IMPORTANT -

The reduction is dependent on high fidelity snapshot --> check that the code that you implement for the high fidelity is correct: so you have to test it!!
- Measure an error using the exact solution (polinomial of degree 4 that we use)
- Computer it with another forcing term and compute an error

--> You have to TEST that the solution of the high fidelity problem --> the erorr has to goes down when you improve the degree of freedom (increment the number of point)

- Order for high fidelity solution: what you want (1 or 2) --> if 1 is a good approximation is ok, otherwise increase at 2
- The mesh size (number of degree of freedom) is not writtten, you have to choose it

Project: Nonlinear Elliptic problem POD vs PINNs

Let us consider the two-dimensional spatial domain $\Omega = (0, 1)^2$. We wan to solve the following parametrized problem: given $\boldsymbol{\mu} = (\mu_0, \mu_1) \in \mathcal{P} = [0.1, 1]^2$, find $u(\boldsymbol{\mu})$ such that

Parametric space of dimension 2

Restirc version of the parametric space wrt the paper
Also for the domain is the same

This is the non linearity: the solution is in the exponential

$$-\Delta u(\boldsymbol{\mu}) + \frac{\mu_0}{\mu_1}(e^{\mu_1 u(\boldsymbol{\mu})} - 1) = g(\boldsymbol{x}; \boldsymbol{\mu}),$$

with homogeneous Dirichlet condition on the boundary. --> 0-Boundary condition

Part 1

Forcing term g_1: fixed in the parametric space, does not depend on the parameters

The source term defined as

$$g(\boldsymbol{x}; \boldsymbol{\mu}) = g_1 = 100 \sin(2\pi x_0) cos(2\pi x_1) \quad \forall \boldsymbol{x} = (x_0, x_1) \in \Omega.$$

Tasks:

1. solve the problem by means of POD-Galerkin method over a Finite Element full order model NON LINEAR PROBLEM: using high fidelity finite element approximation of POD
2. solve the problem with a parametric PINN Find the NN structure to be able to reduce the problem in the righ way
3. compare the two approaches in terms of computational costs and accuracy with respect to the full order model Compare 1. & 2. --> in term of error, time (which one is faster)
4. **Optional**: solve the problem with the POD-NN approach and compare it to the other two strategies Mixed approach: POD-NN and compare with 1&2 wrt time and error as before

This is the solution with the data of the paper, hence the domain is a little bit larger and also the parametric space

This is a sort of periodic function in this domain

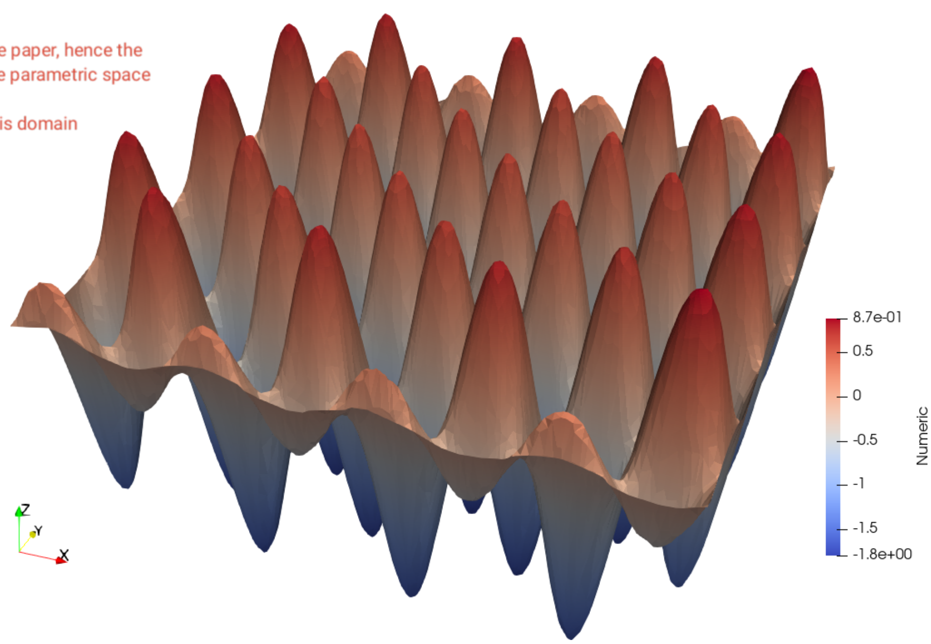

Figure 1 - *Solution on* $\tilde{\Omega} = [0, 4]^2$ *for* $\mu = [3.34, 4.45]$ *with* g_1

Part 2 Change the forcing term: we have the parameter inside the g_2 function
The problem become non linear in the sense of the parameter

The source term defined as

$$g(\boldsymbol{x}; \boldsymbol{\mu}) = g_2 = 100 \sin(2\pi\mu_0 x_0) cos(2\pi\mu_0 x_1) \quad \forall \boldsymbol{x} = (x_0, x_1) \in \Omega.$$

Tasks:

1. solve the problem by means of POD-Galerkin method over a Finite Element full order model Solve the same problem as before, but with a different forcinfg term
2. solve the problem with a parametric PINN using the **same** network structure (number of layers and nodes per layer) of Part 1
3. compare the two approaches in terms of computational costs and accuracy with respect to the full order model
4. **Optional**: solve the problem with the POD-NN approach and compare it to the other two strategies

Do NOT modify the structure of the NN that you implement before
You need to train the network before!

Compare the result as before

Use the net that you have, try again, see the result. Hence you have to to obtain the right structure of the NN

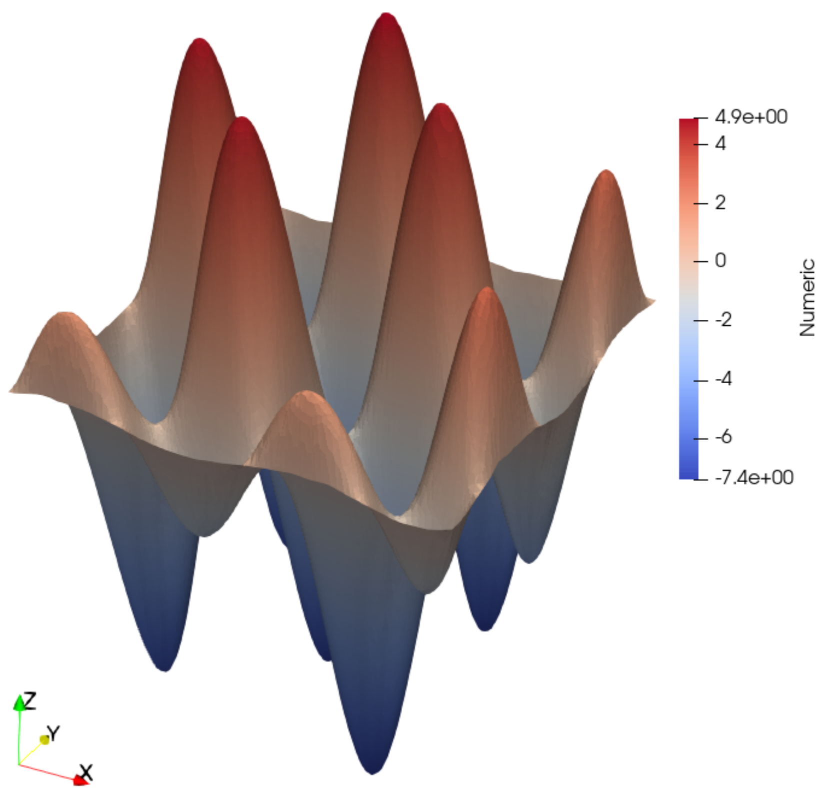

Figure 2 - *Solution on* $\tilde{\Omega} = [0, 4]^2$ *for* $\mu = [0.455, 0.72]$ *with* g_2 *zoom* $0.5x$

Hints! Good loss function to impose in the PINN: boundary part (takes in account the boundary condition) + physical part (takes in account the equation that we want to solve)

Given a neural network $\tilde{w}(\boldsymbol{x}, \boldsymbol{\mu})$, the loss will be of the form

$$MSE \doteq MSE_b^{\boldsymbol{\mu}} + \lambda MSE_p^{\boldsymbol{\mu}}.$$

where the *boundary* MSE is:

$$MSE_b^{\boldsymbol{\mu}} \doteq \frac{1}{N_b} \sum_{k=1}^{N_b} |\tilde{w}(\boldsymbol{x}_k^b, \boldsymbol{\mu}_k^b) - u(\boldsymbol{x}_k^b, \boldsymbol{\mu}_k^b)|^2,$$

Not for all the point (x) you have to select the same parameter

You are NOT setting the same parameter for all the point, you can change it!

for $(\boldsymbol{x}_k^b, \boldsymbol{\mu}_k^b) \in \partial\Omega \times \mathcal{P}$. While the *physical* MSE is

$$MSE_p^{\boldsymbol{\mu}} \doteq \frac{1}{N_p} \sum_{k=1}^{N_p} |\mathcal{R}(\tilde{w}(\boldsymbol{x}_k^p, \boldsymbol{\mu}_k^p))|^2,$$

You can not sum all the summation changing the parameter only when you compute all the operation for all the x, then you can change the parameter in the sum when you cange the point x

You have to tune it to understand which is the best one!

where $(\boldsymbol{x}_k^p, \boldsymbol{\mu}_k^p) \in \Omega \times \mathcal{P}$. Finally, λ is the hyperparameter that can be used to balance the two different contributions.

EXAM - Rules

1. Oral presentation of at most **10** minutes with some slides (10 slides no more!!) of the project
2. **Questions** on the **presentation**
3. **Theoretical questions:** on the notes (mathematically speaking writing!!)

Report. 10 pages

- Desciption of the problem
- Weak formulation of the problem
- Exact solution that you use to solve the high fidelity problem
- Results that you obtain in each point of the 2 parts with some comments on the results
- Report with table and images (ParaView)

Put the report on the portal at least two days before the presentation

NB You have to upload on the portal only the report (not the notebook)