Politecnico di Torino
Financial Engineering-Exam 07-2-2024
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SURNAME and NAME

Student number

All the answers must be clearly motivated, the numerical results are not sufficient. All the answers that will be considered in the correction MUST be written here. If the student number and the name are not filled in the text will not be corrected.

Answers written with a pencil are null.

Exercise 1 (10 points)

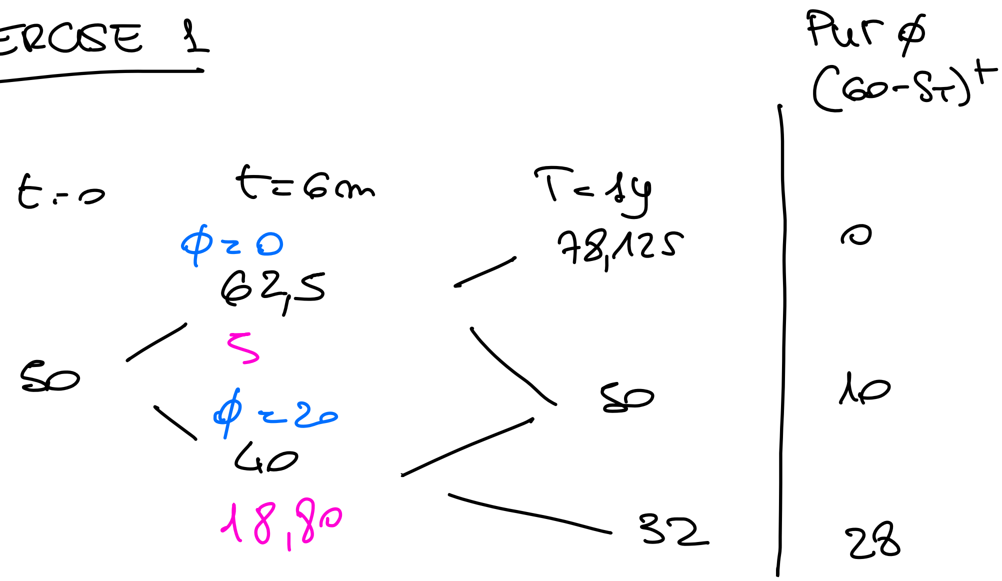
An option is written on a stock without dividends and with current price of 50 euros. At the end of each one of next two semesters the stock price can rise by 25% or fall by 20%, the risk-free (nominal annual) interest rate is 4% convertible 2 times a year and the strike of the option is 60 euros. 1. Find the current price of a European Put option written on this underlying and with maturity T = 1 year.

4 2. Is it optimal to exercise the corresponding (i.e. with the same parameters) American option before maturity? Why? What is its fair value? Comment on the results.

3. Discuss whether there exists a probability measure P under which the price process
3 has costant mean.

3

EXERCISE 1



1. Find the risk neutral measure

N.B. one period r is $0.04/2 = 0.02$

$$q_u = \frac{(1.02) - 0.8}{1.25 - 0.08} = 0.49$$

$$q_d = 1 - 0.49 = 0.51$$

$$P_{6m\,u} = \frac{1}{1{,}02}\left[0 \times 0.49 + 10 \times 0{,}51\right] = 5.$$

$$P_{6m\,d} = \frac{1}{1{,}02}\left[10 \times 0.49 + 28 \times 0{,}51\right] = 18{,}8$$

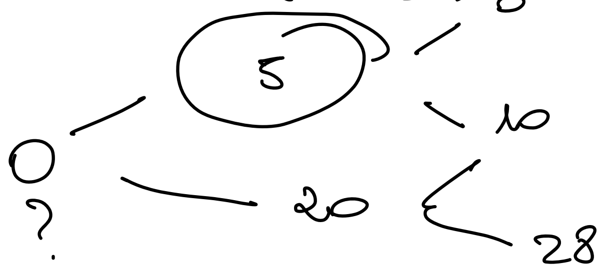
$$P = \frac{1}{1{,}02}\left[5 \times 0{,}49 + 18{,}80 \times 0{,}51\right] = 11{,}8$$

the put price is $P = 11{,}80$

2) the optimal exercise is in node down at time $t = 6$ m because the payoff is 20 while the continuation value is 18.80

the American option dynamics is

$E(S(3)|S(2))_0 =$



$$P^A_0 = \frac{1}{1.02}\left[5 \times 0.49 + 20 \times 0.51\right] = 12.40$$

the american put price is higher because early exercise is optimal

3) p/ $E[S(1)] = p\,62.5 + (1-p)\,40 = 50$

$$22.5p = 10$$

$$\underline{p = 0.44}$$

$$E[S(2)] = 0.44^2 \times 78.125 + 2\,0.44 \times 0.56\ 50 + 0.56^2 \times 32 = 50$$

So $p = 0.44$

Exercise 2 (10 points) Let us consider a European-style put option on a non-dividend-paying asset whose price follows a GBM with drift 10% and volatility 40% (annualized). The risk-free rate is 5% with continuous compounding. The option matures in six months, the current underlying asset price is 40, and the strike is 50.

Sol

3 1. Find the put price.

4 2. Find the probability that the payoff is between 10 and 20.

3 3. You hold a portfolio that consists of a long position in 1000 put options. How many stock shares do you need to make the portfolio delta-neutral?

EXERCISE 2

1) $$d_1 = \frac{\ln S/K + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln 40/50 + \left(0.05 + \frac{0.4^2}{2}\right)1/2}{0.4\sqrt{1/2}} = -0.559$$

$$d_2 = \frac{\ln S/K + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln 40/50 + \left(0.05 - \frac{0.4^2}{2}\right)1/2}{0.4\sqrt{1/2}} = -0.842$$

$$P = Ke^{-rT}N(-d_2) - S_0N(-d_1) =$$

$$= 50e^{-0.05/2}N(+0.842) - 40\,N(+0.559) \cong 10{,}54$$

2) $P(10 \le (K - S_T)^+ \le 20)$

$(K - S_T)^+ \le 20$ ne $S_T \ge K - 20 = 50 - 20 = 30$

$(K - S_T)^+ \ge 10$ ne $S_T \le 50 - 10 = 40$

$$P(10 \le (K - S_T)^+ \le 20) = P(30 \le S_T \le 40) =$$

$$= P(S_T \le 40) - P(S_T \le 30)$$

Since $P(S_T \ge 40)$ is the prob. of exercise of a call option with strike $K' = 40$ we have under the historical measure ($\mu = 0.1$)

$$P(S_T \ge 40) = \Phi\left(\frac{\ln 40/40 + \left(0.1 - \frac{0.4^2}{2}\right)1/2}{0.4\sqrt{1/2}}\right) = 0.5142$$

$N(\tilde{d}_2)$ with drift $\mu = 0.1$ and $K = 40$

Since $P(S_T \geq 30)$ is the prob. of exercise of a call option with strike $K'' = 30$ we have (under the historical measure : $\mu = 0.1$)

$$P(S_T \geq 30) = \phi\left(\frac{\ln 40/30 + \left(0.1 - \frac{0.4^2}{2}\right)1/2}{0.4\sqrt{1/2}}\right) = 0.8537$$

$N(d_2)$ with drift $\mu = 0.1$ and $K = 30$

$$1 - P(S_T \geq 40) - (1 - P(S_T \geq 30)) = P(S_T \geq 30) - P(S_T \geq 40)$$
$$= 0.8537 - 0.5142 = 0.3396$$

3) $+1000\, p + n S = Port$

$\Delta Port = +100\, \Delta p + n$

$\Delta p = N(d_1) - 1 = 0.228 - 1 = -0.71$

$0 = -1000 \times 0{,}71 + n \Rightarrow \boxed{n = +710}$

Exercise 3 (10 points)

The price of a stock on 1^{st} January is 130\$, and it will pay a dividend of 3\$ on 1^{st} June and a dividend of 2\$ on 1^{st} September. The interest rate is 10% per annum. On 1^{st} January the forward price for delivery of the stock on 1^{st} October is 142\$.

3 1. Find if there is an arbitrage opportunity, explaining why.

4 2. If so, illustrate the arbitrage opportunity and compute the arbitrage profit.

3 3. If the stock does not pay dividends and forward price for delivery of the stock on 1^{st} October is still 142\$, is there an arbitrage oppurtunity? Why? Find the value of the interest rate that makes the market arbitrage free.

1 The theoretical forward price for delivery of the stock on 1st October is:

$$F(0,T) = (S(0) - I)\, e^{rT}$$

where I is the present value of the dividends, that is:

$$I = 3 \cdot e^{-0,10 \cdot \frac{5}{12}} + 2 \cdot e^{-0,10 \cdot \frac{8}{12}} = 4,75$$

Therefore:

$$F\left(0, \tfrac{9}{12}\right) = (130 - 4,75)\, e^{0,10 \cdot \frac{9}{12}} = 135$$

and since this value is less than the quoted forward price of 142 \$ there is an arbitrage opportunity.

2. The arbitrage opportunity can be realized as follows:

on 1st January

- borrow 130 \$	+ 130
- buy one stock	- 130
- enter into a short forward position	—
net profit	0

on 1st June

- collect the first dividend	+ 3
- invest the dividend risk-free	- 3
net profit	0

on 1st September

- collect the second dividend	+ 2
- invest the dividend risk-free	- 2
net profit	0

on 1st October

- pay the loan back $\quad - 130 \cdot e^{0,10 \cdot \frac{9}{12}} = -140,12$
- sell the stock forward $\quad = +142$
- collect the result of the $\quad 3 \cdot e^{0,10 \cdot \frac{4}{12}} +$

- sell the stock forward $\quad = +142$

- collect the result of the investment of dividends $\quad 3 \cdot e^{0,10 \cdot \frac{4}{12}} + 2 \cdot e^{0,10 \cdot \frac{1}{12}} = +5,12$

net profit $\quad 7$

3. If $S(t)$ does not pay dividends

$$F(0,T) = S(0)\, e^{rT} = 130\, e^{0.10 \cdot 9/12} = 140{,}13$$

Since 140,13 is still less than $F_0 = 142$

also in this case there is an arbitrage opportunity

the value of r such that F_0 is the risk neutral price is:

$$130\, e^{r\,9/12} = 142$$

$$r\,9/12 = \ln \frac{142}{130}$$

$$r = \frac{12}{9} \underbrace{\ln \frac{142}{130}}_{0{,}88} \cong 0{,}12$$