# 4- LOW FIDELITY MODEL

The idea is to reduce $S_\sigma$ to solve the solution in a cheaper way => we approximate $S_\sigma$ using a space $V_N \subseteq V_\delta$ for which $N = \dim(V_N)$, $N\delta = \dim(V\delta)$ and $N \ll N\delta$. Hence, $S_N$ is the approximation of $S$ that is named as **low fidelity model** in contrapposition to $S_\delta$ that is high fidelity model → cheaper to solve it

Given the manifold $\mathcal{M}_\delta$ we wish to build an approximation in which is cheaper to evaluate $\forall \mu \in P$. For the approximation of $S_\delta$ we imagine to select a subset $\{\mu_1, .., \mu_N\} \subset P$ and to compute the high fidelity solution we use the so called snapshots $\{w_1 = u(\mu_1), .., w_N = u(\mu_N)\} \in V_\delta$ that produce a "surrogare" of $S_\delta$ that we call low-fidelity model $S_N$ or reduced model

Different way to approximare $S_\delta$ -
1. Interpolation – not taken in account
2. Least Square – not taken in account
3. Reduces basis Method RBM => Galerkin approach
4. Neural Network: supervisio of unsupervision learning → later

## 4.1 Reduced basis method RBM

Consider $V_\delta \subset V$, $a(u_\delta, v_\delta; \mu) = F(v_\delta, \mu)$, the subspace of the solution is $\{w_1, .., w_N\} \in V_\delta$ that could be linear indipendent : (N different solution ⊥; ↳ snapshots)

$$\boxed{\sum_{i=1}^{N} a_i w_i = 0 \iff \{a_i\} = 0 \quad \forall i \in \{1, .., N\}}$$ that solutions form a basis, so they build a subspace $V_N \subseteq V_\delta$, $N \ll N_\delta$

$V_N := \text{span}\{w_1, .., w_N\} \subseteq V_\delta$

we formulate a variational problem on $V_N$:

Find $u_N \in V_N$ st $\forall \mu \in P$, $a(u_N, v_N; \mu) = F(v_N; \mu)$ $\forall v_N \in V_N$ => $\boxed{a(u_\delta - u_N, v_N) = 0 \quad \forall v_N \in V_N}$ Galerkin orthogonality in $V_N$

$\langle u_\delta - u_N, v_N \rangle_a = 0$

↳ can be solved because is a bilinear form continuous, coercive, etc

So if the hp holds, it's true $\boxed{\|u_\delta(\mu) - u_N(\mu)\|_V \le \frac{\gamma}{\alpha} \inf_{v_N \in V_N} \|u_\delta(\mu) - v_N\|}$ => it holds the Cea Lemma

we express $u_N(\mu)$ in terms of $\{w_i\}_{i=1}^{N}$

**Generation of the linear system** - We have to select a basis $\{w_1, .., w_N\}$ of $V_N$ and we rewrite the solution in that way to compute the degree of freedom (or coefficient)

- $S_N : \mu \in P \mapsto u_N(\mu) \in V_N \subset V_\delta$
- $\mathcal{M}_N := \{u_N(\mu) \in V_N : \mu \in P\}$

$$\boxed{u_N(\mu) = \sum_{i=1}^{N} u_{N,i}(\mu) w_i \iff u_N(\mu) := [u_{N,1}, .., u_{N,N}] \in \mathbb{R}^N}$$ array of coefficient

↓ Put that in the bilinear form: $a(u_N, v_N; \mu) = F(v_N; \mu)$ => $\sum_{i=1}^{N} a(u_{N,i} w_i, v_N; \mu) = F(v_N, \mu)$ $\forall v_N \in V_N$

↳ we have N unknows, we need N equations

$$\Rightarrow \boxed{\sum_{i=1}^{N} u_{N,i}(\mu)\, a(w_i, w_j; \mu) = F(w_j; \mu) \quad \forall \{w_j\}_{j=1}}$$ set of linear indipendent vector from which we can derive the solution

In matrix form $u_N \in \mathbb{R}^N$, $A_N^\mu \in \mathbb{R}^{N,N}$, $f_N^\mu \in \mathbb{R}^N$ => $\boxed{A_N^\mu u_N = f_N^\mu}$ **Linear system parametric** dimension very low, easy to solve

NB $A_N^\mu := [a(w_i, w_j; \mu)]_{1 \le i,j \le N}$, $f_N^\mu := [F(w_i; \mu)]_{1 \le i \le N}$ ↳ what we want to compute

↓ In this way we can compute the solution in a cheaper way, with a not to much error

$N \ll N_\delta$ ~10 ~10'000

priori formula, exponential decay

**Exercise** - How the matrix $A_N$ is related to $A_\delta$? Consider $A_N^\mu u_N = f_N^\mu$ and $A_\delta^\mu \cdot u_\delta = f_\delta^N$

=> From $A_N^\mu$ we can deduce $\{w_i\}_{i=1}^{N} \in V_N \subset V_\delta$ (basis of $V_N$) and $\{\varphi_j\}_{j=1}^{N_\delta} \in V_\delta$ (basis of $V_\delta$) so $w_i = \sum_{j=1}^{N_\delta} b_{ji} \varphi_j$

NB The computation of $A_N^\mu$ requires element of size $N_\delta$, that is NOT good if $N_\delta \gg N$!

To answer, we can use the definition of the matrix A:

$$(A_N^\mu)_{ji} = a(w_i, w_j; \mu) = a\left(\sum_{k=1}^{N_\delta} b_{ki} \varphi_k, \sum_{e=1}^{N_\delta} b_{ej} \varphi_e; \mu\right) = \sum_{k,e=1}^{N_\delta} b_{ki}\, a(\varphi_k, \varphi_e; \mu)\, b_{ej}$$

$(A_\delta^\mu)_{ek}$ : depends on the parameters, has to compute each time

↓ In matrix form $A_N^\mu = B^T A_\delta^\mu B$, where $(B)_{ki} = b_{ki}$, so the two matrices are similar

$\in \mathbb{R}^{N,N}$, $\in \mathbb{R}^{N_\delta, N_\delta}$, $\in \mathbb{R}^{N_\delta, N}$ : do not depends on the parameters

How to select $\{w_i\}_{i=1}^{N}$ basis of $V_N$? Proper Orthogonal Decomposition POD

## 4.2 How to select the snapshots?

We saw that if we have a set of snapshots $\{w_i\}_{i=1}^{N}$ linearly indipendent we are able to reduce and solve the $VP_{N(\mu)}$

But, how to detect them? $P_H = \{\mu_1, ..., \mu_H\} \subseteq P$

Suppose we know $M \le N_\delta$ snapshots $w_m = u_\delta(\mu_m) \in V_\delta$, $m = \{1, .., M\}$. They may be linearly dependent $\Rightarrow X_M := \text{span}\{w_m\}_{m=1}^M$ may have dimension $< M$.

$\boxed{\text{The idea is to reduce the unnecessary snapshots}}$

↓

## 4.2 Proper orthogonal decomposition POD

→ Also know as PRINCIPAL COMPONENT ANALYSIS

POD → can be seen from the Singular - Value - Decompositio's point of view

Consider $P \in \mathbb{R}^P$ the parameter space, so the subspace $P_M \subseteq P$ with $|P_M| = M$ is defined as $P_M = \{\mu_1, .., \mu_M\} \in P$. Here, we can compute a high fidelity solution $\forall \mu_i \in P_M$ : $w_i = u_\delta(\mu_i)$ (solution) of VPs → for that problem

So we can build $X_M := \text{span}\{w_i\}_{i=1}^M$ that are linear dependent

The main idea is to reduce $X_M$ so that to have an indipendent space. To do that we introduce an operator:

$$\mathcal{C}: X_M \to X_M, \quad v \mapsto \sum_{m=1}^M \langle v, w_m \rangle_V w_m$$

in which $w_m$ is called snapshot and represent a solution

Properties of $\mathcal{C}$.
1. $\mathcal{C}$ is self-adjoint $\forall v, z \in X_M$ $\langle \mathcal{C}v, z\rangle_V = \langle \mathcal{C}z, v\rangle_V$ $\Rightarrow \langle \mathcal{C}v, z\rangle_V = \sum_{m=1}^M \langle v, w_m\rangle_V \langle w_m, z\rangle_V = \langle \mathcal{C}z, v\rangle_V$
2. $\mathcal{C}$ is semi-definite positive $\forall v \in X_M$ : $\langle \mathcal{C}v, v\rangle_V = 0$, so $\mathcal{C}$ has $M$ eigenpairs $\{\lambda_m, \Psi_m\}_{m=1}^M$ (eigenfunctions)
   ↳ so $\{\lambda_m \ge 0\}$, $\lambda_1 \ge \lambda_2 \ge .. \ge \lambda_M \ge 0$ (eigenvalue) and $\|\Psi_m\| = 1$ $\Rightarrow \langle \mathcal{C}v, v\rangle_V = \sum_{m=1}^M \langle v, w_m\rangle^2 \ge 0$ $\quad \mathcal{C}\Psi_m = \lambda_m \Psi_m \quad \forall m = \{1, .., M\}$
3. $\mathcal{C}$ is injective iff $\{w_m\}_{m=1}^M$ are linearly indipendent

Aim. Find linear indipendent snapshot: we need eigenpair decomposition

↓

**Theorem.** We define $(C)_{m,n} := \langle w_m, w_n\rangle_V$ and $C \in \mathbb{R}^{M,M}$ that is the covariance matrix

$\Rightarrow$ The eigenvalue $\boxed{\{\lambda_m\}_{m=1}^M}$ of $C$ are the eigenvalue of the operator $\mathcal{C}$

$\Rightarrow$ The eigenfunctions $\boxed{\Psi_m := \frac{1}{\sqrt{\lambda_m}} \sum_{k=1}^M w_k [a_m]_k}$ $\forall m \in \{1, .., M\}$, where $a_m$ are the eigenvectors of $C$, $[a_m]_k = a_k$

↓

Proof on the lecture notes

More Details on POD on Lecture 4

## 4.3 Affine case

If the bilinear form $a(\cdot, \cdot; \mu)$ admits an affine decomposition s.t. $a(u, v; \mu) = \sum_{q=1}^{Q_a} \vartheta_a^q(\mu) a_q(u, v)$ where $\forall q \in \{1, .., Q_a\}$ $a_q: V \times V \to \mathbb{R}$ and $\vartheta_a^q : P \to \mathbb{R}$, then

$$A_N^\mu = B^t A^\mu B = B^t \left(\sum_{q=1}^{Q_a} \vartheta_a^q(\mu) A_q\right) B = \sum_{q=1}^{Q_a} \vartheta_a^q(\mu) B^t A_q B \in \mathbb{R}^{N \times N}$$

↓

can be computed once for all $\mu \in P$ $\Rightarrow$ off-line phases

and $A_N$ is assemble fastly and cheaply each value of $\mu$ $\Rightarrow$ on-line phases

Similarly, the computation of $f_N^\mu$ can be deduced from $f^\mu$ with $(f_N^\mu) = F(w_i; \mu) = \sum_{j=1}^{N_\delta} b_{ji} F(\varphi_j; \mu) = (B^T f^\mu)_i$ and, if $F(\cdot; \mu)$ admits affine decomposition i.e

$$F(\cdot; \mu) = \sum_{q=1}^{Q_a} \vartheta_f^q(\mu) F_q(v)$$

with $F_q: V \to \mathbb{R}$ and $\vartheta_f^q : P \to \mathbb{R}$ then $f_N^\mu = B^T f^\mu = B^T \sum_{q=1}^{Q_f} \vartheta_f^q(\mu) f_q = \sum_{q=1}^{Q_f} \vartheta_f^q(\mu) (B^T f_q) \in \mathbb{R}^N$ and again $B^T f_q$ is indipendent of $\mu$ (off-line)

More details on Error estimates of RBM on Lecture 3