Politecnico di Torino
Financial Engineering-Exam 07-21-2023
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SURNAME and NAME

Student number

All the answers must be clearly motivated, the numerical results are not sufficient. All the answers that will be considered in the correction MUST be written here. If the student number and the name are not filled in the text will not be corrected.

Answers written with a pencil are null.

Exercise 1 (10 points)

A trader considers the following two contracts:

(i) a 1-year long forward contract on a non-dividend-paying stock;

(ii) a 1-year long futures contract on a commodity that is an investment asset with storage costs.

The stock price relative to the first contract is 60 euros, while the spot price of the commodity relative to the second contract is 30 euros. The risk-free rate of interest is 5% per annum with continuous compounding for all maturities for both contracts.

1) Explain what is a forward contract and what are the forward price and the initial value of a forward contract.

2) Find the forward price and the initial value of the forward contract in point (i).

3) Find the forward price and the value of the forward contract after 6 months, assuming that the price of the stock is now 65 euros and the risk-free interest rate is still 5%.

4) Find the futures price of the commodity for delivery in 1 year assuming the presence of storage costs of 0.80 euros per year payable quarterly in advance. Comment on the difference between the forward price in case of commoditities and in case of stocks.

EXERCISE 1

1) See the theory

2) The forward price in the case of a non-dividend-paying stock is:

$$F_0 = S_0 e^{rT}$$

hence in this case:

$$F_0 = 60 \cdot e^{0,05 \cdot 1} = 63,08$$

while the initial value of the forward contract is always 0.

3) After 6 months the forward price is:

$$F_0 = S_0 e^{rT} = 65 \cdot e^{0,05 \cdot \frac{6}{12}} = 66,65$$

while the value of the forward contract is:

$$f = S_0 - K e^{-rT}$$

that is:

$$f = 65 - 63,08 \cdot e^{-0,05 \cdot \frac{6}{12}} = 3,48$$

4) The futures price in the case of a commodity that is an investment asset with storage costs is:

$$F_0 = (S_0 + U) e^{rT}$$

and in this case the present value of the storage costs for 1 year is:

$$U = 0,20 + 0,20 \cdot e^{-0,05 \cdot \frac{3}{12}} + 0,20 \cdot e^{-0,05 \cdot \frac{6}{12}} + 0,20 \cdot e^{-0,05 \cdot \frac{9}{12}} = 0,79$$

hence the futures price of the commodity is:

$$F_0 = (30 + 0,79) \cdot e^{0,05 \cdot 1} = 32,37$$

Storage costs: see theory.

It is important to specify that storage costs are added because the owner holding the asset (short) means we storage costs. In fact the long part does not pay storage costs until it own the asset, so the short part requires the long part to compensate storage costs and the forward price is higher.

Exercise 2(10 points)

A stock (indexed by A) is available on the market at the current price $S_A(0) = 8$ euros. In one year, the price may increase by 25% or decrease by 25%. Another stock (indexed by B) is also available. Its current price is $S_B(0) = 12$ euros that, in one year, may increase by 25% (when also the price of stock A is increased) or decrease by 25% (when also the price of stock A is decreased). The risk-free interest rate on the market is 4% per year (simple compounding).

1. Consider a European Call option with maturity of one year, with strike of 8 euros and written on stock A. Verify if it is possible to replicate such an option only by means of stock A and of cash invested or borrowed at risk free rate. If yes explain the replicating strategy.

2. Verify if it is possible to replicate the European Call option above by taking long positions in stock A, short positions in stock B and cash. If yes, find a replicating portfolio and compute the cost of the replicating strategy.

3. Verify if it is possible to replicate the European Call option above only by means of stock A, of cash and of one (and only one) short position in stock B. If yes, find the replicating portfolio and compute the cost of the replicating strategy.

SOLUTION EXERCISE 2

$r = 0.04$

$$T = 1 \qquad\qquad \phi(S_A(T)) = (S_A(T) - 8)^+$$

$$\begin{array}{lll}
 & S_A(1) = 10 & 2 \\
 & S_B(1) = 15 & \\
S_A(0) = 8 & & \\
S_B(0) = 12 & & \\
 & S_A(1) = 6 & 0 \\
 & S_B(1) = 9 &
\end{array}$$

$$\begin{array}{ll}
 & S_B(1) = 15 \\
S_B(0) = 12 & \\
 & S_B(1) = 9
\end{array}$$

$$C(0) = 1 \longrightarrow C(1) = 1.04$$

↑ cash

(a)

$P = (h_S, h_B)$

$$V_p(1) = \begin{cases} 2 & \text{if } S_A(1) = 10 \\ 0 & \text{if } S_A(1) = 6 \end{cases} \quad ; \quad V_p(1) = h_S S(1) + h_C (C(1))$$

$$V_p(1) = \begin{cases} h_S 10 + 1.04 \cdot h_C & \text{if } S_A(1) = 10 \\ h_S\, 6 + 1.04 \cdot h_C & \text{if } S_A(1) = 6 \end{cases}$$

$$\begin{cases} h_S\, 10 + h_C \cdot 1.04 = 2 \\ h_S \cdot 6 + h_C \cdot 1.04 = 0 \end{cases}$$

$4 h_S = 2 \qquad h_S = 1/2$ and $h_C = -2{,}88$

$\boxed{\text{buy } \tfrac{1}{2} \text{ share and borrow } 2.88 \text{ euros}}$

(b) long A, short B and cash

$P(h_A, h_B, h_C)$

$V_P(1) = h_A S_A(1) + h_B S_B(1) + h_C S_C(1)$ $\quad \phi(S_A(1))$

$$V_P(1) = \begin{cases} h_A 10 + h_B 15 + h_C 1.04 = 2 \\ h_A \cdot 6 + h_B \cdot 9 + h_C 1.04 = 0 \end{cases}$$

N.B. we want a solution with $h_A > 0$ and $h_B < 0$

~~A~~ not unique

the system has ∞^1 solutions $3-2$

since $\begin{vmatrix} 15 & 1.04 \\ 9 & 1.04 \end{vmatrix} \neq 0 \Rightarrow rg A = rg A^c = 2$

Parametrize with h_B that I can choose $h_B \leq 0$

$h_A 10 + h_B 15 + h_C 1.04 = 2$

$h_A \cdot 6 + h_B \cdot 9 + h_C 1.04 = 0$ subtraction

$4h_A + 6h_B + \quad // \quad = 2$

$\Rightarrow h_A = 1/2 - \frac{3}{2} h_B$

$1.04 h_C = -6h_A - 9h_B = -3 + 9h_B - 9h_B = -3$

and $h_C = -3/1.04 = -2.88$

For example

$h_B = -\frac{1}{3} \Rightarrow h_A = 1$

$$1.04\, h_C = -6 h_A - 9 h_B = -6 + \frac{9}{3} = -3$$

$$h_C = -\frac{3}{1.04} = -2.88$$

Replication cost

$$V_P(0) = h_A S_A(0) + h_B S_B(0) + h_C =$$

$$= \left(+\tfrac{1}{2} - \tfrac{3}{2} h_B\right) 8 + h_B \cdot 12 - 2.88 =$$

$$= 1.12 \text{ euros}$$

c) the answer of this point is yes, you choose

$h_B = -1$

$h_A = \frac{1}{2} + \frac{3}{2} = 2$, $(h_C = -2{,}88)$

and $V_P(0)$ is unchanged since it is a a constant function of h_B

$$V_P(0) = \left(+\tfrac{1}{2} - \tfrac{3}{2} h_B\right) 8 + h_B \cdot 12 - 2.88 = 1.12 \text{ euros}$$

Exercise 3 (10 points) Consider a market model where it is possible to trade on a stock (with current price $S_0 = 20$ euros) and on different European Call written on such a stock. The price of the stock follows a geometric Brownian motion, namely $dS(t) = 0.8S(t)dt + 0.3S(t)dW(t)$ with $S(0) = 20$. Our goal is an investment with one year as horizon of time. The risk free rate is 4% per year continuously compounded. Consider two European Call options $C1$ and $C2$ (both with maturity T = 1 year and written on the stock above), with strikes of $K1 = 20$ euro and $K2 = 25$ euro, respectively.

1. Find the Black and Scholes price of call option $C1$.

2. Find the price of a put option on the same stock with strike $K1$.

3. Assume that the two options have prices: $C1_0 = 2.75$ euro $C2_0 = 1$ euro, respectively.

   Find the the probability that a bear spread formed by the previous options is non-negative.

1) The Black and Scholes price is:

$$C_0^{E} = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} =$$

$$= \frac{\ln(20/20) + \left(0.04 + \frac{0.3^2}{2}\right)1}{0.3\sqrt{1}} = 0.2833$$

and

$$d_2 = d_1 - 0.3\sqrt{1} = -0.0167$$

Thus

$$N(d_1) = 0.6115$$

$$N(d_2) = 0.4934$$

and

$$C_{10} = 20 \cdot 0.6115 - 20\, e^{-0.04 \cdot 1} \cdot 0.4934 =$$

$$= 2.75$$

2) By using put-call parity we have:

$$P_{10} = C_{10} - S_0 + K e^{-rT} \cong 1.96$$

3)

A bear spread can be built - by means of call options - as follows:

- sell the call option with the lower strike
- buy the call option with the higher strike

the payoff is:

	A	B	TOTAL	TOTAL PROFIT
$S_T < k_1 = 20$	0	0	0	$C_0^1 - C_0^2$
$k_1 = 20 < S_1 < k_2 = 25$	$-(S_T - k_1)$	0	$k_1 - S_T$	$k_1 - S_T + C_0^1 - C_0^2$
$S_1 > 25$	$-(S_T - k_1)$	$S_T - k_2$	$k_1 - k_2$	$k_1 - k_2 + C_0^1 - C_0^2$

thus

	BEAR SPREAD
$S_T < 20$	$2.75 - 1 = 1.75$
$20 < S_T < 25$	$20 - S_T + 1.75 = 21.75 - S_T$
$S_T > 25$	$20 - 25 + 1.75 = -5 + 1.75 = -3.25$

$$P(\text{profit} \geq 0) =$$

$$S_T \leq 21.5$$

$$P(\{S_T < 20\} \cup \{20 \leq S_T \leq 25 \cap 21.75 - S_T > 0\})$$

$$= P(S_T < 20) + P(20 \leq S_T \leq 21.75) =$$

$= P(S_1 \leq 21.75) =$
$\underset{T=1}{\uparrow}$

$= P\left(20 e^{(\mu-\frac{1}{2}\sigma^2)\cdot 1 + 0.3 W_1} \leq 21.75\right) =$

$= P\left(0.8 - \frac{1}{2}0.3^2 + 0.3W_1 \leq \ln \frac{21.75}{20}\right) =$ (ln 1.0875)

$= P\left(0.7550 + 0.3W_1 \leq 0.0839\right) =$

$= P\left(W_1 \leq \frac{0.0839 - 0.7550}{0.3}\right) = P\left(W_1 \leq -2.2370\right)$

$= N(-2.2370) \cong 0.012$

Similarly:

$S_1 = S_0 e^{(\mu - \frac{1}{2}\sigma^2) + \sigma W}$

$\ln S_1 = \ln S_0 + (\mu - \frac{1}{2}\sigma^2) + \sigma W$

etc...