Financial Engineering

Springer Semester 2025

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Problem set 9

Topics: Delta hedging, Greek parameters and Value at Risk

Exercise 1

Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

Exercise 2

Find the generic expression for the delta of a European put option on a non-dividend-paying stock, then calculate it in the case of a European put option with the same characteristics of the call option of Exercise 1.

Exercise 3

Consider the equation:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

and show, by substituting for the various terms, that the equation is true for:
(i) A single European call option on a non-dividend-paying stock.
(ii) A single European put option on a non-dividend-paying stock.
(iii) Any portfolio of European put and call options on a non-dividend-paying stock.

Exercise 4

Use the put-call parity relationship to derive, for a non-dividend-paying stock, the relationship between:
(i) The delta of a European call and the delta of a European put.
(ii) The gamma of a European call and the gamma of a European put.
(iii) The vega of a European call and the vega of a European put.
(iv) The theta of a European call and the theta of a European put.

Exercise 5

Evaluate VaR at 95% confidence level for a one-year investment of 100$ in a stock whose price follows a log-normal distribution with $\mu = 12\%$ and $\sigma = 30\%$, if the interest rate for risk-free investments is $r = 8\%$.

Exercise 6

Evaluate VaR at 95% confidence level for a one-year investment of 1000$ into euros if the interest rate for risk-free investments in euros is $r_{EUR} = 4\%$, the interest rate for risk-free investments in dollars is $r_{USD} = 5\%$ and the exchange rate from euros into US dollars follows a log-normal distribution with $\mu = 1\%$ and $\sigma = 15\%$.

Exercise 7

Suppose that 1000\$ are invested in European call options on a stock with current price $S(0) = 60\$$. The options expire after 1 year with strike price $K = 40\$$. Assume that $\sigma = 30\%$, $r = 8\%$ and the expected logarithmic return on stock is 12%. Compute VaR after 1 year at 95% confidence level.

Exercise session 8 - Solutions

Exercise 1

Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

EXERCISE 1

We have:

$$S_0 = K \qquad r = 0{,}10 \text{ per annum}$$
$$T = \frac{6}{12} \qquad \sigma = 0{,}25 \text{ per annum}$$

and for a European call option the delta is:

$$\Delta(\text{call}) = N(d_1)$$

In this case:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} =$$

$$= \frac{\ln\left(\frac{\cancel{K}}{\cancel{K}}\right) + \left(0{,}10 + \frac{0{,}25^2}{2}\right)\cdot\frac{6}{12}}{0{,}25\sqrt{\frac{6}{12}}} =$$

$$= 0{,}3712$$

and then:

$$N(d_1) = N(0{,}3712) = 0{,}6447$$

Therefore the delta of the option is 0,6447 (hence when the price of the stock increases/decreases by a small amount, the price of the option increases/decreases by 64,47% of this amount, i.e. when the stock price changes by ΔS the option price changes by $0{,}6447\,\Delta S$).

Exercise 2

Find the generic expression for the delta of a European put option on a non-dividend-paying stock, then calculate it in the case of a European put option with the same characteristics of the call option of Exercise 1.

EXERCISE 2

From put-call parity we have:

$$p + S = c + Ke^{-r(T-t)}$$

and differentiating with respect to S:

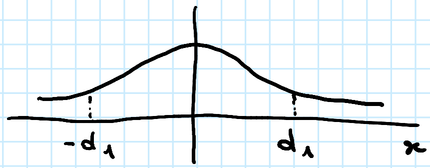
$$\frac{\partial p}{\partial S} + 1 = \frac{\partial c}{\partial S} \rightarrow \frac{\partial p}{\partial S} \rightarrow \frac{\partial c}{\partial S} - 1 \rightarrow \Delta(\text{put}) = \Delta(\text{call}) - 1$$

and since:

$$\Delta(\text{call}) = N(d_1)$$

we have:

$$\Delta\,(put) = N(d_1) - 1 = -\,(1 - N(d_1)) = -\,N(-d_1)$$



For a put option with:

$$S_0 = K \qquad r = 0{,}10$$
$$T = \frac{6}{12} \qquad \sigma = 0{,}25$$
$\rightarrow$ $d_1 = 0{,}3712$ as found in Exercise 1

we then have:

$$\Delta\,(put) = N(d_1) - 1 = N(0{,}3712) - 1 =$$
$$= 0{,}6447 - 1 = -\,0{,}3553$$

and also:

$$\Delta\,(put) = -\,N(-d_1) = -\,N(-\,0{,}3712) = -\,0{,}3553$$

Therefore the delta of the option is $-\,0{,}3553$ (hence when the price of the stock increases/decreases by a small amount, the price of the option decreases/increases by 35,53 % of this amount, i.e. when the stock price changes by ΔS the option price changes by $-\,0{,}3553\,\Delta S$).

Exercise 3

Consider the equation:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

and show, by substituting for the various terms, that the equation is true for:

(*i*) A single European call option on a non-dividend-paying stock.
(*ii*) A single European put option on a non-dividend-paying stock.
(*iii*) Any portfolio of European put and call options on a non-dividend-paying stock.

EXERCISE 3

We have:

For a European call option on a non-dividend-paying stock the Greeks Θ, Δ and Γ are:

$$\Theta = -\,\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT} N(d_2)$$

$$\Delta = N(d_1)$$

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

hence the left-hand side of equation (*) is:

$$-\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) + rS_0 N(d_1) + \frac{1}{2}\sigma^2 S_0^2 \frac{N'(d_1)}{\cancel{S_0\sigma}\sqrt{T}} =$$

$$= -\cancel{\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}}} - rKe^{-rT}N(d_2) + rS_0 N(d_1) + \cancel{\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}}} =$$

$$= r\left[S_0 N(d_1) - Ke^{-rT}N(d_2)\right] =$$

$$= r\Pi$$

and the equation is satisfied.

For a European put option on a non-dividend-paying stock the Greeks Θ, Δ and Γ are:

$$\Theta = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

$$\Delta = N(d_1) - 1 = -N(-d_1)$$

$$\Gamma = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}$$

hence the left-hand side of equation $(*)$ is:

$$-\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) - rS_0 N(-d_1) + \frac{1}{2}\sigma^2 S_0^2 \frac{N'(d_1)}{\cancel{S_0\sigma}\sqrt{T}} =$$

$$= -\cancel{\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}}} + rKe^{-rT}N(-d_2) - rS_0 N(-d_1) + \cancel{\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}}} =$$

$$= r\left[Ke^{-rT}N(-d_2) - S_0 N(-d_1)\right] =$$

$$= r\Pi$$

and the equation is satisfied.

For a portfolio of options the values of Θ, Δ, Γ and Π are the sums of their values for the individual options in the portfolio, hence it follows that equation $(*)$ is true for any portfolio of European put and call options.

EXERCISE 4

We have:

Exercise 4

Use the put-call parity relationship to derive, for a non-dividend-paying stock, the relationship between:

(*i*) The delta of a European call and the delta of a European put.

(*ii*) The gamma of a European call and the gamma of a European put.

(*iii*) The vega of a European call and the vega of a European put.

(*iv*) The theta of a European call and the theta of a European put.

We have:

$$p + S = c + Ke^{-r(T-t)}$$

that is the put-call parity for a non-dividend-paying stock at a general time t. Differentiating with respect to S we have:

$$\frac{\partial p}{\partial S} + 1 = \frac{\partial c}{\partial S} \rightarrow \underbrace{\frac{\partial p}{\partial S}}_{\Delta(\text{put})} = \underbrace{\frac{\partial c}{\partial S}}_{\Delta(\text{call})} - 1$$

that is:

$$\Delta(\text{put}) = \Delta(\text{call}) - 1$$

that shows that the delta of a European put equals the delta of the corresponding European call less 1.

Differentiating again with respect to S we then get:

$$\underbrace{\frac{\partial^2 p}{\partial S^2}}_{\Gamma(\text{put})} = \underbrace{\frac{\partial^2 c}{\partial S^2}}_{\Gamma(\text{call})}$$

that is:

$$\Gamma(\text{put}) = \Gamma(\text{call})$$

that shows that the gamma of a European put equals the gamma of a European call.

Differentiating the put-call parity relationship with respect to σ we have:

$$\underbrace{\frac{\partial p}{\partial \sigma}}_{\nu(\text{put})} = \underbrace{\frac{\partial c}{\partial \sigma}}_{\nu(\text{call})}$$

that is:

$$\nu(\text{put}) = \nu(\text{call})$$

that shows that the vega of a European put equals the vega of a European call.

Differentiating the put-call parity relationship with respect to t, finally, we have:

$$\frac{\partial p}{\partial t} = \frac{\partial c}{\partial t} + rKe^{-r(T-t)}$$

$$\Theta\,(\text{put}) \qquad \Theta\,(\text{call})$$

that is:

$$\Theta\,(\text{put}) = \Theta\,(\text{call}) + rKe^{-r(T-t)}$$

that shows that the theta of a European put equals the theta of the corresponding European call plus $rKe^{-r(T-t)}$.

Exercise 5

Evaluate VaR at 95% confidence level for a one-year investment of 100$ in a stock whose price follows a log-normal distribution with $\mu = 12\%$ and $\sigma = 30\%$, if the interest rate for risk-free investments is $r = 8\%$.

EXERCISE 5

We have:

$$S(0) = 100 \qquad \mu = 0{,}12 \qquad r = 0{,}08$$
$$T = 1 \text{ year} \qquad \sigma = 0{,}30$$

and if we invest 100 \$ at the riskless rate r for 1 year we obtain:

$$S(0) \cdot e^{rt} = 100 \cdot e^{0{,}08 \cdot 1}$$

while if we invest 100 \$ in the stock, after 1 year we have $S(1)$ and the loss is given by:

$$S(0)\,e^{rt} - S(1) = 100 \cdot e^{0{,}08 \cdot 1} - S(1)$$

and VaR at 95% confidence level is the amount such that:

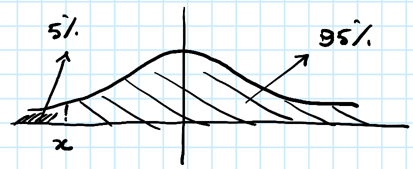
$$P(\text{loss} < VaR) = 95\% \quad \rightarrow \quad P(100 \cdot e^{0{,}08} - S(1) < VaR) = 95\%$$

Since the stock price follows a log-normal distribution, the logarithmic return follows a normal distribution, that is:

$$\ln\left(\frac{S(1)}{S(0)}\right) \sim N(\mu, \sigma^2) \quad \rightarrow \quad \underbrace{\ln\left(\frac{S(1)}{S(0)}\right)}_{k} \sim N(0{,}12;\ 0{,}3^2)$$

hence we have:

$$\underbrace{\frac{k-\mu}{\sigma}}_{z} \sim N(0,1) \quad \rightarrow \quad \frac{k - 0{,}12}{0{,}3} \sim N(0,1)$$



$N(x) = 5\%$ when $x = -1,645$

and with probability 95% we have that:

$$\boxed{Z > -1,645} \rightarrow \frac{k-\mu}{\sigma} > -1,645 \rightarrow \frac{k-0,12}{0,3} > -1,645 \rightarrow$$

$$\rightarrow k > 0,12 - 1,645 \cdot 0,3 = -0,3735$$

hence with probability 95% we have that:

$$\ln\left(\frac{S(1)}{S(0)}\right) > -0,3735 \rightarrow \frac{S(1)}{S(0)} > e^{-0,3735} \longrightarrow$$

$$\rightarrow S(1) > S(0)\,e^{-0,3735} \rightarrow S(1) > 100 \cdot e^{-0,3735} = 68,83$$

As a consequence, with probability 95% we also have:

$$-S(1) < -68,83 \rightarrow S(0)e^{rt} - S(1) < S(0)e^{rt} - 68,83 \rightarrow$$

$$\rightarrow S(0)e^{rt} - S(1) < 100 \cdot e^{0,08 \cdot 1} - 68,83 \rightarrow$$

$$\rightarrow \underbrace{100 \cdot e^{0,08} - S(1)}_{\text{loss}} < \underbrace{39,50}_{\text{VaR}}$$

and therefore:

$$\boxed{VaR = 39,50\ \$}$$

EXERCISE 6

Exercise 6

Evaluate VaR at 95% confidence level for a one-year investment of 1000$ into euros if the interest rate for risk-free investments in euros is $r_{EUR} = 4\%$, the interest rate for risk-free investments in dollars is $r_{USD} = 5\%$ and the exchange rate from euros into US dollars follows a log-normal distribution with $\mu = 1\%$ and $\sigma = 15\%$.

We have:

1'000 \$	$r_{EUR} = 0,04$	$\mu = 0,01$
$T = 1$ year	$r_{USD} = 0,05$	$\sigma = 0,15$

and if we invest 1'000 \$ at the riskless rate $r_{USD} = 5\%$ for 1 year we obtain:

$$\boxed{1'000 \cdot e^{0,05 \cdot 1} = 1'051,27}$$

while if we convert 1'000 \$ into euros, invest the corresponding amount at the riskless rate $r_{EUR} = 4\%$ and convert back into dollars after 1 year we obtain:

$$1'000 \cdot e^{0,04 \cdot 1} \cdot \frac{E(1)}{E(0)}$$ → exchange rate between euros

$$1'000 \cdot e^{0{,}04 \cdot 1} \cdot \frac{E(1)}{E(0)} \rightarrow \text{exchange rate between euros and dollars}$$

and the loss is given by:

$$1'000 \cdot e^{0{,}05} - 1'000 \cdot e^{0{,}04} \cdot \frac{E(1)}{E(0)}$$

and VaR at 95% confidence level is the amount such that:

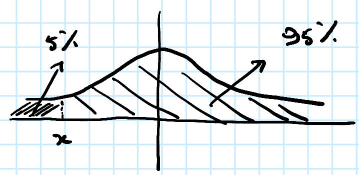
$$P(\text{loss} < \text{VaR}) = 95\% \rightarrow P\left(1'000 \cdot e^{0{,}05} - 1'000 \cdot e^{0{,}04} \cdot \frac{E(1)}{E(0)} < \text{VaR}\right) = 95\%$$

Since the exchange rate follows a log-normal distribution, the logarithmic return on the exchange rate follows a normal distribution:

$$\ln\left(\frac{E(1)}{E(0)}\right) \sim N(\mu, \sigma^2) \rightarrow \underbrace{\ln\left(\frac{E(1)}{E(0)}\right)}_{K} \sim N(0{,}01; 0{,}15^2)$$

hence we have:

$$\underbrace{\frac{K-\mu}{\sigma}}_{Z} \sim N(0,1) \rightarrow \frac{K - 0{,}01}{0{,}15} \sim N(0,1)$$



$N(x) = 5\%$ when $x = -1{,}645$

and with probability 95% we have that:

$$Z > -1{,}645 \rightarrow \frac{K-\mu}{\sigma} > -1{,}645 \rightarrow \frac{K - 0{,}01}{0{,}15} > -1{,}645 \rightarrow$$

$$\rightarrow K > 0{,}01 - 1{,}645 \cdot 0{,}15 = -0{,}2368$$

hence with probability 95% we have that:

$$\ln\left(\frac{E(1)}{E(0)}\right) > -0{,}2368 \rightarrow \frac{E(1)}{E(0)} > e^{-0{,}2368}$$

As a consequence, with probability 95% we also have:

$$1'000 \cdot e^{0{,}04} \cdot \frac{E(1)}{E(0)} > 1'000 \cdot e^{0{,}04} \cdot e^{-0{,}2368} \rightarrow$$

$$\rightarrow -1'000 \cdot e^{0{,}04} \cdot \frac{E(1)}{E(0)} < -1'000 \cdot e^{0{,}04} \cdot e^{-0{,}2368} \rightarrow$$

$$\rightarrow 1'000 \cdot e^{0{,}05} - 1'000 \cdot e^{0{,}04} \cdot \frac{E(1)}{E(0)} < 1'000 \cdot e^{0{,}05} - 1'000 \cdot e^{0{,}04} \cdot e^{-0{,}2368} \rightarrow$$

$E(0)$

$$\to 1000 \cdot e^{0,05} - 1000 \cdot e^{0,04} \cdot \frac{E(1)}{E(0)} < 1000 \cdot e^{0,05} - 1000 \cdot e^{0,04} \cdot e^{-0,2368} \to$$

$$\to \underbrace{1000 \cdot e^{0,05} - 1000 \cdot e^{0,04} \cdot \frac{E(1)}{E(0)}}_{\text{Loss}} < \underbrace{229,88}_{\text{VaR}}$$

and therefore:

$$\text{VaR} = 229,88 \; \$$$

EXERCISE 7

Suppose that 1000$ are invested in European call options on a stock with current price $S(0) = 60\$$. The options expire after 1 year with strike price $K = 40\$$. Assume that $\sigma = 30\%$, $r = 8\%$ and the expected logarithmic return on stock is 12%. Compute VaR after 1 year at 95% confidence level.

We have:

$$S(0) = 60 \qquad \mu = 0,12 \qquad r = 0,08$$
$$K = 40 \qquad \sigma = 0,30 \qquad T = 1$$

Therefore according to the Black-Scholes-Merton formula for the price of a European call option:

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

with:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

hence:

$$d_1 = \frac{\ln\left(\frac{60}{40}\right) + \left(0,08 + \frac{0,3^2}{2}\right) \cdot 1}{0,3 \cdot \sqrt{1}} = 1,7682$$

$$d_2 = 1,7682 - 0,3 \cdot \sqrt{1} = 1,4682$$

and then:

$$c = 60 \cdot N(1,7682) - 40 \cdot e^{-0,08 \cdot 1} \cdot N(1,4682) =$$

$$= 60 \cdot 0,9616 - 40 \cdot e^{-0,08 \cdot 1} \cdot 0,9292 =$$

$$= 23,39$$

and with 1000 $ we purchase:

$$\frac{1000}{\ldots} \simeq 42,75 \text{ options}$$

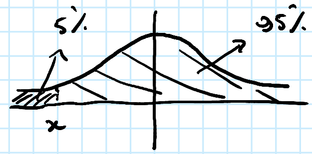
$$\frac{1.000}{23,39} \simeq 42,75 \quad \text{options}$$

Since the stock price follows a log-normal distribution, the logarithmic return follows a normal distribution, that is:

$$\ln\left(\frac{S(1)}{S(0)}\right) \sim N(\mu, \sigma^2) \quad \rightarrow \quad \ln\left(\frac{S(1)}{S(0)}\right) \sim N(0,12;\ 0,3^2)$$

hence we have:

$$\underbrace{\frac{K-\mu}{\sigma}}_{Z} \sim N(0,1) \quad \rightarrow \quad \frac{K-0,12}{0,3} \sim N(0,1)$$



$N(x) = 5\%$ when $x = -1,645$

and with probability 95% we have that:

$$Z > -1,645 \quad \rightarrow \quad \frac{K-\mu}{\sigma} > -1,645 \quad \rightarrow \quad \frac{K-0,12}{0,3} > -1,645 \rightarrow$$

$$\rightarrow K > 0,12 - 1,645 \cdot 0,3 = -0,3735$$

hence with probability 95% we have that:

$$\ln\left(\frac{S(1)}{S(0)}\right) > -0,3735 \quad \rightarrow \quad \frac{S(1)}{S(0)} > e^{-0,3735} \quad \rightarrow$$

$$\rightarrow S(1) > S(0)\, e^{-0,3735} \quad \rightarrow \quad S(1) > 60 \cdot e^{-0,3735} = 41,30$$

As a consequence, with probability 5% the stock price in 1 year will be less than 41,30, and at this price the option will still be exercised, obtaining in the borderline case:

$$\max(S(1) - K;\ 0) = \max(41,30 - 40;\ 0) = 1,30$$

and in total:

$$1,30 \cdot 42,75 = 55,58\ \$$$

This is the situation investing 1.000 \$ in call options, while if we invest the same amount at the riskless rate $r = 8\%$ for 1 year we obtain:

$$1.000 \cdot e^{0,08 \cdot 1} = 1.083,29$$

With probability 95% we have therefore:

$$P\,(\text{loss} < \underbrace{1'083{,}29 - 55{,}58}_{\text{VaR}}) = 95\%$$

and therefore:

$$\text{VaR} = 1'027{,}71 \ \$$$