

Politecnico di Torino
Financial Engineering-Exam 02-01-2024
P. Semeraro

SURNAME AND NAME

Student number

Exercise 1 (10 points)

Consider a one-period (annual) model formed by a bond (paying a risk-free rate of 5% per year) and by two stocks with prices S^1 and S^2 evolving as follows:

$$S_0^1 = 10;$$
$$S_T^1(w) = \begin{cases} 12; & \text{if } w = w_1 \\ 10; & \text{if } w = w_2 \\ 6; & \text{if } w = w_3 \end{cases} \quad (0.1)$$

and

$$S_0^2 = 10;$$
$$S_T^2(w) = \begin{cases} 15; & \text{if } w = w_1 \\ 8; & \text{if } w \in \{w_2, w_3\} \end{cases} \quad (0.2)$$

with $P(w_1), P(w_2), P(w_3) > 0$ and $P(w_1) + P(w_2) + P(w_3) = 1$.

1. Establish if the market is free of arbitrage and complete.
2. Consider the derivative A with maturity T of one year and with

$$\Phi_A = \left(\frac{S_T^1 + S_T^2}{2} - 8 \right)^+ \quad (0.3)$$

and find its price.

3. Discuss whether the market formed only by the bond and by stock S^2 would remain free of arbitrage and complete.

EXERCISE 1

1) s^1 more pole

$$\begin{cases} \mathcal{E}\left[\frac{s^1}{1+s}\right] = \mathcal{S}^1 \\ \mathcal{E}\left[\frac{s^2}{1+s}\right] = \mathcal{S}^2 \end{cases}$$

$$\begin{cases} \frac{12}{1.05} q_1 + \frac{10}{1.05} q_2 + \frac{6}{1.05} q_3 = 10 \\ \frac{15}{1.05} q_1 + \frac{8}{1.05} (q_2 + q_3) = 10 \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 12 q_1 + 10 q_2 + 6 q_3 = 10.5 \\ 15 q_1 + 8 (q_2 + q_3) = 10.5 \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

$$\begin{vmatrix} 12 & 10 & 6 \\ 15 & 8 & 8 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 10 & 6 \\ 8 & 8 \end{vmatrix} - \begin{vmatrix} 12 & 6 \\ 15 & 8 \end{vmatrix} + \begin{vmatrix} 12 & 10 \\ 15 & 8 \end{vmatrix} = \\ = 80 - 48 - 96 + 90 + 96 - 150 = -28 \neq 0$$

the system has only one solution \Rightarrow

Solving the system:

$$\begin{cases} q_1 = 5/16 \\ q_2 = 3/56 \\ q_3 = 3/56 \end{cases}$$

Solution positive $\rightarrow \exists!$ unique non neutral
quintal market arb. free & complete

2]

$$\phi_A = \begin{cases} 5.5 & \text{if } \omega = \omega_1 \\ 1 & \text{if } \omega = \omega_2 \\ 0 & \text{if } \omega = \omega_3 \end{cases}$$

$$E[\phi_A / (1+\pi)] = \frac{1}{1.05} [5.5 q_1 + q_2] = 2.932$$

3] With only asset S^2 : $\tilde{q}_1 = P(\omega_1)$
 $\tilde{q}_2 = P(\{\omega_2, \omega_3\})$

$$\begin{cases} \frac{1.5}{1.05} \tilde{q}_1 + \frac{8}{1.06} \tilde{q}_2 = 0 \\ \tilde{q}_1 + \tilde{q}_2 = 1 \end{cases}$$

$$\tilde{q}_1 = 5/16 \quad \tilde{q}_2 = \frac{36}{56} \quad \text{unique sol.}$$

Complete market (1-Step Binomial model)

Exercise 2 (10 points)

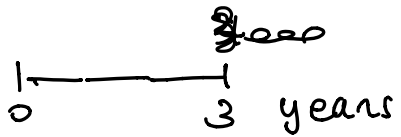
A portfolio consists of a 3-year zero-coupon bond with face value of 3000 \$ and a 5-year zero-coupon bond with face value of 6000 \$. The current yield on all bonds is 6% per annum.

- (a) Compute the duration of the portfolio (using continuous compounding).
- (b) Compute the percentage change in the value of the portfolio in the case of a 0.2% per annum decrease in yields.
- (c) Compare the result obtained in (b) with the *1st* order and *2nd* order approximations based on the use of duration and of convexity, and comment the results.

SOLUTION

EXERCISE 1

(a) We have :



current yield = 0.06 per annum

Duration :

$$D = \frac{\sum_{i=1}^n t_i C_i e^{-yt_i}}{\sum_{i=1}^n C_i e^{-yt_i}} =$$

$$= \frac{3 \cdot 3000 e^{-0.06 \cdot 3} + 5 \cdot 6000 e^{-0.06 \cdot 5}}{3000 e^{-0.06 \cdot 3} + 6000 e^{-0.06 \cdot 5}} =$$

$$= \frac{29741,98}{6950,72} = 4,27$$

$$D = 4,27 \text{ years}$$

(b) Value of the portfolio:

$$V = 3000 e^{-0.06 \times 3} + 6000 e^{-0.06 \times 5} = 6950,72$$

If the yields decreases by 0.2% it becomes 5.8% the value of the portfolio becomes:

$$V' = 3000 e^{-0.058 \cdot 3} + 6000 e^{-0.058 \cdot 5} = 7010,672$$

Percentage change of the portfolio:

$$\frac{V' - V}{V} = \frac{7010,672 - 6950,72}{6950,72} = \frac{59,95}{6950,72} = 0,86\%$$

(c) Using first order approximation based on duration we have

$$\frac{\Delta V}{V} = -D \Delta y = -4,27 (-0.002) = 0,00854 = 0,85\%$$

The convexity of the portfolio is

$$\begin{aligned} C &= \frac{\sum_{i=1}^n t_i^2 C_i e^{-y t_i}}{\sum_{i=1}^n C_i e^{-y t_i}} = \\ &= \frac{3^2 \cdot 3000 e^{-0.06 \times 3} + 5^2 \cdot 6000 e^{-0.06 \times 5}}{6950,72} = \\ &= \frac{133675}{6950,72} = 19,23 \end{aligned}$$

and using the second order approximation based on the convexity we have:

$$\begin{aligned}\frac{\Delta V}{V} &= -D\Delta y + \frac{1}{2}C(\Delta y)^2 = \\ &= -4.27 \cdot (-0.002) + \frac{1}{2} 19.23 (-0.002)^2 = \\ &= 0.00887 = 0.86\% \end{aligned}$$

the approximation based on the convexity is better, since it is a second order Taylor expansion that is always more precise than a first order Taylor expansion, like the one based on duration.

Exercise 3 (10 points)

Let $dS = \mu_t dt + \sigma_t dW$ and risk-free interest rate is 4% per annum (all rates are continuously compounded). (a) When $\sigma_t = 0.01$ for $t \in [0, 4]$ and

$$\mu_t = \begin{cases} 0.02t; & 0 \leq t \leq 2 \\ 0.01(10 - t); & 2 < t \leq 4 \end{cases} \quad (0.4)$$

compute the probability of having at $t = 4$ a profit greater or equal than 0.4.

(b) When $\mu_t = 0.01$ for $t \in [0, 4]$ and $\sigma_t = 0.2$ for $t \in [0, 4]$ compute the probability of having at $t = 4$ a profit greater than or equal to 0.4.

(c) Assuming $\mu_t = 0.03S_t$ and $\sigma_t = 0.01S_t$ for $t \in [0, 1]$, in years, i.e.

$$dS = 0.03S_t dt + 0.01S_t dW$$

and that the risk free rate is 2% per annum, price an European call option with strike price 100, maturing in one year, written on the stock $S(t)$ (no dividends) whose current price $S(0)$ is 98.

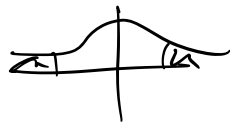
(a) by Ito's formula $dS_t = \mu_t dt + \sigma_t dW_t$
we have

$$\begin{aligned} S_4 - S_0 &= \int_0^4 \mu_s ds + \int_0^4 \sigma_s dW_s = \\ &= \int_0^2 0.02 S_0 dt + \int_2^4 0.01(10 - S) dt + \int_0^4 0.01 dW_s \\ &= 0.02 \left. \frac{S^2}{2} \right|_0^2 + 0.01 \left[10S - \frac{S^2}{2} \right]_2^4 + 0.01 (W_4 - W_0) \\ &= 0.04 + 0.01 \left[40 - 8 - 20 + 2 \right] + 0.01 W_4 \\ &= 0.18 + 0.01 W_4 \sim N(0.18, 0.01^2) \end{aligned}$$

Profit greater or equal to 0.4
means $S_4 - S_0 \geq 0.4$

$$\begin{aligned} P((S_4 - S_0) > 0.4) &= P(0.18 + 0.01 W_4 > 0.4) \\ &= P\left(W_4 > \frac{0.4 - 0.18}{0.01}\right) = \end{aligned}$$

$$= P\left(Z > \frac{0.22}{\sqrt{4} \cdot 0.01}\right)$$

because $W_4 \sim N(0, 4)$ 

$$= P\left(Z > \frac{0.22}{0.02}\right) = 1 - \Phi(11) \approx 0$$

(b)

$$S_4 - S_0 = \int_0^4 \mu_s ds + \int_0^4 \sigma_s dW_s =$$

$$= 0.01s \Big|_0^4 + \int_0^4 0.2 dW_s$$

$$= 0.01 \cdot 4 + 0.2(W_4 - W_0)$$

$$= 0.04 + 0.2 N_4$$

where $N_2 \sim N(0, 4)$ since

$W_t - W_s \sim N(0, t-s)$. Thus

$$0.04 + 0.2 N_4 = 0.04 + N_1,$$

where

$$N \sim N(0, (0,04) \cdot 4) \\ \sim N(0, 0,16)$$

therefore

$$\begin{aligned} P(S_n - S_0 \geq 0,4) &= P(0,04 + N \geq 0,4) \\ &= P\left(Z \geq \frac{0,4 - 0,04}{\sqrt{0,16}}\right) = P(Z \geq 0,9) \\ &= 1 - \Phi(0,9) = 0,18 \end{aligned}$$

(c) We have $S(0) = 98$ $K = 100$ $T = 1$

$b = 0.01$ per annum

$\sigma = 0.02$

According to the Black-Scholes-Merton formula for a European Call option

$$C = S(0) N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$
$$= \frac{\ln(0.98) + \left(0.02 + \frac{0.01^2}{2}\right) \cdot 1}{0.01}$$

$$= -0.0152$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.0152 - 0.01\sqrt{1} =$$
$$= -0.025$$

$$C = 98 N(-0.015) - 100 e^{-0.02} N(-0.025)$$

$$= 0.38$$