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Exercise 1.
Consider a 2-year European call option with a strike price of \$90 on a stock whose current price is \$100. Suppose that there are 2 time steps of 1 year and in each time step the stock price either moves up by a factor of 1.5 or moves down by a factor of 0.5. Suppose that the risk-free interest rate is 1% per annum with simple compounding.

1. Find the price of the option.
2. Find the delta of the option in $t = 1$.
3. Find the price of an option written on the same stock, with the same maturity and strike (K=90) and payoff

$$\phi = \big(S_T - \frac{(K - S_0)^2}{2}\big)^+.$$

Exercise 2.
Let $dS = 0.06Sdt + 0.4SdW$ and risk-free interest rate is 4% per annum (all rates are continuously compounded).

1. Price an European call option with strike price 100, maturing in five months, written on the stock $S(t)$ (no dividends) whose current price $S(0)$ is 90.
2. Establish whether the probability to exercise the call option is greater that the probability to exercise the put with the same strike and maturity and both written on the stock above.
3. Find the greatest strike K^* making the call payoff greater than 10 euros with probability at least of 10%.

Exercise 3 Suppose that $S(t)$ satisfies:

$$dS(t) = \mu_t S(t)dt + \sigma_t S(t)dW(t).$$

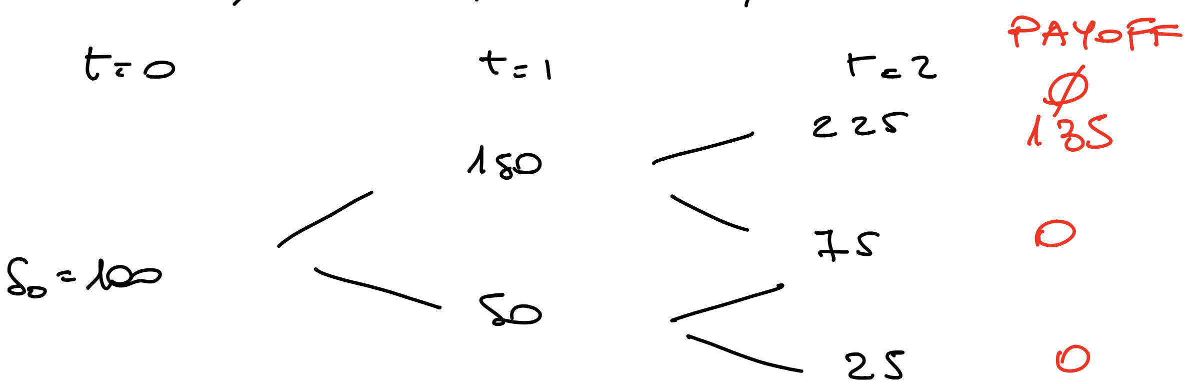
1. Assume that the drift and volatility are picewise constant functions of time (measured in years). For the first three years $\mu_t = 2$ and $\sigma_t = 3$ and for the next three years $\mu_t = 3$ and $\sigma_t = 4$. Assume $S_0 = 6$, what is the probability distribution of $S(6)$?

2. Assume that $\mu_t = \mu t$ and $\sigma_t = \sigma t$. By applying Ito's formula find the dynamics of $Y(t) = \log S(t)$.

Solution

Exercise 1 :

$S_0 = 100$; $K = 90$; 2-years ; $r = 1\%$

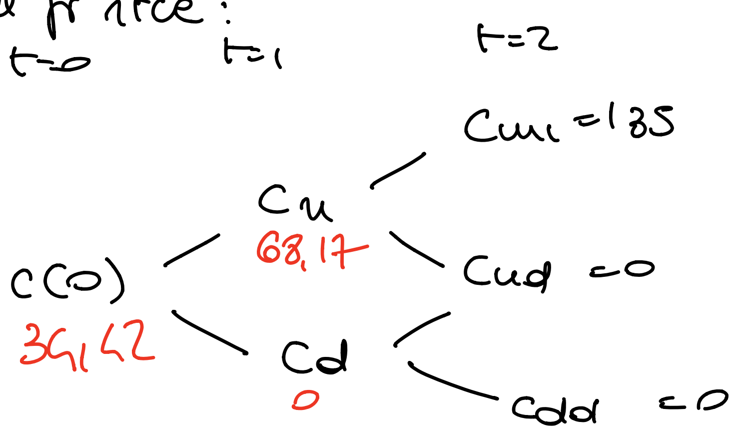


Risk neutral probabilities:

$$q_u = \frac{(1+0.01) - 0,5}{1,5 - 0,5} = \frac{0,51}{1} = 0,51$$

$$q_d = 1 - 0,51 = 0,49$$

Call price:



$$C_u = \frac{1}{1+r}\left(C_{uu}\, q_u + C_{ud}\, q_d\right) = \frac{1}{1{,}01}\left(135 \times 0{,}51 + 0\right) =$$

$= 68{,}17$

$$C_d = \frac{1}{1{,}01}\left(0 \cdot q_u + 0 \cdot p_d\right) = 0$$

$$C(0) = \frac{1}{1+r}\left(C_u\, p_u + C_d\, p_d\right) = \frac{1}{1{,}01}\left(68{,}17 \cdot 0{,}51 + 0\right)$$

$$= 34{,}42$$

The call option price is $C(0) = 35{,}83$

2] in $t = 1$

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{68{,}17}{100} \simeq 0{,}6817$$

3] Payoff $\phi = \left(S_T - \frac{(K - S_0)^2}{2}\right)^+$

$K - S_0 = 90 - 100 = 50$, thus

$\phi = (S_T - 50)^+$

ϕ

$$\begin{tikzcd}
 & & \phi_{uu} = 175 \\
 & \underset{100{,}50}{\phi_u} \arrow[ur] \arrow[dr] & \\
\underset{56{,}87}{\phi(0)} \arrow[ur] \arrow[dr] & & \phi_{ud} = 25 \\
 & \underset{12{,}62}{\phi} \arrow[ur] \arrow[dr] & \\
 & & \phi_{dd} = 0
\end{tikzcd}$$

q_u and q_d are the same of point 1

$$\phi_u = \frac{1}{1{,}01}\left(175 \times 0{,}51 + 25 \times 0{,}49\right) = 100{,}50$$

$$\phi_d = \frac{1}{1{,}01}\left(25 \times 0{,}51\right) = 12{,}62$$

$$\phi(0) = \frac{1}{1{,}01}\left(0{,}51 \times 100{,}50 + 0{,}49 \times 12{,}62\right) = 56{,}87$$

the price of the option is:

$$\boxed{\phi(0) = 56{,}87}$$

EXERCISE 2

$S_0 = 90$; $K = 100$; $T = 5/12$ $r = 0.04$ $\sigma = 0.4$

$e^{-rT} = e^{-0,04 \cdot 5/12} = 0,98$

1]

$$d_1 = \frac{\ln(90/100) + \left(0,04 + \frac{0,4^2}{2}\right) \cdot 5/12}{0,4\sqrt{5/12}} = -0,21$$

$$d_2 = d_1 - 0.4\sqrt{5/12} = -0,47$$

$N(d_1) = 0,415$ $N(d_2) = 0,31$

$$C = S_0 N(d_1) - K e^{-rT} N(d_2) =$$

$$= \underset{36,9}{90 \times 0,41} - \underset{30,38}{100 \cdot 0,98 \cdot 0,31} = 6,52$$

2] Probability to exercise the call option

$P(S_T > K) = P(S_{5/12} > 100) \Rightarrow$

$$\ln S_T \sim N\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T ; \sigma^2 T\right)$$

$$\sim N\left(\ln 90 + \left(0,06 - \frac{0,4^2}{2}\right)5/12, \; 0,4^2\, 5/12\right) =$$

$$= N(4,52 \; , 0,07)$$

$P(\ln S_T > \ln 100) =$

$$= P\left(\frac{\ln S_T - 4{,}52}{0{,}07} > \frac{4{,}61 - 4{,}52}{0{,}07}\right) =$$

$$= 1 - \phi(0{,}34) = 1 - 0{,}63 = 0{,}366$$

Probability to exercise the put

is $P(\ln S_T < \ln 100) = 1 - 0.366 = 0{,}634$

So the probability to exercise the put is higher than the probability to execise the call

3] $(S_T - K)^+ > 10$ iff $S_T > 10 + K$, thus

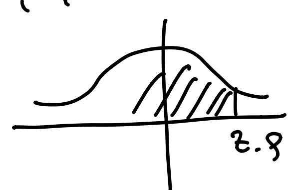
$$P\left((S_T - K)^+ > 10\right) = P\left(S_T \geq K + 10\right)$$

$$= P\left(\ln S_T \geq \ln(K+10)\right)$$

$$P\left(Z \geq \frac{\ln(K+10) - 4{,}52}{0{,}07}\right) =$$

$$= 1 - \phi\left(\frac{\ln(K+10) - 4{,}52}{0{,}07}\right) \geq 0.1$$

$$\phi\left(\frac{\ln(K+10) - 4{,}52}{0{,}07}\right) \leq 0{,}9$$



$$\frac{\ln(K+10) - 4.52}{0{,}26} \leq z_{0.9}$$

$$\frac{\ln(K+10) - 4{,}52}{0{,}26} \leq 1{,}28$$

$$\ln(k+10) \leq 4,85$$

$$k+10 \leq e^{4,85}$$

$$k \leq 118,85$$

$$\boxed{\text{The longer } k \text{ is } k^* = 118,85}$$

EXERCISE 3

1] this is a generalized Wiener process with piecewise coefficients, i.e.

$$S_3 - S_0 \sim N(2\cdot 3,\ 3^2\cdot 3)$$

and

$$S_6 - S_3 \sim N(3\cdot 3,\ 4^2\cdot 3)$$

Since the increments are independent

$S_6 - S_0 = S_3 - S_0 + S_6 - S_3$ still has a Normal distribution with

$$\mu = E(S_6 - S_0) = 2\cdot 3 + 3\cdot 3 = 6 + 9 = 17$$

$$\sigma^2 = V(S_6 - S_0) = 3^2\cdot 3 + 4^2 \cdot 3 = 27 + 48 = 75$$

thus, since $S_0 = 6$

$S_6 \sim N(S_0 + \mu, \sigma^2)$, i.e.

$E[S_6] = 6 + 15 = 21$

$V(S_6) = 75 \quad (V[S_0] = 0)$

$\Rightarrow S_6 \sim N(21, 75)$

It is also accepted the solution obtained by assuming μ_t and σ_t picewise constant:

Since $\ln S_T - \ln S_0 \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$

$$\ln S_6 - \ln S_0 = \underbrace{\ln S_6 - \ln S_3}_{N(-7{,}5,\, 27)} + \underbrace{\ln S_3 - \ln S_0}_{N(-15,\, 48)}$$

$$\ln S_6 = \ln S_0 + N(-22{,}5, 75)$$

$$\Rightarrow N\big(\underbrace{-22{,}5 + 1{,}79}_{-20{,}71}, 75\big)$$

In this case you find the distribution of $Y_6 = \ln S_6 \sim N(-20{,}71, 75)$

2] $Y = \log S$ $\quad \frac{\partial f}{\partial t} = 0 \quad \frac{\partial f}{\partial x} = \frac{1}{x} \quad \frac{\partial^2 f}{\partial x^2} = -1/x^2$

$$dY = \left(\frac{\partial f}{\partial t} + \mu t S \frac{\partial f}{\partial S} + \frac{\sigma^2 t^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} \right) dt +$$

$$+ S \sigma t \frac{\partial f}{\partial S} \, dW_t =$$

$$= \left(0 + \mu t \frac{1}{S} \mu t x + \frac{\sigma^2 t^2 S^2}{2} \left(-\frac{1}{S^2} \right) \right) dt +$$

$$+ \sigma t \frac{S}{S} \, dW_t =$$

$$= \left(\mu t - \frac{1}{2} \sigma^2 t^2 \right) dt + \sigma t \, dW_t$$

$$dY(t) = \left(\mu t - \frac{1}{2} \sigma^2 t^2 \right) dt + \sigma t \, dW_t$$