Politecnico di Torino
Financial Engineering-Exam 06-19-2025
P. Semeraro

SURNAME and NAME

Student number

All the answers must be clearly motivated, the numerical results are not sufficient. All the answers that will be considered in the correction MUST be written here. If the student number and the name are not filled in the text will not be corrected.

Answers written with a pencil are null.

Exercise 1 (10 points)
A non-dividend paying stock price $S(0)$ is \$100, and the risk-free rate of interest is 5% per annum with continuous compounding for all maturities.
(a) A trader observes that the forward price of a six month g forward contract written on S is $F_0 = 105$. Can he make an arbitrage? Why? Construct an arbitrage strategy.
(b) The stock is expected to pay a dividend of \$0.6 per share in 2 months and in 5 months. What are the no arbitrage forward price and the initial value of a six month forward contract on the stock?

Solution
$S_0 = 100 \quad r = 0.05$

(a) $F_0 = 105$

no arbitrage price:

$$\tilde{F}_0 = S_0 e^{rT} = 100 e^{0.05 \cdot 1/2} = 102.5315$$

Since $\tilde{F}_0 < F_0$ there is an arbitrage opportunity strategy:

	$t=0$ cash position	T	cash position (F_0)
short forward	0	sell S_0	$+105$
long S_0	-100		
borrow \$ S_0	$+100$	repay the loan	$-100e^{rT} = -102.5315$
	$//$		$105-102.5315>0$

Since $105-102.5315>0$ there is an arbitrage strategy

(b) $d_1 = 0.6$ in 2 months $t=2/12$

$d_2 = 0.6$ in 5 months $t=5/12$

$$D_1 = 0.6\, e^{-2/12 \cdot 0.05} = 0.5950$$

$$D_2 = 0.6\, e^{-5/12 \cdot 0.05} = 0.5876$$

$$D = D_1 + D_2 = 1.1826$$

The no arbitrage price in this case is

$$\tilde{F}_0 = (S_0 - D)e^{rT} = (100 - 1.1826)e^{0.05 \cdot 1/2} =$$

$$= 101.3190$$

$\tilde{F}_0$ is the no arbitrage price and the initial value of the contract is zero.

Exercise 2(10 points)

A stock (indexed by A) is available on the market at the current price $S_A(0) = 8$ euros. In one year, the price may increase by 25% or decrease by 25% or stay unchanged. Another stock (indexed by B) is also available. Its current price is $S_B(0) = 12$ euros that, in one year, may increase by 25% (when also the price of stock A is increased) or decrease by 25% (when also the price of stock A is decreased or unchaged). The risk-free interest rate on the market is 4% per year (simple compounding).

1. Consider a European put option with maturity of one year, with strike of 8 euros and written on stock A. Verify if it is possible to replicate such an option only by means of stock A and of cash invested or borrowed at risk free rate. If yes explain the replicating strategy, if no explain why.

2. Verify if it is possible to replicate the European put option above investing in stock A, stock B and cash. If yes, find a replicating portfolio and compute the cost of the replicating strategy. Comment on the result.

3. Discuss whether the market (A, B, C), where C is the risk-free asset is complete. It possible finde a risk neutral measure.

Solution

$t=1$ | $t=1$

$S_A = 8$ — 10 ω_1; 8 ω_2; 6 ω_3

$S_B = 12$ — 15 ω_1; 9 $\{\omega_2, \omega_3\}$

$r = 0.04$

$C(0) = 1 \longrightarrow C(1) = 1.04$

$\Phi(S_A) = (8 - S_T)^+$

$S_A = 8$ — 6: 2; 8: 0; 10: 0

(1) $P(h_A, h_C)$ where A asset A and C is cash

$$V^h(1) = h_A S_A(1) + h_C C(1)$$

Cerco h_A, h_C t.c.

$$V^h(1) = \begin{cases} 2 & \omega_1 \\ 0 & \{\omega_2\ \omega_3\} \end{cases}$$

$$\begin{cases} h_A 6 + h_C\ 1.04 = 2 \\ h_A 8 + h_C\ 1.04 = 0 \\ h_A 10 + h_C\ 1.04 = 0 \end{cases}$$

this system does not have solution

since $|A^C| = \begin{vmatrix} 6 & 1.04 & 2 \\ 8 & 1.04 & 0 \\ 10 & 1.04 & 0 \end{vmatrix} =$

$$= 2 \begin{vmatrix} 8 & 1.04 \\ 10 & 1.04 \end{vmatrix} = 2(8-10)1.04 \neq 0$$

$\operatorname{rg} A^C = 3$

and $\operatorname{rg} A = 2$

We cannot replicate the option on S_A only with cash and S_A because the market $\{S_A, C\}$ is not complete!

In fact a risk neutral measure should verify

$$\begin{cases} q_u 12 + q_s 8 + q_d 6 = 8(1.04) \\ q_u + q_s + q_d = 1 \end{cases}$$

that has a solution (the market is also free since

$$0.75 \leq 1.04 \leq 1.25$$

therefore the system admits a solution)

(2) $h = (h_A, h_B, h_C)$

$$V_h(1) = h_A S_A(1) + h_B S_B(1) + h_C C(1)$$

$$\begin{cases} h_A 6 + h_B 9 + h_C 1.04 = 2 \\ h_A 8 + h_B 9 + h_C 1.04 = 0 \\ h_A 10 + h_B 15 + h_C 1.04 = 0 \end{cases}$$

since $\det A = 12.8$

the system admits a unique solution that is

$$h = (-1, 0.33, 4.8)$$

the cost of the strategy is

$$V^h(0) = -S_A(0) + 0.33 S_B(0) - 4.8 = 0.8$$

(3) the system

$$\begin{cases} q_u 10 + q_s 8 + q_d 6 = 8(1.04) \\ q_u 15 + (q_s + q_d) 9 = 12(1.04) \\ q_u + q_s + q_d = 1 \end{cases}$$

has 1 solution $(\det A \neq 0)$ that is

$q_u = 0.58 \quad q_s = 0 \quad q_d = 0.42$

the measure $\mathbb{Q}$ is not equivalent to $\mathbb{P}$

Exercise 3 (10 points)

Consider a market model where it is possible to trade on a stock with current price $S(0) = 20$ euro. Assume that the stock price follows a geometric Brownian motion with $\mu = 0.03$, $\sigma = 0.1$ and the risk free-interest rate is $r = 0.05$.

1. Consider a European call option with strike $K = 20$ euro and maturity in one year. Find its price.

2. Establish how many shares of the stock we need to buy/sell in order to make our short position Delta-neutral.

3. Consider now two European call options written on a stock that follows a geometric Brownian motion with undefined parameters μ and σ. The strikes are $K_1 = 20$ euro and $K_2 = 40$ euro, and the call prices are 6 and 2 euro, respectively. Both options have maturity in one year. Establish under which condition on μ and $\sigma > 0$ the profit of a bear spread from these two call options is positive with a probability of at least 50%.

1) From Black-Scholes, the price of a European CALL option is

$$C = S N(d_1) - K e^{-rT} N(d_2), \quad \text{where}$$

$$d_1 = \left[\ln \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right) T\right] / \sigma \sqrt{T},$$

$$d_2 = d_1 - \sigma \sqrt{T}.$$

Plugging in the numbers, we get $d_1 = 0.55$, $d_2 = 0.45$ and

$$C = 1.3610.$$

2) Consider the portfolio $h = (\underbrace{x_S}_{\#\text{ of shares}}, \underbrace{x_C}_{\#\text{ of CALLS option}})$

If we are short in 1 option, then $x_C = -1$ and the value of portfolio is: $P = x_S \cdot S - C$.
To make it neutral w.r.t. variation of S we ask that $\frac{\partial P}{\partial S} = 0 \Rightarrow x_S - \Delta^C = 0$

$$\Rightarrow x_S = \Delta^C.$$

We then compute $\Delta^C = N(d_1) = 0.7088$, i.e. to make our short position neutral we need to buy 0.7088 shares.

3) Recall that a bear spread is obtained by:

- selling the CALL option with lower strike. (A)
- buying the " " with higher strike. (B)

We have the following scheme:

	A	B	TOTAL PAYOFF	TOTAL PROFIT
			0	$C_1 - C_2$ 4
$S_T \leq K_1$	0	0		
$K_1 < S_T < K_2$	$-(S_T - K_1)$	0	$K_1 - S_T$	$K_1 - S_T + C_1 - C_2 = 24 - S_T$
$S_T \geq K_2$	$-(S_T - K_1)$	$S_T - K_2$	$K_1 - K_2$	$K_1 - K_2 + C_1 - C_2 = -16$

We further know that $S_T \sim S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right)$.

Hence, $0.5 \le \mathbb{P}(\text{PROFIT} \ge 0)$

$$= \mathbb{P}\left(\{0 \le S_T \le 20\}\right) \cup \{20 \le S_T \le 24;\ 24 - S_T \ge 0\}\big)$$

$$= \mathbb{P}\left(\{0 \le S_T \le 24\}\right)$$

$$= \mathbb{P}(S_T \le 24)$$

$$= \mathbb{P}\left(20 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} \le 24\right)$$

$$\overset{T=1}{=} \mathbb{P}\left(\left(\mu - \frac{1}{2}\sigma^2\right)1 + \sigma W_1 \le \ln\frac{24}{20}\right)$$

$$\stackrel{..}{=} \mathcal{N}\left(\frac{\ln(1.2) - \left(\mu - \frac{1}{2}\sigma^2\right)}{\sigma}\right).$$

We now look for a condition on (μ, σ): $\mathcal{N}\left(\frac{\overset{24/20}{\ln(1.2)} - \left(\mu - \frac{1}{2}\sigma^2\right)}{\sigma}\right) \ge 0.5$,

Which implies that $\frac{\ln(1.2) - \left(\mu - \frac{1}{2}\sigma^2\right)}{\sigma} \ge 0 \Rightarrow \boxed{\mu \le \frac{1}{2}\sigma^2 + \ln(1.2)}$

$$\ln(1.2) = 0.182$$