

Politecnico di Torino
Financial Engineering-Exam 02-09-2021
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Student number

Exercise 1. (10 points)

Consider a 2-year European put option with a strike price of \$50 on a stock whose current price is \$60. Suppose that there are 2 time steps of 1 year and in each time step the stock price either moves up by a factor of 1.5 or moves down by a factor of 0.5. Suppose that the risk-free interest rate is 1% per annum with simple compounding.

- (a) Find the delta of the option in $t = 1$.
- (b) Find the replicating portfolio in $t = 1$ assuming to have observed an up movement in the first step.

Exercise 2 (10 points)

A stock price is \$50, and the risk-free rate of interest is 6% per annum with continuous compounding for all maturities.

- (a) A trader observes that the forward price of a six month forward contract is $F_0 = 60$. Can he make an arbitrage? Why? How?
- (b) The stock is expected to pay a dividend of \$0.5 per share in 2 months and in 6 months. What are the no arbitrage forward price and the initial value of a six month forward contract on the stock?

Exercise 3 (12 points)

A stock price follows a geometric Brownian motion with an expected return (equal to the risk-free interest rate) of 12% per annum and a volatility of 30% per annum. The current stock price is 50 \$.

- (a) Calculate the price of a 6-month European call option and of a 6-month European put option on this stock (no dividends), both with strike price 40 \$.
- (b) Verify that the prices found in (a) satisfy the put-call parity.
- (c) Find the probabilities that the call option and the put option considered above will be exercised.

Solution

EXERCISE 1

$$K = 50$$

$$\alpha = 0.01$$

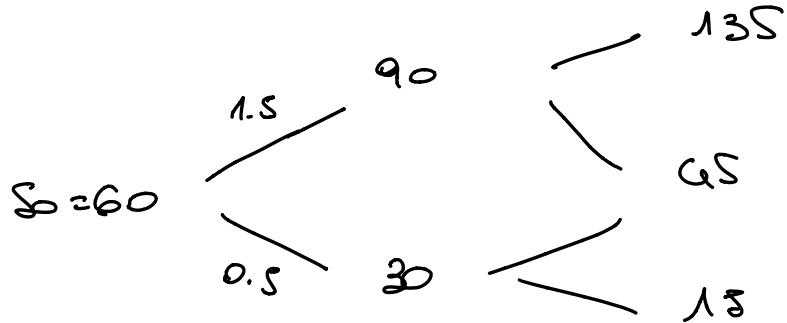
$$T = 2 \text{ years}$$

$$S_0 = 60$$

$$t=0$$

$$t=1$$

$$t=2$$



Risk neutral measure -

$$q_u = \frac{(1+r) - d}{u-d} = \frac{(1+0.1) - 0.5}{1.5 - 0.5} = 0.51$$

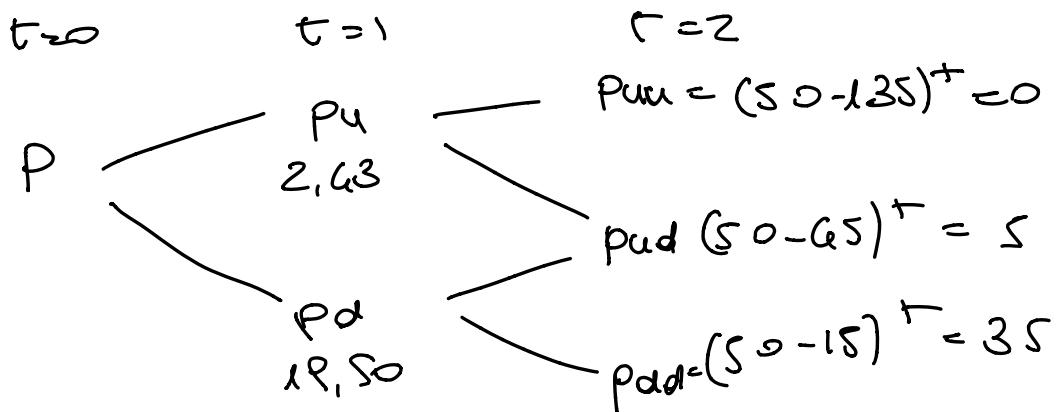
$$q_d = 1 - q_u = 0.49$$

Put price at time $t=1$

$$p_u = \frac{1}{1+r} [p_{uu} q_u + p_{ud} q_d] = \frac{1}{1.01} (5 \cdot 0.51 + 5 \cdot 0.49) = 2.43$$

$$p_d = \frac{1}{1+r} [p_{ud} q_u + p_{dd} q_d] = \frac{1}{1.01} (5 \cdot 0.51 + 85 \cdot 0.49) = \frac{19.7}{1.01} = 19.50$$

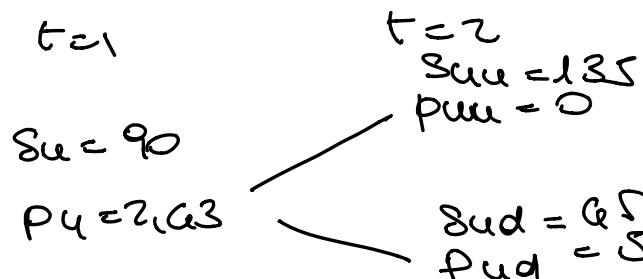
(a) Put option by nodes



$$\Delta_1 = \frac{p_u - p_d}{s_u - s_d} = \frac{2,43 - 19,50}{20 - 19} = -0,28$$

N.B.: It is not necessary to find the price of the put option at $t=0$.

(b) Consider the tree



The hedging portfolio at tree t=1 node k is

$$\Pi_t^k = (x_t(k), q_t(k))$$

↑ ↗
bank account asset-

where

$$x_t(u) = \frac{1}{1+\lambda} \frac{u V_t(k) - d V_t(k+1)}{u-d}$$

$$y_t(k) = \frac{1}{S_{t-1}} \frac{V_t(k+1) - V_t(u)}{u-d}$$

$$V_2(uu) = 0 ; V_2(ud) = 5$$

$$x_2(u) = \frac{1}{1+\lambda} \frac{u V_2(ud) - d V_2(uu)}{u-d} = \\ = \frac{1}{1,01} \frac{1,5 \cdot 5 - 0,5 \cdot 0}{1} = 7,43$$

$$y_2(u) = \frac{1}{q_0} \frac{V_2(uu) - V_2(ud)}{u-d} = \\ = \frac{1}{q_0} \frac{0 - 5}{1} = -0,056$$

Or you can find it directly by:

$$\begin{aligned} V_{uu} &= x_1(u)(1+\lambda) + y_1(u) \cdot S_{uu} \stackrel{135}{=} 0 \\ - V_{ud} &= \underline{x_1(u)(1+\lambda) + y_1(u) \cdot S_{ud} = 5} \\ &\equiv 90y_1 = -5 \\ y_1 &= -\frac{5}{90} = 0,056 \end{aligned}$$

$$x_1(u) + 1,01 + (-0,056) \cdot 135 = 0$$

$$x_1(u) = \frac{135 - 0,056}{1,01} = 7,43$$

$$\boxed{q_{11}(1) = (7,43, -0,056)}$$

Check with the value of $V_2(u)$

$$V_2(u) = 7,43 - 0,056 \cdot 8u = \\ = 7,43 - 0,056 \cdot 90 = 2,43 = \text{pu}$$

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EXERCISE 2)

$$(a) S_0 = \$0$$

$$\pi = 0.06$$

$$F_0 = \$0$$

the risk neutral price for a forward contract is

$$\tilde{F}_0 = e^{0.06 \frac{1}{2} \cdot 0} \cdot S_0 = \$0 e^{0.03} = \$1.52$$

thus, $F_0 > \tilde{F}_0$ and there is an arbitrage opportunity

Arbitrage strategy

	$t=0$	$t=1$
short forward borrow \$ S_0	0	pay the loan $= S_0 e^{\pi T}$
buy asset S_0	$-S_0$	sell the asset $+ F_0$
	<hr/>	<hr/>
		$F_0 - S_0 e^{\pi T}$

$$F_0 - S_0 e^{\pi T} = 60 - 50 e^{0.03} = \\ = 60 - 51.52 > 0$$

(b) $d_1 = 0.5$ w 2 months

$d_2 = 0.5$ w 6 months

$$I_1 = 0.5 \times e^{-0.06 \frac{2}{12}} = 0.495$$

$$I_2 = 0.5 \times e^{-0.06 \frac{6}{12}} = 0.48$$

$$I = I_1 + I_2 = 0.98$$

$$F_0 = (S_0 - I) e^{x\bar{T}} =$$

$$= (50 - 0.98) e^{0.06 \frac{1}{2}} =$$

$$= 50, 51$$

the initial value of a forward contract is zero by definition -

EXERCISE 3

a) We have $S_0 = 50$, $r = 12\%$ per annum
 $k = 40$; $\sigma = 30\%$ per annum

$T = \frac{6}{12}$ 6 month call and 6-month put

According to the Black-Scholes-Merton formula for the price of a European call option

$$C = S_0 N(d_1) - k e^{-rT} N(d_2)$$

with

$$d_1 = \frac{\ln(\frac{S_0}{k}) + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

hence

$$d_1 = \frac{\ln(\frac{50}{40}) + (0.12 + \frac{0.3^2}{2}) \frac{6}{12}}{0.3 \sqrt{\frac{6}{12}}} = 1.4408$$

$$d_2 = 1.4408 - 0.3 \sqrt{\frac{6}{12}} = 1.2287$$

then

$$\begin{aligned} C &= 50 \cdot N(1.4408) - 40 e^{-0.12 \frac{6}{12}} N(1.2287) \\ &= 50 \cdot 0.9252 - 40 e^{-0.12 \frac{6}{12}} \cdot 0.8904 \\ &= 12.72 \end{aligned}$$

The price of the call option is $C = 12.72$

According to the Black-Scholes-Merton formula for the price of a European put option

$$P = k e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where d_1 and d_2 are the same as above, hence:

$$\begin{aligned} P &= 40 e^{-0,12 \frac{6}{12}} N(-1,2287) - 50 N(-1,4608) = \\ &= 40 e^{-0,12 \frac{6}{12}} \cdot 0,1096 - 50 \cdot 0,0748 = \\ &= 0,38 \end{aligned}$$

The price of the put option is $P = 0,38$

b) The put-call parity relation is:

$$C + k e^{-rT} = P + S_0$$

$$C + k e^{-rT} = 12,72 + 40 e^{-0,12 \frac{6}{12}} = 50,38$$

$$P + S_0 = 0,38 + 50 = 50,38$$

Therefore the relation is satisfied

c) The probability that the European call option will be exercised is the probability that $S_T > k$, i.e.

$$P(S_{6/12} > 40)$$

Since the expected return $\mu = 0,12 = r$

$$\text{then } P(S_{6/12} > 40) = N(d_2) = N(1,228) = 0,8903$$

This can also be solved directly as follows.

S follows a geometric Brownian motion:

$$\ln S_T \sim N\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T; \sigma^2 T\right]$$

$$\ln S_T \sim N\left[\ln S_0 + \left(\mu_{12} - \frac{q_3^2}{2}\right)\frac{6}{12}; q_3^2 b_{12}\right]$$

$$\ln S_T \sim N(3,9485; 0,045)$$

and then:

$$P(S_T > 40) = P(\ln S_T > \ln 40) = P(\ln S_T > 3,689)$$

that is

$$\begin{aligned} P\left(\frac{\ln S_T - \mu}{\sigma} > \frac{3,689 - 3,9485}{\sqrt{0,045}}\right) &= P(Z > -1,228) \\ &= P(Z < 1,228) = N(1,228) = 0,8903 \\ &\quad \text{thus } \uparrow \\ &\quad \text{thus is } N(d_2)! \end{aligned}$$

hence the probability that the option is exercised is 89,03%

The probability that the European put option is exercised is

$$P(S_T < K) = P(S_{6/12} < 40) =$$

$$= 1 - P(S_{6/12} > 40) =$$

$$= 1 - 0,8903 = 0,1097$$

hence the probability that the put is exercised is 10,97%.

