Politecnico di Torino
Financial Engineering-Exam 07-21-2023
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SURNAME and NAME

Student number

All the answers must be clearly motivated, the numerical results are not sufficient. All the answers that will be considered in the correction MUST be written here. If the student number and the name are not filled in the text will not be corrected.

Answers written with a pencil are null.

Exercise 1

We take a long position in a European Call option on a stock S with strike $K = 80$ euros and maturity $T = 1$ year. The price of the underlying follows a geometric Brownian motion with $S_0 = 80$ euros, $\mu = 0.2$ and $\sigma = 0.4$ per year. The risk-free interest rate available on the market is $r = 0.04\%$ per year continuously compunded.

1. Find the price of the Call option.

1.Compute the Delta Δ of the Call option and, accordingly, establish how many shares of stock S we need to buy/sell in order to make our long position Delta-neutral.

3. Compute the Delta Δ of a put option with the same underlying, maturity and strike and, accordingly, establish how many shares of stock S we need to buy/sell in order to make our short position Delta-neutral.

SOLUTION

$K1 = 80 \quad T = 1 \text{ year} \quad S_0 = 80 \quad r = 0.04 \quad \sigma = 0.4$

1] The Black and Scholes price is:

$$C_0^{E} = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} =$$

$$= \frac{\ln(80/80) + \left(0.04 + \frac{0.4^2}{2}\right) \cdot 1}{0.4\sqrt{1}} = 0.3$$

and

$$d_2 = d_1 - 0.4\sqrt{T} = -0.1$$

Thus

$N(d_1) = 0.6179$

$N(d_2) = 0.4602$

$$C_0 = 80 \cdot 0.618 - 80 e^{-0.04} \, 0.4602 = 14.062$$

2] $\Delta_1 = N(d_1) = 0.6179$

Short on the call : $h(x_S, x_C) = P$ with $x_C = 1$

$$h(\underset{\substack{\uparrow \\ S(t)}}{x_S}, \underset{\substack{\uparrow \\ \text{Call}}}{x_C})$$

$P = x_S S + 1 \cdot C$

$\Delta_P = + x_S + \Delta_1$

To have P Δ neutral $\Delta_1 = -x_S$

we have to short 0.6179 shares to make a long position on the call Δ neutral.

3] $\Delta_1^P = \Delta_C - 1 = 0.618 - 1 = -0.382$

$h = (x_S, x_P) = (x_S, -1)$

$P = x_S S - P$

$\Delta P = x_S + 0.382 \to x = -0.382$

we have to short 0.382 shares

Exercise 2

Your current wealth is 100,000 euro, which you invest in a financial portfolio S whose value process follows a geometric Brownian motion with drift coefficient 7% and volatility coefficient 25%. The risk-free rate (with continuous compounding) is 3%.

1. Your target is to double your wealth in 10 years. Find the shortfall probability, i.e., the probability you will not achieve your target.

2. Define the loss of a financial portfolio (use logreturns) its value at risk.

3. Find the value at risk of the portfolio loss after 10 years at level 0.95.

SOLUTION

$\mu = 0.07 \qquad \sigma = 0.25$

1] $P(T) = P(0)\, e^{(\mu - \frac{\sigma^2}{2})T + \sigma W(T)}$ with $W(T)$ standard brownian motion

The target is $P(T) = 2P(0)$, thus

$$P(P(T) \leq 2P(0)) = P\left(P(0)\, e^{(\mu - \sigma^2/2)T + \sigma W(T)} \leq 2P(0)\right) =$$

$$= P\left(e^{(\mu - \sigma^2/2)T + \sigma W(T)} \leq 2\right) =$$

$$= P\left((\mu - \sigma^2/2)T + \sigma W(T) \leq \ln 2\right) =$$

$$= P\left(W(T) \leq \frac{\ln 2 - (\mu - \sigma^2/2)T}{\sigma}\right) =$$

$$= P\left(W(1) \leq \frac{\ln 2 - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\right) =$$

$$= \phi\left(\frac{\ln 2 - (0.07 - \frac{0.25^2}{2})10}{0.25 \cdot \sqrt{10}}\right) = \phi(0.38661) = 0.65048$$

0.7905

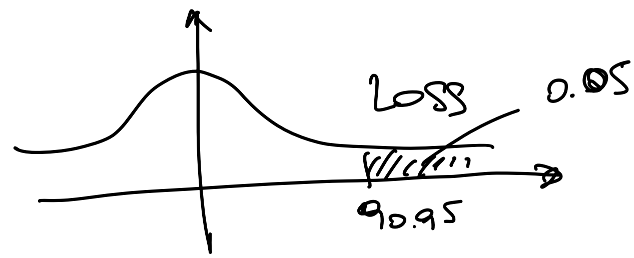
the shortfall probability is 0.65048

2] R portfolio log return $R(t+h) = \ln \frac{V(t+h)}{V(t)}$

Loss $\quad L(t+h) = -R(t+h)$

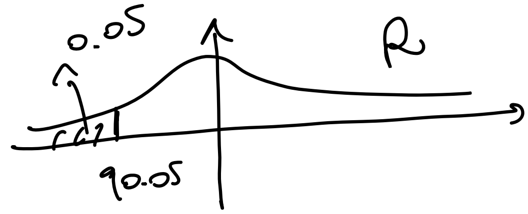
VaR : see theory

3] $t = 0 \quad h = 10$ years $\quad L(10) = -R(10) = -\ln \frac{V(10)}{V(0)}$



$P(L(10) \leq y) = 0.95$

$R(10) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right) = (0.3875,\ 0.25^2 \cdot 10)$



$\Rightarrow \quad L(10) \leq y$

$L(10) \sim N(-0.3875,\ 0.25^2 \cdot 10)$

$\mathrm{VaR}_{0.95}(L(10)) = -0.3875 + 0.25\,(1.64485)\sqrt{10} = 0.91694$

$= \mu_L + \sigma_L Q(\alpha) = \mu_L + \sigma_L Q(0.95)$

μ_L is the mean of the loss $L(10)$

σ_L " " st. dev. " " " "

You then consider $\mathrm{VaR}(\tilde{L}(10))$ with

$\tilde{L}(10) = V(0) \cdot L(10) = V(0) \cdot (-R(10)) \cong 91694$

Exercise 3

Let $W(t)$ be a standard Wiener process.

1. Show that
$$cov(W(t), W(s)) = min\{s, t\} \tag{0.1}$$
and that
$$\rho(W(t), W(s)) = \sqrt{\frac{min\{s, t\}}{max\{s, t\}}}$$

2. Write down the SDE for the process
$$X(t) = W(t)^4$$

3. Show that
$$Var(Y(t)) = \frac{t^3}{3},$$
where
$$Y(t) = \int_0^t (t - s)\ dW(s)$$

Solution -

1. See the theory: main passages below

$$Cov(W(t), W(s)) = E[W(t)W(s)] - \underbrace{E[W(t)]}_{0}\underbrace{E[W(s)]}_{0} = E[W(t)W(s)]$$

assume w.l.o.g. $s \leq t$

$$E[W(t)W(s)] = E[W(t)W(s) - W(s)^2 + W^2(s)] =$$

$$= E[W(s)(W(t) - W(s))] + E[W^2(s)] =$$

$$= \underbrace{E[W(s)]}_{0} E[W(t) - W(s)] + s = s$$

s because $W(s) \sim N(0, s)$

$\Rightarrow Cov(W(t), W(s)) = s$

Similarly $t \le s$ $\quad Cov(W(t), W(s)) = t$

$$\rho = \frac{Cov(W(t), W(s))}{\sqrt{var(W(t))}\sqrt{var W(s)}} = \frac{\min\{s,t\}}{\sqrt{t}\sqrt{s}} = \text{se } t < s$$

$$= \frac{t}{\sqrt{t}\sqrt{s}} = \sqrt{t}/\sqrt{s} = \frac{\sqrt{\min\{s,t\}}}{\sqrt{\max\{s,t\}}}$$

se $s < t$ analogo

2) let $f(t,x) = x^4$

$$\partial_t f = 0$$
$$\partial_x f = 4x^3$$
$$\partial_{xx} f = 12x^2$$

$$df(t, W(t)) = f_t(t, W(t))\,dt + f_x(t, W(t))\,dW(t) +$$
$$+ \frac{1}{2} f_{xx}(t, W(t))\,dt$$
$$= 0 \cdot dt + 4W^3 dW(t) + \frac{1}{2} 12 W^2(t)\,dt =$$
$$= 6W^2 dt + 4W^3 dW$$

3) by Ito Isometry

$$E[Y_t^2] = E\left(\int_0^t (t-s)\,dW_s\right)^2 = \int_0^t (t-s)^2\,ds$$

$$= -\frac{(t-s)^3}{3}\Big|_0^t = -\frac{0}{3} + \frac{t^3}{3} = \frac{t^3}{3}$$

since $E(Y_t) = E\left[\int_0^t (t-s)\,dW_s\right] = 0$

we have

$$V(Y_t) = EY_t^2 = \frac{t^3}{3}$$