Nonlinear Elliptic Problem

Model Order Reduction and Machine Learning
Master's Degree in Mathematical Engineering

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Problem definition Given $\Omega = (0,1)^2$, given $\mu = (\mu_0, \mu_1) \in \mathcal{P} = [0.1, 1]^2$, find $u(\mu)$ such that

$$-\Delta u(\mu) + \frac{\mu_0}{\mu_1}(e^{\mu_1 u(\mu)} - 1) = g(x;\mu)$$

with homogeneous Dirichlet condition on the boundary. The source term g is defined as:

1. For NEP1:
$$g(x;\mu) = g_1 = 100\sin(2\pi x_0)\cos(2\pi x_1), \quad \forall x = (x_0, x_1) \in \Omega.$$
2. For NEP2:
$$g(x;\mu) = g_2 = 100\sin(2\pi \mu_0 x_0)\cos(2\pi \mu_0 x_1), \quad \forall x = (x_0, x_1) \in \Omega.$$

Weak formulation and Newton scheme Integrating on the domain, multiplying by a general function $v \in V$ and recalling the boundary condition, we get the weak formulation: given $\mu \in \mathcal{P}$, find $u(\mu) \in V$ such that for every $v \in V$

$$F(u)[v] = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega \frac{\mu_0}{\mu_1}(e^{\mu_1 u} - 1) v \, dx - \int_\Omega g v \, dx = 0.$$

To solve $F(u)[v] = 0$ at each Newton iteration, we solve for δu

$$\left(\int_\Omega \nabla \delta u \cdot \nabla v \, dx + \int_\Omega \mu_0 e^{\mu_1 u_k} \delta u v \, dx\right) \delta u = -\left(\int_\Omega \nabla u_k \cdot \nabla v \, dx + \int_\Omega \frac{\mu_0}{\mu_1}(e^{\mu_1 u_k} - 1) v \, dx - \int_\Omega g v \, dx\right)$$

and update $u_{k+1} = u_k + \delta u$.

Check of theoretical results

We know from theory that for mesh size h:

$$Err_{L^2}(h) = Err_{L^2}(h_0)\left(\frac{h}{h_0}\right)^{s+1},$$
$$Err_{H^1}(h) = Err_{H^1}(h_0)\left(\frac{h}{h_0}\right)^{s}$$

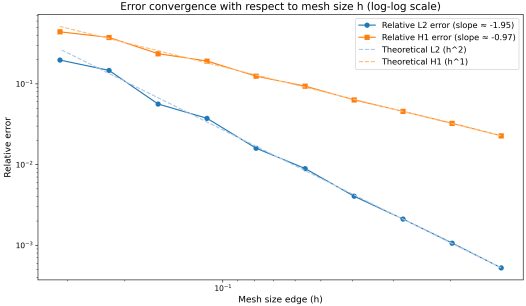
We check if this expected behavior is observed experimentally.

Choice of the mesh size

The most suitable mesh sizes are 0.00312 and 0.00019. We evaluate the trade-off between accuracy and cost:

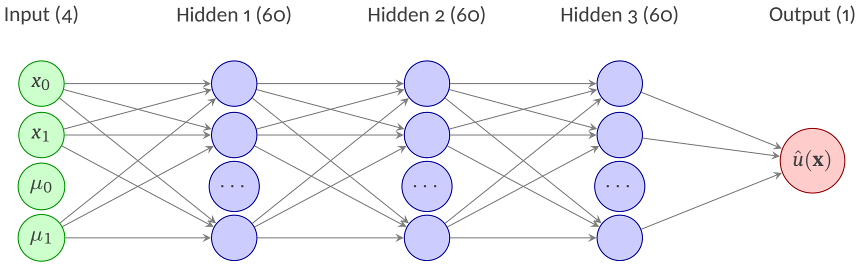
Performance metrics for different mesh sizes

Metric	Mesh = 0.00312	Mesh = 0.00019
Avg. snapshot time (s)	0.5948	11.2261
Rel. error (L^2 Norm)	0.0089	0.0005
Rel. error (H^1 Norm)	0.0937	0.0224



Experimental error decay aligns with theoretical predictions

1. **POD**: the reduced dimension for NEP1 is $N = 3$ and $N = 9$ for NEP2.
2. **PINN**: trained in an unsupervised manner by minimizing the PDE residual, using *Adam* optimizer (`lr`=0.0001) followed by *L-BFGS* (`lr`=1) for fine-tuning, and enforcing Dirichlet conditions exactly through a multiplicative ansatz.



PINNHardBC Architecture Diagram

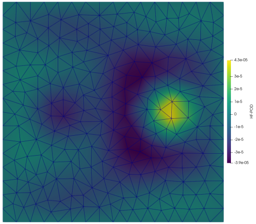
3. **POD-NN**: fully connected network with 4 hidden layers of 40 neurons, $\tanh$ activation, *Adam* optimizer (`lr`=0.001), up to 500,000 epochs, early stopping at 10^{-6}.

Table of Contents

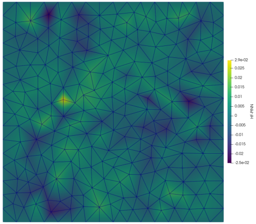
Performance comparison: Accuracy vs computational cost for NEP1

NEP1 Summary		POD (N=3)	PINN	PODNN
Error w.r.t. HF	L2 relative	2.77×10^{-5}	4.19×10^{-2}	6.05×10^{-4}
	H1 relative	3.07×10^{-5}	2.18×10^{-1}	6.04×10^{-4}
Execution Time	Avg. eval. time (s)	8.04×10^{-4}	1.10×10^{-3}	2.18×10^{-4}
	Avg. speed-up vs HF	15.66	8.42	68.62
Training	Iterations	-	10,689	119,274
	Training time (min)	-	~ 12	~ 2

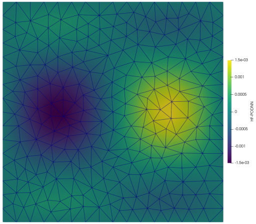
Plots



(a) Differences HF and POD solution



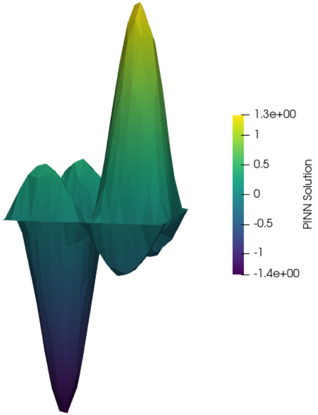
(b) Differences HF and PINN solution



(c) Differences HF and POD-NN solution

Differences between High Fidelity Solution and (a) POD, (b) PINN, (c) POD-NN for NEP1

Animated plot



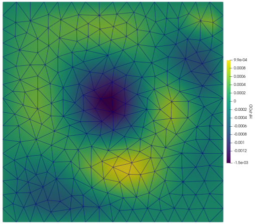
Comparison of High Fidelity and PINN solutions

Performance comparison: Accuracy vs computational cost for NEP2

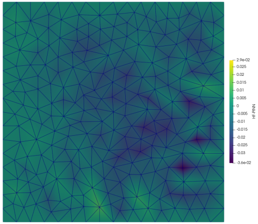
NEP2 Summary		POD (N=9)	PINN	PODNN
Error w.r.t. HF	L2 relative	5.1432×10^{-4}	2.4579×10^{-2}	2.8012×10^{-2}
	H1 relative	9.8641×10^{-4}	1.5484×10^{-1}	2.5705×10^{-2}
Execution Time	Avg. eval. time (s)	5.5382×10^{-4}	1.1493×10^{-3}	1.8587×10^{-4}
	Avg. speed-up vs HF	22.327	10.0273	70.8458
Training	Iterations	-	$17{,}415$	$500{,}000$
	Training time (min)	-	~ 18	~ 10

Plots

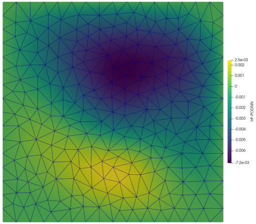
4 Comparison of Methods for NEP2



(a) Differences HF and POD solution

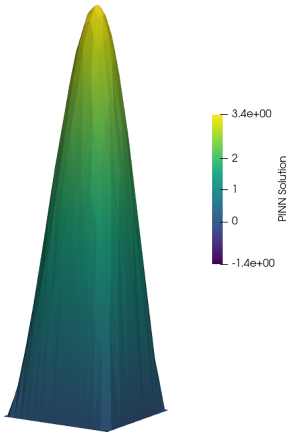


(b) Differences HF and PINN solution



(c) Differences HF and POD-NN solution

Differences between High Fidelity Solution and (a) POD, (b) PINN, (c) POD-NN for NEP2



Comparison of High Fidelity and PINN solutions

Thank you for your attention!