frac{s^2}{2}\Big|_0^1 + 0.01 \left[10s - \frac{s^2}{2} \right]_1^2 + 0.01 (W_2 - W_0)$$

$$= \frac{1}{2} 0.02 + 0.01 \Big[\underbrace{20 - 2 - (10 - \tfrac{1}{2})}_{8.5} \Big] + 0.01 W_2$$

$$= 0{,}095 + 0.01 W_2 \sim N(0{,}095, (\sqrt{2}\, 0.01)^2)$$

Profit greater or equal to 0

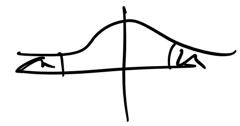
means $S_2 - S_0 \geq 0{,}2$

$$P((S_2 - S_0) > 0.2) = P(0{,}095 + 0.01 W > 0{,}2)$$

$$= P\left(W > \frac{0{,}2 - 0{,}095}{0{,}01} \right) =$$

$$= P\left(Z > \frac{0,105}{\sqrt{2}\; 0.01}\right)$$

because $W_2 \sim N(0,2)$



$$= P(Z > 7,42) = 1 - \phi(7,42) \simeq 0$$

(b)

$$S_4 - S_0 = \int_0^2 \mu_s\, ds + \int_0^2 \sigma_s\, dW_s =$$

$$= 0.01s \Big|_0^2 + \int_0^1 0.2\, dW_s + \int_1^2 0.4\, dW_s =$$

$$= 0.01 \cdot 2 + 0.2(W - W_0) + 0.4(W_2 - W_1)$$

$$= 0.01 + 0.2\, N_1 + 0.4\, \tilde{N}_1$$

where $N_1 \sim N(0,1)$ and $\tilde{N}_1 \sim N(0,1)$

$W_t - W_s \sim N(0, t-s)$. thus

$$0.01 + 0.2 N_1 + 0.4 \tilde{N}_1 = 0.04 + N,$$

because N_1 and $\tilde{N}_1$ are independent where

$$N \sim N(0, (0{,}04 + 0{,}16))$$

$$\sim N(0; 0{,}2)$$

because if X, Y are indep. and

$X \sim N(0, \sigma_X)$ $Y \sim N(0, \sigma_Y)$, we have

$$aX + bY \sim N(0, a^2\sigma_X^2 + b^2\sigma_Y^2)$$

therefore

$$P(S_2 - S_0 \geq 0.2) = P(0.02 + N \geq 0.2)$$

$$= P\left(Z \geq \frac{0.2 - 0.02}{\sqrt{0{,}2}} \right) = P(Z \geq 0{,}4024)$$

$$= 1 - \phi(0{,}4024)$$

(c) We have $S(0) = 98$ $K = 100$ $T = 1Y$

$\sigma = 0.01$ per annum

$r = 0.02$

According to the Black-Scholes-Merton formula for a European Call option

$$C = S(0)\,N(d_1) - Ke^{-rT}N(d_2)$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)\,T}{\sigma\sqrt{T}} =$$

$$= \frac{\ln(0{,}98) + \left(0.02 + \frac{0.01^2}{2}\right)\cdot 1}{0.01} =$$

$$= -0{,}0152$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.0152 - 0.01\sqrt{1} =$$
$$= -0{,}025$$

$$C = 98\,N(-0.015) - 100\,e^{-0.02}N(-0{,}025)$$

$$= 0{,}38$$