

**Politecnico di Torino**  
**Financial Engineering-Exam 02-01-2024**  
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SURNAME AND NAME

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Student number

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**Exercise 1** (10 points)

Consider a one-period (annual) model formed by a bond (paying a risk-free rate of 5% per year) and by two stocks with prices  $S^1$  and  $S^2$  evolving as follows:

$$S_0^1 = 10;$$

$$S_T^1(w) = \begin{cases} 12; & \text{if } w = w_1 \\ 10; & \text{if } w = w_2 \\ 6; & \text{if } w = w_3 \end{cases} \quad (0.1)$$

and

$$S_0^2 = 10;$$

$$S_T^2(w) = \begin{cases} 15; & \text{if } w = w_1 \\ 8; & \text{if } w \in \{w_2, w_3\} \end{cases} \quad (0.2)$$

with  $P(w_1), P(w_2), P(w_3) > 0$  and  $P(w_1) + P(w_2) + P(w_3) = 1$ .

1. Establish if the market is free of arbitrage and complete.
2. Consider the derivative A with maturity T of one year and with

$$\Phi_A = \left( \frac{S_T^1 + S_T^2}{2} - 8 \right)^+ \quad (0.3)$$

and find its price.

3. Discuss whether the market formed only by the bond and by stock  $S^2$  would remain free of arbitrage and complete.

## EXERCISE 1]

1]  $\delta^1$  More n pole

$$\left\{ \begin{array}{l} \bar{E}\left(\frac{s_1^1}{1+r}\right) = S^1 \\ \bar{E}\left(\frac{s_1^2}{1+r}\right) = S^2 \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{12}{1.05} Q_1 + \frac{10}{1.05} Q_2 + \frac{6}{1.05} Q_3 = 10 \\ \frac{15}{1.05} P_1 + \frac{8}{1.05} (Q_2 + Q_3) = 10 \\ Q_1 + Q_2 + Q_3 = 1 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} 12P_1 + 10Q_2 + 6Q_3 = 10.5 \\ 15P_1 + 8(Q_2 + Q_3) = 10.5 \\ P_1 + Q_2 + Q_3 = 1 \end{array} \right.$$

$$\begin{vmatrix} 12 & 10 & 6 \\ 15 & 8 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 10 & 6 \\ 8 & 8 \end{vmatrix} - \begin{vmatrix} 12 & 6 \\ 15 & 8 \end{vmatrix} + \begin{vmatrix} 12 & 10 \\ 15 & 8 \end{vmatrix} = \\ = 80 - 48 - 96 + 90 + 96 - 150 = -28 \neq 0$$

the system has only one solution  $\Rightarrow$

Solving the system:

$$\begin{cases} q_1 = \frac{5}{16} \\ q_2 = \frac{33}{56} \\ q_3 = \frac{3}{56} \end{cases}$$

Solve for  $\omega$   $\rightarrow$  3! more now needed  
quimod moments arb. free & complete

2]

$$\phi_A = \begin{cases} 5.5 & \text{if } \omega = \omega_1 \\ 1 & \text{if } \omega = \omega_2 \\ 0 & \text{if } \omega = \omega_3 \end{cases}$$

$$E(\phi_A)_{(1+\pi)} = \frac{1}{1.05} [5.5 q_1 + q_2] = 2.932$$

3) With only asset  $S^2$ :  $\tilde{q}_1 = P(\omega_1)$   
 $\tilde{q}_2 = P(\{\omega_2, \omega_3\})$

$$\begin{cases} \frac{15}{1.05} \tilde{q}_1 + \frac{8}{1.05} \tilde{q}_2 = 10 \\ \tilde{q}_1 + \tilde{q}_2 = 1 \end{cases}$$

$$\tilde{q}_1 = \frac{5}{16} \quad \tilde{q}_2 = \frac{36}{56} \quad \text{unique sol.}$$

Complete market (1-step binomial model)

**Exercise 2** (10 points)

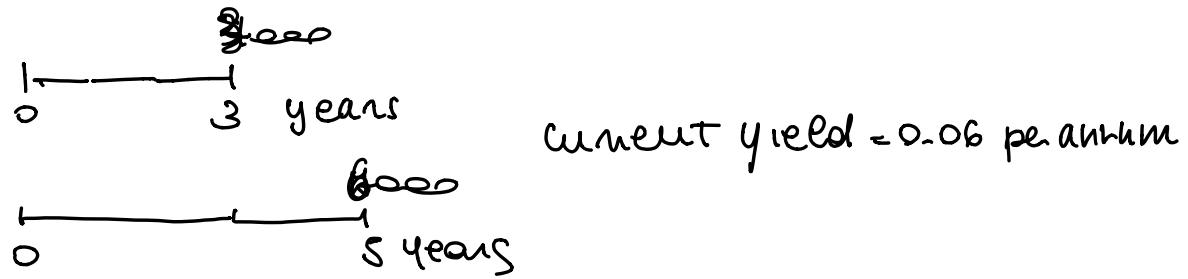
A portfolio consists of a 3-year zero-coupon bond with face value of 3000 \\$ and a 5-year zero-coupon bond with face value of 6000 \\$. The current yield on all bonds is 6% per annum.

- (a) Compute the duration of the portfolio (using continuous compounding).
- (b) Compute the percentage change in the value of the portfolio in the case of a 0.2% per annum decrease in yields.
- (c) Compare the result obtained in (b) with the 1st order and 2nd order approximations based on the use of duration and of convexity, and comment the results.

## SOLUTION

### EXERCISE 1

(a) We have :



Duration :

$$D = \frac{\sum_{i=1}^m t_i c_i e^{-yt_i}}{\sum_{i=1}^n c_i e^{-yt_i}} =$$

$$= \frac{3 \cdot 3000 e^{-0.06 \cdot 3} + 5 \cdot 6000 e^{-0.06 \cdot 5}}{3000 e^{-0.06 \cdot 3} + 6000 e^{-0.06 \cdot 5}} =$$

$$= \frac{29241,98}{6950,72} = 4,27$$

$$D = 4,27 \text{ years}$$

(b) Value of the portfolio:

$$V = 3000 e^{-0.06 \times 3} + 6000 e^{-0.06 \times 5} = 6950,72$$

If the yields decreases by 0,2% it becomes 5,8%. The value of the portfolio becomes:

$$V' = 3000 e^{-0,058 \cdot 3} + 6000 e^{-0,058 \cdot 5} = 7010,672$$

Percentage change of the portfolio:

$$\frac{V' - V}{V} = \frac{7010,67 - 6950,72}{6950,72} = \frac{59,75}{6950,72} = 0,86\%$$

(c) Using first order approximation based on duration we have

$$\frac{\Delta V}{V} = -D \Delta y = -4,27 (-0,002) = 0,0083 = \\ = 0,85\%$$

The convexity of the portfolio is

$$C = \frac{\sum_{i=1}^n t_i^2 c_i e^{-y t_i}}{\sum_{i=1}^n c_i e^{-y t_i}} = \\ = \frac{3^2 \cdot 3000 e^{-0.06 \times 3} + 5^2 \cdot 6000 e^{-0.06 \times 5}}{6950,72} = \\ = \frac{133675}{6950,72} = 19,23$$

and using the second order approximation based on the convexity we have :

$$\begin{aligned}\frac{\Delta V}{V} &= -D \Delta y + \frac{1}{2} C (\Delta y)^2 = \\ &= -9.27 \cdot (-0.002) + \frac{1}{2} 19.23 (-0.002)^2 = \\ &= 0.00887 = 0.86\%\end{aligned}$$

The approximation based on the convexity is better, since it is a second order Taylor expansion that is always more precise than a first order Taylor expansion, like the one based on intuition.

**Exercise 3** (10 points)

Let  $dS = \mu_t dt + \sigma_t dW$  and risk-free interest rate is 4% per annum (all rates are continuously compounded). (a) When  $\sigma_t = 0.01$  for  $t \in [0, 4]$  and

$$\mu_t = \begin{cases} 0.02t; & 0 \leq t \leq 2 \\ 0.01(10 - t); & 2 < t \leq 4 \end{cases} \quad (0.4)$$

compute the probability of having at  $t = 4$  a profit greater or equal than 0.4.

(b) When  $\mu_t = 0.01$  for  $t \in [0, 4]$  and  $\sigma_t = 0.2$  for  $t \in [0, 4]$  compute the probability of having at  $t = 4$  a profit greater than or equal to 0.4.

(c) Assuming  $\mu_t = 0.03S_t$  and  $\sigma_t = 0.01S_t$  for  $t \in [0, 1]$ , in years, i.e.

$$dS = 0.03S_t dt + 0.01S_t dW$$

and that the risk free rate is 2% per annum, price an European call option with strike price 100, maturing in one year , written on the stock  $S(t)$  (no dividends) whose current price  $S(0)$  is 98.

(a) by integrating  $dS_t = \mu_t dt + \sigma_t dW_t$   
 we have

$$\begin{aligned}
 S_4 - S_0 &= \int_0^4 \mu_s ds + \int_0^4 \sigma_s dW_s = \\
 &= \int_0^2 0.02 s ds + \int_2^4 0.01(10-s) ds + \int_0^4 0.01 dW_s \\
 &= 0.02 \frac{s^2}{2} \Big|_0^2 + 0.01 \left[ 10s - \frac{s^2}{2} \right]_2^4 + 0.01 (W_4 - W_0) \\
 &= 0.04 + 0.01 \left[ 40 - 8 - 20 + 2 \right] + 0.01 W_4 \\
 &= 0.18 + 0.01 W_4 \sim N(0, 18, 0.01^2)
 \end{aligned}$$

Profit greater or equal to 0.6

means  $S_4 - S_0 \geq 0.6$

$$\begin{aligned}
 P((S_4 - S_0) \geq 0.6) &= P(0.18 + 0.01 W_4 \geq 0.4) \\
 &= P(W_4 \geq \frac{0.4 - 0.18}{0.01}) =
 \end{aligned}$$

$$= P\left(Z > \frac{0.22}{\sqrt{0.1}}\right)$$

because  $W_4 \sim N(0, 4)$



$$= P\left(Z > \frac{0.22}{0.2}\right) = 1 - \phi(1) \approx 0$$

(b)

$$S_{t+1} - S_0 = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s =$$

$$= 0.01s \int_0^4 + \int_0^4 0.2 dW_s$$

$$= 0.01 \cdot 4 + 0.2(W_4 - W_0)$$

$$= 0.04 + 0.2 N_4$$

where  $N_2 \sim N(0, 4)$  since

$W_t - W_s \sim N(0, t-s)$ . thus

$$0.04 + 0.2 N_4 = 0.04 + N,$$

where

$$N \sim N(0, (0, 0.4) \cdot 4)$$

$$\sim N(0, 0.16)$$

therefore

$$P(S_n - S_0 > 0.4) = P(0.04 + N \geq 0.4)$$

$$= P\left(Z \geq \frac{0.4 - 0.04}{\sqrt{0.16}}\right) = P(Z \geq 0.9)$$

$$= 1 - \phi(0.9) = 0.18$$

(C) We have  $S(0) = 98$   $k = 0.02$   $T = 14$

$b = 0.01$  per annum  
 $\sigma = 0.02$

According to the Black-Scholes-Merton formula for a European call option

$$C = S(0) N(d_1) - k e^{-rT} N(d_2)$$

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S}{k}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \\ &\approx \frac{\ln(0.98) + \left(0.02 + \frac{0.01^2}{2}\right) \cdot 1}{0.01} \\ &\approx -0.0152 \end{aligned}$$

$$d_2 = d_1 - \sigma \sqrt{T} \approx -0.0152 - 0.01 \sqrt{1} = -0.025$$

$$\begin{aligned} C &= 98 N(-0.015) - 100 e^{-0.02} N(-0.025) \\ &\approx 0.38 \end{aligned}$$