3- FULL ORDER NUMICAL MODEL

not reduced

NOT computable

we select a finite subspace of V, V_δ and we create a VP promblem on V_δ

$VP(\mu)$ can not be solved automatically forall parameter $\Rightarrow$ we solve it numerically. To do so, we must discretize the space: «δ» is the discretization parameter for the discrete problem **GALERKIN METHOD**

related to the finite dimension of V_δ — family of subspaces of V

$VP_\delta(\mu)$: find $u_\delta \in V_\delta \subseteq V$ st $\forall \mu \in P$, $a(u_\delta, v_\delta; \mu) = F(v_\delta, \mu)$ $\quad \forall v_\delta \in V_\delta$ → It is NOT garanted that there is a solution

Consider $\boxed{\{V_\delta\}_{\delta>0}}$ subspaces of V st the **consistency assumptions** hold: $\forall \mu \in V \Rightarrow$ $\boxed{dist_V(u, V_\delta) = \min_{v_\delta \in V_\delta} \|\mu - v_\delta\| \xrightarrow[\delta \to 0]{} 0}$

$\delta \to 0 \Rightarrow V_\delta \to V$ How fast $dist_V(u, V_\delta) \to 0$? we chose V_δ so that $\exists$ an unique solution

We take $\{V_\delta\}_{\delta>0} \subseteq V$ such that $\exists Z \subset V$ normed space such that $\exists\ \Psi_Z: \mathbb{R} \to \mathbb{R}$, $\Psi_Z(\delta) \xrightarrow[\delta \to 0]{} 0$, ① $\boxed{dist_V(w, Z) \leq \Psi_Z(\delta) \|w\|_Z \ \forall w \in Z, u \in Z}$

controls the distance of the consistency assumptions so that we know how much fast goes $dist_V(u, V_\delta) \to 0$

NB Tells us that $u_\delta(\mu) \in V_\delta$, $\forall \mu \in P$ is the BEST APPROXIMATION we can obtain of $u \in V$ fixing $\{V_\delta\}_{\delta>0}$

Lemma - Cea If u_δ solution of $VP_\delta(\mu)$ and u is a solution of $VP(\mu)$

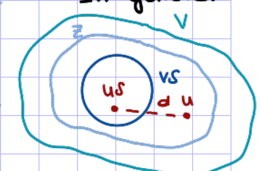
$\Rightarrow \boxed{\|u(\mu) - u_\delta(\mu)\|_V \leq \frac{\gamma(u)}{\alpha(u)} \boxed{\inf_{v_\delta \in V_\delta} \|u(\mu) - v_\delta\|_V} \quad \forall \mu \in P}$ ②

distance controlled by continuity and cohercivity — for consistency assumption, if we chose the right δ, goes to zero

Theorem If $\{V_\delta\}_{\delta>0}$ satisfy 1. and 2. $\Rightarrow$ $\boxed{\|u(\mu) - u_\delta(\mu)\|_V \leq \frac{\gamma(u)}{\alpha(u)} \Psi_Z(\delta) \|u(\mu)\|_Z}$ → by hypothesis $u \in Z$

↳ Combined 1+2 $\Rightarrow$ A-priori error estimation

In general



Example Consider $Z = H^r(\Omega)$, $r > 1$ $\forall \mu \in P$, $V = H^1(\Omega)$ and $Z \subset V$.

1. $H^r \Rightarrow \Psi_Z(\delta) = \delta^{r-1}$: $dist_V(w, V_\delta) \leq \delta^{r-1} \|w\|_{H^r}$ $\quad \forall \mu \in P \Rightarrow \|u(\mu) - u_\delta(\mu)\|_V \leq \frac{\gamma(u)}{\alpha(u)} \delta^{r-1} \|u(\mu)\|_{H^r}$
2. $\nabla(K \cdot \nabla \mu) = f$ $\quad u \in H^2 = Z$, $\Psi_Z(\delta) = \delta^1 \Rightarrow \|u(\mu) - u_\delta(\mu)\|_V \leq \frac{\gamma(u)}{\alpha(u)} \delta \|u(\mu)\|_{H^2}$

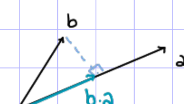
$VP_\delta(\mu)$: $a(u_\delta, v_\delta) = F(v_\delta)$ $\quad \forall v_\delta \in V_\delta$ $\quad \Rightarrow$ but $V_\delta \subset V$, so $\forall v_\delta \in V_\delta$: $a(u_\delta, v_\delta) - a(u, v) = F(v_\delta) - F(v) = 0$

$VP(\mu)$: $a(u, v) = F(v)$ $\quad \forall v \in V$ $\qquad a(u_\delta - u, v_\delta) = 0 \quad \forall v_\delta \in V_\delta$

If $a(\cdot,\cdot)$ is symmetric, since is also bilinear and cohercive, it's a **scalar product**

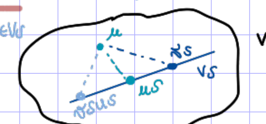
$\Rightarrow a(u_\delta - u, v_\delta) = \langle u_\delta - u, v \rangle_a = 0 \Rightarrow u_\delta - u \perp v$

$\Rightarrow$ u_δ is the **orthogonal projection** of u on V_δ with respect to a-product



Subtracking $VP_\delta(\mu)$ from $VP(\mu)$: $a(u, v_\delta) - a(u_\delta, v_\delta) = F(v_\delta) - F(v_\delta) = 0$ $\forall \mu \in P \Rightarrow a(u - u_\delta, v_\delta; \mu) = 0$ $\forall v_\delta \in V_\delta$

Galerkin orthogonalization Consider $a(u - u_\delta, v_\delta) = 0$ $\quad \forall v_\delta \in V_\delta$



$\Rightarrow \|u - u_\delta\|_a = \sqrt{\langle u - u_\delta, u - u_\delta \rangle_a} = \sqrt{a(u - u_\delta, u - u_\delta)} = \|u - v_\delta\|_a$ $\quad \forall v_\delta \in V_\delta, \forall \mu \in P$

↳ being it the orthogonal projection, it's the element of V closest to u $\Rightarrow$ all the other $v_\delta \in V_\delta$ are at a greater distance

3) **How to find $u_\delta(\mu)$ $\forall \mu \in P$?** Consider $V_\delta \subset V$ finite, $N_\delta = \dim(V_\delta)$. Let's take $\{\varphi_k\}_{k=1}^{N_\delta}$ basis of V_δ,

$\varphi_k: \delta \to \mathbb{R}$ $\Rightarrow \forall v_\delta \in V_\delta$ $\quad v_\delta(\mu) = \sum_{n=1}^{N_\delta} \varphi_k \cdot v_{\delta n}(\mu)$ $\quad \forall \mu \in P$ — ↳ basis functions

Putting this expression in $VP_\delta(\mu)$, considering $v_\delta = \varphi_j$ $\quad \in \mathbb{R}$

we want to find $u_\delta \in V_\delta$ st $a(u_\delta, v_\delta; \mu) = F(v_\delta, \mu)$ $\quad \forall v_\delta \in V_\delta$

$\Rightarrow a\left(\sum_{k=1}^{N_\delta} u_{\delta k} \varphi_k, v_\delta; \mu\right) = F(v_\delta; \mu)$ $\quad \Rightarrow \sum_{k=1}^{N_\delta} a(u_{\delta,k} \varphi_k, v_\delta; \mu) = F(v_\delta; \mu)$

$\Rightarrow \sum_{k=1}^{N_\delta} u_{\delta,k} \cdot a(\varphi_k, v_\delta; \mu) = F(v_\delta; \mu)$ $\quad \forall v_\delta \in V_\delta$ where N_δ and $u_{\delta k}$ unknowns

What v_δ we take? We can take $v_\delta = \varphi_j$ for every $j \in \{1, .., N_\delta\}$ — one for every basis function φ_j

$\Rightarrow$ it holds $\sum_{k=1}^{N_\delta} u_{\delta k}\, a(\varphi_k, \varphi_j; \mu) = \boxed{F(\varphi_j; \mu)}$ $\longrightarrow$ N_δ equations and N_δ unknowns: I can solve it for $u_{\delta,k}$

In matrix form $\quad u_\delta^u = \begin{bmatrix} u_{\delta,1} \\ \\ u_{\delta,N_\delta} \end{bmatrix}$ $\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} f_1 \\ \\ f_{N_\delta} \end{bmatrix} = \begin{bmatrix} F(\varphi_1; \mu) \\ \\ F(\varphi_{N_\delta}; \mu) \end{bmatrix}$ $\Rightarrow$ $\boxed{\begin{bmatrix} A^\mu \end{bmatrix} \begin{bmatrix} u_\delta^\mu \end{bmatrix} = \begin{bmatrix} f^\mu \end{bmatrix}}$ **full problem** / **High fidelity problem** $A^\mu u_\delta^\mu = f^\mu$

A^μ st $A_{jk} = a(\varphi_k, \varphi_j; \mu)$ $\qquad \in \mathbb{R}^{N_\delta, N_\delta}$ $\quad \in \mathbb{R}^{N_\delta}$ $\quad \in \mathbb{R}^{N_\delta}$

So that we obtain a discrete map and a discrete manifold

$S_\delta: P \to V_\delta$

$\mu \mapsto u_\delta(\mu)$

$\mathcal{M}_\delta = \{u_\delta(\mu) \in V_\delta : \mu \in P\} \subset V_\delta$