Bond's sensitivity to changes of the yield:

Explaination:

From the past lecture we know that the price of a Z.C.B. at time t is: $B(t,T) = e^{-y(T-t)}$

With $t=0$ we get the Present Value : $B(0,T) = e^{-yT}$

With coupon bonds instead: $B(0,T) = \sum_{i=1}^{n} C_i \, e^{-y t_i}$ with $t_n = T \wedge C_n =$ coupon + face value.

(bonds yield)

$\parallel$

$B(y)$ <- For convenience we keep the couple 0,T fixed but we write B as a function of the bond's yield.

Taylor approximation:

In order to study the sensitivity of the bond's price to changes in the bond's yield we simply analyze the first order derivative w.r.t. y of the bond's price B(y). To obtain the first order derivative information we use the Taylor expansion formula that allows us to approximate B(y).

First order Taylor approximation:

$$B(y) = B(y_0) + B'(y_0)\,\Delta y + B''(y_0)\cdot\frac{1}{2}\cdot(\Delta y)^2 + \dots$$

(← we stop here, at the first order term of the polynomial)

(This leads to)

$$\Leftrightarrow \quad B(y) - B(y_0) = \Delta B(y) = B'(y_0)\,\Delta y$$

$$\Leftrightarrow \quad \frac{dB(y)}{dy} = \frac{d}{dy}\sum_{i=1}^{n} C_i \, e^{-t_i y} = -\sum_{i=1}^{n} C_i \, t_i \, e^{-t_i y}$$

(For really small perturbations)

(measure of sensitivity)

DURATION of a BOND:

Definition (Duration):

The duration of a bond is a measure of the perceived waiting time before receiving a "sufficiently large" amount of money.

Formula:

$$D = \frac{\sum_{i=1}^{n} C_i t_i e^{-y t_i}}{\underbrace{B}_{\text{Bond price}}} = \underbrace{\sum_{i=1}^{n} t_i \left(\frac{C_i e^{-y t_i}}{B} \right)}$$

Convex combination because:

- $\frac{C_i e^{-y t_i}}{B} \leq 1 \quad \forall i$
- $\sum_{i=1}^{n} C_i e^{-y t_i} = B$

(1) OBS. This is a convex combination of the times when I get paid, this is why can be thought as an expected value.

"Expected value of time waited before receiving a sufficiently large amount of money"

Duration as a measure of sensitivity:

Formula:

- $B' = -\sum_{i=1}^{n} C_i t_i e^{-t_i y} = -D(y) B$

- $\frac{dB(y)}{dy} = -D(y) B(y) \quad \Rightarrow \quad \frac{\Delta B(y)}{B(y)} = -D(y) \Delta y$ for small Δy (first order approx)

≤ 0

$\updownarrow$

(2)

$$\frac{B'(y)}{B(y)} = -D(y)$$

($\rightsquigarrow$ The duration measures the sensitivity of the bond's price, at the varying of the bond's yield y, all w.r.t. the actual bond's price $B(y)$.)

(3) (The duration might be considered a measure of risk because it reflects how much an investment in bonds is exposed to changes in interest rates. When the interest rates rises, the value of the already existing bonds tends to decrease (pensaci ha senso), as new instruments offer higher returns. Conversely, when interest rates decrease, the value of existing bonds tend to increase for the same reason.)

Convexity:

Convexity: 2nd order Taylor approx.

$$B(y) = B(y_0) - D\,B(y_0)\Delta y + \frac{1}{2}\,C\,B(y_0)\Delta y^2 + \dots$$

$$+ B'(y_0)\,dy + \tfrac{1}{2}\,B''(y_0)\,dy^2 + \dots$$

Since $B'(y) = -D(y)\,B(y_0)$

$\wedge$

$B''(y) = C \cdot B(y_0)$

Definition / Formula:

$$C := \frac{d^2 B(y)}{dy^2} \cdot \frac{1}{B} = \frac{\sum_{i=1}^{n} t_i^2\, c_i\, e^{-y t_i}}{B}$$

Portfolio of BONDS :

Definition (Portfolio of bonds):

Set of bonds owned by a single individual or a group. Indicated as:

$$P = (a_1, a_2, \dots, a_n)$$

a_1: (exposure w.r.t. B_1) → Exposure = capitale investito nel bond.

Value of the portfolio:

$B_i(y)$:= i-th bond of the portfolio.

$$P(y) = \sum_{i=1}^{n} a_i B_i(y) \quad (\longleftarrow \text{Value of the portfolio } P)$$

Sensitivity of the portfolio to changes in yield.

$$P'(y) = -D_P(y)\, P(y) \qquad (1)$$

$$P'(y) = \sum_{i=1}^{n} a_i \cdot B_i'(y) \qquad \left(\text{since} \quad B_i'(y) = -\underbrace{D_i(y)}_{\text{Duration of bond } i} B_i(y)\right)$$

$$P'(y) = -\sum_{i=1}^{n} a_i \cdot D_i(y) \cdot B_i(y) \qquad (2)$$

Imposing that (1) = (2):

$$-D_P(y)\, P(y) = -\sum_{i=1}^{n} a_i \cdot D_i(y) \cdot B_i(y)$$

$$D_P(y) = \frac{\sum_{i=1}^{n} a_i \cdot D_i(y) \cdot B_i(y)}{P(y)} = \sum_{i=1}^{n} D_i(y) \overbrace{\frac{a_i \cdot B_i(y)}{P(y)}}^{w_i}$$

convex combination: $\sum_{i=1}^{n} a_i B_i(y) = P(y)$

$$\boxed{D_P(y) = \sum_{i=1}^{n} w_i D_i(y)}$$

Example:

Suppose you want to invest in bonds $B_1, \dots, B_n$ and you know their duration $D_1, \dots, D_n$. Construct a portfolio P with duration D_0 (our target)

$$w_i = \frac{Q_i B_i(y)}{P(y)} \qquad Q_i = \frac{W_i \cdot P(y)}{B_i(y)}$$

- Solution -

$$D_0 = \sum_{i=1}^{n} W_i \cdot D_i(y) = \sum_{i=1}^{n} \frac{Q_i \cdot B_i(y)}{P(y)} \cdot D_i(y)$$

○ known

○ unkown

(↳ more than 1 solution)

(Hedging = copertura)

Dynamic Hedging

Definition (Dynamic Hedging):

It is a strategy used by investors to reduce or mitigate the risk of adverse scenario due to uncertainty.

Scenario:

Even if a portfolio is selected with duration matching the desired investment lifetime, this will only be valid at the initial instant, since duration changes with time as well as with the interest rate.

As an example, take a 5-year bond with 10 \$ annual coupons and 100 \$ face value. If $y = 10\%$ then the duration will be 4.16 years, and we want to keep the bond for exactly 4.16 years, so the duration matches the lifetime of our investment. Before the first coupon is paid the duration decreases in line with time. After 6 months it will be 3.66 and after 9 months $4.16 - 0.75 = 3.31$. As soon as the first coupon is paid after one year, the bond will become a 4-year one, and while the lifetime of our investment is 3.16, now the duration is 3.43. So they no longer matches. (OBS. If the duration matches our investment's lifetime and the interest rate do not change, no action will be necessary until a coupon becomes payable).

OBS.: •) $D(y, t)$ ↳ in a general setting the duration might be function of both yield and time.

•) Duration of a ZCB ($B(0,2)$ with continuous compounding) $D(y) = \dfrac{t_1 \cdot \cancel{1 \cdot e^{-2 \cdot 0.1}}}{\cancel{1 \cdot e^{-2 \cdot 0.1}}} = t_1 = 2$

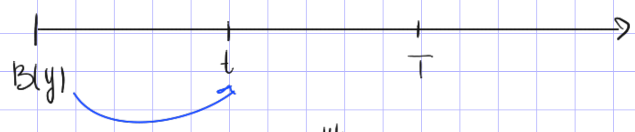
↳ it does not depend on the yield.

↳ $D_{ZCB} = T \quad (B(0,T))$

Why do we want duration to be equal to the investment lifetime.

A portfolio with duration matching the investment lifetime is not sensitive to small changes of interest rate.

→ We invest in a bond and we want to close out the position in t.



$y = y(0)$ $\qquad$ $B(y)\, e^{yt}$

↑ Value of $B(y)$ at time t

The sensitivity of the investment in t at changes of y.

$$\frac{d\left(B(y)\, e^{yt}\right)}{dy} = B'(y)\, e^{yt} + t\, B(y)\, e^{yt} \qquad B'(y) = -D(y)\, B(y)$$

$$\left(B(y)\, e^{yt}\right)' = -D(y)\, B(y)\, e^{yt} + t\, B(y)\, e^{yt} = B(y)\, e^{yt}\, (t - D(y))$$

<u>OBS.</u> If $D(y) = t$ than $\left(B(y)\, e^{yt}\right)' = 0$ $\left(\leftarrow \text{The bond here is } \underline{\underline{\text{not}}} \text{ sensitive at changes of } y.\right)$

TARGET:

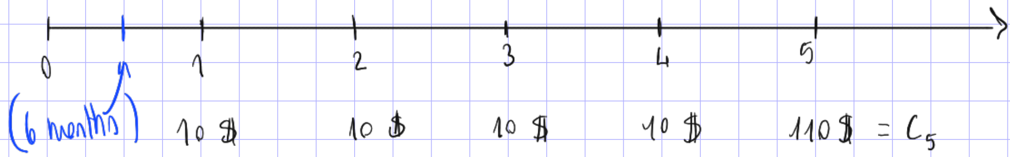
- The idea is to construct a portfolio with duration = the investment's lifetime.
  - → There are 2 problems: [Duration changes with time as well with the interest y.]

    ↑ (Dynamic hedging)

Example (1): (The same as the one explained in the scenario)

1) Duration changes in time , $y = \text{cost.}$

Take a 5 years bond with 10 \$ annual coupon and face value with 100 \$,
Let $y = 10\%$



$$B(0,5) = \sum_{j=1}^{4} 10\, e^{-0.1\, j} + 110\, e^{-0.1 \cdot 5} = 98.06944$$

$$D(0) = \frac{\sum_{j=1}^{5} C_i\, e^{-0.1 \cdot j} \cdot j}{B(0,5)} = 4.16$$

Let's look for duration after 6 months:

$$D(6m) = \frac{10\, e^{-0.1 \cdot 0.5}\, 0.5 + 10\, e^{-0.1 \cdot 1.5}\, 1.5 + \ldots + 110\, e^{-0.1 \cdot 4.5}\, 4.5}{B(6m,5)} = 3.66$$

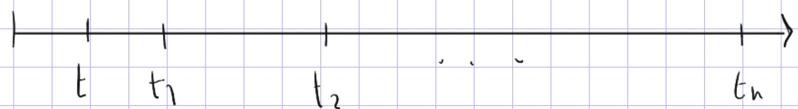
$$D(6m) - D(0) = 0.5$$

$$D(9m) = 3.31 = 4.17 - 0.75$$

$9m = 75\%$ of a year

Before the payment of C, $D(t)$ decreases linearly in time, indeed:

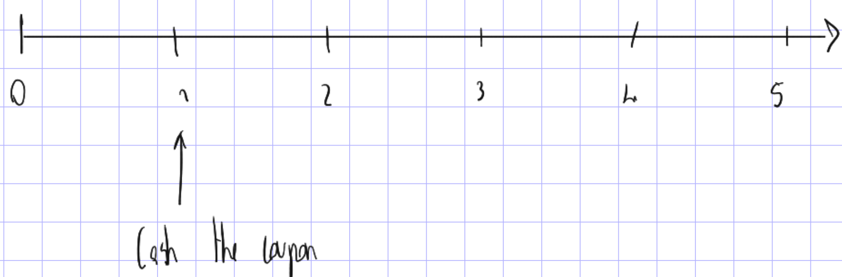
If $t \in [0, t_1]$ $\quad D(t) = D(0) - t$



$$D(t) = \frac{\sum_{i=1}^{n} (t_i - t)\, e^{-y(t_i - t)} \cdot C_i}{\sum_{i=1}^{n} C_i\, e^{-y(t_i - t)}} = \frac{\sum_{i=1}^{n} t_i \cdot C_i\, e^{-y(t_i - t)} - t \sum_{i=1}^{n} C_i\, e^{-y(t_i - t)}}{\sum_{i=1}^{n} C_i\, e^{-y(t_i - t)}} =$$

$$= \frac{\sum_{i=1}^{n} t_i \cdot C_i\, e^{-y(t_i - t)}}{\sum_{i=1}^{n} C_i\, e^{-y(t_i - t)}} - t = \frac{\cancel{e^{yt}} \sum_{i=1}^{n} t_i \cdot C_i \cdot e^{-yt_i}}{\cancel{e^{yt}} \sum_{i=1}^{n} C_i\, e^{-yt_i}} - t = D(0) - t$$

After the payment of the first coupon Bond becomes a 4-year coupon bond.



$$B(1,5) = 10 \cdot e^{-0.1 \cdot 1} + 10 \cdot e^{-0.1 \cdot 2} + \dots + 110 \cdot e^{-0.1 \cdot 4}$$

$$D(1) = \frac{10 \cdot 1 \cdot e^{-0.1 \cdot 1} + 10 \cdot 2 \cdot e^{-0.1 \cdot 2} + \dots + 110 \cdot 4 \cdot e^{-0.1 \cdot 4}}{B(1,5)} = 3.48$$

$\neq 4.16 - 1 = 3.16$

Example (2): 2) Duration changes with yield y.

$y = 10\%$ $\quad D(y) = 4.16$

1) $y \to y' = 6\%$ $\quad D(y') = 4.23$

2) $y \to y' = 14\%$ $\quad D(y') = 4.08$

Theory: (Example of Hedging Strategy)

Duration will now be applied to design an investment strategy immune to interest rate changes. This will be done by monitoring the position at the end of each year, or more frequently if needed. For clarity of exposition we restrict ourselves to an example.

Set the lifetime of the investment to be 3 years and the target value to be \$100,000. Suppose that the interest rate is 12% initially. We invest \$69,767.63, which would be the present value of \$100,000 if the interest rate remained constant.

↳ $V(0) = 100000\, e^{-3 \cdot 0.12} = 69767,633$

We restrict our attention to two instruments, a 5-year bond A with \$10 annual coupons and \$100 face value, and a 1-year zero-coupon bond B with the same face value. We assume that a new bond of type B is always available. In subsequent years we shall combine it with bond A.

At time 0 the bond prices are \$90.27 and \$88.69, respectively. We find $D_A \cong 4.12$ and the weights $w_A \cong 0.6405$, $w_B \cong 0.3595$ which give a portfolio with duration 3. We split the initial sum according to the weights, spending \$44,687.93 to buy $a \cong 495.05$ bonds A and \$25,079.70 to buy $b \cong 282.77$ bonds B. Consider some possible scenarios of future interest rate changes.

1. After one year the rate increases to 14%. The value of our portfolio is the sum of:
   - the first coupons of bonds A: \$4,950.51, ← incasso i coupon al tempo 1
   - the face value of cashed bonds B: \$28,277.29, ← incasso il face value degli zero coupon bond
   - the market value of bonds A held, which are now 4-year bonds selling at \$85.65: \$42,403.53. (OSSERVIAMO CHE IL PREZZO È SCESO PER GLI A) ← (calcolo il valore degli A ora che sono 4-year coupon bonds)

   This gives \$75,631.32 altogether. The duration of bonds A is now 3.44. The desired duration is 2, so we find $w_A \cong 0.4094$ and $w_B \cong 0.5906$ and arrive at the number of bonds to be held in the portfolio: 361.53 bonds A and 513.76 bonds B. (This means that we have to sell 133.52 bonds A and buy 513.76 new bonds B.)

   a) After two years the rate drops to 9%. To compute our wealth we add:
      - the coupons of A: \$3,615.30,
      - the face values of B: \$51,376.39,
      - the market value of A, selling at \$101.46: \$36,682.22. (il valore di A è salito)

      The result is \$91,673.92. We invest all the money in bonds B, since the required duration is now 1. (The payoff of these bonds is guaranteed next year.) We can afford to buy 1,003.07 bonds B selling at \$91.39. The terminal value of the investment will be about **\$100,307**.

   b) After two years the rate goes up to 16%. We cash the same amount as above for coupons and zero-coupon bonds, but bonds A are now cheaper, selling at \$83.85, so we have less money in total: \$85,305.68. However, the zero-coupon bonds are now cheap as well, selling at \$85.21, and we can afford to buy 1,001.07 of them, ending up with **\$100,107**.

Example of Hedging Strategy:

(We will analyse a scenario to show how effective is dynamic Hedging.)

We consider a portfolio $P(a_A, a_B)$ with two assets.

A: 5-year CB with \$10 annual coupons and $F = 100$

B: 1-year ZCB $F = 100$

We assume we can always buy B.

AIM: target \$ 100'000 in three years $(t = 3Y)$ $y = 12\%$

$$P = 100'000 \, e^{-0.12 \cdot 3} = \underline{69'767,63}$$

·) We can invest this amount of money, if the rate was constant this would be enough to obtain 100'000 with bonds. unfortunately y is not constant, Here we can show the power of Hedging.

time 0: (Prices of entities)

$B_A(0,5) = 90.27$ $\qquad$ $B_B(0,1) = 88.69$

$D_A = 4.12$ $\qquad$ $D_B = 1$

1) Consider a portfolio P s.t. $D_P = 3$

2) $P = Q_A \cdot B_A + Q_B B_B$, Q_A, Q_B

$P(0) = 69'767,63$

$D_P = 3$

·) look for W_A, W_B : $\begin{cases} W_A + W_B = 1 \\ W_A \cdot 4.12 + W_B \cdot 1 = 3 \end{cases} \Rightarrow$ $\quad W_A = 0.6410$, $W_B = 0.3590$

$$W_A = \frac{q_A B_A}{P} \qquad W_B = \frac{q_B B_B}{P}$$

$$\downarrow$$

$$q_A = \frac{P W_A}{B_A} = \frac{0.6410 \cdot 69'767,63}{90.27} \cong 495,05$$

$$q_B = \frac{P W_B}{B_B} = \frac{0.3590 \cdot 69'767.63}{88.69} \cong 282,77$$

$495,05 \cdot 90.27 = 44'687,93$ $\to$ Buy A

$282.77 \cdot 88.69 = 25'079,70$ " " B

<u>Future Scenario, after 1 year.</u> (Let's assume y increases to 14%)

y increases to 14%

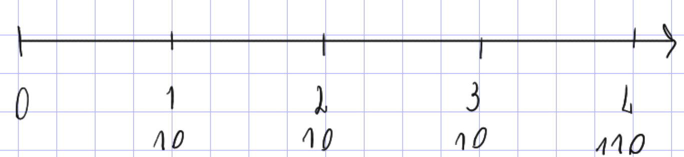
1) We cash the first coupon of A

$\$ 10 \cdot q_A = 10 \cdot 495.05 = 4950.50$

2) We cash the face value of B

$\$ 100 \cdot 282.77 = 28'277,29$

3) Market value of A $\to$ 4 year coupon bond



$$B_A(1,5) = 10\, e^{-0.14 \cdot 1} + 10^{-0.14 \cdot 2} + \dots + 110\, e^{-0.14 \cdot 4} = 86.65$$

$86.65 \cdot 495.05 = 42'403,53$

↳ number of bonds a in P

Portfolio value ① + ② + ③ = 75'631,32

$D_A = 3.44$ → and this is the duration of P

$D = 2$ years (lifetime left)

We now solve $\begin{cases} W_A + W_B = 1 \\ W_A \cdot 3.44 + W_B \cdot 1 = 2 \end{cases}$ → $W_A = 0.9094$, $W_B = 0.596$

→ As we did above we can find $\underset{\substack{\| \\ 361,63 \\ \text{(Bonds A)}}}{Q_A}$ and $\underset{\substack{\| \\ 513,76 \\ \text{(Bonds B)}}}{Q_B}$

We have to sell 495 - 361 = 133 bonds of type A and buy 513 bonds of type B.

After two years $y' = 9\%$: (Again an assumption)

1) Coupon of A 3615,3

2) Face value of B 51376,39

3) Market value A

$$B_A(3,5) = 10 \cdot e^{-0.09 \cdot 1} + 110\, e^{-0.09 \cdot 2} = 101,46$$

$$101,46 \cdot Q_A = 36.682$$

① + ② + ③ 100'370 > 100'000 in this scenario we ended up satisfied.

3 Recap Lezione 3 – Duration and Interest Rate Risk

Idea generale della lezione La lezione analizza il **rischio di tasso di interesse** associato ai bond e introduce strumenti analitici per misurare la **sensibilità del prezzo di un titolo obbligazionario alle variazioni del rendimento**. Il concetto centrale è la **duration**, ottenuta a partire da un'approssimazione di primo ordine del prezzo del bond, e il suo utilizzo nella gestione del rischio e nella costruzione di strategie di copertura.

Bond price come funzione del rendimento Il prezzo di un bond può essere visto come una funzione del rendimento y,

$$B(y) = \sum_{i=1}^{n} c_i e^{-yt_i},$$

dove c_i sono i flussi di cassa pagati ai tempi t_i.

Sviluppo di Taylor del prezzo del bond Si considera lo sviluppo di Taylor del prezzo del bond attorno a un valore y_0,

$$B(y) = B(y_0) - D(y_0)B(y_0)\Delta y + \frac{1}{2}C(y_0)B(y_0)(\Delta y)^2 + \dots$$

dove $D(y_0)$ e $C(y_0)$ rappresentano rispettivamente duration e convexity valutate in y_0.

Duration La **duration** è definita come

$$D(y) = -\frac{1}{B(y)}\frac{dB(y)}{dy} = \sum_{i=1}^{n} t_i \left(\frac{c_i e^{-yt_i}}{B(y)}\right).$$

La duration ammette le seguenti interpretazioni

- È una **combinazione convessa dei tempi di pagamento**, poiché i pesi $\frac{c_i e^{-yt_i}}{B(y)}$ sono positivi e sommano a uno
- Misura la **sensitività del prezzo del bond** rispetto a variazioni del rendimento secondo

$$\frac{\Delta B}{B} \approx -D(y)\,\Delta y$$

- Costituisce una **misura di rischio di tasso di interesse**, quantificando l'esposizione del valore del bond a shock sui tassi

Convexity La **convexity** è definita come

$$C(y) = \frac{1}{B(y)}\frac{d^2B(y)}{dy^2} = \sum_{i=1}^{n} t_i^2 \left(\frac{c_i e^{-yt_i}}{B(y)}\right).$$

Portfolio of bonds Si considera un portafoglio di n bond con quantità q_i e prezzi $B_i(y)$, per cui il valore del portafoglio è

$$P(y) = \sum_{i=1}^{n} q_i B_i(y).$$

Imponendo che la duration del portafoglio soddisfi

$$-D_p(y)P(y) = \frac{dP(y)}{dy} = -\sum_{i=1}^{n} q_i D_i(y) B_i(y), \qquad D_p(y) = \frac{\sum_{i=1}^{n} q_i D_i(y) B_i(y)}{P(y)} = \sum_{i=1}^{n} D_i(y)\left(\frac{q_i B_i(y)}{P(y)}\right).$$

Si definiscono quindi

$$w_i = \frac{q_i B_i(y)}{P(y)}, \qquad D_p(y) = \sum_{i=1}^{n} w_i D_i(y).$$

Dynamic hedging La duration fornisce una copertura valida localmente. Poiché il rendimento e il tempo influenzano il valore della duration, è necessario un aggiornamento continuo del portafoglio. Il **dynamic hedging** consiste nel ribilanciare dinamicamente le posizioni per mantenere sotto controllo l'esposizione al rischio di tasso.