Nonlinear Elliptic Problem

Model Order Reduction and Machine Learning
Master's Degree in Mathematical Engineering

Lucia Ghezzi, Elisabetta Roviera

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Problem definition Given $\Omega = (0,1)^2$, given $\mu = (\mu_0, \mu_1) \in \mathcal{P} = [0.1, 1]^2$, find $u(\mu)$ such that

$$-\Delta u(\mu) + \frac{\mu_0}{\mu_1}(e^{\mu_1 u(\mu)} - 1) = g(x; \mu)$$

with homogeneous Dirichlet condition on the boundary. The source term g is defined as:

1. For NEP1:

$$g(x;\mu) = g_1 = 100 \sin(2\pi x_0) \cos(2\pi x_1), \quad \forall x = (x_0, x_1) \in \Omega.$$

2. For NEP2:

$$g(x;\mu) = g_2 = 100 \sin(2\pi \mu_0 x_0) \cos(2\pi \mu_0 x_1), \quad \forall x = (x_0, x_1) \in \Omega.$$

Weak formulation and Newton scheme Integrating on the domain, multiplying by a general function $v \in V$ and recalling the boundary condition, we get the weak formulation: given $\mu \in \mathcal{P}$, find $u(\mu) \in V$ such that for every $v \in V$

$$F(u)[v] = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega \frac{\mu_0}{\mu_1}(e^{\mu_1 u} - 1) v \, dx - \int_\Omega g v \, dx = 0.$$

To solve $F(u)[v] = 0$ at each Newton iteration, we solve for δu

$$\left(\int_\Omega \nabla \delta u \cdot \nabla v \, dx + \int_\Omega \mu_0 e^{\mu_1 u_k} \delta u v \, dx \right) \delta u = -\left(\int_\Omega \nabla u_k \cdot \nabla v \, dx + \int_\Omega \frac{\mu_0}{\mu_1}(e^{\mu_1 u_k} - 1) v \, dx - \int_\Omega g v \, dx \right)$$

and update $u_{k+1} = u_k + \delta u$.

Preliminary Domain Analysis

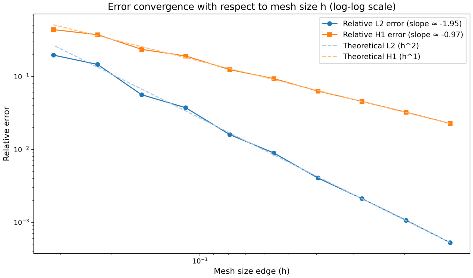
Check of theoretical results

We know from theory that for mesh size h:

$$Err_{L^2}(h) = Err_{L^2}(h_0)\left(\frac{h}{h_0}\right)^{s+1},$$

$$Err_{H^1}(h) = Err_{H^1}(h_0)\left(\frac{h}{h_0}\right)^{s}$$

We check if this expected behavior is observed experimentally.



Experimental error decay aligns with theoretical predictions

Choice of the mesh size

The most suitable mesh sizes are 0.00312 and 0.00019. We evaluate the trade-off between accuracy and cost:

Performance metrics for different mesh sizes

Metric	Mesh = 0.00312	Mesh = 0.00019
Avg. snapshot time (s)	0.5948	11.2261
Rel. error (L^2 Norm)	0.0089	0.0005
Rel. error (H^1 Norm)	0.0937	0.0224

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Methods

1. **POD**: the reduced dimension for NEP1 is $N = 3$ and $N = 9$ for NEP2.
2. **PINN**: trained in an unsupervised manner by minimizing the PDE residual, using Adam followed by L-BFGS for fine-tuning, and enforcing Dirichlet conditions exactly through a multiplicative ansatz.

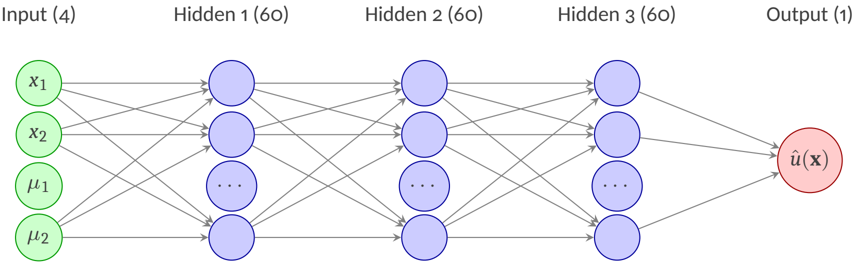

Figure: PINNHardBC Architecture Diagram.

3. **POD-NN**: fully connected network with 4 hidden layers of 40 neurons, $\tanh$ activation, Adam optimizer (`lr`=0.001), up to 500,000 epochs, early stopping at 10^{-6}.

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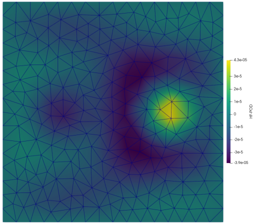
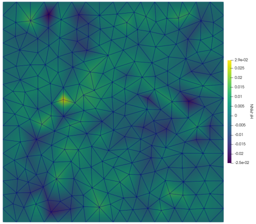
Comparison of Methods – NEP1

Performance comparison: Accuracy vs computational cost for NEP1

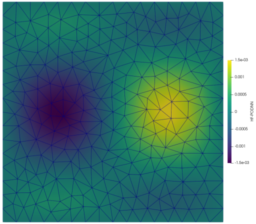
NEP1 Summary		POD (N=3)	PINN	PODNN
Error w.r.t. HF	L2 relative	2.77×10^{-5}	4.19×10^{-2}	6.05×10^{-4}
	H1 relative	3.07×10^{-5}	2.18×10^{-1}	6.04×10^{-4}
Execution Time	Avg. eval. time (s)	8.04×10^{-4}	1.10×10^{-3}	2.18×10^{-4}
	Avg. speed-up vs HF	15.66	8.42	68.62
Training	Iterations	–	10,689	119,274
	Training time (s)	–	718.11	133.36



(a) Differences HF and POD solution



(b) Differences HF and PINN solution (b) Differences HF and PINN solution



(c) Differences HF and POD-NN solution (c) Differences HF and POD-NN solution

Figure: Differences between High Fidelity Solution and (a) POD, (b) PINN, (c) POD-NN for NEP1

Animated plot

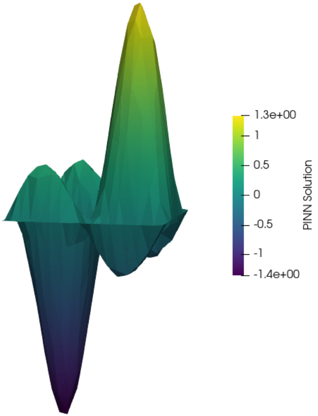


Figure: Comparison of High Fidelity and PINN solutions

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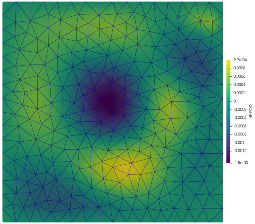
Comparison of Methods – NEP2

Performance comparison: Accuracy vs computational cost for NEP2

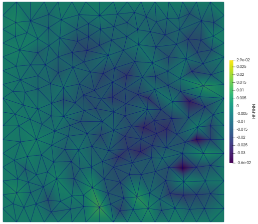
NEP2 Summary		**POD (N=9)**	**PINN**	**PODNN**
Error w.r.t. HF	L2 relative	5.1432×10^{-4}	2.4579×10^{-2}	2.8012×10^{-2}
	H1 relative	9.8641×10^{-4}	1.5484×10^{-1}	2.5705×10^{-2}
Execution Time	Avg. eval. time (s)	5.5382×10^{-4}	1.1493×10^{-3}	1.8587×10^{-4}
	Avg. speed-up vs HF	22.327	10.0273	70.8458
Training	Iterations	–	17,415	500,000
	Training time (s)	–	1068.91	570.41

Plots

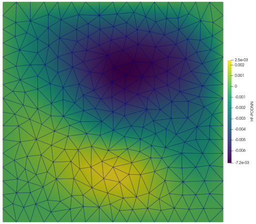
4 Comparison of Methods for NEP2



(a) Differences HF and POD solution



(b) Differences HF and PINN solution



(c) Differences HF and POD-NN solution

Figure: Differences between High Fidelity Solution and (a) POD, (b) PINN, (c) POD-NN for NEP2

Animated plot

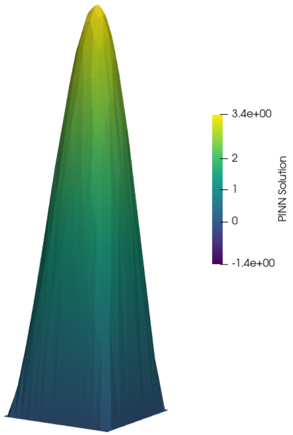


Figure: Comparison of High Fidelity and PINN solutions

Thank you for your attention!