② PARAMETRIC VARIATIONAL PROBLEMS

Consider $P \subset \mathbb{R}^p$, $\underline{\mu} \in P$ defined as vector of parameters $\underline{\mu} = [\mu_1, \dots, \mu_p]$. Now a $VP(\mu)$ looks for $u(x;\underline{\mu}) \in V$ s.t. $a(u, v; \underline{\mu}) = F(v, \underline{\mu})$ $\forall \underline{v} \in V$, $\forall \underline{\mu} \in P$ where $a: V \times V \times P \to \mathbb{R}$, $F: V \times P \to \mathbb{R}$

↳ $u(\mu)$

Theorem - Lax Milgram for $VP(\mu)$ If

I. $a(\cdot,\cdot;\mu)$ is uniformly continuous: $\exists \gamma \in \mathbb{R} < +\infty$ s.t. $\forall \underline{\mu} \in \mathbb{R}$ $|a(u,v;\mu)| \leq \gamma(\underline{\mu}) \cdot \|u\|_V \cdot \|v\|_V \leq \gamma \|u\|_V \cdot \|v\|_V < +\infty$
where $\gamma = \sup_{\underline{u}, \underline{v} \in V} \frac{|a(u,v,\mu)|}{\|u\|_V \cdot \|v\|_V} \leq \gamma < \infty$

II. $a(\cdot,\cdot;\mu)$ is uniformly coercive: $\exists \alpha \in \mathbb{R} < +\infty$ s.t. $\forall \underline{\mu} \in P$ $a(v,v;\mu) \geq \alpha(\underline{\mu}) \|v\|_V^2 \geq \alpha \|v\|_V^2$, $\alpha(\mu) = \inf_{v \in V} \frac{a(v,v;\mu)}{\|v\|_V} \geq \alpha > 0$

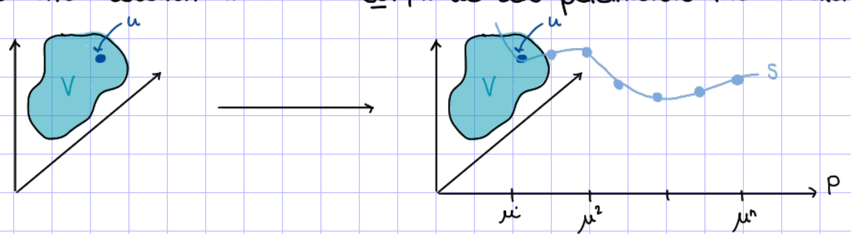
III. F is uniformly continuous: $\exists \rho \in \mathbb{R} < +\infty$ s.t. $|F(v,\mu)| \leq \rho(\mu) \|v\|_V \leq \rho \|v\|_V$, $\rho(\mu) = \sup_{v \in V} \frac{|F(v,\mu)|}{\|v\|_V} = \|F(\cdot,\mu)\|_{V'} \leq \rho < +\infty$

$\Rightarrow$ $VP(\mu)$ has a unique solution $\forall \underline{\mu} \in P$

we can find a solution for every value of μ

Example. We had one solution in V $\longrightarrow$ But, if we add parameters the solution Manifold is $\mathcal{M} = \{u(\mu) \in V : \mu \in P\} \subset V$

So we define a curve of solutions



$\boxed{S:}$ $P \to V$ — solution mapping of $VP(\mu)$
$\mu \mapsto u(\mu)$
In general non linear

Proposition. If $a(\cdot,\cdot;\mu)$ and $F(\cdot,\mu)$ depend smoothly on $\underline{\mu} \in P$ $\Rightarrow$ $S: P \to V$ is smooth

↳ Proof on the notes if you're interested in

Example. Consider $VP(\mu)$, $\int_{\Omega_p} K(\mu) \nabla u \nabla v = \int_\Omega f v + \int_{\Gamma_N} \psi v$ $\forall v \in V$ and $K(x;\mu) = K_0 + \sum_{i=1}^{p} K_i(x) \mu_i$,
$\underline{\mu} = (\mu_1, \dots, \mu_p) \in [0,1]^p = P$

with $K_0, K_i \in L^\infty(\Omega)$ s.t.
- $K_0(x) + \sum_{i=1}^{p} |K_i(x)| \leq K_* < +\infty$ $\forall x \in \Omega$
- $K_0(x) - \sum_{i=1}^{p} |K_i(x)| \geq K_* > 0$ $\forall x \in \Omega$

$\Big\}$ for the uniformity property of a