

Politecnico di Torino
Financial Engineering-Exam 02-09-2021
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Student number

Exercise 1. (10 points)

Consider a 2-year European put option with a strike price of \$50 on a stock whose current price is \$60. Suppose that there are 2 time steps of 1 year and in each time step the stock price either moves up by a factor of 1.5 or moves down by a factor of 0.5. Suppose that the risk-free interest rate is 1% per annum with simple compounding.

- (a) Find the delta of the option in $t = 1$.
- (b) Find the replicating portfolio in $t = 1$ assuming to have observed an up movement in the first step.

Exercise 2 (10 points)

A stock price is \$50, and the risk-free rate of interest is 6% per annum with continuous compounding for all maturities.

- (a) A trader observes that the forward price of a six month forward contract is $F_0 = 60$. Can he make an arbitrage? Why? How?
- (b) The stock is expected to pay a dividend of \$0.5 per share in 2 months and in 6 months. What are the no arbitrage forward price and the initial value of a six month forward contract on the stock?

Exercise 3 (12 points)

A stock price follows a geometric Brownian motion with an expected return (equal to the risk-free interest rate) of 12% per annum and a volatility of 30% per annum. The current stock price is 50 \$.

- (a) Calculate the price of a 6-month European call option and of a 6-month European put option on this stock (no dividends), both with strike price 40 \$.
- (b) Verify that the prices found in (a) satisfy the put-call parity.
- (c) Find the probabilities that the call option and the put option considered above will be exercised.

Solution

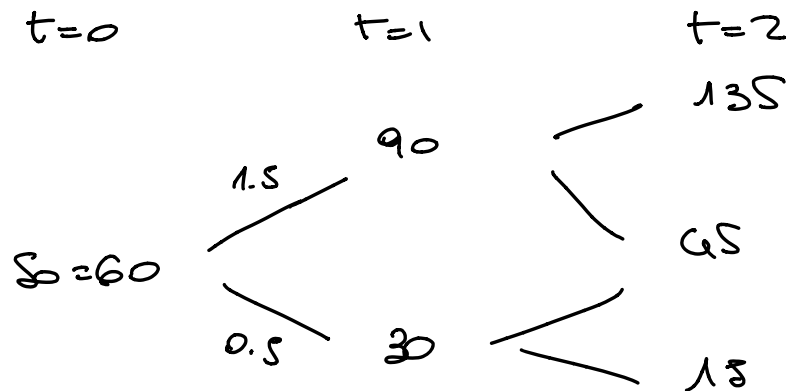
EXERCISE 1

$$K = 50$$

$$r = 0.01$$

$$T = 2 \text{ years}$$

$$S_0 = 60$$



Risk neutral measure -

$$q_u = \frac{(1+r) - d}{u - d} = \frac{(1+0.01) - 0.5}{1.5 - 0.5} = 0.51$$

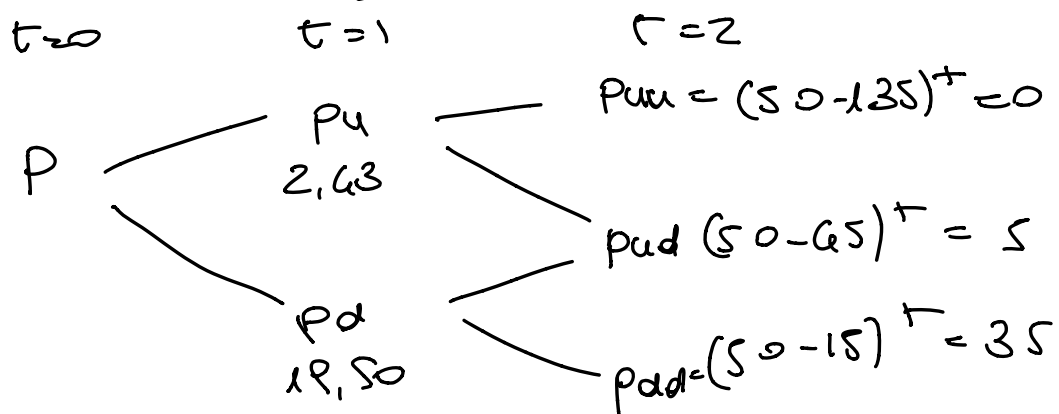
$$q_d = 1 - q_u = 0.49$$

Put price at time $t=1$

$$P_u = \frac{1}{1+r} [P_{uu} q_u + P_{ud} q_d] = \frac{1}{1.01} (0 \cdot q_u + 5 \cdot 0.49) = 2.43$$

$$P_d = \frac{1}{1+r} [P_{du} q_u + P_{dd} q_d] = \frac{1}{1.01} (5 \cdot 0.51 + 35 \cdot 0.49) = \frac{19.7}{1.01} = 19.50$$

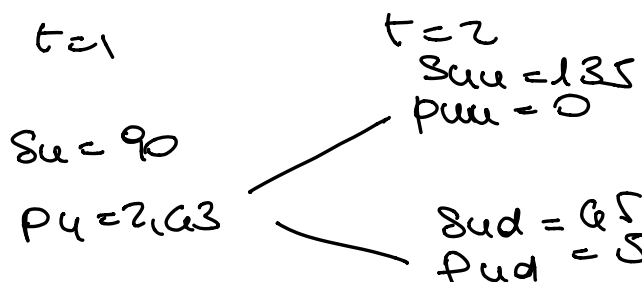
(a) Put option dynamics



$$\Delta_1 = \frac{P_u - P_d}{S_u - S_d} = \frac{2.43 - 19.50}{90 - 30} = -0.28$$

N.B.: It is not necessary to find the price of the put option at $t=0$.

(b) Consider the tree



The hedging portfolio at time t is made of

$$\mathbf{h}_t = (x_t(u), \varphi_t(u))$$

\uparrow \nwarrow
 bank account asset.

where

$$\left\{ \begin{aligned} x_t(u) &= \frac{1}{1+r} \frac{uV_t(u) - dV_t(u+1)}{u-d} \\ y_t(u) &= \frac{1}{S_{t-1}} \frac{V_t(u+1) - V_t(u)}{u-d} \end{aligned} \right.$$

$$V_2(uu) = 0 ; \quad V_2(ud) = 5$$

$$\begin{aligned} x_2(u) &= \frac{1}{1+r} \frac{uV_2(ud) - dV_2(uu)}{u-d} = \\ &= \frac{1}{1.01} \frac{1.5 \cdot 5 - 0.5 \cdot 0}{1} = 7.43 \end{aligned}$$

$$\begin{aligned} y_2(u) &= \frac{1}{90} \frac{V_2(uu) - V_2(ud)}{u-d} = \\ &= \frac{1}{90} \frac{0 - 5}{1} = -0.056 \end{aligned}$$

Or you can find it directly by:

$$\begin{aligned} V_{uu} &= x_1(u)(1+r) + y_1(u) \cdot S_{uu}^{135} = 0 \\ - V_{ud} &= x_1(u)(1+r) + y_1(u) \cdot S_{ud}^{45} = 5 \\ \hline &90y_1 = -5 \\ y_1 &= -5/90 = -0.056 \end{aligned}$$

$$x_1(u) = 1.01 + (-0.056) \cdot 135 = 0$$

$$x_1(u) = \frac{135 \cdot 0,056}{1,01} = 7,43$$

$$\boxed{q_1(1) = (7,43, -0,056)}$$

check with the value of $V_2(u)$

$$V_2(u) = 7,43 - 0,056 \cdot 86 =$$

$$= 7,43 - 0,056 \cdot 90 = 2,43 = p_u$$

□

EXERCISE 2)

(a) $S_0 = 50$
 $r = 0.06$

$$F_0 = 60$$

the risk neutral price for a forward contract is

$$\tilde{F}_0 = e^{0.06 \cdot \frac{1}{2}} \cdot 50 = 50 e^{0.03} = 51.52$$

thus, $F_0 > \tilde{F}_0$ and there is an arbitrage opportunity

Arbitrage strategy

	$t=0$		$t=1$
short forward	0		
borrow \$ S_0	$+S_0$	→	pay the loan $= S_0 e^{rt}$
buy asset S_0	$-S_0$		sell the asset $+ F_0$
	<u>0</u>		<u>$F_0 - S_0 e^{rt}$</u>

$$\begin{aligned} F_0 - S_0 e^{rt} &= 60 - 50 e^{0.03} < \\ &= 60 - 51.52 > 0 \end{aligned}$$

(b) $d_1 = 0.5$ in 2 months

$d_2 = 0.5$ in 6 months

$$I_1 = 0.5 \times e^{-0.06 \frac{2}{12}} = 0.495$$

$$I_2 = 0.5 \times e^{-0.06 \frac{6}{12}} = 0.48$$

$$I = I_1 + I_2 = 0.98$$

$$\begin{aligned} F_0 &= (S_0 - I) e^{rT} = \\ &= (50 - 0.98) e^{0.06 \frac{1}{2}} = \\ &= 50.81 \end{aligned}$$

The initial value of a forward contract is zero by definition.

EXERCISE 3

a) We have $S_0 = 50$, $r = 12\%$ per annum
 $K = 40$; $\sigma = 30\%$ per annum

$T = 6/12$ 6 months call and 6-month put

According to the Black-Scholes-Merton formula for the price of a European call option

$$C = S_0 N(d_1) - K e^{-rT} N(d_2)$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

hence

$$d_1 = \frac{\ln\left(\frac{50}{40}\right) + \left(0.12 + \frac{0.3^2}{2}\right) \frac{6}{12}}{0.3 \sqrt{\frac{6}{12}}} = 1.4408$$

$$d_2 = 1.4408 - 0.3 \sqrt{\frac{6}{12}} = 1.2284$$

then

$$\begin{aligned} C &= 50 \cdot N(1.4408) - 40 e^{-0.12 \frac{6}{12}} N(1.2284) \\ &= 50 \cdot 0.9252 - 40 e^{-0.12 \frac{6}{12}} \cdot 0.8904 \\ &= 12.72 \end{aligned}$$

The price of the call option is $C = 12.72$

According to the Black-Scholes-Merton formula for the price of a European put option

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where d_1 and d_2 are the same as above, hence:

$$\begin{aligned} p &= 40 e^{-0.12 \frac{6}{12}} N(-1.2287) - 50 N(-1.4408) = \\ &= 40 e^{-0.12 \frac{6}{12}} \cdot 0.1096 - 50 \cdot 0.0748 = \\ &= 0.39 \end{aligned}$$

the price of the put option is $p = 0.39$

b) The put-call parity relation is:

$$C + Ke^{-rT} = p + S_0$$

$$C + Ke^{-rT} = 12.72 + 40 e^{-0.12 \frac{6}{12}} = 50.39$$

$$p + S_0 = 0.39 + 50 = 50.39$$

Therefore the relation is satisfied

c) The probability that the European call option will be exercised is the probability that $S_T > K$, i.e.

$$P(S_{6/12} > 40)$$

Since the expected return $\mu = 0.12 = r$

then $P(S_{6/12} > 40) = N(d_2) = N(1.228) = 0.8903$.

This can also be solved directly as follows.

S follows a geometric Brownian motion:

$$\ln S_T \sim N \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T ; \sigma^2 T \right]$$

$$\ln S_T \sim N \left(\ln 80 + (0,12 - \frac{0,3^2}{2}) \frac{6}{12} ; 0,3^2 \frac{6}{12} \right)$$

$$\ln S_T \sim N (3,9495 ; 0,045)$$

and then:

$$P(S_T > 40) = P(\ln S_T > \ln 40) = P(\ln S_T > 3,689)$$

that is

$$P \left(\frac{\ln S_T - \mu}{\sigma} > \frac{3,689 - 3,9495}{\sqrt{0,045}} \right) = P(Z > -1,228)$$

$$= P(Z < 1,228) = N(1,228) = 0,8903$$

↑
this is $N(d_2)$!

hence the probability that the option is exercised is 89,03%

The probability that the European put option is exercised is

$$P(S_T < K) = P(S_{6/12} < 40) =$$

$$= 1 - P(S_{6/12} > 40) =$$

$$= 1 - 0,8903 = 0,1097$$

hence the probability that the put is exercised is 10,97%.

