1- INTRODUCTION TO PDE

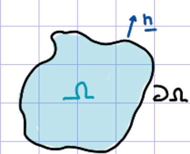
Notation. Consider $x \in \mathbb{R}^d$, $\underline{x} := (x_1, \dots, x_d)$, $A \in \mathbb{R}^{m,n}$, V an Hilbert space and $u, v \in V$. The scalar product is $\langle u, v \rangle_V$ and the norm $\|u\|_V^2 = \langle u, u \rangle_V$. The dual space of V is $V' := \{F : V \to \mathbb{R}$ linear and continuous$\}$ and $\langle F, u \rangle_{V', V} = F(u) \in \mathbb{R}$.

Example. Consider
1. $V = L^2(\Omega)$, with $\Omega \subset \mathbb{R}^d \Rightarrow \langle u, v \rangle_V = \int_\Omega u \cdot v$ and $\|u\|_V^2 = \int_\Omega u^2$ $= \langle u, u \rangle_V$
2. $V = H^1(\Omega) \Rightarrow \langle u, v \rangle_V = \int_\Omega u \cdot v + \int_\Omega \nabla u \cdot \nabla v$ and $\|u\|_V^2 = \int_\Omega u^2 + \int_\Omega \nabla u \cdot \nabla u$

> The idea is to reduce the order of partial differential equations so that finding an approximate solution (whose distance from the real one is theoretically controlled) is much easier for the reduced problem

Example. Second order partial differential equations in $\mathbb{R}$ - PDE -

$x \in \mathbb{R}^d$
$$\begin{cases} -\Delta u(\underline{x}) = f(\underline{x}) & \text{on } \Omega \subset \mathbb{R}^d \\ u(\underline{x}) = 0 & \text{on } \partial\Omega \subset \end{cases}$$
$\Rightarrow$ required $u \in \mathcal{C}^2$, twice differentiable, where $\Delta u(x) = \frac{\partial^2 u}{\partial x^2}$



The solution is a function $u : \mathbb{R}^d \to \mathbb{R}$ and $f : \mathbb{R}^d \to \mathbb{R}$

We reformulate $-\Delta u = f$ as: search $u \in V$ such that

$-\int_\Omega \Delta u \cdot v = \int_\Omega f \cdot v \quad \forall v \in V$ — **weak formulation** or variational form

$\Rightarrow$ We don't want to solve it directly (needed too many hypothesis). So we change its formulation so that the solutions requires less hypothesis

$\downarrow$ by parts ↳ since less hy are need to find u

$$\boxed{\int_\Omega \nabla u \cdot \nabla v - \int v \nabla u \cdot \underline{n} = \int_\Omega f \cdot v}$$

$\Rightarrow V = \{ \forall v, v \in L^2(\Omega), \nabla v \in L^2(\Omega) \} = H^1(\Omega)$ and $f \in L^2(\Omega)$ Hilbert space

Then we want to find $u \in H^1(\Omega)$ s.t. $\forall v \in H^1(\Omega)$, $f \in L^2(\Omega)$, so in general we introduce a ↳ hypotesis to integrate f

$\Rightarrow a(u, v) = \int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} v \nabla u \cdot \underline{n} \quad a : V \times V \to \mathbb{R}$ $\Rightarrow$ We need u only once differentiable

Hence, the problem becomes: finding $u \in V$ s.t. $\boxed{a(u, v) = F(v) \quad \forall v \in V}$ that is the **variational problem VP**

↳ $F(v) : V \to \mathbb{R}$

Theorem - Lax Milgram If a is bilinear and continuous ($\exists \gamma \in \mathbb{R} < \infty$ s.t. $\forall u, v \in V$ $|a(u, v)| \le \gamma \|u\|_V \cdot \|v\|_V$ with $\gamma = \sup_{\substack{u, v \in V \\ \neq 0}} \frac{|a(u, v)|}{\|u\|_V \|v\|_V}$) and a is cohercive ($\exists \alpha \in \mathbb{R} > 0$ s.t. $\forall v \in V$ $a(v, v) \ge \alpha \|v\|_V^2$, with $\alpha = \inf_{v \in V \neq 0} \frac{a(v, v)}{\|v\|_V^2} > 0$) and F is linear and continuous ($\exists \rho \in \mathbb{R} < \infty$ s.t. $\forall v \in V$ $|F(v)| \le \rho \|v\|_V$, with $\rho = \sup_{v \in V \neq 0} \frac{|F(v)|}{\|v\|_V}$, $F \in V'$ $\Rightarrow$ (VP) has a unique solution and $\boxed{\|u\|_V \le \frac{1}{\alpha} \|F\|_{V'}}$

↳ $\rho = \|F\|_{V'}$

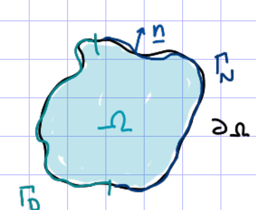
- VP is well-posed if $\forall F \in V'$ $\exists! u$ and u is controlled by F and α → controlled by data

Proof. Assume u exists and is unique, we want to prove $\|u\|_V \le 1/\alpha \|F\|_{V'}$

- We know $a(u, v) = F(v) \ \forall v \in V$. Take $v = u$: $a(u, u) = F(u) \Rightarrow \alpha \|u\|_V^2 \le a(u, u) = F(u) \le \|F\|_{V'} \cdot \|u\|_V \Rightarrow \|u\|_V \le \frac{1}{\alpha} \|F\|_V$
- To prove uniqueness, assume $u_1, u_2 \in V$ solutions of VP: $a(u_1, v) = F(v)$ and $a(u_2, v) = F(v) \ \forall v \in V$
  $\Rightarrow a(u_1 - u_2, v) = 0$ with $w = u_1 - u_2$ $\forall v \in V$ is VP, with solution w
  $\Rightarrow \|w\|_V \le \frac{1}{\alpha} \|\tilde{F}\|_{V'} = 0$, $\|u_1 - u_2\|_{V'} = 0 \Rightarrow u_1 = u_2$ □

Observation. Tacking $F_1 \neq F_2$ and u_1 such that $a(u_1, v) = F_1(v)$ and u_2 s.t. $a(u_2, v) = F_2(v) \ \forall v \in V$
$\Rightarrow a(u_1 - u_2, v) = F_1(v) - F_2(v)$ is VP_2 and it has as solution $w = u_1 - u_2$
Moreover, we know $\|u_1 - u_2\|_V = \|w\|_V \le \frac{1}{\alpha} \|F_1 - F_2\|_{V'}$, in which $1/\alpha$ is proportional to the condition number of VP fixed $a(\cdot, \cdot)$
↳ related to cohercivity of $a(\cdot,\cdot)$, the smaller α, the harder is to solve (ill-conditioned)
$\downarrow$ a small perturbation on F is reflected on a small perturbation on u if α is NOT too small

Example. Consider the system
$$\begin{cases} -\nabla(K(x) \nabla u(x)) = f(x) & \text{on } \Omega \subset \mathbb{R}^d \\ u(x) = 0 & \text{on } \Gamma_D \\ K(x) \nabla u(x) \cdot \underline{n} = \Psi & \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D \end{cases}$$
→ before we have $-\nabla(\nabla u)$, now we add $K(x)$
} boundary conditions
↳ diffusion parameter



↳ i. $u(x) = 0$ on Γ_D is the **Dirichlet boundary Condition**
ii. $K(x) \nabla u(x) \cdot \underline{n} = \Psi$ on Γ_N is the **Neumann boundary Condition**

Let's compute the weak formulation: $\forall v \in V$ find $u \in V$ such that $-\int_\Omega \nabla(K(x)\nabla u(x))\, v = \int_\Omega f v$, $v \in V$

$\Rightarrow$ we must define V such that it satisfies one boundary condition: we assume $V = \{ v_{|\Gamma_D} = 0 \}$

by parts $\displaystyle \int_\Omega K(x)\nabla u(x)\nabla v(x) - \underbrace{\int_{\Gamma_D} v K(x)\nabla u\cdot n}_{=0 \text{ since } v_{|\Gamma_D}=0} - \underbrace{\int_{\Gamma_N} v K(x)\nabla u \cdot n}_{=\psi} = \int_\Omega f v \overset{\text{if } v=0 \text{ on } \Gamma_D}{\Rightarrow} \underbrace{\int_\Omega K(x)\nabla u \nabla v}_{a(u,v)} = \underbrace{\int_\Omega f v + \int_{\Gamma_N} v \psi}_{F(v)}$ VP

$\|v\|_V = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{[L^2(\Omega)]^d}$

$\|v\|_V = \|\nabla v\|_{[L^2(\Omega)]^d}$

NB. Dirichlet boundary condition is imposed to the whole space (strong boundary condition)

Neumann boundary condition is imposed in the VP equations (weak boundary condition)

$\hookrightarrow$ $V = \{ v \in L^2(\Omega), \nabla v \in L^2(\Omega) \} \cup \{ v = 0 \text{ on } \Gamma_D \} = \boxed{H^1(\Omega) \cup \{v = 0 \text{ on } \Gamma_D\}} = H^1_0(\Omega)$ and $f \in L^2(\Omega)$, $\psi \in L^2(\Omega)$

To have $\int_\Omega u \nabla v < \infty$, $\int v \nabla u < \infty$, $\int f v < \infty$ we need V defined

sol is 0 on the Dirichlet boundary

$\hookrightarrow$ $K \in L^\infty(\Omega)$ and $\exists K_*, K^*$: $0 < \underset{\alpha \text{ of } a(u,v)}{K_*} \le K(x) \le \underset{\gamma \text{ of } a(u,v)}{K^*} < +\infty$ $\forall x \in \Omega$

Poincaré Inequality. If $v \in H^1_0(\Omega) \Rightarrow \exists\, C_1 \in \mathbb{R} < +\infty$ s.t. $\boxed{\|v\|_{L^2(\Omega)} \le C_1 \|\nabla v\|_{L^2(\Omega)^d}}$, where $L^2(\Omega)^d = \overbrace{L^2(\Omega) \times \cdots \times L^2(\Omega)}^{d \text{ times}}$, then if $v \in H^1_0(v) \Rightarrow \|v\|_{H^1_0(\Omega)} = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)^d} \le \tilde{C} \|\nabla v\|_{L^2(\Omega)^d}$ and $\|\nabla v\|_{L^2(\Omega)^d} \le \|v\|_{H^1_0(\Omega)} = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)^d}$

$\Rightarrow$ $\|\cdot\|_{L^2(\Omega)^d}$ and $\|\cdot\|_{H^0_1(\Omega)}$ are equivalent and $\|v\|_{H^1(\Omega)} \simeq \|\nabla v\|_{L^2(\Omega)^d}$ if $v \in H^1_0(\Omega)$

If $v \in H^1_0(\Omega) \Rightarrow \exists\, C_2 \in \mathbb{R} < +\infty$ s.t. $\boxed{\|v\|_{L^2(\Gamma_N)} \le C_2 \|v\|_{H^1_0(\Omega)}}$. We can define the operator $\boxed{\gamma : V \to W \equiv H^{1/2}(\Gamma_N), \quad v \mapsto \gamma(v) = v_{|\Gamma_N}}$ the **continuous trace operator**, then:

$V = H^1_0(\Omega)$, $f \in L^2(\Omega)$, $\psi \in L^2(\Omega)$ and $K \in L^\infty(\Omega)$ $(\exists K_*, K^*$ s.t. $0 < K_* \le K(x) \le K^* < \infty)$