Politecnico di Torino
Financial Engineering-Exam 02-01-2024
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SURNAME AND NAME

Student number

Exercise 1 (10 points)
Consider a one-period (annual) model formed by a bond (paying a risk-free rate of 5% per year) and by two stocks with prices S^1 and S^2 evolving as follows:

$$S_0^1 = 10;$$

$$S_T^1(w) = \begin{cases} 12; \text{if } w = w_1 \\ 10; \text{if } w = w_2 \\ 6; \text{if } w = w_3 \end{cases} \tag{0.1}$$

and

$$S_0^2 = 10;$$

$$S_T^2(w) = \begin{cases} 15; \text{if } w = w_1 \\ 8; \text{if } w \in \{w_2, w_3\} \end{cases} \tag{0.2}$$

with $P(w_1), P(w_2), P(w_3) > 0$ and $P(w_1) + P(w_2) + P(w_3) = 1$.

1. Establish if the market is free of arbitrage and complete.
2. Consider the derivative A with maturity T of one year and with

   $$\Phi_A = (\frac{S_T^1 + S_T^2}{2} - 8)^+ \tag{0.3}$$

   and find its price.
3. Discuss whether the market formed only by the bond and by stock S^2 would remain free of arbitrage and complete.

EXERCISE 1]

1] S^1 martingale

$$\begin{cases} \overset{q}{E}\left[S_1^1 / 1+r \right] = S_0^1 \\ \overset{q}{E}\left[S_1^2 / 1+r \right] = S_0^2 \end{cases}$$

$$\begin{cases} \frac{12}{1.05} q_1 + \frac{10}{1.05} q_2 + \frac{6}{1.05} q_3 = 10 \\ \frac{15}{1.05} q_1 + \frac{8}{1.05}(q_2 + q_3) = 10 \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 12 q_1 + 10 q_2 + 6 q_3 = 10.5 \\ 15 q_1 + 8 (q_2 + q_3) = 10.5 \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

$$\begin{vmatrix} 12 & 10 & 6 \\ 15 & 8 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 10 & 6 \\ 8 & 8 \end{vmatrix} - \begin{vmatrix} 12 & 6 \\ 15 & 8 \end{vmatrix} + \begin{vmatrix} 12 & 10 \\ 15 & 8 \end{vmatrix} =$$

$$= 80 - 48 - 96 + 90 + 96 - 150 = -28 \neq 0$$

the system has only one solution $\Rightarrow$

Solving the system:

$$\begin{cases} q_1 = 3/14 \\ q_2 = \frac{33}{56} \\ q_3 = 3/56 \end{cases}$$

Soluzione positiva $\rightarrow \exists!$ misura risk neutral quindi mercato Arb. free e completo.

2]

$$\phi_A = \begin{cases} 5.5 & \text{if } \omega = \omega_1 \\ 1 & \text{if } \omega = \omega_2 \\ 0 & \text{if } \omega = \omega_3 \end{cases}$$

$$E[\phi_A/(1+r)] = \frac{1}{1.05}[5.5\, q_1 + q_2] = 2.432$$

3] With only asset S^2: $\tilde{q}_1 = P(\omega_1)$

$\tilde{q}_2 = P(\{\omega_2, \omega_3\})$

$$\begin{cases} \frac{15}{1.05}\tilde{q}_1 + \frac{8}{1.05}\tilde{q}_2 = 10 \\ \tilde{q}_1 + \tilde{q}_2 = 1 \end{cases}$$

$\tilde{q}_1 = 5/14 \qquad \tilde{q}_2 = \frac{36}{56}$ unique sol.

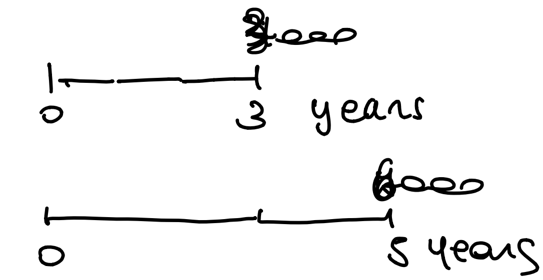
Complete market (1-step Binomial model)

Exercise 2 (10 points)
A portfolio consists of a 3-year zero-coupon bond with face value of 3000 \$ and a 5-year zero-coupon bond with face value of 6000 \$. The current yield on all bonds is 6% per annum.
(a) Compute the duration of the portfolio (using continuous compounding).
(b) Compute the percentage change in the value of the portfolio in the case of a 0.2% per annum decrease in yields.
(c) Compare the result obtained in (b) with the $1st$ order and $2nd$ order approximations based on the use of duration and of convexity, and comment the results.

SOLUTION

EXERCISE 2

(a) We have:



current yield = 0.06 per annum

Duration:

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-y t_i}}{\sum_{i=1}^{n} c_i e^{-y t_i}} =$$

$$= \frac{3 \cdot 3000\, e^{-0.06 \cdot 3} + 5 \cdot 6000\, e^{-0.06 \cdot 5}}{3000\, e^{-0.06 \cdot 3} + 6000\, e^{-0.06 \cdot 5}} =$$

$$= \frac{29741{,}98}{6950{,}72} = 4{,}27$$

$D = 4{,}27$ years

(b) Value of the portfolio:

$$V = 3000\, e^{-0.06\times 3} + 6000\, e^{-0.06\times 5} = 6950,72$$

If the yields decreases by 0,2% it becomes 5,8% the value of the portfolio becomes:

$$V' = 3000\, e^{-0,058\cdot 3} + 6000\, e^{-0,058\cdot 5} = 7010,472$$

Percentage change of the portfolio:

$$\frac{V'-V}{V} = \frac{7010,47 - 6950,72}{6950,72} = \frac{59,75}{6950,72} = 0,86\%$$

(c) Using first order approximation based on duration we have

$$\frac{\Delta V}{V} = -D\Delta y = -4.\cancel{27}(-0.002) = 0,0085 =$$
$$= 0,85\%$$

the convexity of the portfolio is

$$C = \frac{\sum_{i=1}^{n} t_i^2 c_i e^{-y t_i}}{\sum_{i=1}^{n} c_i e^{-y t_i}} =$$

$$= \frac{3^2 \cdot 3000 e^{-0.06\times 3} + 5^2 \cdot 6000 e^{-0.06\cdot 5}}{6950,72} =$$

$$= \frac{133675}{6950,72} = 19,23$$

and using the second order approximation based on the convexity we have:

$$\frac{\Delta V}{V} = -D\Delta y + \frac{1}{2}C(\Delta y)^2 =$$

$$= -4{,}27 \cdot (-0{,}002) + \frac{1}{2}\, 19{,}23\,(-0{,}002)^2 =$$

$$= 0{,}00857 = 0{,}86\,\%$$

the approximation based on the convexity is better, since it is a second order Taylor expansion that is always more precise than a first order Taylor expansion, like the one based on duration.

Exercise 3 (10 points)
Let $dS = \mu_t dt + \sigma_t dW$ and risk-free interest rate is 4% per annum (all rates are continuously compounded). (a) When $\sigma_t = 0.01$ for $t \in [0, 4]$ and

$$\mu_t = \begin{cases} 0.02t;\ 0 \leq t \leq 2 \\ 0.01(10-t);\ 2 < t \leq 4 \end{cases} \tag{0.4}$$

compute the probability of having at $t = 4$ a profit greater or equal than 0.4.

(b) When $\mu_t = 0.01$ for $t \in [0, 4]$ and $\sigma_t = 0.2$ for $t \in [0, 4]$ compute the probability of having at $t = 4$ a profit greater than or equal to 0.4.

(c) Assuming $\mu_t = 0.03S_t$ and $\sigma_t = 0.01S_t$ for $t \in [0, 1]$, in years, i.e.

$$dS = 0.03S_t dt + 0.01S_t dW$$

and that the risk free rate is 2% per annum, price an European call option with strike price 100, maturing in one year , written on the stock $S(t)$ (no dividends) whose current price $S(0)$ is 98.

EXERCISE 3

(a) by integrating $dS_t = \mu_t dt + \sigma_t dW_t$ we have

$$S_4 - S_0 = \int_0^4 \mu_s ds + \int_0^4 \sigma_s dW_s =$$

$$= \int_0^2 0.02 s ds + \int_2^4 0.01(10-s) ds + \int_0^4 0.01 dW_s$$

$$= 0.02 \frac{s^2}{2}\Big|_0^2 + 0.01\left[10s - \frac{s^2}{2}\right]_2^4 + 0.01(W_4 - W_0)$$

$$= 0.04 + 0.01\left[\overset{14}{40-8-20+2}\right] + 0.01 W_4$$

$$= 0{,}18 + 0.01 W_4 \sim N(0{,}18, 0{,}01^2)$$

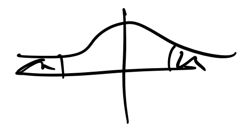
Profit greater or equal to 0.4 means $S_4 - S_0 \geq 0.4$

$$P((S_4 - S_0) > 0.4) = P(0.18 + 0.01 W_4 > 0.4)$$

$$= P\left(W_4 > \frac{0{,}4 - 0{,}18}{0{,}01}\right) =$$

$$= P\left(Z > \frac{0,22}{\sqrt{4}\,0.01}\right)$$

because $W_4 \sim N(0,4)$



$$= P\left(Z > \frac{0,22}{0,02}\right) = 1 - \phi(11) \approx 0$$

(b)

$$S_4 - S_0 = \int_0^4 \mu_s\,ds + \int_0^4 \sigma_s\,dW_s =$$

$$= 0.01s\Big|_0^4 + \int_0^4 0.2\,dW_s$$

$$= 0.01 \cdot 4 + 0.2(W_4 - W_0)$$

$$= 0.04 + 0.2\,N_4$$

where $N_2 \sim N(0,4)$ since

$W_t - W_s \sim N(0, t-s)$. thus

$$0.04 + 0.2N_4 \quad = 0.04 + N,$$

where

$$N \sim N(0,(0,04)\cdot 4)$$

$$\sim N(0;0,16)$$

therefore

$$P(S_4 - S_0 \geq 0.4) = P(0.04 + N \geq 0.4)$$

$$= P\left(Z \geq \frac{0.4-0.04}{\sqrt{0,16}}\right) = P(Z \geq 0.9)$$

$$= 1 - \phi(0,9) = 0,18$$

(c) We have $S(0)=98$ $K=100$ $T=1Y$

$\sigma=0.01$ per annum

$r=0.02$

According to the Black-Scholes-Merton formula for a European Call option

$$C=S(0)N(d_1)-Ke^{-rT}N(d_2)$$

$$d_1=\frac{\ln(S0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}=$$

$$=\frac{\ln(0{,}98)+\left(0.02+\frac{0.01^2}{2}\right)\cdot 1}{0.01}=$$

$$=-0{,}0152$$

$$d_2=d_1-\sigma\sqrt{T}=-0.0152-0.01\sqrt{1}=$$

$$=-0{,}025$$

$$C=98N(-0.015)-100e^{-0.02}N(-0{,}025)$$

$$=0{,}38$$