Politecnico di Torino
Financial Engineering-Exam 06-27-2023
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Student number

All the answers must be clearly motivated, the numerical results are not sufficient.

Answers written with a pencil are null.

Exercise 1 (10 points)

The current price of a stock is $S(0) = 8$ euros. In one year, such price may move up to 12 euros with probability 2/3 or down to 7 euros with probability 1/3.

1. Find a range for the risk-free interest rate r that makes the market composed by the stock S and a riskless bond B arbitrage free and motivate the answer. Choose $r = 4\%$ and show that the market is arbitrage free and complete.

2. Find the price of a call option with payoff strike $K = 11$ and maturity $T = 2$ and the probability that the option is exercised.

3. Assume now that in one year, S may move up to 16 euros or to 12 euros, or move down to 8 euros. Each event may happen with probability 1/3. Suppose now that on the market it is possible to sell (at the price of one euro) a European Call option written on the stock above, with maturity of 1 year and with strike of 11 euros.

   Consider the following strategy:

   - buy one share of the stock;
   - sell the Call;
   - borrow 7 euros at the risk-free rate.

   Verify that such a strategy represents an arbitrage opportunity.

SOLUTION

$S_1 = 12 \quad S_0 = 8 \quad S_1^u / S_0 = \frac{12}{8} = 1.5 \Rightarrow u = 1.5$

$S_1^d / S_0 = 7/8 = 0.875 \Rightarrow d = 0.875$

1) Arbitrage free means that there are not arbitrage opportunities in the market

the market is arbitrage free iff $1+r \in [u, d]$

therefore if

$$0.875 < 1+r < 1.5$$

Otherwise the future asset price is always higher than the risk free asset or viceversa and these two scenarios allows us to construct arbitrage strategies → you should show one example

We can prove this condition since it is equivalent to require that there is a risk neutral martingale measure, i.e

$Q: \quad \mathbb{E}^Q\left[\frac{S(1)}{1+r}\right] = S(0)$ that means

that the system

$$\begin{cases} u q_u + d q_d = 1+r \\ q_u + q_d = 1 \end{cases}$$

must have a positive solution

If $r = 0.04$ we have

$$1+r = 1{,}04 \in (0{,}875; 1{,}5) \Rightarrow$$ the market is arbitrage free and it is also complete because the binomial model is complete means that every contr. claim is reachable and we have a unique price.

In fact if we look for a risk neutral Q we look for $Q = (q_u, q_d)$ so that

$$\begin{cases} u q_u + d q_d = 1+r \\ q_u + q_d = 1 \end{cases}$$

this system has a (unique) solution iff

$$\begin{vmatrix} u & d \\ 1 & 1 \end{vmatrix} = u - d \neq 0$$, and this is the case

then $$\begin{cases} q_u = \dfrac{(1+r) - d}{u-d} \\ q_d = \dfrac{u - (1+r)}{u-d} \end{cases}$$

and this is positive iff

$$d < 1+r < u$$, thus
$$0.875 < 1+r < 1.5$$

In this case, since the solution is unique the market is also complete.

If $r = 0.04$, we have

$$0.875 < 1.04 < 1.5$$

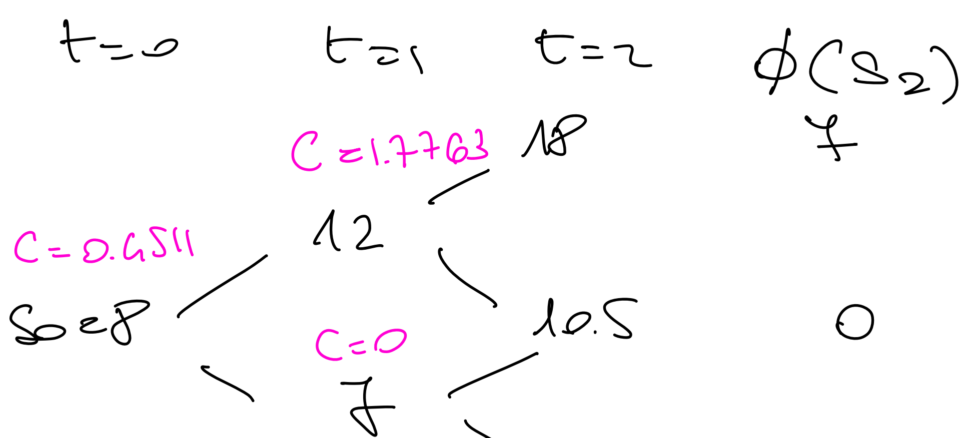
then

$$\begin{cases} q_u = \dfrac{(1+r) - d}{u-d} = 0.2640 \\[2ex] q_d = \dfrac{u - (1+r)}{u-d} = 0.7360 \end{cases}$$

is positive.

Therefore the market is arbitrage free (there is a positive solution) and complete (the solution is unique).

2] The binomial tree:



$$C_0 = \frac{1}{(1+r)^2} \sum_{k=0}^{2} \binom{2}{k} q_u^k q_d^{T-k} \phi(S u^k d^{3-k}) =$$

$$= \frac{1}{(1.04)^2} (0.2640)^2 \cdot 7$$

$$= 0.4511$$

the option is exercised if $S_2 > 11$

$$P(S_2 \geq 11) = P(S_2 = 18) = p^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

3] Consider the value of the strategy at time $t=0$ and $t=1$ year

$t=0$		$t=1$	
buy one stock	$-S_0 = -8$	sell the stock	S_T
sell the call	$+C_0 = 1$	call payoff	$-(S_T - 11)^+$
borrow 7 euro (to buy the stock you need 7 euro)	$+7$	return the loan	$-7(1+0.04)$
	//		$S_T - (S_T - k)^+ - 7.28$

$$S_T - (S_T - \overset{11}{\uparrow}{k})^+ - 7.28 = \begin{cases} 16 - 5 - 7.28 = 3.72 & 1/3 \\ 12 - 1 - 7.28 = 3.72 & 1/3 \\ 8 - 0 - 7.28 = 0.72 & 1/3 \end{cases} \quad \mathbb{P}$$

$\Rightarrow$ ARBITRAGE

Exercise 2 (10 points)

A 4-year bond with a yield of 5% (continuously compounded) pays a 6% coupon at the end of each year.

1) Determine the current price of the bond.

2) Find the duration and the convexity of the bond.

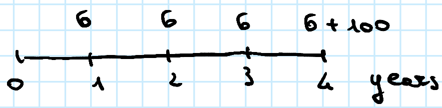
3) Use first only the duration, then both the duration and the convexity, to calculate the effect on the bond's price of a 0.3% decrease in its yield.

4) Ricalculate the bond's price on the basis of the new yield and compare the result with that obtained at point 3), commenting on it.

Solutions

EXERCISE 2

1) In this case the cash-flows are:



and the current price of the bond is:

$$P = 6 \cdot e^{-0,05 \cdot 1} + 6 \cdot e^{-0,05 \cdot 2} + 6 \cdot e^{-0,05 \cdot 3} + 106 \cdot e^{-0,05 \cdot 4} =$$

$$= 103,09$$

2) The duration of the bond is:

$$D = \frac{1 \cdot 6 \cdot e^{-0,05 \cdot 1} + 2 \cdot 6 \cdot e^{-0,05 \cdot 2} + 3 \cdot 6 \cdot e^{-0,05 \cdot 3} + 4 \cdot 106 \cdot e^{-0,05 \cdot 4}}{103,09} =$$

$$= 3,68 \text{ years}$$

and the convexity is:

$$C = \frac{1^2 \cdot 6 \cdot e^{-0,05 \cdot 1} + 2^2 \cdot 6 \cdot e^{-0,05 \cdot 2} + 3^2 \cdot 6 \cdot e^{-0,05 \cdot 3} + 4^2 \cdot 106 \cdot e^{-0,05 \cdot 4}}{103,09} =$$

$$= 14,19$$

3) If the yield decreases by 0,3% the effect on the price of the bond, using only the duration, is:

$$(\Delta P)_I = - D \cdot \Delta y \cdot P \rightarrow \Delta P = - 3,68 \cdot (-0,003) \cdot 103,09 = 1,13$$

hence the bond price becomes:

$$P' = P + (\Delta P)_I = 103,09 + 1,13 = 104,22$$

The effect on the price of the bond using also the convexity is:

$$(\Delta P)_{II} = - D \cdot \Delta y \cdot P + \frac{1}{2} \cdot C \cdot (\Delta y)^2 \cdot P =$$

$$= - 3,68 \cdot (-0,003) \cdot 103,09 + \frac{1}{2} \cdot 14,19 \cdot (-0,003)^2 \cdot 103,09 =$$

$$= 1,14$$

hence the bond price becomes:

$$P' = P + (\Delta P)_{\sigma} = 103,09 + 1,14 = 104,23$$

4) If the yield decreases by 0,3%, i.e. from 5% to 4,7%, the price of the bond becomes:

$$P' = 6 \cdot e^{-0,047 \cdot 1} + 6 \cdot e^{-0,047 \cdot 2} + 6 \cdot e^{-0,047 \cdot 3} + 106 \cdot e^{-0,047 \cdot 4} =$$

$$= 104,23$$

That is consistent with the results found above (in particular, the 2^{nd} order approximation is more precise than the 1^{st} order one).

Exercise 3 (10 points)

An investor considers the possibility of creating both a bull spread and a bear spread using put options on a stock. The strike prices of the options are 40 euros and 45 euros and the costs of these options are 5 euros and 8 euros, respectively.

1) Describe how the investor can create a bull spread, then find the payoff and the profit for this spread.

2) Describe how the investor can create a bear spread, then find the payoff and the profit for this spread.

3) Determine for which prices of the underlying stock at maturity is possible to obtain a positive profit in the case of the bull spread and in the case of the bear spread.

EXERCISE 3

1) The investor can create a bull spread by:

- buying 1 put option with strike price $K_1 = 40$ that costs $p_1 = 5$
- selling 1 put option with strike price $K_2 = 45$ that costs $p_2 = 8$

The initial inflow is:

$$p_2 - p_1 = 8 - 5 = 3$$

and the payoff and the profit are:

Stock price range	Payoff from long put with strike K_1	Payoff from short put with strike K_2	Total payoff	Profit
$S_T \leq 40$	$40 - S_T$	$-(45 - S_T)$	-5	-2
$40 < S_T < 45$	0	$-(45 - S_T)$	$S_T - 45$	$S_T - 42$
$S_T \geqslant 45$	0	0	0	3

2) The investor can create a bear spread by:

- selling 1 put option with strike price $K_1 = 40$ that costs $p_1 = 5$
- buying 1 put option with strike price $K_2 = 45$ that costs $p_2 = 8$

The initial investment is:

$$p_2 - p_1 = 8 - 5 = 3$$

and the payoff and the profit are:

Stock price range	Payoff from long put with strike K_2	Payoff from short put with strike K_1	Total payoff	Profit
$S_T \leq 40$	$45 - S_T$	$-(40 - S_T)$	5	2
$40 < S_T < 45$	$45 - S_T$	0	$45 - S_T$	$42 - S_T$
$S_T \geq 45$	0	0	0	-3

3) In the case of the bull spread the profit is positive when:

$$S_T - 42 \geq 0 \quad \rightarrow \quad S_T > 42$$

while in the case of the bear spread the profit is positive when:

$$42 - S_T > 0 \quad \rightarrow \quad S_T < 42$$